"REAL" ANALYSIS Is A DEGENERATE CASE of DISCRETE ANALYSIS

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The ICDEA Conferences: An Asymptotically Stable Recurrence

In one of yesterday's invited talks, Gerry Ladas outlined briefly the history of the previous conferences, and how successful they were. I totally agree. But no recursive sequence can exist without *initial conditions*. Hence, special credit and thanks should go to Saber Elaydi, whose initial idea it was, in 1994. Well done, Saber!

The current term in this sequence, $ICDEA_7$, is a huge success, thanks to the efficient and *friendly* organization of Bernd Aulbach and his gang of young assistants.

Discrete Analysis: Yet Another Cinderella Story

There are many ways to divide mathematics into *two-culture* dichotomies. An important one is the Discrete vs. the Continuous. Until almost the end of the 20th century, the continuous culture was dominant, as can be witnessed by notation. An important family of Banach spaces of *continuous* functions is denoted by L^p , with a *Capital L*, while their discrete analogs are denoted by the lowercase counterpart l^p. A function of a *continuous* variable is denoted by $f(x)$, where the continuous output, f, is written at the same level as the continuous input x, but if the input is discrete, then the function is given the derogatory name *sequence*, and written a_n, where the continuous output, a, looks down on the discrete input, n.

Indeed, the conventional wisdom, fooled by our misleading "physical intuition", is that the real world is *continuous*, and that discrete models are necessary evils for approximating the "real" world, due to the innate discreteness of the digital computer.

Ironically, the opposite is true. The

REAL REAL WORLDS (Physical and MATHEMATICAL) ARE DISCRETE .

Continuous analysis and geometry are just degenerate approximations to the discrete world, made necessary by the very limited resources of the human intellect. While discrete analysis is conceptually simpler (and truer) than continuous analysis, technically it is (usually) much more difficult. Granted, real geometry and analysis were necessary simplifications to enable humans to make progress in science and mathematics, but now that the *digital* Messiah has arrived, we can start to study discrete math in greater depth, and do *real*, i.e. discrete, analysis.

When we watch a movie we have the *appearance* of continuity, but in fact it consists of a discrete

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sequence of frames. When we look at a photograph, we have the *semblance* of a continuous image, but it is really a collection of *discrete* pixels. On a more fundamental level, we now know that energy and matter and probably time and space too, are discrete, as described so charmingly in Professor Trigiante's invited talk given in this conference two days ago.

Don't Worry, the Continuous Heritage is not a Total Waste

I will show later that while the efforts of Cauchy, Weierstrass, Dedekind and many others for a 'rigorous' foundation of analysis were misguided, a lot (and perhaps most) of continuous analysis can be salvaged as a special degenerate case of "discrete symbolic analysis".

My Perhaps Not So Foolish 'April Fool's Jokes'

As Dr. Peter Menacher, the eloquent and erudite *Oberbürgermeister* of Augsburg, said in yesterday's lovely reception at the magnificent (and mathematically tiled!) City Hall, Augsburg has seen many royalties, starting with its namesake, Emperor Augustus. Now each self-respecting king or duke had a *court jester*, also known as the *fool*. Of course, that 'fool' was usually the least foolish person in the whole kingdom, but his position enabled him to get away with much more freedom of speech than any other subject, since it was all 'in jest'.

Analogously, my own best ideas, far surpassing anything in my 'serious' papers, are contained in my annual *April Fool's* jokes, sent to my E-correspondents and posted on my website. This way I can express my 'off the wall' ideas without being considered a crackpot.

For example (2001), the idea of computerizing Tim Gowers's plan for studying the asymptotics of the Ramsey numbers $R(n, n)$, published in **Ekhad and Zeilberger's personal journal**, `http://math.rutgers.edu/~zeilberg/pj.html`, or my idea (1995) for proving the Riemann Hypothesis, also published there. But the most promising idea is in my 1999 'joke', entitled: 'Mathematical Genitalysis: A Powerful New Combinatorial Theory that Obviates Mathematical Analysis', that was also published in the 'Personal Journal'.

