On the Existence of Physically Valid Magnetotelluric Data for General (3-D) Conductivity Distributions, Part II: Formulation a Practical Test

A. Tzanis*

Abstract

In a prequel to this paper, it was shown that Schmucker response function derived from the characteristic values of the impedance tensor can be cast into a simple Cauer form (expansion). This representation epitomizes the properties of the magnetotelluric responses, which are a direct consequence of its sensu stricto causality. It may also form the basis for a practical means to test a measured magnetotelluric response function for realizability, i.e. physical validity and origin in a real and recoverable Earth structure. Moreover, owing to its analyticity, the Cauer representation may also be used to interpolate (reconstruct) distorted portions of the observed response functions. A procedure and algorithm to realize these objectives is presented herein and its effectiveness is demonstrated with a number of applications to synthetic and measured MT data.

1 Introduction

In the prequel paper [1], it was shown that the generalized Schmucker response function of the form $\mathbf{c}(\omega) = \mathbf{M}/i\omega\mu_0$, where

$$\mathbf{M} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

contains the characteristic values (principal components) of the magnetotelluric impedance tensor, admits the Cauer representation (expansion)

$$\mathbf{c}(\omega) = \begin{bmatrix} b_1(0) & 0 \\ 0 & b_2(0) \end{bmatrix} + \int_{-\infty}^{+\infty} \frac{1 - i\omega\lambda}{\lambda + i\omega} \begin{bmatrix} db_1(\lambda) & 0 \\ 0 & db_2(\lambda) \end{bmatrix} = \mathbf{b}(0) + \int_{-\infty}^{\infty} \frac{1 - i\omega\lambda}{\lambda + i\omega} d\mathbf{b}(\lambda) \qquad (1)$$

where the integral should be read in the Lebesgue - Stieltjes sense and $\mathbf{b}(\lambda)$, $\mathbf{b}(0)$, $\mathbf{b}(1)$ are real and positive. Natural EM fields at ELF and ULF frequencies are very weak and notoriously susceptible to distortion by noise of various denominations. Consequently, an important question arising in the interpretation of measured (incomplete and inconsistent) MT field data is that of the existence of realizable Earth response functions, i.e. ones that are physically valid (*sensu stricto* causal) and owe their origin to real and recoverable Earth structure. Herein, I will present a procedure and practical algorithm to examine MT data for validity, based on representation (1). Another important question is whether it is possible to reconstruct (interpolate) distorted or missing parts of the observational bandwidth with a faithful method, i.e. one consistent with the analytical properties of the MT response function. The analyticity of (1) offers this opportunity as a by-product of the examination procedure. The effectiveness of the proposed method is demonstrated with a number of examples.

*Department of Geophysics and Geothermy, University of Athens, Panepistimiopoli, Zografou 157 84, Greece; e-mail: atzanis@geol.uoa.gr

2 Existence

A realizable MT response function, i.e. one representing true earth structure, has very restrictive properties: it must be positive real and must have characteristic values with singularities confined to the positive imaginary frequency axis. These requirements are expressed by the Cauer form (1). Conversely, an experimental (observed) MT response function is realizable, i.e. it represents true earth structure, only if it has characteristic values that admit to representation by the Cauer form (1). However, (1) is a continuous frequency expression, whereas practical realizations of the MT response functions allow only for a discrete set $\mathbf{c}(\omega_m)$ over a finite and discrete frequency bandwidth of size M (incomplete data). In this case (1) can be re-written as

$$\mathbf{c}(\omega_m) = \mathbf{b}(0) + \int_0^\infty \frac{1 - i\omega_m \lambda}{\lambda + i\omega_m} d\mathbf{b}(\lambda), \quad m = 1, 2, \ldots M \tag{2}$$

