Sub-Gaussian directions of isotropic convex bodies

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Abstract

Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. A direction $\theta \in S^{n-1}$ is called sub-Gaussian for $K$ with constant $b > 0$ if $||\langle \cdot, \theta \rangle||_{L_{\psi_2}(K)} \leq b ||\langle \cdot, \theta \rangle||_2$. We show that if $K$ is isotropic then most directions are sub-Gaussian with a constant which is logarithmic in the dimension. More precisely, for any $a > 1$ we have

$$||\langle \cdot, \theta \rangle||_{L_{\psi_2}(K)} \leq C \left(\log n\right)^{3/2} \max\left\{\sqrt{\log n}, \sqrt{a}\right\} L_K$$

for all $\theta$ in a subset $\Theta_a$ of $S^{n-1}$ with $\sigma(\Theta_a) \geq 1 - n^{-a}$, where $C > 0$ is an absolute constant.

1 Introduction

Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$; we say that $K$ is centered if it has its barycenter at the origin. A direction $\theta \in S^{n-1}$ is a $\psi_\alpha$-direction (where $1 \leq \alpha \leq 2$) for $K$ with constant $b > 0$ if

$$||\langle \cdot, \theta \rangle||_{L_{\psi_\alpha}(K)} \leq b ||\langle \cdot, \theta \rangle||_2,$$

where

$$||\langle \cdot, \theta \rangle||_{L_{\psi_\alpha}(K)} : = \inf\left\{ t > 0 : \int_K \exp\left(||\langle x, \theta \rangle||/t\right)^\alpha dx \leq 2 \right\}.$$ 

From Markov’s inequality it is clear that if $K$ satisfies a $\psi_\alpha$-estimate with constant $b$ in the direction of $\theta$ then for all $t \geq 1$ we have $|\{x \in K : ||\langle x, \theta \rangle|| \geq t ||\langle \cdot, \theta \rangle||_2\}| \leq 2e^{-ct^\alpha/n}$. Conversely, it is a standard fact that tail estimates of this form imply that $\theta$ is a $\psi_\alpha$-direction for $K$.

From the Brunn-Minkowski inequality it follows that every $\theta \in S^{n-1}$ is a $\psi_1$-direction for $K$ with an absolute constant $C$. The starting point of this note is a question posed by V. Milman: is it true that there exists an absolute constant $C > 0$ such that every $K$ has at least one sub-Gaussian direction ($\psi_2$-direction)
Proposition 1.1. Exploiting Klartag’s ideas we give a short proof of an analogous statement. The existence of at least one sub-Gaussian direction for centroid bodies is provided in Section 2). An immediate consequence of (1.1) is the absolute constants (background information on isotropic convex bodies and their δ-images of the same volume. Then, for any t ⩾ 1, where a = 3 (equivalently, ∥⟨·, θ⟩∥L2(K) ⩽ C(log n)a∥⟨·, θ⟩∥2). This estimate was later improved by Giannopoulos, Paouris and Valettas in [8] and [9] (see also [7]). They considered the body Ψ2(K) with support function y ↦ ∥⟨·, y⟩∥L2(K) and showed that for every centered convex body K of volume 1 in Rn, one has
\[
c_1 \leq \left( \frac{∥Ψ_2(K)∥}{∥Z_2(K)∥} \right)^{1/n} \leq c_2 \sqrt{\log n},
\]
where \{Z_q(K)\}q⩾1 is the family of the Lq-centroid bodies of K, and c1,c2 > 0 are absolute constants (background information on isotropic convex bodies and their centroid bodies is provided in Section 2). An immediate consequence of (1.1) is the existence of at least one sub-Gaussian direction for K with constant b ⩽ C√log n.

A natural question that arises is to consider a suitable position T(K), T ∈ SL(n), of the body K and to study the distribution of the ψ2-norm ∥⟨·, θ⟩∥L2ψ,K with respect to the rotationally invariant probability measure σ on the sphere. Klartag [11] offers a result of this type: if K has volume 1 and barycenter at the origin, then there exists T ∈ SL(n) such that the body K1 = T(K) has the following property: there exists Θ ⋐ Sn−1 with measure σ(Θ) ⩾ \frac{1}{8} such that, for every θ ∈ Θ and every t ⩾ 1,
\[
\{x ∈ K1 : |⟨x, θ⟩| ⩾ ct∥⟨·, θ⟩∥1\} \leq \exp \left( -\frac{ct^2}{\log^2 n \log^5 (t + 1)} \right).
\]
Exploiting Klartag’s ideas we give a short proof of an analogous statement.

**Proposition 1.1.** Let K be a centered convex body of volume 1 in Rn, with barycenter at the origin, such that Ψ2(K) has minimal mean width among all its linear images of the same volume. Then, for any δ ∈ (0, 1) we may find Θδ ⋐ Sn−1 with measure σ(Θδ) ⩾ 1 − δ such that every θ ∈ Θδ is a ψ2-direction for K with constant Cδ−1(log n)3/2.

