On the Existence of Physically Valid Magnetotelluric Data for General (3-D) Conductivity Distributions, Part I: Analytical Structure and Representations of the Response Function

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Abstract

On the assumption of causality, it is shown that for general (three-dimensional) conductivity distributions integrable over any cuboid region of the Earth, the magnetotelluric field possesses very restrictive analytic properties: the singularities of the electric and magnetic field components are all simple zeros confined on the positive imaginary axis of the complex frequency plane. This means that transfer functions comprising simple ratios of orthogonal electric and magnetic field components should also have simple poles and zeros located on the positive imaginary axis. Three-dimensional impedance tensors can be reduced to diagonal or anti-diagonal forms with elements comprising simple ratios of orthogonal field components, using such methods, as the Canonical Decomposition or the SVD, which can be shown to constitute 3-D rotations. Then, it can be shown that the Schmucker Response function derived from the characteristic (singular) values of the impedance tensor can be cast into a simple Cauer form (expansion), completely analogous to the one derived for one-dimensional Earth structures by Parker [3]. This Cauer representation epitomizes the properties of the magnetotelluric responses, which are a direct consequence of its *sensu stricto* causality.

1 Introduction

Natural EM fields at ELF and ULF frequencies are very weak and notoriously susceptible to distortion by noise of various denominations. Consequently, an important question arising in the interpretation of magnetotelluric (MT) field data is that of the existence of solutions to the inverse problem. For a 1-D conductivity structure the Schmucker [1] response function $c(\omega) = E_x(\omega)/i\omega\mu_0 H(\omega) = Z(\omega)/i\omega\mu_0$ has simple zeros located on the positive imaginary frequency axis and admits the *Cauer* [2] representation [3].

$$c(\omega) = b(0) + \int_0^\infty \left(\frac{1 - i\omega\lambda}{\lambda + i\omega}\right) db(\lambda), \quad b(\lambda) > 0, \quad b(0), > 0 \ . \tag{1}$$

Parker [3] has shown that for an observed response function, compliance with the Cauer representation is a necessary and sufficient condition for the existence of 1-D data. Herein I will investigate the extension of the specific form (1) to responses obtained over 3-D geoelectric structures. Furthermore, in a follow up paper I will attempt to show how this formulation can be used to address the problem of existence of a realizable geoelectric structure from measured (incomplete and inconsistent) MT observations.

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2 Analytic structure of the magnetotelluric field in the complex ω-plane

The source-free diffusion of the EM field in the quasi-static approximation can be described by the frequency domain Maxwell's equations, from which the simultaneous vector differential equations are readily derived

$$\nabla^2 \mathbf{E} = i\omega\mu_0\sigma\mathbf{E} \qquad \text{and} \qquad \nabla^2 \mathbf{H} + \sigma^{-1}\left[(\nabla \times \mathbf{H}) \times \nabla\sigma)\right] = i\omega\mu_0\sigma\mathbf{H} \tag{2}$$

In shorthand matrix operational form one can write $\sigma^{-1}(\nabla \times \mathbf{H}) \times \nabla\sigma = \sigma^{-1}\mathbf{\Delta}_h\mathbf{H} = \mathbf{\Delta}_\sigma\mathbf{E}$, and upon letting $\mathbf{F} = [\mathbf{E}\ \mathbf{H}]^T$, equations (2) can be recast into the equivalent compact form,

$$\nabla^2 \mathbf{F} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma^{-1}\mathbf{\Delta}_h \end{bmatrix} \mathbf{F} = i\omega\mu_0\sigma\mathbf{F} \tag{3}$$

