Vector-Valued Maclaurin Inequalities

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Abstract

We investigate a Maclaurin inequality for vectors and its connection to an Aleksandrov-type inequality for parallelepipeds.

1 Introduction

The classical Maclaurin inequality compares consecutive symmetric sums for any sequence of positive real numbers:

Theorem 1 (Maclaurin inequality). *For any sequence of positive real numbers $x_1, \dots, x_m$ and any $1 < k \leqslant m$, the following inequality holds:*

$$\left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[m]} x_{i_1}\cdots x_{i_k}}{\binom{m}{k}} \right)^{\frac{1}{k}} \leqslant \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\}\subset[m]} x_{i_1}\cdots x_{i_{k-1}}}{\binom{m}{k-1}} \right)^{\frac{1}{k-1}}.$$

This was first proved in [15] and a proof of this result using elementary methods can be found also in [8]. The Maclaurin inequality can be seen as a refinement of the arithmetic-geometric mean inequality by noting that the geometric mean and arithmetic mean appear as the smallest and largest quantities respectively in the following chain of inequalities:

$$(x_1\cdots x_m)^{\frac{1}{m}} \leqslant \left(\frac{\sum_{\{i_1,\dots,i_{m-1}\}\subset[m]} x_{i_1}\cdots x_{i_{m-1}}}{\binom{m}{m-1}} \right)^{\frac{1}{m-1}} \leqslant \cdots \leqslant \left(\frac{\sum_{\{i,j\}\subset[m]} x_i x_j}{\binom{m}{2}} \right)^{\frac{1}{2}} \leqslant \frac{\sum_{i=1}^m x_i}{m},$$

which follows directly from Theorem 1.

In this note, we explore a variant of this classical result, with sequences of numbers replaced by families of vectors, and the standard product replaced by the wedge product operator. More precisely, let $v_1, \dots, v_m \in \mathbb{R}^d$ with $d \leqslant m$, and for any $1 \leqslant i_1 < \cdots < i_k \leqslant m$ with $k \leqslant d$ denote by $|v_{i_1} \wedge \cdots \wedge v_{i_k}|$ the k-dimensional volume of the parallelotope spanned by $v_{i_1}, \dots, v_{i_k}$. We are interested in "vector-valued" inequalities of the form

$$\left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|^p}{\binom{m}{k}} \right)^{\frac{1}{kp}} \leqslant \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|^p}{\binom{m}{k-1}} \right)^{\frac{1}{(k-1)p}}, \tag{1}$$

with $p \in [0, \infty]$ and $2 \leqslant k \leqslant d$. Note that if $m = d$ and $v_1, \dots, v_m$ are orthogonal, then each term $|v_{i_1} \wedge \cdots \wedge v_{i_k}|^p$ will just be equal to a k-fold product of numbers, namely $\|v_{i_1}\|^p \cdots \|v_{i_k}\|^p$. It follows that (1) reduces to a special case of the classical Maclaurin inequality for any $p \in (0, \infty)$.

Given a general family of vectors $v_1, \dots, v_m \in \mathbb{R}^d$, the value of p plays a more important role. Using elementary results from linear algebra, we are able to establish (1) for $p = 2$ and $m = d$:

Theorem 2. *For any d-tuple of vectors $v_1, \dots, v_d \in \mathbb{R}^d$ and any $1 < k \leqslant d$, the following inequality holds:*

$$\left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|^2}{\binom{d}{k}} \right)^{\frac{1}{k}} \leqslant \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\}\subset[d]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|^2}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}.$$

Moreover, taking limits as $p \to \infty$, we can write

$$\lim_{p\to\infty} \left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|^p}{\binom{m}{k}} \right)^{\frac{1}{p}} = \max_{\{i_1,\dots,i_k\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|.$$

Again using purely linear algebra, namely Szasz's inequality for subdeterminants, we are able to prove the following endpoint case:

Theorem 3. *Fix vectors $v_1, \dots, v_m \in \mathbb{R}^d$ with $1 \leqslant d \leqslant m$. Then, for any $1 < k \leqslant d$, the following inequality holds:*

$$\left(\max_{\{i_1,\dots,i_k\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \right)^{\frac{1}{k}} \leqslant \left(\max_{\{i_1,\dots,i_{k-1}\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}| \right)^{\frac{1}{k-1}}. \tag{2}$$

By a similar argument, we also prove (1) for $p = 0$:

Theorem 4. *Fix vectors $v_1, \dots, v_m \in \mathbb{R}^d$ with $1 \leqslant d \leqslant m$. Then, for any $1 < k \leqslant d$, the following inequality holds:*

$$\left(\prod_{\{i_1,\dots,i_k\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \right)^{\frac{1}{\binom{m}{k}k}} \leqslant \left(\prod_{\{i_1,\dots,i_{k-1}\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}| \right)^{\frac{1}{\binom{m}{k-1}(k-1)}}.$$

It is not difficult to verify that (1) fails to hold in general for negative values of p.

If we take $p = 1$, it seems more difficult to establish the desired inequality for a general family of vectors. However, in the case where $m = d$, we have some partial results:

Theorem 5. *If $m = d$, then for any $v_1, \dots, v_m \in \mathbb{R}^d$, inequality (1) holds with $p = 1$ and $k = 2, 3, d$ in all dimensions d.*

Using a certain duality between families of vectors, one can also prove the case for $p = 1$ and $k = d-1$, which includes the only remaining case in $\mathbb{R}^5$ where $k = 4$ see [13], Section 6 for details. To prove Theorem 5, in each case we essentially construct a new family of orthogonal vectors $\tilde{v}_1, \dots, \tilde{v}_m \in \mathbb{R}^d$ such that

$$\left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|}{\binom{m}{k}} \right)^{\frac{1}{k}} \leqslant \left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[m]} |\tilde{v}_{i_1} \wedge \cdots \wedge \tilde{v}_{i_k}|}{\binom{m}{k}} \right)^{\frac{1}{k}},$$

and

$$\left(\frac{\sum_{\{i_1,\dots,i_{k-1}\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|}{\binom{m}{k-1}} \right)^{\frac{1}{k-1}} \geqslant \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\}\subset[m]} |\tilde{v}_{i_1} \wedge \cdots \wedge \tilde{v}_{i_{k-1}}|}{\binom{m}{k-1}} \right)^{\frac{1}{k-1}}.$$

The desired result then follows by a simple application of Theorem 1.

In light of these results, we conjecture that the vector-valued Maclaurin inequalities should hold in the full range $0 \leqslant p \leqslant \infty$:

Conjecture 1 (Vector-valued Maclaurin inequality). *Fix vectors $v_1, \dots, v_m \in \mathbb{R}^d$ with $1 \leqslant d \leqslant m$. Then for all $p \in [0, \infty]$ and $2 \leqslant k \leqslant d$, the following inequality holds:*

$$\left(\frac{\sum_{\{i_1,\dots,i_k\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_k}|^p}{\binom{m}{k}} \right)^{\frac{1}{kp}} \leqslant \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|^p}{\binom{m}{k-1}} \right)^{\frac{1}{(k-1)p}},$$

with equality if and only if $m = d$ and the vectors v_i form an orthonormal basis.

