Local h-polynomials, uniform triangulations and real-rootedness

BIU Combinatorics Seminar

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Introduction

A polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x]$$

with nonnegative coefficients is said to be

- symmetric (or palindromic), with center of symmetry $n/2$, if $a_i = a_{n-i}$ for $0 \leq i \leq n$
- unimodal if

  $$a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$$

  for some $0 \leq k \leq n$

- **γ-positive**, with center of symmetry $n/2$, if

$$f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$$

for some $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor} \geq 0$

- **real-rooted** if $f(x) \equiv 0$, or all complex roots of $f(x)$ are real.

Note

- real-rootedness $\Rightarrow$ unimodality

- $\left\{\begin{array}{l}\text{symmetry and} \\ \text{real-rootedness}\end{array}\right.$ $\Rightarrow$

  $\Rightarrow$ γ-positivity

  $\Rightarrow$ $\left\{\begin{array}{l}\text{symmetry and} \\ \text{unimodality.}\end{array}\right.$

Example

We let

$[n] = \{1, 2, \ldots, n\}$

$\mathfrak{S}_n$ = group of permutations of $[n]$

$\mathfrak{D}_n$ = set of permutations $w \in \mathfrak{S}_n$ without fixed points

and for $w \in \mathfrak{S}_n$

$$\mathrm{des}(w) = \#\{i \in [n-1] : w(i) > w(i+1)\}$$
$$\mathrm{exc}(w) = \#\{i \in [n-1] : w(i) > i\}$$

be the number of descents and excedances of w, respectively.

(α) The n^{th} Eulerian polynomial

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} = \sum_{w \in \mathfrak{S}_n} x^{\mathrm{exc}(w)}$$

is

- symmetric, with center of symmetry $(n-1)/2$
- γ-positive (Foata-Schützenberger 1970)
- real-rooted (Frobenius 1910).

$$A_n(x) = \begin{cases} 1, & n=1 \\ 1+x, & n=2 \\ 1+4x+x^2, & n=3 \\ 1+11x+11x^2+x^3, & n=4 \\ 1+26x+66x^2+26x^3+x^4, & n=5 \end{cases}$$

$$= \begin{cases} 1, & n=1 \\ 1+x, & n=2 \\ (1+x)^2+2x, & n=3 \\ (1+x)^3+8x(1+x), & n=4 \\ (1+x)^4+22x(1+x)^2+16x^2, & n=5. \end{cases}$$

(b) The n^{th} derangement polynomial

$$d_n(x) = \sum_{w \in D_n} x^{exc(w)}$$

is

- symmetric, with center of symmetry $n/2$, and unimodal (Brenti 1990, Stembridge 1992)
- γ-positive (several authors)
- real-rooted (Zhang 1995, Haglund-Zhang 2019, Brändén-Solus 2021).

$$d_n(x) = \begin{cases} 0, & n=1 \\ x, & n=2 \\ x+x^2, & n=3 \\ x+7x^2+x^3, & n=4 \\ x+21x^2+21x^3+x^4, & n=5 \end{cases}$$

$$= \begin{cases} 0, & n=1 \\ x, & n=2 \\ x(1+x) & n=3 \\ x(1+x)^2+5x^2, & n=4 \\ x(1+x)^3+18x(1+x), & n=5. \end{cases}$$

(c) The n^{th} binomial Eulerian polynomial

$$\tilde{A}_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} A_k(x)$$

is

- symmetric, with center of symmetry $n/2$ (several authors)
- γ-positive (Postnikov–Reiner–Williams 2008, Shareshian–Wachs 2020, Brändén–Jochemko 2022)
- real-rooted (Haglund–Zhang 2019, Brändén–Jochemko 2022).

$$\tilde{A}_n(x) = \begin{cases} 1+x, & n=1 \\ 1+3x+x^2, & n=2 \\ 1+7x+7x^2+x^3, & n=3 \\ 1+15x+33x^2+15x^3+x^4, & \\ 1+31x+131x^2+131x^3+31x^4+x^5, & \end{cases}$$

$$= \begin{cases} 1+x, & n=1 \\ (1+x)^2+x, & n=2 \\ (1+x)^3+4x(1+x), & n=3 \\ (1+x)^4+11x(1+x)^2+5x^2, & n=4 \\ (1+x)^5+26x(1+x)^3+43x^2(1+x), & n=5. \end{cases}$$

Question. Can algebraic-geometric combinatorics shed light into these properties?