The main thrust of that article was the concept of 'symbolic discretization', akin to, but much more powerful than, 'numeric discretization'. I believe that this *crazy* idea has a great potential. But, even more important, it suggests a truly *rigorous* and *honest* foundation for the whole of mathematics.

Towards a FINITE (and hence RIGOROUS) Foundation of Mathematics

(i) The mathematical (and physical) universe is a huge (but FINITE) **DIGITAL** computer.

(ii) the traditional real line is a meaningless concept. Instead the *real* **REAL** 'line', is neither real, nor a line. It is a *discrete* necklace! In other words $R = hZ_p$, where p is a huge and unknowable (but fixed!) prime number, and h is a tiny, but *not* infinitesimal , 'mesh size'.

Hence even the *potential infinity* is a meaningless concept.

Since h is so tiny, and p is so large, and both are unknowable, they should be denoted by symbols, like h and c in physics, and π and e in math. This also explains why traditional real analysis did so well in modeling nature, the same way that Newtonian physics approximated nature so well, as long as you didn't travel too fast, or penetrated with too high energy.

It is probably possible to *deconstruct* the whole of traditional mathematics along finitism, but I doubt whether it is worth the effort. Let's just redo a few basic definitions.

The True Derivative

Leibnitz and Newton defined the derivative by

$$Df(x) = \frac{f(x+h) - f(x)}{h} \quad ,$$

where h is *infinitesimal*, whatever that means. Then Cauchy and Weierstrass found a 'rigorous' definition:

$$Df(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad ,$$

using the notion of *limit*, whatever it means. But the only TRUE definition is:

$$Df(x) := \frac{f(x+h) - f(x)}{h} \quad ,$$

where h is the Fundamental mesh size, a Mathematical Universal constant, that unlike Planck's constant, we will never know, but it is very tiny. Since it is so tiny, we keep it as a *symbol*, but remember that it *signifies* a fixed constant.

When Einstein discovered General Relativity he already had the mathematical framework for it, Riemannian Geometry. Luckily, discrete calculus also already exits, but there $D = \Delta_h$. (Speaking of Δ, I love the logo of this conference that is a graphic pun featuring the finite difference operator Δ turned into the Pascal triangle mod 2 fractal.)

Let's recall the

Product Rule:

$$D(fg) = f(Dg) + (Df)g + h(Df)(Dg) \quad ,$$

which implies Leibnitz's rule:

$$D^n(fg) = \sum_{i+j+r=n} \frac{n!}{i!j!r!}(D^{i+r}f)(D^{j+r}g)h^r \quad .$$

Integration is *not* a 'limit' of Riemann sums, but rather *is* a Riemann sum:

$$\int_a^b f(x)dx := h \cdot \sum_{i=0}^{(b-a)/h} f(a+ih) \quad .$$

REAL (i.e. discrete) analysis is conceptually simpler than traditional 'real' (continuous) analysis, and of course is much truer. But it is, on the whole, technically more difficult. Hence 'Naked Brain' humans had no choice but to pursue the latter kind.

My First Love: DISCRETE Analytic Functions

These are functions defined on the lattice $hZ + ihZ$, satisfying:

$$\frac{f(m+(1+i)h) - f(m)}{(1+i)h - 0} = \frac{f(m+ih) - f(m+h)}{ih - h} \quad . \qquad (Duffin)$$

In other words, the two "derivatives" along the two diagonals of any unit square $\{m, m+h, m+(1+i)h, m+ih\}$ are the same. Now $(Duffin)$ can be rewritten as

$$\frac{f(m) + f(m+h)}{2} \cdot h + \frac{f(m+h) + f(m+h+ih)}{2} \cdot ih +$$

$$\frac{f(m+h+ih) + f(m+ih)}{2} \cdot (-h) + \frac{f(m+ih) + f(m)}{2} \cdot (-ih) \;=\; 0 \quad ,$$

which means that the "integral" of an analytic function around any fundamental square is zero, and since the "integral" over any closed simple (discrete) 'curve' is a (finite!) sum of integrals over fundamental squares, we have immediately both Cauchy's and Morera's theorems!