It should be noted that for $p = 1$, the vector-valued Maclaurin inequality is of particular interest, as it turns out to be closely related to the far-reaching Aleksandrov–Fenchel inequality from convex geometry. By a simple argument one can deduce the classical Maclaurin inequality as a consequence of Newton's inequality, and similarly one would be able to deduce the vector-valued Maclaurin inequality for $p = 1$ from a corresponding vector-valued version of Newton's inequality of the following form

$$\left(\frac{\sum_{\{i_1,\dots,i_k\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{m}{k}} \right)^2 \geqslant \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\binom{m}{k-1}} \right) \left(\frac{\sum_{\{i_1,\dots,i_{k+1}\} \subset [m]} |v_{i_1} \wedge \dots \wedge v_{i_{k+1}}|}{\binom{m}{k+1}} \right), \tag{3}$$

where $2 \leqslant k \leqslant d - 1$. It is worth noting the explicit connection between Newton's inequality and the Aleksandrov–Fenchel inequality. Up until this point, in the literature, the Aleksandrov–Fenchel inequality has been referred to as a Newton-type inequality, simply because it is of the same form (square greater than a product).

To illustrate the connection between the vector-valued Maclaurin inequality and the Aleksandrov–Fenchel inequality, let us denote by P the following Minkowski sum of line segments

$$P = \sum_{j=1}^{m} \frac{1}{2}[-v_j, v_j].$$

Using this notation, Conjecture 1 with $p = 1$ exactly states that for $1 < k \leqslant d$ we have

$$\left(\frac{V_k(P)}{\binom{m}{k}} \right)^{\frac{1}{k}} \leqslant \left(\frac{V_{k-1}(P)}{\binom{m}{k-1}} \right)^{\frac{1}{k-1}}, \tag{4}$$

or

$$\left(\frac{V_k(P)}{V_k(C_m)} \right)^{\frac{1}{k}} \leqslant \left(\frac{V_{k-1}(P)}{V_{k-1}(C_m)} \right)^{\frac{1}{k-1}},$$

where V_k denotes the k-th intrinsic volume and C_m is the m-dimensional unit cube. This is a general isoperimetric-type inequality. For example, the case $m = d$ and and $k = d$, which is proved here, says that among all parallelepipeds (non necessarily orthogonal) with the same volume, the cube has the smallest surface area. This particular case was first proved by Hadwiger in [7] (see also [6]) using Steiner symmetrisation. Moreover, if (4) holds for an arbitrary sum of line segments, then one would recover the dimension free estimate of McMullen [11] restricted to the class of zonoids. Furthermore, (4) is related to isoperimetric-type inequalities proved in [9].

Despite the fact that we don't have a proof for the sharp inequality (4), we are able to prove it with a constant that it is bounded by an absolute constant that doesn't depend on the dimension.

Theorem 6. *For any d-tuple of vectors $v_1, \dots, v_d \in \mathbb{R}^d$ and any $2 < k \leqslant d$, the following inequality holds:*

$$\left(\frac{\sum_{\{i_1,\dots,i_k\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \leqslant \frac{2(d-k+1)}{(d-k+2)} \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\} \subset [d]} |v_{i_1} \wedge \dots \wedge v_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}$$

Note that the constant appearing on the right-hand side is greater than 1, but smaller than 2.

Structure of paper

In Section 2 we introduce some relevant notation and terminology. Section 3 is dedicated to proving Theorems 2, 3 and 4. In Section 4, we discuss a general approach for attempting to establish (1) with $p = 1$, and prove the special cases listed in Theorem 5. At the beginning of Section 5, we introduce tools from convex geometry which allow us to rewrite the vector-valued Maclaurin inequality with $p = 1$ in terms of convex bodies and mixed volumes. In Section 5.1, we then use these tools to establish Theorem 6.

2 Notation and background information

We work in $\mathbb{R}^d$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$ and we fix an orthonormal basis $\{e_1, \dots, e_d\}$. We denote by B_2^d and S^{d-1} the Euclidean unit ball and sphere in $\mathbb{R}^d$ respectively. We write σ for the normalised rotationally invariant probability measure on S^{d-1} and ν for the Haar probability measure on the orthogonal group $O(d)$. Let $G_{d,k}$ denote the Grassmannian of all k-dimensional subspaces of $\mathbb{R}^d$. Then, $O(d)$ equips $G_{d,k}$ with a Haar probability measure $\nu_{d,k}$. The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$.

Let $\mathcal{K}_d$ denote the class of all non-empty compact convex subsets of $\mathbb{R}^d$. If $K \in \mathcal{K}_d$ has non-empty interior, we will say that K is a convex body. If $A \in \mathcal{K}_d$, we will denote by $|A|$ the volume of A in the appropriate affine subspace unless otherwise stated. The volume of B_2^d is denoted by κ_d. We say that a convex body K in $\mathbb{R}^d$ is symmetric if $x \in K$ implies that $-x \in K$, and that K is centred if its centre of mass $\frac{1}{|K|}\int_K x\,dx$ is at the origin. The support function of a convex body K is defined by $h_K(y) = \max\{\langle x, y \rangle : x \in K\}$. For any $E \in G_{d,k}$ we denote by $E^\perp$ the orthogonal subspace of E, i.e. $E^\perp = \{x \in \mathbb{R}^d : \langle x, y \rangle = 0 \text{ for all } y \in E\}$. In particular, for any $u \in S^{d-1}$ we define $u^\perp = \{x \in \mathbb{R}^d : \langle x, u \rangle = 0\}$. The section of $K \in \mathcal{K}_d$ with a subspace E of $\mathbb{R}^d$ is $K \cap E$, and the orthogonal projection of K onto E is denoted by $P_E(K)$.

Mixed volumes are introduced by a classical theorem of Minkowski which describes the way volume behaves with respect to the operations of addition and multiplication of compact convex sets by non-negative reals: if $K_1, \dots, K_N \in \mathcal{K}_d$, $N \in \mathbb{N}$, then the volume of $t_1 K_1 + \cdots + t_N K_N$ is a homogeneous polynomial of degree d in $t_i \geqslant 0$ (see [3] and [16]):

$$\left|t_1 K_1 + \cdots + t_N K_N\right| = \sum_{1 \leqslant i_1, \dots, i_d \leqslant N} V(K_{i_1}, \dots, K_{i_d}) t_{i_1} \dots t_{i_d}, \tag{5}$$

where the coefficients $V(K_{i_1}, \dots, K_{i_d})$ are invariant under permutations of their arguments. The coefficient $V(K_{i_1}, \dots, K_{i_d})$ is called the mixed volume of the d-tuple $(K_{i_1}, \dots, K_{i_d})$. We will often use the fact that V is positive, linear with respect to each of its arguments, and that $V(K, \dots, K) = |K|$ (the d-dimensional Lebesgue measure of K) for all $K \in \mathcal{K}_d$.