Face enumeration of simplicial complexes

Let

Δ = simplicial complex of dimension $n-1$

$f_k(\Delta)$ = # k-dimensional faces of Δ.

Definition. The f,h-polynomials of Δ are defined as

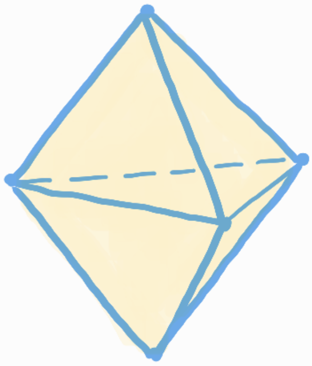
$$f(\Delta,x) = \sum_{k=0}^{n} f_{k-1}(\Delta)\, x^k$$

$$h(\Delta,x) = \sum_{k=0}^{n} f_{k-1}(\Delta)\, x^k (1-x)^{n-k}$$

$$= (1-x)^n f\left(\Delta, \frac{x}{1-x}\right).$$

Note. $h(\Delta,1) = f_{n-1}(\Delta)$.

Example. (a)



$n = 3$

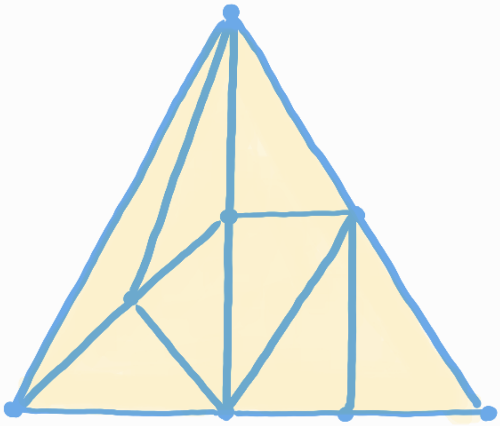
Δ

$f_0(\Delta) = 5, \ f_1(\Delta) = 9, \ f_2(\Delta) = 6$

- $f(\Delta, x) = 1 + 5x + 9x^2 + 6x^3$
- $h(\Delta, x) = (1-x)^3 + 5x(1-x)^2 + 9x^2(1-x) + 6x^3$

$$= 1 + 2x + 2x^2 + x^3.$$

(b)



$n = 3$

Δ

$f_0(\Delta) = 8, \quad f_1(\Delta) = 15, \quad f_2(\Delta) = 8$

- $f(\Delta, x) = 1 + 8x + 15x^2 + 8x^3$
- $h(\Delta, x) = (1-x)^3 + 8x(1-x)^2 + 15x^2(1-x) + 8x^3$

$= 1 + 5x + 2x^2.$

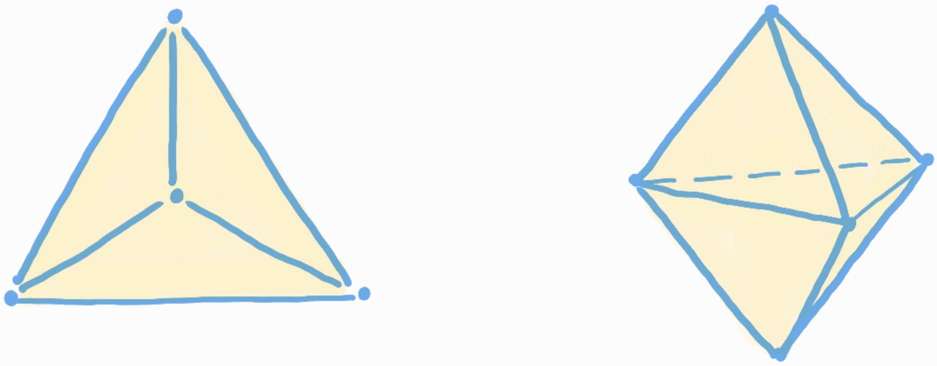
Theorem. The polynomial $h(\Delta, x)$

(a) has nonnegative coefficients if Δ triangulates a ball or a sphere (Reisner, Stanley 1976)

