The Characteristic States of the Magnetotelluric Impedance Tensor for General Conductivity Distributions II: Analytic Properties and Utility

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SUMMARY

In a prequel presentation it was shown that the MT impedance tensor admits an anti-symmetric generalized eigenstate – eigenvalue representation. Herein, the analytic structure of the generalized eigenvalues (eigen-impedances) for general conductivity distributions. It is shown that in the absence of sources, the eigen-impedances are analytic in the entire lower-half complex frequency plane with their singularities confined on the positive imaginary frequency axis. These properties can be violated only if there are powerful extrinsic or intrinsic electromagnetic processes with time dependence different than the time dependence of passive induction. The expected passivity of the eigen-impedances can be an effective means of appraising measured tensors for compliance with the tenets of the MT method. As an example, it is shown that anomalous phases and electric field reversals are not necessarily associated with violation of causality and that it is possible to assess whether tensors exhibiting anomalous phases can be used for interpretation.

Keywords: Impedance Tensor, Decomposition, Causality, Passivity, Anomalous Phase

INTRODUCTION

In a prequel presentation (Tzanis, 2014) it was shown that the Magnetotelluric (MT) impedance tensor **Z** admits an anti-symmetric generalized eigenstate – eigenvalue decomposition of the form

$$\mathbf{Z} = \mathcal{E}(\theta_E,\varphi_E)\cdot\tilde{\mathbf{Z}}\cdot\mathcal{H}^{\dagger}(\theta_H,\varphi_H), \qquad \tilde{\mathbf{Z}} = \begin{bmatrix} 0 & \zeta_1 \\ -\zeta_2 & 0 \end{bmatrix},$$

with ζ_1, ζ_2 being the maximum and minimum generalized eigenvalues (*eigen-impedances*) of the tensor and $\mathcal{E}(\theta_E,\varphi_E)$, $\mathcal{H}(\theta_H,\varphi_H)$ rotation operators of the SU(2) group. By substituting the decomposition in relationship **E** = **ZH** one obtains the generalized eigenstates of the MT field

$$\begin{bmatrix} E_1(\theta_E,\varphi_E) \\ E_2(\theta_E,\varphi_E+\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & \zeta_1 \\ -\zeta_2 & 0 \end{bmatrix}\cdot\begin{bmatrix} H_1(\theta_H,\varphi_H) \\ H_2(\theta_H,\varphi_H+\frac{\pi}{2}) \end{bmatrix},$$

or $\tilde{\mathbf{E}} = \tilde{\mathbf{Z}}\cdot\tilde{\mathbf{H}}$, where E_1, E_2 respectively are the maximum and minimum eigenvalues of the electric field and H_1, H_2 the maximum and minimum eigenvalues of the total magnetic field. The angles (θ_E, φ_E) define the orientation of the *characteristic* coordinate frame $\{x_E, y_E, z_E\}$ of the *electric eigen-field* $\tilde{\mathbf{E}}$, relative to the experimental coordinate frame. Likewise, the angles (θ_H, φ_H) define the orientation of the characteristic eigen-frame $\{x_H, y_H, z_H\}$ of the *magnetic eigen-field* $\tilde{\mathbf{H}}$. Each eigen-frame contains orthogonal, *linearly polarized* components.

The present work investigates the analytic properties of the eigen-impedances and their utility in the analysis of MT data. Impedance is constrained by the requirement of positive energy dissipation and is bound to be passive (positive real). Accordingly, the tenet of causality is intuitive and has *ab initio* been thought to hold for any impedance tensor at the surface of any source-free Earth structure. This, however, has been disputed and for good reasons. Egbert (1990) argued that causality can be violated and postulated that an indication of violation would be the observation of *anomalous phases*; he argued that a principal cause of such effects is the distortion of the electric field with particular reference current channelling and orientation reversals. Anomalous phase observations have been reported by several authors and have mostly been explained with 3-D induction in near-surface elongate conductors that may or may not be coupled with other forms of channelling structures (e.g. faults, ocean etc.). An unfortunate corollary of passivity violation is that non-passive impedances are not interpretable in terms of Earth structure. However, some recent research indicates that anomalous phases may be part of the Earth response, (e.g. Selway et al. 2012; Ichihara and Mogi, 2009; Heise and Pous, 2003), the empirical consequence being that anomalous phases may not necessarily signify the breakdown of causality!