Steiner's formula is a special case of Minkowski's theorem. If $K \in \mathcal{K}_d$ then the volume of $K + tB_2^d$, $t > 0$, can be expanded as a polynomial in t:

$$|K + tB_2^d| = \sum_{k=0}^{d} \binom{d}{k} W_k(K) t^k, \tag{6}$$

where $W_k(K) := V(K[d-k], B_2^d[k])$ is the k-th quermassintegral of K. Moreover, for $k = 1, \dots, d$, the k-th intrinsic volume of a convex body $L \subset \mathbb{R}^d$ is defined as

$$V_k(L) = \binom{d}{k} \frac{V(L[k], B_2^d[d-k])}{\kappa_{d-k}}.$$

The Aleksandrov-Fenchel inequality states that if $K, L, K_3, \dots, K_d \in \mathcal{K}_d$, then

$$V(K, L, K_3, \dots, K_d)^2 \geqslant V(K, K, K_3, \dots, K_d) V(L, L, K_3, \dots, K_d). \tag{7}$$

In particular, this implies that the sequence $(W_0(K), \dots, W_d(K))$ is log-concave. From the Aleksandrov-Fenchel inequality one can recover the Brunn-Minkowski inequality as well as the following generalisation for the quermassintegrals:

$$W_k(K+L)^{\frac{1}{d-k}} \geqslant W_k(K)^{\frac{1}{d-k}} + W_k(L)^{\frac{1}{d-k}}, \qquad k = 0, \dots, d-1. \tag{8}$$

We write $S(K)$ for the surface area of K. From Steiner's formula and the definition of surface area we see that $S(K) = dW_1(K)$. Finally, let us mention Kubota's integral formula

$$W_k(K) = \frac{\kappa_d}{\kappa_{d-k}} \int_{G_{d,d-k}} |P_E(K)| \, d\nu_{d,d-k}(E), \qquad 1 \leqslant k \leqslant d-1. \tag{9}$$

The case $k = 1$ is Cauchy's surface area formula

$$S(K) = \frac{\kappa_d}{d\kappa_{d-1}} \int_{S^{d-1}} |P_{u^\perp}(K)| \, d\sigma(u). \tag{10}$$

We refer to the books [4] and [16] for basic facts from the Brunn-Minkowski theory and to the books [1] and [2] for basic facts from asymptotic convex geometry.

3 Principal minors and Szasz's inequality

In this section we introduce some elementary tools from linear algebra, and use them to prove Theorems 2, 3 and 4.

Vector-valued Maclaurin inequality with $p = 2$ Firstly, for any family of vectors $v_1 \dots, v_d \in \mathbb{R}^d$, we write the square of each k-dimensional volume $|v_{i_1} \wedge \cdots \wedge v_{i_k}|^2$ as the determinant of a $k \times k$ submatrix of some fixed $d \times d$ matrix. Let us introduce the notion of a principal minor:

Definition 1 (Principal minors)**.** Let M be an $n \times n$ matrix. For $S \subset [n]$, define M_S to be submatrix constructed by removing rows and columns with indices not in S. The set of principal submatrices of M is defined as $\{M_S \,|\, S \subset [n]\}$. Furthermore, the set of principal minors of M is defined as $\{\det(M_S) \mid S \subset [n]\}$. For $1 \leqslant k \leqslant n$, we define the principal k-submatrices and principal k-minors of M by adding the condition that $|S| = k$.

Let A denote the square $d \times d$ matrix with columns $v_1, \dots, v_d$ and let B be the $d \times k$ matrix with columns $v_1, \dots, v_k$ for some $1 \leqslant k \leqslant d$, then we can write

$$|v_1 \wedge \cdots \wedge v_k|^2 = \det(B^T B).$$

A short proof of this identity can be found in [10]. The matrix $B^T B$ is a principal k-submatrix of $A^T A$, and can be constructed by removing the last $(d-k)$ rows and columns. In this way we see that the sum of terms $|v_{i_1} \wedge \cdots \wedge v_{i_k}|^2$ over all $1 \leqslant i_1 < \cdots < i_k \leqslant d$ is equal to the sum of all principal k-minors of $A^T A$. The next lemma allows us to work with the sum of all principal minors of $A^T A$:

Lemma 1 (Sum of principal minors)**.** *Let M be a $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct). For $1 \leqslant k \leqslant n$ we have that*

$$\sum_{|S|=k} \det(M_S^T M_S) = \sum_{|S|=k} \prod_{i \in S} \lambda_i^2.$$

A proof of this lemma can be found in [12]. Now we are ready to establish the vector-valued Maclaurin inequality with $p = 2$:

Proof of Theorem 2. As before, let A denote the square matrix with columns $v_1, \dots, v_d$ and (not necessarily distinct) eigenvalues $\lambda_1, \dots, \lambda_d$. Then for $1 \leqslant i_1 < \cdots < i_k \leqslant d$, define $B_{i_1,\dots,i_k}$ to be the $d \times k$

matrix with columns $v_{i_1}, \dots, v_{i_k}$. Now we use Lemma 1 along with Theorem 1, which is precisely the classical Maclaurin inequality, to write

$$\begin{aligned}
\left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|^2}{\binom{d}{k}}\right)^{\frac{1}{k}} &= \left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[d]} \det\left(B_{i_1,\dots,i_k}^T B_{i_1,\dots,i_k}\right)}{\binom{d}{k}}\right)^{\frac{1}{k}} \\
&= \left(\frac{\sum_{\{i_1,\dots,i_k\}\subset[d]} \lambda_{i_1}^2 \cdots \lambda_{i_k}^2}{\binom{d}{k}}\right)^{\frac{1}{k}} \\
&\leqslant \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\}\subset[d]} \lambda_{i_1}^2 \cdots \lambda_{i_{k-1}}^2}{\binom{d}{k-1}}\right)^{\frac{1}{k-1}} \\
&= \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\}\subset[d]} \det\left(B_{i_1,\dots,i_{k-1}}^T B_{i_1,\dots,i_{k-1}}\right)}{\binom{d}{k-1}}\right)^{\frac{1}{k-1}} \\
&= \left(\frac{\sum_{\{i_1,\dots,i_{k-1}\}\subset[d]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|^2}{\binom{d}{k-1}}\right)^{\frac{1}{k-1}} .
\end{aligned}$$

This proves the desired result. $\square$

Endpoint cases $p = 0$ and $p = \infty$ To begin, let us state a result of Szasz regarding principal minors:

Lemma 2 (Szasz's inequality). *Let M be some $n \times n$ matrix. For $1 < k < n$ the following inequality holds*

$$\left(\prod_{|A|=k} \det(M_A)\right)^{\frac{1}{\binom{n-1}{k-1}}} \leqslant \left(\prod_{|B|=k-1} \det(M_B)\right)^{\frac{1}{\binom{n-1}{k-2}}} .$$

A proof of this using elementary methods can be found in [10]. Simple applications of this lemma allow us to deduce the vector-valued Maclaurin inequality with $p = 0$ and $p = \infty$.

Proof of Theorem 4. Let A denote the $d \times m$ matrix with columns $v_1, \dots, v_m$ and let $A_{i_1,\dots,i_k}$ be the $d \times k$ matrix with columns $v_{i_1}, \dots, v_{i_k}$ for some $1 \leqslant k \leqslant d$, then we have

$$|v_{i_1} \wedge \cdots \wedge v_{i_k}|^2 = \det(A_{i_1,\dots,i_k}^T A_{i_1,\dots,i_k}).$$

As we mentioned earlier, $A_{i_1,\dots,i_k}^T A_{i_1,\dots,i_k}$ can also be seen as a principal submatrix of $A^T A$, which is an $m \times m$ matrix. Hence, by Szasz's lemma we have

$$\left(\prod_{\{i_1,\dots,i_k\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|^2\right)^{\frac{1}{\binom{m-1}{k-1}}} \leqslant \left(\prod_{\{i_1,\dots,i_{k-1}\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|^2\right)^{\frac{1}{\binom{m-1}{k-2}}} .$$

Taking square roots, and noting that $\binom{m-1}{k-1} = \binom{m}{k} \cdot \frac{k}{m}$ and $\binom{m-1}{k-2} = \binom{m}{k-1} \cdot \frac{k-1}{m}$, the above inequality can be written as

$$\left(\left(\prod_{\{i_1,\dots,i_k\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|\right)^{\frac{1}{\binom{m}{k}}}\right)^{\frac{1}{k}} \leqslant \left(\left(\prod_{\{i_1,\dots,i_{k-1}\}\subset[m]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|\right)^{\frac{1}{\binom{m}{k-1}}}\right)^{\frac{1}{k-1}} , \tag{11}$$

as required. $\square$

Remark 1. In (11) we see that we have geometric means appearing inside the first set of parenthesis on both sides. Intriguingly, simply replacing these geometric means by arithmetic means reveals the vector valued Maclaurin inequality with $p = 1$. So, if we think of the vector-valued Maclaurin inequality with $p = 1$ as a chain of inequalities for a sequence of arithmetic means, then (11) can be thought of as an analogous chain of inequalities for the corresponding geometric means.