where $\mathbf{0}$ is a 3×3 null matrix. Denoting by $x_{\pm\infty}$ and $y_{\pm\infty}$ horizontal dimensions large with respect to the wavelength, it is possible to represent the 3-D conducting crust as a cuboid $R = (x_{-\infty}, x_{+\infty}) \times (y_{-\infty}, y_{+\infty}) \times (0, h_{max})$. For h_{max}, either a radiation boundary condition may be assumed (i.e. $\mathbf{F} \to 0$ as $z \to \infty$), or a perfect conductor may be placed at some great depth, so as to absorb all the energy incident from above and below. Note also that if homogeneous Neumann boundary conditions are applied at the surface, then a provision must be made for a possible discontinuity of $\mathbf{H}$ at z=0, in cases of simple (e.g. layered) conductivity distribution functions, hence, R is assumed to be open at z=0. The scalar conductivity distribution function σ is real and positive and therefore integrable over any cuboid $V = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2] \subseteq R - h_{max}$, in the sense

$$\Sigma = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} \sigma dz = \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} dz \int_{x_1}^{x_2} \sigma dx = \int_{z_1}^{z_2} dz \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} \sigma dy > 0,$$

meaning that the conductance of any given volume should be independent of how one chooses to integrate.

Location of singularities: In general, (2) cannot be solved for arbitrary σ. However, the mere objective of the present analysis is to establish the analytic properties of the MT response functions; this does not require the knowledge of the exact analytic form of the solution, so long as a solution exists. Postulating that a solution always exists for a realisable σ, The location of the singularities can be derived with simple, physical rather than mathematical arguments: Irrespective of the boundary conditions and exact conductivity distribution, in the absence of sources or sinks within a finite crustal volume V, if the electromagnetic energy flux entering and exciting the volume is finite, then $\mathbf{E}$ and $\mathbf{H}$ must be finite and exhibit finite and positive energy dissipation. Writing

$$\int_V \mathbf{J}^* \cdot \mathbf{E} dv = \int_V \sigma\mathbf{E}^* \cdot \mathbf{E} dv = \int_V \mathbf{E} \cdot (\nabla \times \mathbf{H}^*)\, dv$$

and using the vector identity $\nabla\cdot(\mathbf{E}\times\mathbf{H}^*) = \mathbf{H}^*\cdot(\nabla\times\mathbf{E})$-$\mathbf{E}\cdot(\nabla\times\mathbf{H}^*)$ and Faraday's law, one obtains

$$\int_V \sigma\mathbf{E}^* \cdot \mathbf{E} dv + i\omega \int_V \mu_0 \mathbf{H}^* \cdot \mathbf{H} dv = -\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) dv.$$

This is an energy conservation statement saying that the work done by the electric field plus the rate of the energy stored in the magnetic field within V is equal to the negative of the energy flowing out through the boundary surfaces. It follows immediately that $\mathbf{E}$ and $\mathbf{H}$ (hence $\mathbf{F}$) cannot have poles anywhere in the four-dimensional domain $\{x,y,z,\omega\}$, hence in the complex ω-plane. However, they may have zeros. Consequently, the measure of the singularities of $\mathbf{E}$ and $\mathbf{H}$ will always add up to zero, and they can be defined as functions continuous almost everywhere and Lebesgue integrable over any V. Moreover, it is apparent that there are no zeros of $\mathbf{E}$ and $\mathbf{H}$ unless ω is on the positive imaginary axis, since it is there and only there where the LHS may vanish.

Multiplicity of zeros: In the following, use will be made of the shorthand notation $\partial_x = \partial/\partial x$, $\partial_{xx} = \partial^2/\partial x^2$, $\partial_{xy} = \partial^2/\partial x \partial y$ etc. On the positive imaginary axis set $\omega = i\lambda$, $\lambda > 0$. The multiplicity of the zeros depends on the behaviour of the derivative $\partial \mathbf{F}(i\lambda)/\partial\lambda$ denoted by $\partial_\lambda \mathbf{F}(i\lambda)$. Specifically, the zeros will be simple if $\partial \mathbf{F}(i\lambda)/\partial\lambda \neq 0$. Consider equation (5b) which, on the positive imaginary axis becomes:

$$\nabla^2 \mathbf{F}(i\lambda) + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma^{-1}\mathbf{\Delta}_h \end{bmatrix} \mathbf{F}(i\lambda) = -\lambda\mu_0\sigma\mathbf{F}(i\lambda) \Leftrightarrow \quad \nabla^2\mathbf{F}(i\lambda) + \aleph\mathbf{F}(i\lambda) = -\lambda\mu_0\sigma\mathbf{F}(i\lambda) \tag{4}$$