At any rate, impedance must be passive: if a function is not passive it is not impedance and the only reason for this to occur is for sources to exist in the Earth. “Source” is *any* EM effect with sufficient power and time dependence significantly different than that of passive induction. This is not simple: because the time-dependence is important, secondary inductive (reactive) effects generated within the Earth cannot be ruled out as sources but time-independent effect (e.g. galvanic) might! Therefore, the generality of passivity, the conditions of passivity violation and their association with anomalous phases are still in need of clarification. Because the characteristic state formulation offers a *succinct* means to characterize the tensor, the *expected* analytic properties of the characteristic states are examined from first principles. It is established that the characteristic states are *expected* to be passive and offer rigorous means to appraise the compliance of measured data with the tenets of the MT method.

CAUSALITY AND PASSIVITY

Let the part of the Earth's crust probed by **E** and **H** be represented by a cuboid $V = [x_1, x_2] \times [y_1, y_2] \times [0, z_2]$ and characterized by a conductivity function $\sigma \equiv \sigma(x,y,z) > 0$, which is arbitrary but defines a *linear* medium. Let $\mathbf{H}=\mathbf{H}_i+\mathbf{H}_S$, where $\mathbf{H}_i$ is the internal (induced) magnetic field and $\mathbf{H}_S$ is a *uniform* source (external) magnetic field. Finally, let there be no internal to *V* sources of EM field, so that **H**=0 and **E**=0 when $\mathbf{H}_S=0$.

The vector Helmholz equation governing the diffusion of the electric field is

$$\nabla\times\nabla\times\mathbf{E} = -i\omega\mu_0\sigma\mathbf{E} - i\omega\mu_0\mathbf{J}_S \quad (1)$$

with $\mathbf{J}_S$ being the source (extrinsic) current density, located at some distance $z=z_S<0$ above the surface of *V* (z=0). In the region $z \geq 0^+$ the eigenvalues of the electric field (electric eigen-fields) are by definition solutions of Equation 1. On substituting $\tilde{\mathbf{E}}$ and taking the inner product with $\tilde{\mathbf{E}}^*$,

$$\nabla\times\nabla\times\tilde{\mathbf{E}}\bullet\tilde{\mathbf{E}}^* = -i\omega\mu_0\sigma\tilde{\mathbf{E}}\bullet\tilde{\mathbf{E}}^* - i\omega\mu_0\mathbf{J}_S\bullet\tilde{\mathbf{E}}^* \quad (2)$$

Considering that $\nabla\times\tilde{\mathbf{E}} = -i\omega\mu_0\tilde{\mathbf{H}}$, the LHS of Equation 2 is $\nabla\times(-i\omega\mu_0\tilde{\mathbf{H}})\bullet\tilde{\mathbf{E}}^* = -i\omega\mu_0(\tilde{\mathbf{E}}^*\bullet\nabla\times\tilde{\mathbf{H}})$ and given that $\nabla\bullet(\tilde{\mathbf{E}}^*\times\tilde{\mathbf{H}}) = \tilde{\mathbf{H}}\bullet(\nabla\times\tilde{\mathbf{E}}^*) - \tilde{\mathbf{E}}^*\bullet(\nabla\times\tilde{\mathbf{H}})$, it reduces to

$$\tilde{\mathbf{H}}\bullet(\nabla\times\tilde{\mathbf{E}}^*) - \nabla\bullet(\tilde{\mathbf{E}}^*\times\tilde{\mathbf{H}}) = \sigma\tilde{\mathbf{E}}\bullet\tilde{\mathbf{E}}^* + \mathbf{J}_s\bullet\tilde{\mathbf{E}}^*. \quad (3)$$