The case for $p = \infty$ also follows directly from Szasz's inequality.

Proof of Theorem 3. Let $i_1 < \cdots < i_k \leqslant m$ be the indices where the left hand side is maximised and let B be the matrix with columns $v_{i_1}, \ldots, v_{i_k}$. Now, using Szasz's inequality for $M = B^T B$, $n = k$ and taking square root to both sides, we get

$$\begin{aligned} |v_{i_1} \wedge \cdots \wedge v_{i_k}| &\leqslant \left(\prod_{j=1}^{k} |v_{i_1} \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_{i_k}| \right)^{\frac{1}{k-1}} \\ &\leqslant \left(\prod_{j=1}^{k} \left(\max_{\{i_1,\ldots,i_{k-1}\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}| \right) \right)^{\frac{1}{k-1}} \\ &= \left(\max_{\{i_1,\ldots,i_{k-1}\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}| \right)^{\frac{k}{k-1}}, \end{aligned}$$

which concludes the proof. □

4 Partial results for $p = 1$ and monotonicity argument

In the next section we prove the special cases of vector-valued Maclaurin inequalities with $p = 1$ and $m = d$ listed in Theorem 5. Our method is somewhat inspired by a monotonicity argument given in [8] to prove the classical Maclaurin inequality. Given vectors $v_1, \ldots, v_d \in \mathbb{R}^d$, we attempt to construct a second family of orthogonal vectors $\tilde{v}_1, \ldots, \tilde{v}_d \in \mathbb{R}^d$, such that

$$\left(\frac{\sum_{\{i_1,\ldots,i_k\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \leqslant \left(\frac{\sum_{\{i_1,\ldots,i_k\} \subset [d]} |\tilde{v}_{i_1} \wedge \cdots \wedge \tilde{v}_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}},$$

and

$$\left(\frac{\sum_{\{i_1,\ldots,i_{k-1}\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}} \geqslant \left(\frac{\sum_{\{i_1,\ldots,i_{k-1}\} \subset [d]} |\tilde{v}_{i_1} \wedge \cdots \wedge \tilde{v}_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}.$$

If such an orthogonal family exists, then applying Theorem 1 with positive numbers $\|v_1\|, \ldots, \|v_d\|$, we can write

$$\begin{aligned}
\left(\frac{\sum\limits_{\{i_1,\ldots,i_k\}\subset[d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|}{\binom{d}{k}}\right)^{\frac{1}{k}} &\leqslant \left(\frac{\sum\limits_{\{i_1,\ldots,i_k\}\subset[d]} |\tilde{v}_{i_1} \wedge \cdots \wedge \tilde{v}_{i_k}|}{\binom{d}{k}}\right)^{\frac{1}{k}} \\
&= \left(\frac{\sum\limits_{\{i_1,\ldots,i_k\}\subset[d]} \|\tilde{v}_{i_1}\| \cdots \|\tilde{v}_{i_k}\|}{\binom{d}{k}}\right)^{\frac{1}{k}} \\
&\leqslant \left(\frac{\sum\limits_{\{i_1,\ldots,i_{k-1}\}\subset[d]} \|\tilde{v}_{i_1}\| \cdots \|\tilde{v}_{i_{k-1}}\|}{\binom{d}{k-1}}\right)^{\frac{1}{k-1}} \\
&= \left(\frac{\sum\limits_{\{i_1,\ldots,i_{k-1}\}\subset[d]} |\tilde{v}_{i_1} \wedge \cdots \wedge \tilde{v}_{i_{k-1}}|}{\binom{d}{k-1}}\right)^{\frac{1}{k-1}} \\
&\leqslant \left(\frac{\sum\limits_{\{i_1,\ldots,i_{k-1}\}\subset[d]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|}{\binom{d}{k-1}}\right)^{\frac{1}{k-1}}.
\end{aligned}$$

In order to streamline the argument slightly, we introduce the following convenient notation

$$S_k(v_1, \ldots, v_d) := \sum_{\{i_1,\ldots,i_k\}\subset[d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|.$$

Notice that by symmetry, instead of constructing a whole family of orthogonal vectors, it suffices to construct $\tilde{v}_1 \in \{v_2, \ldots, v_d\}^{\perp}$ such that

$$S_k(v_1, \ldots, v_d) \leqslant S_k(\tilde{v}_1, v_2, \ldots, v_d),$$

and

$$S_{k-1}(v_1, \ldots, v_d) \geqslant S_{k-1}(\tilde{v}_1, v_2, \ldots, v_d).$$

Since we only need to worry about the terms involving v_1, these last two conditions can be written respectively as

$$\sum_{\{i_1,\ldots,i_{k-1}\}\subset[d]\backslash 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}| \leqslant \sum_{\{i_1,\ldots,i_{k-1}\}\subset[d]\backslash 1} |\tilde{v}_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|,$$

and

$$\sum_{\{i_1,\ldots,i_{k-2}\}\subset[d]\backslash 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-2}}| \geqslant \sum_{\{i_1,\ldots,i_{k-2}\}\subset[d]\backslash 1} |\tilde{v}_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-2}}|.$$

Note that $\tilde{v}_1$ is orthogonal to the vectors $v_2, \ldots, v_d$, so for any $\{v_{i_1}, \ldots, v_{i_r}\} \subset \{v_2, \ldots, v_d\}$

$$|\tilde{v}_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_r}| = \|\tilde{v}_1\| |v_{i_1} \wedge \cdots \wedge v_{i_r}|.$$

Hence we can rewrite the previous two inequalities as follows,

$$\frac{\sum\limits_{\{i_1,\ldots,i_{k-1}\}\subset[d]\backslash 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|}{\sum\limits_{\{i_1,\ldots,i_{k-1}\}\subset[d]\backslash 1} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|} \leqslant \|\tilde{v}_1\| \leqslant \frac{\sum\limits_{\{i_1,\ldots,i_{k-2}\}\subset[d]\backslash 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-2}}|}{\sum\limits_{\{i_1,\ldots,i_{k-2}\}\subset[d]\backslash 1} |v_{i_1} \wedge \cdots \wedge v_{i_{k-2}}|}.$$

So the question is can we choose a length $\|\tilde{v}_1\|$ satisfying the above? Let us summarise what we have just derived with the following result:

Theorem 7. *Fix $1 < k \leqslant d$. Suppose that for any $v_1, \ldots, v_d \in \mathbb{R}^d$*

$$\frac{\sum\limits_{\{i_1,\ldots,i_{k-1}\}\subset[d]\setminus 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|}{\sum\limits_{\{i_1,\ldots,i_{k-1}\}\subset[d]\setminus 1} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|} \leqslant \frac{\sum\limits_{\{i_1,\ldots,i_{k-2}\}\subset[d]\setminus 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-2}}|}{\sum\limits_{\{i_1,\ldots,i_{k-2}\}\subset[d]\setminus 1} |v_{i_1} \wedge \cdots \wedge v_{i_{k-2}}|}, \tag{12}$$

where we interpret the $k = 2$ case as

$$\frac{\sum\limits_{i\in[d]\setminus 1} |v_1 \wedge v_i|}{\sum\limits_{i\in[d]\setminus 1} \|v_i\|} \leqslant \|v_1\|.$$