Right multiply (4) by $\partial_\lambda \mathbf{F}\prime(i\lambda)$, where the prime denotes the transposed but not conjugated vector. Differentiate the transpose of (4) with respect to λ and right multiply with $\mathbf{F}(i\lambda)$. Integrate the difference over $[z_1, z_2]$ and integrate the second vertical derivative term by parts to obtain

$$\begin{gathered} \partial_\lambda \mathbf{F}'(i\lambda)\cdot\partial_z\mathbf{F}(i\lambda)|_{z_1}^{z_2} - \partial_{z\lambda}\mathbf{F}'(i\lambda)\cdot\mathbf{F}(i\lambda)|_{z_1}^{z_2} = \int_{z_1}^{z_2}\mu_0\sigma\mathbf{F}'(i\lambda)\cdot\mathbf{F}(i\lambda)dz - \\ \left(\int_{z_1}^{z_2}[\partial_\lambda\mathbf{F}'(i\lambda)\partial_{xx}\mathbf{F}(i\lambda) - \partial_{xx\lambda}\mathbf{F}'(i\lambda)\mathbf{F}(i\lambda)] + [\partial_\lambda\mathbf{F}'(i\lambda)\partial_{yy}\mathbf{F}(i\lambda) - \partial_{yy\lambda}\mathbf{F}'(i\lambda)\mathbf{F}(i\lambda)]dz\right) \end{gathered} \tag{5}$$

Equation (5) holds for *all* x and y, so that when $\omega = i\lambda_n$ is the n^{th} zero of $\mathbf{F}(x_0, y_0, z_2, i\lambda)$,

$$\begin{gathered} \partial_\lambda\mathbf{F}'(z_2, i\lambda_n)\partial_z\mathbf{F}(z_2, i\lambda_n) = \begin{vmatrix} \partial_\lambda\mathbf{F}'(z_1, i\lambda_n) & \mathbf{F}(z_1, i\lambda_n) \\ \partial_{z\lambda}\mathbf{F}'(z_1, i\lambda_n) & \partial_z\mathbf{F}(z_1, i\lambda_n) \end{vmatrix} + \int_{z_1}^{z_2}\mu_0\sigma\mathbf{F}'(i\lambda_n)\cdot\mathbf{F}(i\lambda_n)dz \\ -\left(\int_{z_1}^{z_2}[\partial_\lambda\mathbf{F}'(i\lambda_n)\partial_{xx}\mathbf{F}(i\lambda_n) - \partial_{xx\lambda}\mathbf{F}'(i\lambda_n)\mathbf{F}(i\lambda_n)] + [\partial_\lambda\mathbf{F}'(i\lambda_n)\partial_{yy}\mathbf{F}(i\lambda_n) - \partial_{yy\lambda}\mathbf{F}'(i\lambda_n)\mathbf{F}(i\lambda_n)]dz\right) \end{gathered} \tag{6}$$

It is straightforward to verify that the integral terms in the RHS of (6) are non-zero and moreover, $\partial_{z\lambda}\mathbf{F}(x_0,y_0,z_1,i\lambda_n) \neq \partial_\lambda\mathbf{F}(x_0,y_0,z_1,i\lambda_n) \neq \partial_z\mathbf{F}(x_0,y_0,z_1,i\lambda_n) \neq \mathbf{F}(x_0,y_0,z_1,i\lambda_n)$. Therefore, $\partial_\lambda\mathbf{F}(x_0,y_0,z_2,i\lambda_n) \neq 0$. Likewise, for the m^{th} zero $\mathbf{F}(x_0,y_0,z_1, i\lambda_m)$, it can be shown that $\partial_\lambda\mathbf{F}(x_0,y_0,z_1,i\lambda_m) \neq 0$. Although it should be noted that terms evaluated at z_1 may need to be appropriately adjusted when boundary conditions at the surface are taken into consideration (z_1=0), the result still shows that all the zeros are simple.