Substituting Faraday's law in Equation 3, rearranging terms and integrating over *V* yields

$$\int_V \sigma\tilde{\mathbf{E}}\bullet\tilde{\mathbf{E}}^* dv + i\omega\int_V \mu_0\tilde{\mathbf{H}}\bullet\tilde{\mathbf{H}}^* dv =$$

$$= -\int_V \nabla\bullet(\tilde{\mathbf{E}}^*\times\tilde{\mathbf{H}})dv - \int_V \mathbf{J}_S\bullet\tilde{\mathbf{E}}^* dv \quad (4)$$

The second term in the RHS of Equation 4 is zero because the integration takes place in the region $z \geq 0^+$, where $\mathbf{J}_s = 0$. Thus, Equation 4 reduces to

$$\int_V \sigma\tilde{\mathbf{E}}\bullet\tilde{\mathbf{E}}^* dv + i\omega\int_V \mu_0\tilde{\mathbf{H}}\bullet\tilde{\mathbf{H}}^* dv = -\oiiint_{\partial V} (\tilde{\mathbf{E}}^*\times\tilde{\mathbf{H}})\bullet d\mathbf{s}\ )$$

This is a global energy conservation statement saying that the work done by the induced electric field in *V* plus the rate of the energy stored in the total magnetic field in *V* is equal to the negative of the energy flowing out through the boundary surfaces. Because all the integrands are finite, $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ cannot have poles in the complex ω-plane: they may only have *simultaneous zeros*. It is also apparent that the LHS can vanish *only* for ω on the positive imaginary ω-axis, where *all* the zeros are necessarily confined. As a direct consequence of the absence of sources, the electric and magnetic eigen-fields are causal, freely decaying waves.

Because the eigen-impedances are simple ratios of electric and magnetic eigen-fields with zeros confined on the positive imaginary ω-axis, their singularities are also confined on the positive imaginary ω-axis and they are analytic in the entire lower-half ω-plane and the real ω-axis. Therefore, the energy transferred from the magnetic to the electric eigen-field can only exhibit positive dissipation and the eigen-impedances are automatically classified as *positive real* (passive) functions. As a corollary, the impedance tensor **Z** is expected to be causal and passive as it is generated by isometric transformation of its passive eigen-impedances.

The above results imply that for any conductivity distribution, if there are no internal to the Earth sources, the eigen-impedances are passive. They do *not* imply that the eigen-impedances must be passive at all times and for any conductivity distribution because the absence of sources, (including secondary inductive effects), is not guaranteed. There *can* be circumstances in which passivity breaks down for measured tensors; the most significant mechanisms by which this can occur are enumerated as follows:

(a) Extrinsic noise (natural or anthropogenic).

(b) Distortion of the electric field due to *local* internal effects (Chave and Smith, 1994). The locally distorted electric field $\mathbf{E}_l$ is given by $\mathbf{E}_l=\mathbf{C}\cdot\mathbf{E}$, where **C** is a rank 2 tensor with elements:

- Complex valued/frequency-dependent if the inductive component of distortion is significant, meaning that it has time dependence of its own. In this case passivity may be violated because distortion will not only deform the shape, but will also modify the phase of the electric field, introducing delays to levels possibly unsustainable by the passive induction process.
- Real valued/frequency-independent if the inductive component is negligible (galvanic limit). In this case passivity *cannot* be violated because **C** will linearly superimpose undistorted field components thus deforming the electric field. However, the linear superposition of passive processes is necessarily passive and if the undistorted electric field is passive, the galvanically distorted electric field will also be passive and the eigen-impedances will continue to be

positive real.