Then, for any $w_1, \ldots, w_d \in \mathbb{R}^d$, we have

$$\left(\frac{S_k(w_1,\ldots,w_d)}{\binom{d}{k}}\right)^{\frac{1}{k}} \leqslant \left(\frac{S_{k-1}(w_1,\ldots,w_d)}{\binom{d}{k-1}}\right)^{\frac{1}{k-1}}.$$

4.1 Proof of Theorem 5

Now we will prove various special cases of (12) which then imply the corresponding cases listed in Theorem 5:

Lemma 3. *If $k = 2$ or $k = d$, then (12) holds for arbitrary $v_1, \ldots, v_d \in \mathbb{R}^d$.*

Proof. To begin, let us deal with the case where $k = d$. Observe that for $2 \leqslant i \leqslant d$ we have

$$\frac{|v_1 \wedge \cdots \wedge v_d|}{|v_2 \wedge \cdots \wedge v_d|} = \|P_{\{v_2,\ldots,v_d\}^\perp} v_1\| \leqslant \|P_{\{v_2,\ldots,\widehat{v_i},\ldots,v_d\}^\perp} v_1\| = \frac{|v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d|}{|v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d|}.$$

Rearranging this gives

$$|v_1 \wedge \cdots \wedge v_d||v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d| \leqslant |v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d||v_2 \wedge \cdots \wedge v_d|.$$

Summing over i yields

$$\sum_{i=2}^{d} |v_1 \wedge \cdots \wedge v_d||v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d| \leqslant \sum_{i=2}^{d} |v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d||v_2 \wedge \cdots \wedge v_d|$$

which implies that

$$\frac{|v_1 \wedge \cdots \wedge v_d|}{|v_2 \wedge \cdots \wedge v_d|} \leqslant \frac{\sum_{i=2}^{d} |v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d|}{\sum_{i=2}^{d} |v_2 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_d|},$$

which is exactly (12) for $k = d$. Now for the case where $k = 2$. Simply note that

$$\sum_{i=2}^{d} |v_1 \wedge v_i| \leqslant \|v_1\| \sum_{i=2}^{d} \|v_i\|,$$

which immediately gives

$$\frac{\sum_{i=2}^{d} |v_1 \wedge v_i|}{\sum_{i=2}^{d} \|v_i\|} \leqslant \|v_1\|,$$

which concludes the proof. $\square$

Next we deal with the case $k = 3$, which requires a little more work:

Lemma 4. *If $k = 3$, then (12) holds in all dimensions d.*

In order to prove Lemma 4, we first prove the following variant of (12):

Lemma 5.

$$\frac{\sum\limits_{\{i_1,\ldots,i_{d-2}\}\subset[d]\setminus 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|}{\sum\limits_{\{i_1,\ldots,i_{d-2}\}\subset[d]\setminus 1} |v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|} \leqslant \frac{\sum\limits_{i=2}^{d} |v_1 \wedge v_i|}{\sum\limits_{i=2}^{d} \|v_i\|}. \tag{13}$$

Of course when $d = 4$, (13) is exactly (12) with $k = 3$.

To prove Lemma 5 we need to use an elementary fact regarding barycentric coordinates with respect to a simplex. The next result is originally due to Möbius [14]:

Proposition 1 (Barycentric coordinates with respect to a simplex). *Let $v_1, \ldots, v_d \in \mathbb{R}^{d-1}$ be the vertices of a $(d-1)$-simplex Δ. Given a vector $u \in \Delta$, there exists a unique d-tuple $(\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$ with $\sum_{j=1}^{d} \beta_j = 1$, satisfying the following identity*

$$\sum_{j=1}^{d} \beta_j v_j = u.$$

Furthermore, we can write such numbers $\beta_1, \ldots, \beta_d$ explicitly using the following formula

$$\beta_j = \frac{|(v_1 - u) \wedge \cdots \wedge \widehat{(v_j - u)} \wedge \cdots \wedge (v_d - u)|}{2\mathrm{Vol}(\Delta)}.$$

Proof of Lemma 5. By rearranging it suffices to prove

$$\left(\sum_{\{i_1,\ldots,i_{d-2}\}\subset[d]\setminus 1} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|\right)\left(\sum_{i=2}^{d} \|v_i\|\right) \leqslant \left(\sum_{i=2}^{d} |v_1 \wedge v_i|\right) \cdot \left(\sum_{\{i_1,\ldots,i_{d-2}\}\subset[d]\setminus 1} |v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|\right). \tag{14}$$

For $j = 1, \ldots, d-2$, it is always true that

$$\frac{|v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|}{|v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|} = \left\|P_{\{v_{i_1},\ldots,v_{i_{d-2}}\}^{\perp}} v_1\right\| \leqslant \left\|P_{v_{i_j}{}^{\perp}} v_1\right\| = \frac{|v_1 \wedge v_{i_j}|}{\|v_{i_j}\|}.$$

Rearranging this we get

$$|v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|\|v_{i_j}\| \leqslant |v_1 \wedge v_{i_j}||v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|. \tag{15}$$

Summing over $\{i_1, \ldots, i_{d-2}\} \subset [d]$ and $j \in \{1, \ldots, d-2\}$, we get

$$\sum_{\{i_1,\ldots,i_{d-2}\}\subset[d]\setminus 1} \sum_{j=1}^{d-2} |v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|\|v_{i_j}\| \leqslant \sum_{\{i_1,\ldots,i_{d-2}\}\subset[d]\setminus 1} \sum_{j=1}^{d-2} |v_1 \wedge v_{i_j}||v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|. \tag{16}$$

After multiplying out the brackets in (14), we see that it suffices to prove the following estimate involving terms $|v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_{d-2}}|\|v_i\|$ with $i \notin \{i_1, \ldots, i_{d-2}\}$:

$$\sum_{j=2}^{d} |v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_d|\|v_j\| \leqslant \sum_{j=2}^{d} |v_1 \wedge v_j||v_2 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_d|.$$

Without loss of generality we may assume that

$$\frac{|v_1 \wedge v_2|}{\|v_2\|} \leqslant \cdots \leqslant \frac{|v_1 \wedge v_d|}{\|v_d\|},$$

then, bearing this in mind, we can apply a particular choice of (15) to each term on the left hand side to get

$$\sum_{j=2}^{d} |v_1 \wedge v_2 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_d| \|v_j\| \leqslant \frac{|v_1 \wedge v_2|}{\|v_2\|} \left(\sum_{j=3}^{d} |v_2 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_d| \|v_j\| \right) + \frac{|v_1 \wedge v_3|}{\|v_3\|} |v_3 \wedge \cdots \wedge v_d| \|v_2\|.$$

Supposing we know that

$$\sum_{j=3}^{d} |v_2 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_d| \|v_j\| \geqslant |v_3 \wedge \cdots \wedge v_d| \|v_2\|,$$

then by a simple application of the rearrangement inequality for numbers, along with our assumption on the ordering of $\frac{|v_1 \wedge v_2|}{\|v_2\|}, \ldots, \frac{|v_1 \wedge v_d|}{\|v_d\|}$, we see that

$$\begin{aligned}
&\frac{|v_1 \wedge v_2|}{\|v_2\|} \left(\sum_{j=3}^{d} |v_2 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_d| \|v_j\| \right) + \frac{|v_1 \wedge v_3|}{\|v_3\|} |v_3 \wedge \cdots \wedge v_d| \|v_2\| \\
&\leqslant \frac{|v_1 \wedge v_2|}{\|v_2\|} |v_3 \wedge \cdots \wedge v_d| \|v_2\| + \frac{|v_1 \wedge v_3|}{\|v_3\|} \left(\sum_{j=3}^{d} |v_2 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_d| \|v_j\| \right) \\
&\leqslant \sum_{j=2}^{d} |v_1 \wedge v_j| |v_2 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_d|,
\end{aligned}$$

which is what we want. It suffices to prove the following claim:

Claim 1. For any $u_1, \ldots, u_d \in \mathbb{R}^d$ we have

$$\sum_{j=2}^{d} |u_1 \wedge \cdots \wedge \hat{u_j} \wedge \cdots \wedge u_d| \|u_j\| \geqslant |u_2 \wedge \cdots \wedge u_d| \|u_1\|.$$

Proof of Claim 1. Supposing that all the vectors lie in a $(d-1)$-dimensional subspace, then it is clear that they must be linearly dependent. In particular, one can find coefficients $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$, which are not all equal to zero, such that

$$\sum_{j=1}^{d} \alpha_j u_j = \mathbf{0}. \tag{17}$$

By assumption, we have $\sum_{j=1}^{d} |\alpha_j| \neq 0$, so by setting

$$\beta_j = \frac{|\alpha_j|}{\sum_{j=1}^{d} |\alpha_j|},$$

and

$$\xi_j = \operatorname{sgn}(\alpha_j) u_j,$$

we can rewrite (17) as

$$\sum_{j=1}^{d} \beta_j \xi_j = \mathbf{0}. \tag{18}$$

By definition, we have

$$\sum_{j=1}^{d} \beta_j = 1,$$

so applying Proposition 1 we can write

$$\beta_j = \frac{|\xi_1 \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_d|}{\mathrm{Vol}\,(\Delta_{\xi_1,\ldots,\xi_d})} = \frac{|u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d|}{\mathrm{Vol}\,(\Delta_{\xi_1,\ldots,\xi_d})},$$

where $\Delta_{\xi_1,\ldots,\xi_d}$ denotes the simplex with vertices $\xi_1, \ldots, \xi_d$. So, (18) becomes

$$\sum_{j=1}^{d} \frac{|u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d|}{\mathrm{Vol}\,(\Delta_{\xi_1,\ldots,\xi_d})}\xi_j = \mathbf{0},$$

and then multiplying through by $\mathrm{Vol}\,(\Delta_{\xi_1,\ldots,\xi_d})$ we get

$$\sum_{j=1}^{d} |u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d|\xi_j = \mathbf{0}.$$

By the reverse triangle inequality, we have

$$\begin{aligned} 0 &= \left\| \sum_{j=1}^{d} |u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d|\xi_j \right\| \\ &\geqslant |u_2 \wedge \cdots \wedge u_d| \|\xi_1\| - \left\| \sum_{j=2}^{d} |u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d|\xi_j \right\|. \end{aligned}$$

Now by simply rearranging and applying the standard triangle inequality, we see that

$$\begin{aligned} |u_2 \wedge \cdots \wedge u_d| \|u_1\| = |u_2 \wedge \cdots \wedge u_d| \|\xi_1\| &\leqslant \left\| \sum_{j=2}^{d} |u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d|\xi_j \right\| \\ &\leqslant \sum_{j=2}^{d} |u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d| \|\xi_j\| \\ &= \sum_{i=2}^{d} |u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d| \|u_j\|. \end{aligned}$$

Suppose that the vectors $u_1, \ldots, u_d$ do not lie in a $(d-1)$-dimensional subspace. Let us define a new family $w_1, \ldots, w_d$ of vectors from the original family by simply projecting u_1 onto the subspace spanned by $u_2, \ldots, u_d$ and scaling appropriately. More precisely define

$$w_1 := \frac{\|u_1\|}{\left\|P_{\mathrm{span}\{u_2,\ldots,u_d\}} u_1\right\|} u_1,$$

and for $j = 2, \ldots, d$ set $w_j := u_j$. Clearly

$$|w_2 \wedge \cdots \wedge w_d| \|w_1\| = |u_2 \wedge \cdots \wedge u_d| \|u_1\|.$$

Applying the previous case, it suffices to prove that for $j = 2, \ldots, d$

$$|w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_d| \|w_j\| \leqslant |u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d| \|u_j\|,$$

which follows from the fact that

$$\frac{\left|\left(P_{\mathrm{span}\{u_2,\ldots,u_d\}} u_1\right) \wedge u_2 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d\right|}{\left\|P_{\mathrm{span}\{u_2,\ldots,u_d\}} u_1\right\|} \leqslant \frac{|u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_d|}{\|u_1\|}$$

□

This concludes the proof of Lemma 5. □

By a simple argument we can now establish (12) for $k = 3$ in all dimensions:

Proof of Lemma 4. Directly applying Lemma 5 with $d = 4$, for any $v_1, \ldots, v_4 \in \mathbb{R}^4$ we have

$$\frac{\sum_{\{i,j\} \subset \{2,3,4\}} |v_1 \wedge v_i \wedge v_j|}{\sum_{\{i,j\} \subset \{2,3,4\}} |v_i \wedge v_j|} \leqslant \frac{\sum_{k \in \{2,3,4\}} |v_1 \wedge v_k|}{\sum_{k \in \{2,3,4\}} \|v_k\|}.$$

Multiplying out the denominators and expanding the brackets, this last inequality can be rewritten as

$$\sum_{\{i,j\} \subset \{2,3,4\}} \sum_{k \in \{2,3,4\}} |v_1 \wedge v_i \wedge v_j| \|v_k\| \leqslant \sum_{\{i,j\} \subset \{2,3,4\}} \sum_{k \in \{2,3,4\}} |v_i \wedge v_j| |v_1 \wedge v_k|. \tag{19}$$

For higher dimensions simply note that using (19), we can deduce that

$$\begin{aligned}
\sum_{\{i,j\} \subset [d] \setminus 1} \sum_{k \in [d] \setminus 1} |v_1 \wedge v_i \wedge v_j| \|v_k\| &= \sum_{\{a,b,c\} \subset [d] \setminus 1} \sum_{\{i,j\} \subset \{a,b,c\}} \sum_{k \in \{a,b,c\}} |v_1 \wedge v_i \wedge v_j| \|v_k\| \\
&\leqslant \sum_{\{a,b,c\} \subset [d] \setminus 1} \sum_{\{i,j\} \subset \{a,b,c\}} \sum_{k \in \{a,b,c\}} |v_i \wedge v_j| |v_1 \wedge v_k| \\
&= \sum_{\{i,j\} \subset [d] \setminus 1} \sum_{k \in [d] \setminus 1} |v_i \wedge v_j| |v_1 \wedge v_k|,
\end{aligned}$$

which directly implies that

$$\frac{\sum_{\{i,j\} \subset [d] \setminus 1} |v_1 \wedge v_i \wedge v_j|}{\sum_{\{i,j\} \subset [d] \setminus 1} |v_i \wedge v_j|} \leqslant \frac{\sum_{k \in [d] \setminus 1} |v_1 \wedge v_k|}{\sum_{k \in [d] \setminus 1} \|v_k\|}.$$

□

5 Connection to intrinsic volumes

As mentioned in the introduction, the vector-valued Maclaurin inequality with $p = 1$ can be rewritten as a sequence of inequalities between intrinsic volumes of certain polytopes. Firstly, let us state a useful formula for calculating the mixed volume of a zonoid:

Theorem 8 (Theorem 5.3.2 in [16])**.** *For $1 \leqslant j \leqslant d$ and $1 \leqslant i \leqslant j$ let Z_i be a generalised zonoid with generating measure ρ_i, and let $K_1, \ldots, K_{d-j} \subset \mathbb{R}^d$ be convex bodies. Then*

$$V(Z_1, \ldots, Z_j, K_1, \ldots, K_{d-j}) =$$
$$\frac{2^j (d-j)!}{d!} \int_{\mathbb{S}^{d-1}} \cdots \int_{\mathbb{S}^{d-1}} |u_1 \wedge \cdots \wedge u_j| \, v^{(d-j)} \left(P_{\{u_1, \ldots, u_j\}^\perp} K_1, \ldots, P_{\{u_1, \ldots, u_j\}^\perp} K_{d-j} \right) d\rho_1(u_1) \cdots d\rho_j(u_j),$$

where $v^{(d-j)}$ denotes the j-dimensional mixed volume.