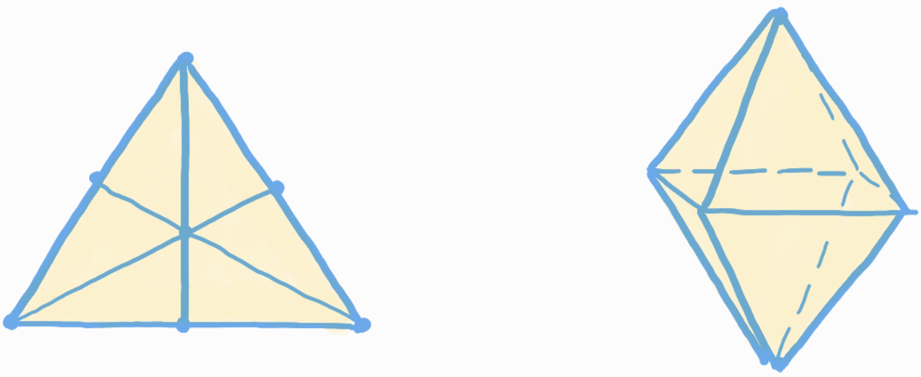
(b) is symmetric if Δ triangulates a sphere (Klee 1964)

(c) is unimodal if Δ is the boundary complex of a simplicial polytope (Stanley 1980).

Note. Recall that a simplicial complex Δ is **flag** if every clique in its one-skeleton is a face of Δ.



not flag



flag

Conjecture (Gal 2005). The polynomial $h(\Delta,x)$ is γ-positive for every flag triangulation Δ of the sphere.

Theorem (Stanley 1994, Karu 2006, Gal 2005) The polynomial $h(\Delta,x)$ γ-positive if Δ is the barycentric subdivision of a CW-regular subdivision of the sphere.

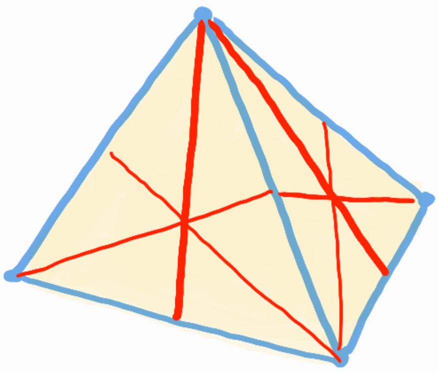
Example. We let

V = n-element set

2^V = abstract simplex on the vertex set V

$\partial(2^V)$ = boundary complex of 2^V

Δ = first barycentric subdivision of $\partial(2^V)$.



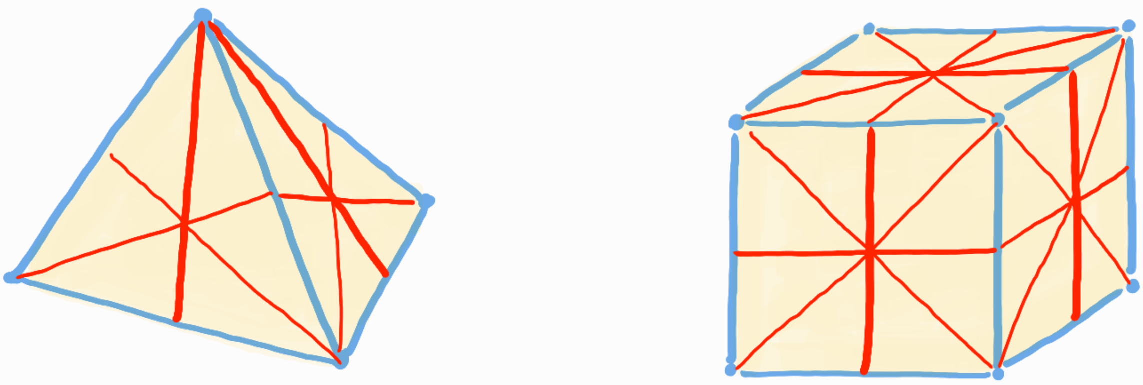
Then, $h(\Delta, x) = A_n(x)$.