(c) Non-linear Earth response. One should not exclude the possibility of linear Earth conductivity configurations that generate strong, large-scale secondary inductive effects, such as reactive eddy currents. These may the response to be non-passive or even or non-causal and have generally not been studied adequately

ANOMALOUS PHASES ARE NOT ALWAYS ANOMALOUS

This Section presents an example of the diagnostic value of the eigen-impedances in cases of complex response functions and entails the analysis of a synthetic impedance tensor with severe anomalous phase behaviour. The discussion is inevitably associated with the problem of appraising Earth responses for compliance with the tenets of the MT method and consistency with realizable geoelectric structures. In this respect, the appraisal of impedance tensor elements has customarily been based on their phases. Phases in the 1st or 3rd quadrants (*in-quadrant*) are taken to indicate causal response, consistent with realizable conductivity structures. Phases in the 2nd or 4th quadrants (*out of quadrant*), are deemed anomalous and the response non-causal. Along this vein, passivity is appraised by testing if *both* eigen-impedances are in-quadrant, in which case the tensor is strictly passive, or either is out of quadrant, in which case it is not. This criterion ultimately expresses the positive real property.

Synthetic impedance tensors with large anomalous phase variations have been presented by Heise and Pous (2003) on the basis of a 2-D model comprising a stack of two azimuthally anisotropic structures with orthogonal anisotropy strikes. The model is described in detail sufficient to warrant faithful as possible reproduction (not shown for conciseness) and was solved with the algorithm of Pek and Verner (1997). The synthetic response used herein was obtained at a location corresponding to *Site 4* of Heise and Pous.

The impedance tensor is shown in Figure 1, in the form apparent resistivities and phases. The phases of Z_{xx} and Z_{yx} are clearly seen to rotate from the 1st via the 2nd to the 3rd quadrant, exhibiting negative real parts during their transition through the second quadrant. The phases of Z_{xy} and Z_{yy} are stable and confined in the 1st quadrant. Heise and Pous (2003) demonstrated that the interaction between the anisotropic block and layer produces *complete* reversal in the orientation of the E_y component and attributed the anomalous phases, at least in part, to this reversal. Note also that all tensor elements have passive attributes at periods < 1s and the passive right column elements are dominant at all periods > 1s. The characteristic states are shown in Figure 2. The phases of the maximum and minimum eigen-impedances (top-right) are both defined in the 1st quadrant, meaning that they, as well as the tensor, are passive.

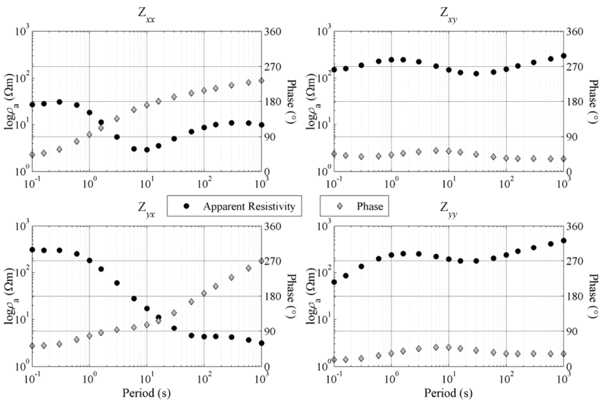

Figure 1. The noise-free almost exact impedance tensor "observed" at *Site 4* of the 2-D anisotropic model of Heise and Pous (2003).

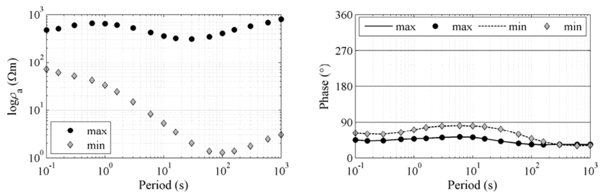

Figure 2. The eigen-impedances of the tensor obtained at *Site 4* of the 2-D anisotropic model of Heise and Pous (2003).