The theory of Discrete Analytic functions was initiated by Jacqueline Ferrand-Lelong[2] . Dick Duffin made it into a full-fledged theory, and it was further developed by myself (in my Ph.D. thesis), and several others.

The main stumbling block in the further development of the theory of Discrete Analytic Functions is the fact that the property of being discrete-analytic is not preserved under multiplication. But using the discrete Leibnitz rule one can express the derivative of a product, and then the product is "almost analytic". So I am sure that the full arsenal of *continuous* complex analysis can be discretized, but the details might be too complicated for humans.

Continuous Analysis is a DEGENERATE (not LIMITING) case of Discrete Symbolic Analysis

So one should be able to develop a full theory for discrete analysis, with an *arbitrary* mesh-size h. But *now* we declare that h does not represent a specific quantity (the mesh-size), but rather represents itself, i.e. stays as a symbol. Then continuous analysis is just the *degenerate* case $h = 0$ of the full h-theory.

So the following is the only valid definition of the classical derivative

$$f'(x) := \frac{f(x+h) - f(x)}{h}\Big|_{h=0} \quad ,$$

[2] A brilliant mathematician. She was the classmate of Roger Apéry, and tied with him for first-place, but unlike Apéry, Ferrand was a `tala` (one of those that `von(t a la) messe`).

which in Maple would read: `subs(h=0, simplify((f(x+h)-f(x))/h))`.

For example,

$$(x^2)' = \frac{(x+h)^2 - x^2}{h}|_{h=0} = \frac{2xh + h^2}{h}|_{h=0} = 2x + h|_{h=0} = 2x \quad .$$

Another example is

$$(a^x)' = \frac{a^{x+h} - a^x}{h}|_{h=0} = a^x \cdot \frac{a^h - 1}{h}|_{h=0} = a^x \ln a$$

where, *by definition*

$$\ln a := \frac{a^h - 1}{h}|_{h=0} \quad .$$

Using this definition, we can recover all the properties of ln, for example:

$$\ln(ab) = \frac{(ab)^h - 1}{h}|_{h=0} = \frac{((ab)^h - b^h) + (b^h - 1)}{h}|_{h=0} = \frac{b^h(a^h - 1) + (b^h - 1)}{h}|_{h=0} =$$

$$b^0 \cdot \frac{a^h - 1}{h}|_{h=0} + \frac{b^h - 1}{h}|_{h=0} = \ln a + \ln b \quad .$$

Neo-Pythagoreanism or: Anaxogoras deserved to be drowned

It is utter nonsense to say that $\sqrt{2}$ is *irrational*, because this presupposes that it exists, as a *number* or *distance*. The truth is that there is no such number or distance. What does exist is the *symbol*, which is just shorthand for an *ideal* object x that satisfies $x^2 = 2$. In Maple notation: `sqrt(2)=RootOf(x**2=2)`.

The fundamental metric in plane geometry is not *length* but *area*. In the discrete plane $(hZ)^2$, the area of a region is simply the number of lattice points in the interior of that region. So the *true* Pythagorean theorem is not $c^2 = a^2 + b^2$, but rather $c^2 = a^2 + b^2 + O(h)$. The notion of *distance* is usually not defined. What *does* make sense is *distance squared* between point A and point B, which, *by definition*, is the area (i.e. the number of lattice points in the interior) of the square one of whose sides is AB.

Interface with Numerics: Interval Arithmetics

Whenever one wants to do fully rigorous analytical calculations on the computer, one uses *interval arithmetics*, where one represents a 'real' number by a closed (or open) interval it is known to belong to. While this is done for computational reasons, we can also do it philosophically, and only talk about intervals $[a, b]$, where a and b are *rational* (and hence meaningful) numbers.