Note that any zonotope $Z = \sum_{i=1}^{m} \alpha_i [-v_i, v_i]$ has a support function defined by

$$h_Z(u) = \sum_{i=1}^{m} \alpha_i \, |\langle u, v_i \rangle| = \int_{\mathbb{S}^{d-1}} |\langle u, v \rangle| \, d\rho(v),$$

where ρ is concentrated at $\pm v_i$ and assigns mass $\frac{\alpha_i}{2}$ to each of these points. So by Theorem 8, given zonoids $Z_1 = \sum_{k_1} \alpha_{k_1} [-v_{k_1}, v_{k_1}], \ldots, Z_j = \sum_{k_j} \alpha_{k_j} [-v_{k_j}, v_{k_j}]$ and a fixed convex body $K \subset \mathbb{R}^d$, we have

$$V(Z_1, \ldots, Z_j, K[d-j]) = \frac{2^j (d-j)!}{d!} \sum_{k_1, \ldots, k_j} \alpha_{k_1} \cdots \alpha_{k_j} \left| v_{k_1} \wedge \cdots \wedge v_{k_j} \right| \left| P^\perp_{\{v_{k_1}, \ldots, v_{k_j}\}} K \right|$$

In particular, if P is a centred parallelotope with edges of lengths $\|v_1\|, \ldots, \|v_d\|$ in the directions of $\frac{v_1}{\|v_1\|}, \ldots, \frac{v_d}{\|v_d\|}$, then we have

$$h_P(u) = \sum_{i=1}^{d} \frac{1}{2} \left|\langle u, v_i\rangle\right|.$$

It follows that

$$\begin{aligned} V_k(P) = \binom{d}{k} \frac{V(P,k;B_2^d,d-k)}{\kappa_{d-k}} &= \frac{\frac{2^k(d-k)!}{d!} \sum\limits_{\{i_1,\ldots,i_k\}\subset[d]} \left(\frac{1}{2}\right)^k |v_{i_1} \wedge \cdots \wedge v_{i_k}| \left|P_{\{v_{i_1},\ldots,v_{i_k}\}^\perp} B_2^d\right|}{\kappa_{d-k}} \\ &= \binom{d}{k} \frac{k!(d-k)!}{d!} \sum_{1\leqslant i_1<\cdots<i_k\leqslant d} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \\ &= \sum_{1\leqslant i_1<\cdots<i_k\leqslant d} |v_{i_1} \wedge \cdots \wedge v_{i_k}|. \end{aligned}$$

In this language, Conjecture 1 for $p=1$ states that for all parallelotopes $P \subset \mathbb{R}^d$ and $1 < j \leqslant d$, the following inequality holds

$$\left(\frac{V_j(P)}{\binom{d}{j}}\right)^{\frac{1}{j}} \leqslant \left(\frac{V_{j-1}(P)}{\binom{d}{j-1}}\right)^{\frac{1}{j-1}}, \tag{20}$$

with equality if and only if P is a cube. Suppose instead we consider the zonotope $Z = \sum\limits_{i=1}^{m} \frac{1}{2}[-v_i, v_i] \subset \mathbb{R}^d$, then by the same argument we have

$$\begin{aligned} V_k(Z) = \binom{d}{k} \frac{V(Z,k;B_2^d,d-k)}{\kappa_{d-k}} &= \frac{\frac{2^k(d-k)!}{d!} \sum\limits_{\{i_1,\ldots,i_k\}\subset[m]} \left(\frac{1}{2}\right)^k |v_{i_1} \wedge \cdots \wedge v_{i_k}| \left|P_{\{v_{i_1},\ldots,v_{i_k}\}^\perp} B_2^d\right|}{\kappa_{d-k}} \\ &= \binom{d}{k} \frac{k!(d-k)!}{d!} \sum_{1\leqslant i_1<\cdots<i_k\leqslant m} |v_{i_1} \wedge \cdots \wedge v_{i_k}| \\ &= \sum_{1\leqslant i_1<\cdots<i_k\leqslant m} |v_{i_1} \wedge \cdots \wedge v_{i_k}|. \end{aligned}$$

So for $k \leqslant d$ we can also write Conjecture 1 as

$$\left(\frac{V_j(Z)}{\binom{m}{j}}\right)^{\frac{1}{j}} \leqslant \left(\frac{V_{j-1}(Z)}{\binom{m}{j-1}}\right)^{\frac{1}{j-1}}. \tag{21}$$

Rearranging this gives

$$\frac{V_{j-1}(Z)^j}{V_j(Z)^{j-1}} \geqslant \frac{\binom{m}{j-1}^j}{\binom{m}{j}^{j-1}}. \tag{22}$$

Now, (22) implies log-concavity,

$$V_j(Z)^2 \geqslant \frac{\binom{m}{j}^2}{\binom{m}{j-1}\binom{m}{j+1}} V_{j+1}(Z)V_{j-1}(Z),$$

which can be simplified to

$$V_j(Z)^2 \geqslant \frac{(j+1)(m-j+1)}{j(m-j)} V_{j+1}(Z)V_{j-1}(Z). \tag{23}$$

Note that (22) is an Aleksandrov-type inequalitity. Indeed, the Aleksandrov inequalities for quermass-integrals are given by:

$$\left(\frac{W_i(K)}{|B_2^n|}\right)^{\frac{1}{n-i}} \geqslant \left(\frac{W_j(K)}{|B_2^n|}\right)^{\frac{1}{n-j}} \qquad n > i > j \geqslant 0.$$

Taking into account that $V_j(K) = \binom{n}{j}\frac{W_{n-j}(K)}{\kappa_{n-j}}$ we can rewrite the last one as

$$\frac{V_{j-1}(K)^j}{V_j(K)^{j-1}} \geqslant \frac{V_{j-1}(B_2^d)^j}{V_j(B_2^d)^{j-1}}. \tag{24}$$

Therefore (22) is a restriction of the general (24), in the class of zonotopes and it is also sensitive to the number of segments that make the zonotope, while (24) is not.

Moreover, (23) can be compared with an inequality of McMullen, who proved the following dimension-free bound for the intrinsic volumes. Namely,

$$V_j(K)^2 \geqslant \frac{j+1}{j} V_{j+1}(K)V_{j-1}(K).$$

The factor that appears in the right-hand side of (23) is always at least $\frac{j+1}{j}$, which implies McMullen's inequality. Moreover, we attain this bound as $m \to \infty$. This was exprected, since we can find a sequence of zonotopes that converge to B_2^d as $m \to \infty$.