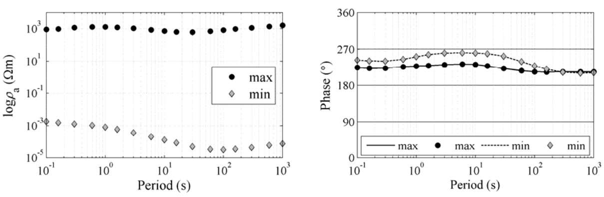

Figure 3. The eigen-impedances of the distorted tensor (see text for details).

The result is not difficult to explain: although there are reactive processes that cause apparent violation of passivity in individual elements, it is the dominance of the passive processes that determines the overall characteristics of the tensor and the properties of the eigen-impedances thereof. In other words, the isometric transformation from the full impedance tensor to the eigen-impedance tensor superimposes the passive and non-passive processes and assigns the result with the property of the dominant (passive) process. This is a direct consequence of the parallel filter rule known from systems theory (e.g. Claerbout, 1976).

The effect of galvanic distortion can be studied by artificially deforming the electric field with a real operator **C** = **T·S·A**, such that **T** is a twist (SO(2) rotation) operator, **S** is the shear tensor and **A** is a splitting tensor (Groom and Bailey, 1989). Figure 3 illustrates the characteristic states obtained from a distorted tensor, obtained by twisting the electric field through an angle of 110°, shearing it by an angle of 44° and letting **A** = **I** (no splitting). The effect of distortion is: (i) severe shifting of the maximum and minimum apparent resistivities; (ii) transfer of the phases from the 1st to the 3rd quadrant. There is no effect on the analytic properties of the eigen-impedances (and the distorted tensor thereof), which remain strictly passive.

This result is also not difficult to explain. Twisting by as much as ±110° causes one of the axes of the experimental coordinate frame to reverse orientation. The electric field is a *polar* vector with *odd* parity: reversals change the reference frame of the output electric field in the sense $\mathbf{E}(\mathbf{x}) \rightarrow -\mathbf{E}(-\mathbf{x})$, even if it is sheared. This introduces a symmetry of π in the phase of the electric field and a corresponding shift in the phase of the tensor elements and eigen-impedances. In the distorted tensor, negative real parts of individual tensor elements may appear! These, however, are defined in a frame different than that of the measurements and the violation of passivity is only apparent! Moreover, a galvanic distortion tensor like the one used above merely forces a weighted linear superposition of the electric field components but does not shift their phases. This certainly deforms the distorted tensor elements but if the dominant electric field components are passive, their superposition will still be passive by virtue of the parallel filter rule and will be manifested in the analytic properties of the eigen-impedances.

To conclude, it is apparent that internal reactive processes corollary to passive induction may generate anomalous phases, but if the dominance of passive induction is not overall challenged the passivity of the tensor is also not challenged and the anomalous phases are inconsequential. Such tensors convey valuable information about the geoelectric structure and can be used for interpretation. For the example in question, this was empirically demonstrated by Heise and Pous (2003).

CONCLUSION

The impedance tensor and its generalized eigenvalues are expected to be passive; this property can be violated only in the presence of sources in the Earth, with dissipation characteristics and time dependence sufficiently different than the respective characteristics of passive induction in a linear medium. This includes all extrinsic effects (noise) and, possibly, secondary inductive effects generated by realistic conductivity configurations. However, it does *not* include time-independent phenomena taking place in a passive induction context, such as galvanic distortion and electric field twists and reversals.

The constraints imposed by passivity comprise fundamental tests of the compliance of measured tensors with the fundamental properties expected of impedance functions (physical validity). In this respect they can be important factors in the process interpretation. For instance, due to the combined action of linear superposition and reference frame reversals, anomalous phases of individual elements are not trustworthy indicators of the tensor's validity. On the other hand, one may refer the tensor to its intrinsic coordinate frames evaluate the resulting eigen-impedances. If they are passive, then the anomalous phases are only apparent and inconsequential. If they turn up with negative real parts, passivity has been violated and the measured function does not constitute impedance!

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