Proposition (Gal 2005). There exists a flag triangulation Δ of the 5-dimensional sphere for which $h(\Delta, x)$ is not real-rooted.

Question. For which (flag) triangulations Δ of the sphere is $h(\Delta, x)$ real-rooted?

Conjecture (Brenti-Welker 2008)

The polynomial $h(\Delta, x)$ is real-rooted if Δ is the barycentric subdivision of the boundary complex of a polytope.



Theorem (Brenti-Welker 2008).

The conjecture holds for simplicial polytopes.

We let

V = n-element set

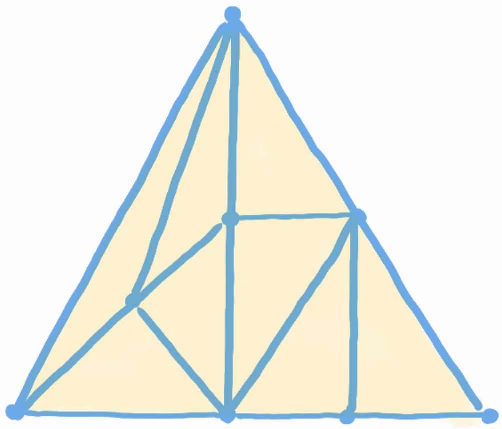
Γ = triangulation of 2^V

Γ_F = restriction of Γ on $F \in 2^V$.

Definition (Stanley 1992). The local h-polynomial of Γ (with respect to V) is defined as

$$\ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{n-|F|} h(\Gamma_F, x).$$

Example.



Γ

- $\ell(\Gamma_v, x) = (1+5x+2x^2) - (1+2x) - (1+x) - 1 + 1 + 1 + 1 - 1$

$= 2x + 2x^2.$

Theorem (Stanley 1992). For every triangulation Δ' of a pure simplicial complex Δ

$$h(\Delta', x) = \sum_{F \in \Delta} \ell_F(\Delta'_F, x)\, h(\mathrm{link}_\Delta(F), x).$$

Theorem (Stanley 1992). The polynomial $\ell_V(\Gamma, x)$

- is symmetric, with center of symmetry $n/2$, for every triangulation Γ of the simplex 2^V,
- has nonnegative coefficients for every triangulation Γ of the simplex 2^V,
- is unimodal for every regular triangulation Γ of the simplex 2^V.

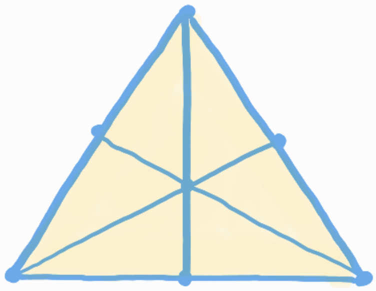
Conjecture (A 2012). The polynomial $\ell(\Gamma_V, x)$ is γ-positive for every flag triangulation Γ of 2^V.

Theorem (Kubitzke-Murai-Seig 2019) The polynomial $\ell(\Gamma_V, x)$ is γ-positive if Γ is the barycentric subdivision of a CW-regular subdivision of 2^V.