Consider, now, the approximation of the Stieltjes integral by a real finite sum

$$c_k(\omega_n) = [b_k]_0 + \sum_{l=1}^{L} \Gamma_{nl} \Delta[b_k]_l, \quad [b_k]_0 > 0, \quad \Delta[b_k]_l > 0, \quad n = 1, \ldots, 2M, \quad k = 1, 2 \tag{3}$$

where

$$c_k(\omega_m) = c_k(\omega_{2m-1}) + ic_k(\omega_{2m}), \quad m = 1, 2, \ldots M, \qquad k = 1, 2 \ .$$

The elements of Γ_{nl} are real and $\Delta[b_k]_l = b_k(\lambda_{l+1}) - b_k(\lambda_l)$ with $\lambda_{l+1} = \lambda_l + \Delta\lambda$, λ_1=0, $\Delta\lambda > 0$; L becomes large and $\Delta\lambda$ becomes small in such a way, that $L\Delta\lambda$ grows without bound. Then, the fundamental theorem of linear programming states that when the matrix Γ_{nl} has rank $2M$, if there is any set of numbers $[b_k]_0$, $\Delta[b_k]_l \geq 0$ satisfying the linear constraints (3), then there must be a set in which at most $2M$ of them are non-zero. Given also that the nominator in the integral in (2) is real and positive since λ is imaginary, the Cauer representation for incomplete data can be written in the much more convenient form:

$$c_k(\omega_m) = [a_k]_0 + \sum_{l=1}^{L} \frac{[a_k]_l}{\lambda_l + i\omega_m}, \quad [a_k]_0 \geq 0, \quad [a_k]_l \geq 0, \quad m = 1, \ldots, M, \quad L \leq M, \quad k = 1, 2 \tag{4}$$

The above argument (the origins of which can be traced to Parker [2]), supposes that Γ_{nl} is of full rank. Nevertheless, equation (4) also holds when Γ_{nl} is rank deficient; demonstration of this property is possible using the bracketing approach of Sabatier [3], but will not be attempted herein for the sake of brevity. As it turns out, the validity of MT data can be verified by testing it for compliance with (4).

3 Implementation

For measured data $\hat{c}_k(\omega_m)$ with variances $s_m^2 = var\left[\hat{c}_k(\omega_m)\right]$ equation (4) can be solved readily by minimizing

$$\chi^2 = \sum_{m=1}^{M} \frac{1}{s_m^2} \left| \left(a_k(0) + \sum_{l=1}^{L} \frac{[a_k]_l}{\lambda_l + i\omega_m} \right) - \hat{c}_k(\omega_m) \right|^2 = \sum_{m=1}^{M} \frac{1}{s_m^2} \left| \left(a_k(0) + \sum_{l=1}^{L} \Lambda_l(\omega_m)[a_k]_l \right) - \hat{c}_k(\omega_m) \right|^2 \tag{5}$$

subject to M positivity constraints $a_k(0)]$, $[a_k]_l \geq 0$. This is an exercise in quadratic programming and can be done in a manner similar to [4]. In this way, the test of validity and existence reduces to finding the solution of (5) with the smallest possible χ^2 misfit!

Equation (5) can be solved for each of the characteristic values independently. However, in the general case λ_l should simultaneously satisfy both diagonal elements of $\mathbf{c}(\omega)$. To address this detail, one must either set up a cumbersome block matrix system, or circumnavigate the complication by defining λ_l from the trace invariant, since:

$$\frac{1}{2}\left[\hat{c}_1(\omega_m) + \hat{c}_2(\omega_m)\right] = \frac{1}{2}\left(a_1(0) + a_2(0)\right) + \frac{1}{2}\sum_{l=1}^{L} \frac{[a_1]_l + [a_2]_l}{\lambda_l + i\omega_m}$$

Thus, to obtain the analytic expansion (4), one may first obtain $\Lambda_l(\omega_m)$ by solving an 1-D problem on the trace invariant and then use the fixed $\Lambda_l(\omega_m)$ to solve for $[a_k]_l$ of each diagonal element.