The statements $e = 2.718281828...$ and $\pi = 3.14159...$ are *meaningless*, while $e = 2.718281828$, $\pi = 3$, $\pi^2 = 10$, and $\pi = 355/113$, while wrong, are at least *meaningful*. On the other hand $\pi \in [31415/10000, 31416/10000]$ is correct and meaningful.

One has the obvious rules: $[a, b] + [c, d] = [a + c, b + d]$ etc.

Blessed Are The ∆iffernce Equations for They Shall Inherit Math

Project 1: Use Difference Equations to prove the Riemann Hypothesis.

I believe that the fundamental equations, both theoretically and practically, are *difference* equations rather than *differential* equations. And indeed they are all over mathematics, and will become more and more prominent with the advent of computers, both as substitutes to differential equations and for their own sake.

Recall that the prime number theorem $\pi(x) - x/\log(x) = o(x/\log(x))$ is equivalent to $R(x) := \psi(x) - x = o(x)$, where

$$\psi(x) := \sum_{p^m \leq x} \log p \quad .$$

Tchebychev proved that, for large x, $A_1 x \leq \psi(x) \leq A_2 x$, by using the extremely simple *recurrence*:

$$(\log 2) \cdot x + O(log(x)) = \psi(x) - \psi(x/2) + \psi(x/3) - \psi(x/4) + \dots \quad ,$$

that yields $A_1 = \log 2$ and $A_2 = 2 \log 2$.

This was considerably improved by Tchebychev himself and James Joseph Sylvester, who found other, more complicated, but still linear, recurrences, that brought A_1 up and A_2 down.

The next step was realized by Erdös and Selberg, who combined two simple recursive inequalities for $R(x) := \psi(x) - x$. The first one is linear (in $|R(x)|$):

$$|R(x)| \leq \frac{1}{\log x} \sum_{n \leq x} |R(x/n)| + O(\frac{x \log\log x}{\log x}) \quad ,$$

while the second one is quadratic:

$$\sum_{n \leq x} \frac{\log n}{n} R(n) = - \sum_{n \leq x} \frac{1}{n} R(n) R(x/n) + O(x) \quad ,$$

from which it follows that $|R(x)| \leq \sigma x$ for $x > x_\sigma$, for every $\sigma > 0$. Hence $R(x) = o(x)$.

Exercise: Find more powerful recurrences (alias *difference equations*) that would imply the stronger statement $R(x) = O(x^{1/2+\epsilon})$, for every $\epsilon > 0$. Collect \$1000000.

Project 2: Find a Rigorous Proof of Fermat's Last Theorem

Andrew Wiles's alleged 'proof' of FLT, while a crowning *human* achievement, is not rigorous, since it uses continuous analysis, which is meaningless. I do believe that it is possible to convert it into a rigorous proof in an analogous way to converting a proof in combinatorics that uses *convergent* (and hence meaningless) power series into a proof that uses *formal* (and hence fully kosher) power series. But the end-result would be very artificial.

I hope that one of you would be able to find a completely elementary (and hence fully rigorous) proof of FLT using recurrences, possibly along the following lines.

Let's define

$$W(n;a,b,c) := (a^n + b^n - c^n)^2 \quad .$$

W satisfies lots of recurrences, e.g.

$$\Delta_a^{2n+1} W = 0 \quad .$$

I am almost sure that there exists a *polynomial*, discoverable by computer, with *positive coefficients* such that

$$W(n;a,b,c) = P(W(n;a-1,b,c)\,,\,W(n;a,b-1,c)\,,\,\dots\,,\,W(n-1;a,b,c),\dots)$$

for $n > 3$. Since $W > 0$ for $n = 3$ and $abc > 0$, FLT would follow. Of course it suffices that P is a rational function both whose numerator and denominator have positive coefficients.