Example. If

V = n-element set

Δ = first barycentric subdivision of 2^V



then

- $$\ell_V(\Gamma, x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k(x)$$

$$= d_n(x).$$

Proposition (Kubitzke-Murai-Seig 2019)

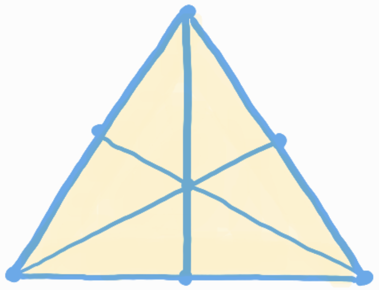
$$d_n(x) = \sum_{k=0}^{n-2} \binom{n}{k} (x + x^2 + \cdots + x^{n-k-1}) \, d_k(x)$$

Note. The proof applies Stanley's locality formula.

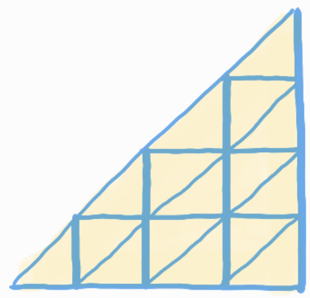
Proposition (A 2016). There exists a flag triangulation Γ of the 7-dimensional simplex 2^V for which $\ell(\Gamma_V, x)$ is not real-rooted.

Question. For which (flag) triangulations Γ of the simplex 2^V is $\ell_V(\Gamma, x)$ real-rooted?

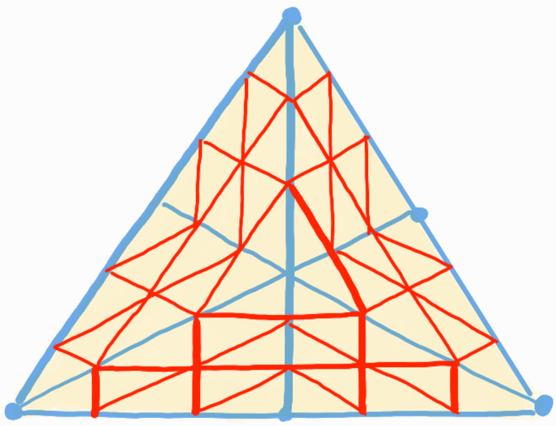
Some answers. Yes for



barycentric subdivision

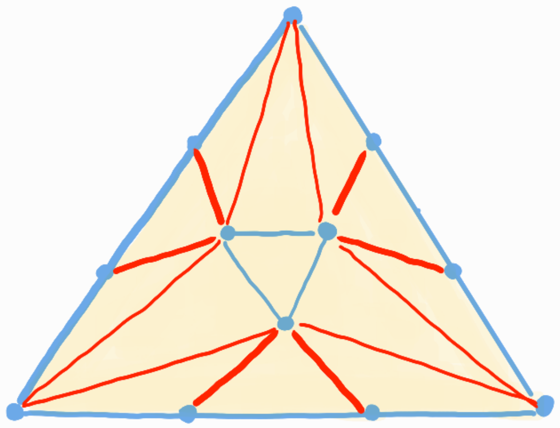


r-fold edgewise subdivision (Zhang 2019)



r-colored barycentric subdivision (Brändén-Solus 2021, Gustafsson-Solus 2020)

Conjecturally (A-Kubitzke-Brunink 2022) yes for the antiprism triangulation.



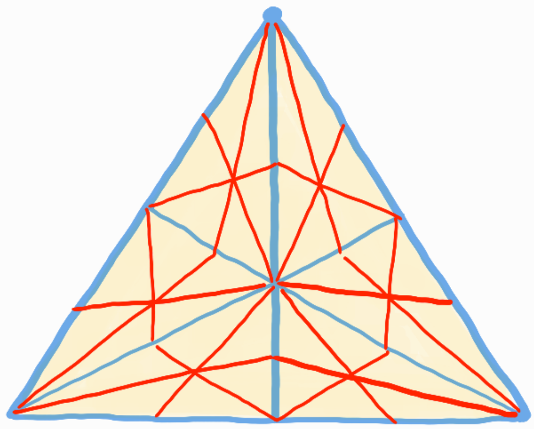
Note. In this case

$$[x^k]\,\ell_V(\Gamma, x) = \binom{n}{k} \#\{ w \in \mathcal{D}_n : \mathrm{Exc}(w) = [k] \}.$$

Notation. Let $sd^{(k)}(\Delta)$ be the k^{th} barycentric subdivision of a complex Δ.