3 Cauer representation of the magnetotelluric response

The analytic structure of the MT field implies that (scalar) impedance functions comprising simple ratios of electric and magnetic field components will have simple poles and zeros, and, therefore a simple(r) analytic representation. On the other hand, a 3-D impedance tensor is generally obtained from a linear superposition of fields induced by different source polarizations. For example, in MT modelling it is customary to consider two linearly independent polarizations of the source field with outcome $E_x^{(k)}, E_y^{(k)}, H_x^{(k)}, H_y^{(k)}$, k =1,2, so that the tensor elements Z_{xx} and Z_{xy} can be computed from the equations $E_x^{(k)} = Z_{xx}H_x^{(k)} + Z_{xy}H_y^{(k)}, \quad k = 1,2$ and similarly for Z_{yx} and Z_{yy}. In this case, while the singularities of the impedance tensor elements will remain on the upper half frequency plane, (the function remains causal), it is unknown whether their location will still remain on the positive imaginary axis. This problem can be circumnavigated by considering the principal components (characteristic states) of the impedance tensor, obtained by diagonalizing it with isometric transformations.

Tensor decomposition methods such as the *Singular Value Decomposition* [4] and the *Canonical Decomposition* [6] reduce the impedance tensor to the form

$$\mathbf{Z} = \mathbf{U}_E(\Theta_E, \Phi_E)\cdot\begin{bmatrix}\mu_1 & 0 \\ 0 & \mu_2\end{bmatrix}\cdot\mathbf{U}_H^+(\Theta_H, \Phi_H) \tag{7}$$

where $\mathbf{U}_E$ and $\mathbf{U}_H$ are unitary, of the form

$$\mathbf{U}(\Theta,\Phi) = \begin{bmatrix} \cos\Theta\cos\Phi - i\sin\Theta\sin\Phi & -\cos\Theta\sin\Phi + i\sin\Theta\cos\Phi \\ \cos\Theta\sin\Phi + i\sin\Theta\cos\Phi & \cos\Theta\cos\Phi + i\sin\Theta\sin\Phi \end{bmatrix}$$

Respectively, (Φ_E, Φ_H) are the (not necessarily perpendicular) azimuths of the electric and magnetic field, (Θ_E, Θ_H) the ellipticities of the magnetic and electric field and (μ_1, μ_2) the *true* characteristic values of the impedance tensor. It can easily be shown, but will not be attempted herein, that (7) is a proper Euler rotation in three dimensions. At any location $\{x_0, y_0, 0\}$ on the surface, the impedance tensor can be re-written as

$$\mathbf{U}_E^{+}\mathbf{Z} = \mathbf{M}\mathbf{U}_H^{+} \Rightarrow \begin{bmatrix} E_1(\Theta_E,\Phi_E) \\ E_2(-\Theta_E,\Phi_E+\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \cdot \begin{bmatrix} H_1(\Theta_H,\Phi_H) \\ H_2(-\Theta_H,\Phi_H+\frac{\pi}{2}) \end{bmatrix} \Rightarrow \mathbf{M} = \begin{bmatrix} E_1/H_1 & 0 \\ 0 & E_2/H_2 \end{bmatrix}$$

with $\{E_1(\Theta_E,\ \Phi_E),\ H_1(\Theta_H,\ \Phi_H)\}$ comprising the *maximum characteristic state* and $\{E_2(\Theta_E,\ \Phi_E + \pi/2),\ H_2(\Theta_H,\ \Phi_H + \pi/2)\}$ the *minimum state* of the electromagnetic field. Therefore, the maximum and minimum characteristic values are simple ratios of the maximum and minimum state fields respectively; in consequence, their singularities should be simple and confined on the positive imaginary axis!