4 Examples

The effectiveness of the procedure described in Section 2 above, is demonstrated with an application to synthetic 3-D MT data. The 3-D model is shown in Figure 1 (left) and comprises an interface between a conductive (1 Ωm) and a resistive (100 Ωm) brick, embedded in a relatively conductive domain (10 Ωm) and located above a conducting halfspace (0.01 Ωm). The 3-D effects are accentuated by a protrusion of the conductive brick that wedges into the resistive brick. The site of 'measurements' is marked with a down arrow; it is located on the resistive brick, at the immediate vicinity of the interface. Forward modeling was carried out with the finite difference algorithm of Mackie and coworkers [5, 6]. The 'observed' characteristic values are shown in Figure 1 (top right) as *discrete* lines and are assigned a variance of $(s_m)^2 = 1$. The Cauer expansion of these characteristic values, as approximated with equation (5), is shown with continuous lines in Figure 1 (bottom right). It faithfully reproduces the observations as expected, with an RMS of 0.26 for the maximum characteristic value and 0.06 for the minimum, thus demonstrating the effectiveness and applicability of the generalized Cauer representation to a 3-D MT response function.

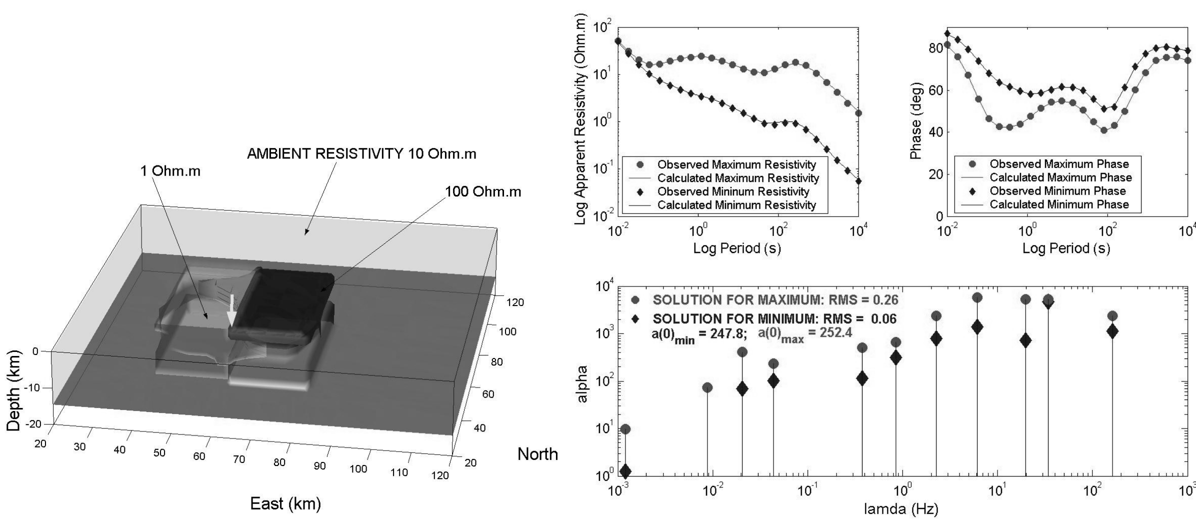

Figure 1: The left panel illustrates the 3-D model used to generate the synthetic data employed in demonstrating the applicability of the Cauer representation. The arrow points to the location of the 'site' yielding the impedance tensor, whose characteristic values, (in the form of maximum and minimum apparent resistivities and phases), are illustrated in the right panel (discrete lines). The Cauer approximation to the characteristic values is shown with continuous lines. The lower right panel illustrates the residues ($[a_k]_l$) of the maximum and minimum characteristic values as a function of the (common) eigenfrequency λ_l.