Question (A 2016) Find a combinatorial interpretation of

$$\ell_V(sd^{(2)}(2^V), x).$$



Is this polynomial real-rooted?

Note. $e_{\bar{V}}(sd^{(2)}(2^V), 1) =$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k!)^2$$

$$= \#\{(u,v) \in \mathfrak{S}_n \times \mathfrak{S}_n : u, v \text{ have no common fixed point}\}.$$

Recall that

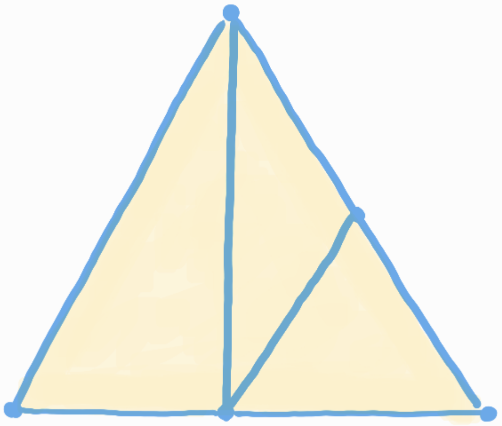
$sd(\Delta)$ = barycentric subdivision of Δ.

Theorem (A 2024+)

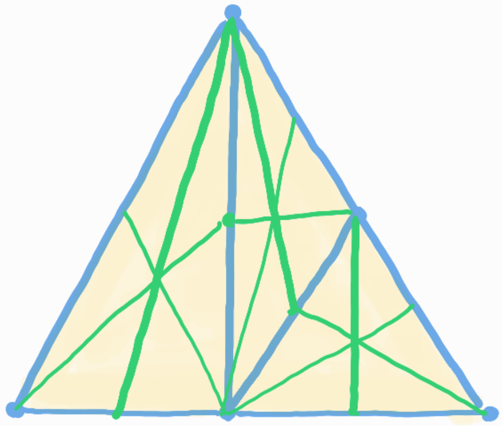
(a) The polynomial $\ell_V(\mathrm{sd}(\Gamma), x)$ is real-rooted for every triangulation Γ of 2^V.

(b) Same for the r-fold edgewise subdivision of Γ, if $r \geqslant n = |V|$.

Conjecture (A 2024+) The polynomial $\ell_V(\mathrm{sd}(\Gamma), x)$ is real-rooted for every CW-regular subdivision Γ of 2^V.



Γ



$sd(\Gamma)$

Basic method: Express $\ell_V(sd(\Gamma), x)$ as a nonnegative linear combination of real-rooted polynomials with nonnegative coefficients which have a common interleaver.

Recall the for real-rooted polynomials $p(x), q(x) \in \mathbb{R}[x]$ with roots

- $\cdots \leq \alpha_2 \leq \alpha_1 \leq 0$
- $\cdots \leq \beta_2 \leq \beta_1 \leq 0$

we say that $p(x)$ interlaces $q(x)$ if $\cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1 \leq 0$ and write $p(x) \prec q(x)$.