Herein, for reasons of continuity with previous treatments of the same problem for layered structures [3, 6, 7], use will be made of a generalised Schmucker [1] response function, of the form $\mathbf{c}(\omega) = \mathbf{M}/i\omega\mu_0$ which represents the apparent skin depth to non-uniform diffusion waves propagating vertically. Because the impedance tensor, hence its characteristic values and $\mathbf{c}(\omega)$ are positive real, (*sensu stricto* causal), and because the singularities of $\mathbf{c}(\omega)$ are confined on the positive imaginary frequency axis, the Cauer representation assumes the form

$$\mathbf{c}(\omega) = \begin{bmatrix} b_1(0) & 0 \\ 0 & b_2(0) \end{bmatrix} - i\omega \begin{bmatrix} b_1(1) & 0 \\ 0 & b_2(1) \end{bmatrix} + \int_{-\infty}^{+\infty} \frac{1 - i\omega\lambda}{\lambda + i\omega} \begin{bmatrix} db_1(\lambda) & 0 \\ 0 & db_2(\lambda) \end{bmatrix}$$

$$= \mathbf{b}(0) - i\omega\mathbf{b}(1) + \int_{-\infty}^{\infty} \frac{1 - i\omega\lambda}{\lambda + i\omega} d\mathbf{b}(\lambda) \qquad (8)$$

where the integral should be read in the Lebesgue - Stieltjes sense and $\mathbf{b}(\lambda)$, $\mathbf{b}(0)$, $\mathbf{b}(1)$ are real and positive. Due to the location and simplicity of the singularities, $\mathbf{b}(\lambda)$ varies with jumps corresponding to the poles of $\mathbf{c}(\omega)$ with positive residues. With arguments identical to those of Parker [3], we may deduce that: **(1)** The lower limit of the integral can be made, zero, i.e. $i\omega \in [0, \infty)$, because unless $\mathbf{b}(\lambda)$ is constant for $\lambda < 0$, so that the integral over $(-\infty, 0]$ vanishes, then $\mathbf{c}(\omega)$ would not have a positive real part in the lower half-plane. **(2)** For large ω the integral diminishes - it is $O(|\omega^{-1}|)$ - and unless $\mathbf{b}(1)$ vanishes, $\mathbf{c}(\omega)$ will increase without limit, which is absurd. Thus, (8) reduces to

$$\mathbf{c}(\omega) = \mathbf{b}(0) + \int_{0}^{\infty} \frac{1 - i\omega\lambda}{\lambda + i\omega} d\mathbf{b}(\lambda) \qquad (9)$$

Equation (9) is *mutatis mutandis* equivalent to the form (1) and it completely describes the analytic properties of the characteristic states of a 3-D magnetotelluric response function; together with (Φ_E, Φ_H) and (Θ_E, Θ_H) they completely describe the propagation of the MT field in the Earth and the impedance tensor.

4 Conclusion

It was shown that for general (3-D) conductivity distributions, the characteristic (singular) values of the impedance tensor can be cast into a simple Cauer form (expansion), completely analogous to the representation derived for 1-D Earth structures by Parker [3]. This striking similarity in the properties of the 1-D and 3-D responses is a direct consequence of the *sensu stricto* causality of an impedance function, which prescribes the nature (zeros), location and multiplicity of the singularities of the magnetotelluric field. Moreover, the Cauer representation suggests a means of addressing the problem of existence of realizable geoelectric structures from measured (incomplete and noisy) MT observations. This involves a practical means of testing the compliance of measured MT data to the Cauer for (9) and will be discussed in a sequel paper (Part II).

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