Analogy:

Theorem (Askey-Gasper 1977)

Let $A(m,n,k)$ be the Maclaurin coefficients of $R := (1-x-y-z+4xyz)^{-1}$, i.e.:

$$\frac{1}{1-x-y-z+4xyz} = \sum_{m,n,k \geq 0} A(m,n,k) x^m y^n z^k \quad ,$$

then $A(m,n,k) > 0$ for all $m,n,k \geq 0$.

The obvious recurrence

$$A(m,n,k) = A(m-1,n,k) + A(m,n-1,k) + A(m,n,k-1) - 4A(m-1,n-1,k-1) \quad ,$$

is *useless* because of the *minus* sign on the right side, but Gillis and Kleeman (1979) came up with another recurrence:

$$mA(m,n,k) = (m+n-k)A(m-1,n,k) + 2(m-n+k-1)A(m-1,n,k-1) \quad ,$$

which immediately implies positivity, by induction, in $m \geq n \geq k \geq 0$, and by symmetry, for all $m,n,k \geq 0$. This recurrence is ingenious, but once found, is completely routine. Just check (or let Maple do it if you are too lazy) that

$$\frac{\partial}{\partial x} R = (1+2z)(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 1)R + 2(y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z})R \quad .$$

Now this differential equation satisfied by R is not only routine to *verify*, it is also routine to *discover*, once you tell the computer what to look for. Let's hope that a similar recurrence would be found for FLT.

Philosophical Conclusion

I am not a professional philosopher of mathematics, nor an expert logician or foundationalist, but I think that the philosophy that I am advocating here is called *ultrafinitism.* If I understand it correctly, the ultrafinitists deny the existence of *any* infinite, not even the potential infinity, but their motivation is 'naturalistic', i.e. they believe in a 'fade-out' phenomenon when you keep counting.

Myself, I don't care so much about the natural world. I am a platonist, and I believe that *finite* integers, *finite* sets of *finite* integers, and all *finite* combinatorial structures have an existence of their own, regardless of humans (or computers). I also believe that *symbols* have an independent existence. What is completely meaningless is any kind of *infinite*, actual or potential.

So I deny even the existence of the Peano axiom that every integer has a successor. Eventually we would get an overflow error in the big computer in the sky, and the sum and product of any two integers is well-defined only if the result is less than p, or if one wishes, one can compute them modulo p. Since p is so large, this is not a practical problem, since the overflow in our earthly computers comes so much sooner than the overflow errors in the big computer in the sky.

However, one can still have 'general' theorems, provided that they are interpreted correctly. The phrase 'for *all* positive integers' is meaningless. One should replace it by: 'for finite or symbolic integers'. For example, the statement: "$(n+1)^2 = n^2+2n+1$ holds for all integers" should be replaced by: "$(n+1)^2 = n^2+2n+1$ holds for finite or symbolic integers n" . Similarly, Euclid's statement: 'There are infinitely many primes' is meaningless. What is true is: if $p_1 < p_2 < \ldots < p_r < p$ are the first r finite primes, and if $p_1 p_2 \ldots p_r + 1 < p$, then there exists a prime number q such that $p_r + 1 \leq q \leq p_1 p_2 \ldots p_r + 1$. Also true is: if p_r is the 'symbolic r^{th} prime', then there is a symbolic prime q in the discrete symbolic interval $[p_r + 1, p_1 p_2 \ldots p_r + 1]$.

By hindsight, it is not surprising that there exist undecidable propositions, as meta-proved by Kurt Gödel. Why should they be decidable, being meaningless to begin with! The tiny fraction of first-order statements that are decidable are exactly those for which either the statement itself, or its negation, happen to be true for *symbolic* integers. A priori, every statement that starts "for every integer n" is completely meaningless.

I hope to expand this line of thought that may be called 'ultrafinite computerism' or 'ansatz-centric formalism' (as opposed to Hilbert's *logocentric* formalism) in the future.