Proposition (Brenti-Welker, 2008)

For every simplicial complex Δ of dimension $n-1$

$$h(sd(\Delta), x) = \sum_{k=0}^{n} h_k(\Delta)\, p_{n,k}(x),$$

where

$$p_{n,k}(x) = \sum_{w \in \mathfrak{S}_{n+1} \,:\, w(1)=k+1} x^{des(w)}$$

for $n \in \mathbb{N}$. Equivalently,

$$\frac{p_{n,k}(x)}{(1-x)^{n+1}} = \sum_{m \geqslant 0} m^k (1+m)^{n-k} x^m .$$

Note.

$$\begin{pmatrix} p_{n,0}(x) \\ p_{n,1}(x) \\ \vdots \\ p_{n,n}(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x & 1 & \cdots & 1 \\ x & x & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x & x & & x \end{pmatrix} \begin{pmatrix} p_{n-1,0}(x) \\ p_{n-1,1}(x) \\ \vdots \\ p_{n-1,n-1}(x) \end{pmatrix}$$

and hence $(p_{n,k}(x))_{0 \leq k \leq n}$ is an interlacing sequence, meaning that $p_{n,i}(x) \prec p_{n,j}(x)$ for $1 \leq i < j \leq n$.

Corollary For every $(n-1)$-dimensional Cohen-Macaulay simplicial complex Δ, $h(sd(\Delta), x)$ is real-rooted and is interlaced by the Eulerian polynomial $A_n(x) = p_{n,0}(x)$.

For $0 \leq k \leq n+1$, $0 \leq j \leq n$ we let

$$d_{n,k,j}(x) = \sum_{\substack{w \in \mathfrak{S}_{n+1} : \mathrm{Fix}(w) \subseteq [n+1-k] \\ w^{-1}(1) = j+1}} x^{\mathrm{exc}(w)}.$$

Note.

- $d_{n,0,j}(x) = p_{n,j}(x)$
- $d_{n,k,0}(x) = d_{n,k}(x)$

$$:= \sum_{w \in \mathfrak{S}_n : \mathrm{Fix}(w) \subseteq [n-k]} x^{\mathrm{exc}(w)}.$$

Theorem (A 2024+) For any triangulation Γ of the $(n-1)$-dimensional simplex 2^V there exist nonnegative integers $c_{k,j}(\Gamma)$, for $k+j \leq n$, such that

$$\ell_V(sd(\Gamma), x) = \sum_{k+j \leq n} c_{k,j}(\Gamma)\, d_{n,k,j}(x).$$

Specifically,

$$c_{k,j}(\Gamma) = \sum_{F \subseteq V : |F| = n-k} [x^j]\, \ell_F(\Gamma_F, x).$$

Theorem (A 2024+) The polynomial $d_{n,k,j}(x)$ is real-rooted and is interlaced by $A_n(x)$ for $k+j \leq n$.

Let

$$D_{n,k} = \#\{ w \in D_n : \operatorname{exc}(w) = k \}.$$

Corollary (A 2024+).

$$\ell_V(sd^{(2)}(2^V), x) = \sum_{k+j \leq n} \binom{n}{k} D_{n-k,j}\, d_{n,k,j}(x)$$

where $n = |V|$. Equivalently, the coefficient of x^i in $\ell_V(sd^{(2)}(2^V), x)$ equals the number of

$$(u,v) \in \mathfrak{S}_n \times \mathfrak{S}_{n+1} : \begin{cases} \operatorname{Fix}(v) \subseteq [n - \operatorname{fix}(u) + 1] \\ v^{-1}(1) = \operatorname{exc}(u) \\ \operatorname{exc}(v) = i. \end{cases}$$

Open. Find a combinatorial inter-
pretation of the γ-polynomial associated to $e_V(sd^{(2)}(\mathcal{Q}^V), x)$.

Proposition (A 2023+).

$$d_{n,k}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi^{+}_{n,k,i}\, x^{i} (1+x)^{n-2i} +$$

$$\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \xi^{-}_{n,k,i}\, x^{i} (1+x)^{n-1-2i}$$

where

$\xi^{+}_{n,k,i}$ = # $w \in \mathfrak{S}_n$: $w(1) > n-k$ and w has i decreasing runs, none of size one.

$\xi^{-}_{n,k,i}$ = # $w \in \mathfrak{S}_n$: $w(1) \leq n-k$ and w has i decreasing runs, none (except possibly the first) of size one.

Thank you for your attention!