As shown above, incomplete but exact MT data can be *faithfully* represented by the finite Cauer expansion (4). However, measured MT data is not only incomplete, but is also inconsistent. Apart from random noise, the data can easily be contaminated by multiple-coherent noise of various forms, producing a variety of frequency-dependent spurious effects and bias; it may also be distorted by galvanic (non-inductive) natural fields. In such a case, testing for compliance with the Cauer representation may assist in discriminating useful data from rubbish, and, due to the analyticity nature of the representation, interpolate and recover distorted parts of the observational bandwidth.

Two examples of such an application are presented in Figure 2. The left panel in Figure 2 illustrates a case in which the maximum and minimum impedances were estimated with a small error margin and appeared to be excellent in, quality. Yet, this data set defied all interpretation attempts: 2-D inversion failed to converge at this site. Testing for compliance with Cauer representation revealed that the phase of the minimum impedance is severely distorted by low amplitude noise that produced a phase shift (downward bias) in the band 10^{-1}s – 10s. Distortion is also observed at the long period end of observational bandwidth, as well as in the 3-8s band of the maximum phase. This weak noise source has pushed the singularities of the minimum characteristic value away from the positive imaginary axis. Nevertheless, it is possible to recover a physically valid (causal) response exhibiting minimum phase. The right panel in Figure 2, illustrates a data set well estimated everywhere, except between the band 2s – 11s where obvious frequency-local distortion of the maximum and minimum apparent resistivities and phases is observed. In this case the identification of noise and recovery of the missing data is straightforward.

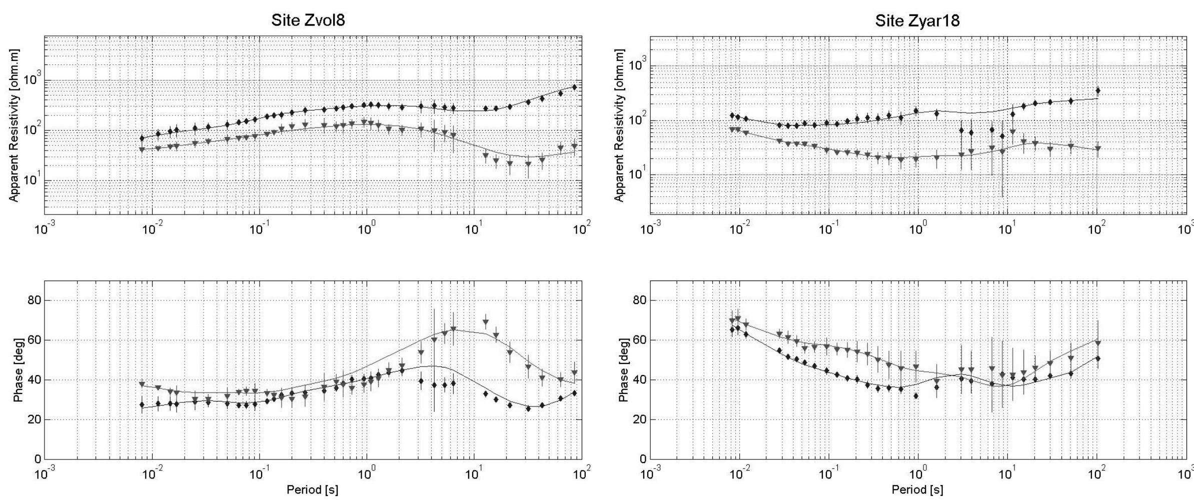

Figure 2: Application of the Cauer representation in testing the validity (realizability) of measured MT data and recovering distorted or missing parts of the observational spectrum.

5 Conclusion

The Cauer expansion of the characteristic values of the impedance tensor is a direct consequence of their *sensu stricto* causality and epitomizes their analytic properties [1]. Herein it was shown that it may also form the basis for a practical means to test a measured magnetotelluric response function for realizability (physical validity and origin in a real and recoverable Earth structure). It was also shown that owing to its analyticity, the Cauer representation may be used to interpolate and reconstruct distorted portions of the observed response functions. The procedures developed in the present analysis may come very handy in the interpretation of measured MT data, which are notoriously susceptible to distortion by noise of various denominations.

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