Note also that in [9] the following inequalities were proved: for zonoids Z

$$\sup_{\Lambda \in GL(d)} \frac{V_j(\Lambda Z)}{V_1(\Lambda Z)^j} \geqslant \frac{1}{d^j}\binom{d}{j} \tag{25}$$

if $\dim(Z) = d$ and $j \geqslant 2$, with equality if and only if Z is a parallelotope. These are reverse, in a sense, to a consequence of (24), namely for any convex body K

$$\frac{V_j(K)}{V_1(K)^j} \leqslant \frac{V_j(B_2^d)}{V_1(B_2^d)^j}. \tag{26}$$

The equality case in (25) exactly says that for any parallelotope $P \subset \mathbb{R}^d$, the following inequality holds:

$$\left(\frac{V_j(P)}{\binom{d}{j}}\right)^{\frac{1}{j}} \leqslant \frac{V_1(P)}{\binom{d}{1}}. \tag{27}$$

This is of course a special case of (20). If this more general equality case can be established, one might hope to prove the corresponding generalisation to (25).

5.1 A non-sharp vector-valued Maclaurin inequality with $p = 1$

Let us see how we can use this language borrowed from convex geometry to reinterpret the reduction described at the beginning of Section 4. In particular, let us try to rewrite (12) purely in terms of intrinsic volumes. Firstly, we fix $v_1, \ldots, v_{m-1} \in \mathbb{R}^{m-1} \subset \mathbb{R}^m$ and $u \in \mathbb{R}^{m-1} \times \mathbb{R} \cong \mathbb{R}^m$, then setting $Z = \sum_{i=1}^{m-1} \frac{1}{2}[-v_i, v_i] \subset \mathbb{R}^{m-1}$, we can apply Theorem 8 to get

$$\begin{aligned}
V_k(P_{u^\perp}Z) &= \binom{m-1}{k}\frac{V\left(P_{u^\perp}Z, k; B_2^{m-1}, m-k-1\right)}{\kappa_{m-k-1}} \\
&= \binom{m}{m-1}\binom{m-1}{k}\frac{V\left(Z, k; B_2^{m}, m-k-1; \frac{1}{2}[-u,u]\right)}{\kappa_{m-k-1}} \\
&= \frac{(m)!}{(m-k-1)!k!}\frac{\left(\frac{2^{k+1}(m-k-1)!}{m!}\right)\sum\limits_{\{i_1,\ldots,i_k\}\subset[m-1]}\left(\frac{1}{2}\right)^{k+1}|v_{i_1}\wedge\cdots\wedge v_{i_k}\wedge u|\left|P_{\{v_{i_1},\ldots,v_{i_k},u\}^\perp}B_2^m\right|}{\kappa_{m-k-1}} \\
&= \sum_{\{i_1,\ldots,i_k\}\subset[m-1]} |v_{i_1}\wedge\cdots\wedge v_{i_k}\wedge u|.
\end{aligned}$$

For any $u_1, \ldots, u_m \in \mathbb{R}^m$, if we set $Z = \sum_{i=2}^{m} \frac{1}{2}[-u_i, u_i]$, then using the calculation we have just made along with Theorem 8, inequality (12) can be rewritten as

$$\frac{V_{k-1}\left(P_{u_1^\perp} Z\right)}{V_{k-1}(Z)} \leqslant \frac{V_{k-2}\left(P_{u_1^\perp} Z\right)}{V_{k-2}(Z)} \tag{28}$$

for $1 < k \leqslant m-1$. Theorem 1.2 in [5] implies the following result: let $K \subset \mathbb{R}^d$ be a convex body, then for any $u \in \mathbb{R}^d$ we have

$$\frac{V_{k-1}(P_{u^\perp} K)}{V_{k-1}(K)} \leqslant \frac{2(d-k+1)}{d-k+2} \frac{V_{k-2}(P_{u^\perp} K)}{V_{k-2}(K)}, \tag{29}$$

for all $3 \leqslant k \leqslant d$. In the special case where $m = d+1$ and u_1 lies in the span of $u_2, \ldots, u_m$, one can view (28) as the sharp version of (29) when we restrict ourselves to zonotopes. Using an analogous derivation to the one given at the beginning of Section 4, we can deduce the following result as a consequence of (29):

Theorem 9. *For any d-tuple of vectors $v_1, \ldots, v_d \in \mathbb{R}^d$ and any $2 < k \leqslant d$, the following inequality holds:*

$$\left(\frac{\sum_{\{i_1,\ldots,i_k\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_k}|}{\binom{d}{k}} \right)^{\frac{1}{k}} \leqslant \frac{2(d-k+1)}{(d-k+2)} \left(\frac{\sum_{\{i_1,\ldots,i_{k-1}\} \subset [d]} |v_{i_1} \wedge \cdots \wedge v_{i_{k-1}}|}{\binom{d}{k-1}} \right)^{\frac{1}{k-1}}$$

Note that the constant appearing on the right-hand side is greater than 1, but smaller than 2. As a consequence of Theorem 5, we know that the sharp constant is equal to 1 in dimensions $d = 3, 4$. Thus, it seems likely that the constant given in Theorem 9 is suboptimal.

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References

[1] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Part I*, Amer. Math. Soc., Mathematical Surveys and Monographs **202** (2015).

[2] S. Brazitikos, A. Giannopoulos, P. Valettas and B-H. Vritsiou, *Geometry of isotropic convex bodies*, Amer. Math. Society, Mathematical Surveys and Monographs **196** (2014).

[3] Y. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin-New York (1988).

[4] R. J. Gardner, *Geometric Tomography*, Second Edition, Encyclopedia of Mathematics and its Applications **58**, Cambridge University Press (2006).

[5] A. Giannopoulos, M. Hartzoulaki and G. Paouris, *On a local version of the Aleksandrov-Fenchel inequality for the quermassintegrals of a convex body*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2403–2412.

[6] P. M. Gruber, J. M. Wills, , G. M. Ziegler. Handbook of convex geometry. Jahresbericht der Deutschen Mathematiker Vereinigung **98** (1996), no. 4, 40–40.

[7] H. Hadwiger , *Vorlesungen Über Inhalt, Oberfläche und Isoperimetrie*, Die Grundlehren der Mathematischen Wissenschaften book series **93**, Springer-Verlag Berlin Heidelberg 1975.

[8] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed. London: Cambridge University Press (1964).

[9] D. Hug and R. Schneider, *Reverse inequalities for zonoids and their application*, Adv. Math. **228** (2011), no. 5, 2634–2646.

[10] E. Gover and N. Krikorian, *Determinants and the volumes of parallelotopes and zonotopes*, Linear Algebra Appl. **433** (2010), no. 1, 28–40.

[11] P. McMullen, *Inequalities between intrinsic volumes*, Monatsh. Math. **111** (1991), no. 1, 47–53.

[12] C. Meyer, *Matrix analysis and applied linear algebra*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2000).

[13] F. D. McIntyre, *Multilinear Geometric Inequalities*, PhD thesis in preparation, University of Edinburgh, 2021.

[14] A. F. Möbius *Der barycentrische Calcul ein neues Hülfsmittel zur analytischen Behandlung der Geometrie dargestellt und insbesondere auf die Bildung neuer Classen von Aufgaben und die Entwickelung mehrerer Eigenschaften der Kegelschnitte angewendet von August Ferdinand Mobius Professor der Astronomie zu Leipzig*, Verlag von Johann Ambrosius Barth (1827)

[15] I. Newton, *Arithmetica universalis sive de compositione et resolutione arithmetica liber*, vol. 1, apud Marcum Michaelem Rey (1761).

[16] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Second expanded edition. Encyclopedia of Mathematics and Its Applications 151, Cambridge University Press, Cambridge (2014).

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