Note that Ψ2(T(K)) = T(Ψ2(K)) for all T ∈ SL(n), and hence there exists a position K1 = T(K) of K such that Proposition 1.1 applies for K1. A more natural and interesting case to consider is when K is in the isotropic position. Our main result provides logarithmic bounds for ∥⟨·, θ⟩∥L2ψ,K with probability polynomially close to 1.

**Theorem 1.2.** Let K be an isotropic convex body in Rn. Then, for any a > 1 we have
\[
∥⟨·, θ⟩∥L2ψ(K) ≤ C(log n)^{3/2} \max \left\{ \sqrt{\log n}, \sqrt{a} \right\} L_K
\]
for all θ in a subset Θa of Sn−1 with σ(Θa) ⩾ 1 − n−a, where C > 0 is an absolute constant.
Theorem 1.2 shows that \( \|\langle \cdot, \theta \rangle\|_{L^2(K)} \leq C(\log n)^2 L_K \) with probability greater than \( 1 - \frac{1}{n} \). This allows us to estimate the expectation of \( \|\langle \cdot, \theta \rangle\|_{L^2(K)} \) on \( S^{n-1} \).

**Theorem 1.3.** Let \( K \) be an isotropic convex body in \( \mathbb{R}^n \). Then,

\[
\int_{S^{n-1}} \|\langle \cdot, \theta \rangle\|_{L^2(K)} d\sigma(\theta) \leq C(\log n)^2 L_K,
\]

where \( C > 0 \) is an absolute constant.

The previously known general estimate was \( \mathbb{E}_{\sigma}(\|\langle \cdot, \theta \rangle\|_{L^2(K)}) \leq C\sqrt{n}L_K \) (see [9]). Regarding the optimal expected result, it is useful to mention a number of sharp results for special classes of convex bodies. Bobkov and Nazarov (see [2] and [3]) have proved that if \( K \) is an isotropic unconditional convex body in \( \mathbb{R}^n \) then, for every \( \theta \in \mathbb{R}^n \),

\[
\|f_\theta\|_{L^2(K)} \leq c\sqrt{n}\|\theta\|_{\infty},
\]

where \( c > 0 \) is an absolute constant. It follows that

\[
\int_{S^{n-1}} \|\langle \cdot, \theta \rangle\|_{L^2(K)} d\sigma(\theta) \leq C\sqrt{\log n} \tag{1.2}
\]

in the unconditional case. In particular, the upper bound of (1.2) holds true the normalized \( \ell^n_1 \)-balls \( \overline{B}_p^n \) for all \( 1 \leq p \leq \infty \). The estimate is sharp in the case of the normalized \( \ell^n_1 \)-ball \( \overline{B}_\infty^n \); one has \( \mathbb{E}_{\sigma}(\|\langle \cdot, \theta \rangle\|_{L^2(\overline{B}_\infty^n)}) \simeq \sqrt{\log n} \). Therefore, the estimate of Theorem 1.3 is best possible up to the power of \( \log n \), and one cannot expect a general upper bound independent from the dimension. A very precise description of the behavior of linear functionals on the \( \ell^n_1 \)-balls, for all \( 1 \leq p \leq \infty \), can be found in the article [1] of Barthe, Guédon, Mendelson and Naor; in particular, they show that in the case \( 2 \leq p \leq \infty \) one has \( \|\langle \cdot, \theta \rangle\|_{L^2(\overline{B}_p^n)} \leq C \) for every \( \theta \in S^{n-1} \), where \( C > 0 \) is a constant independent from \( p \) and \( n \).

The main new tool for the proof of Theorem 1.2 and Theorem 1.3 is a recent result of E. Milman on the mean width \( w(Z_q(K)) \) of the \( L_q \)-centroid bodies \( Z_q(K) \) of an isotropic convex body \( K \) in \( \mathbb{R}^n \).

**Theorem 1.4 (E. Milman [15]).** Let \( K \) be an isotropic convex body in \( \mathbb{R}^n \). Then, for all \( q \geq 1 \) one has

\[
w(Z_q(K)) \leq C \log(1 + q) \max \left\{ \frac{q \log(1 + q)}{\sqrt{n}}, \sqrt{q} \right\} L_K
\]

where \( C > 0 \) is an absolute constant.

The proofs of the main results are given in Section 3.
2 Notation and background information

We work in \( \mathbb{R}^n \), which is equipped with a Euclidean structure \( \langle \cdot, \cdot \rangle \). We denote the corresponding Euclidean norm by \( \| \cdot \|_2 \), and write \( B_2^n \) for the Euclidean unit ball, \( S^{n-1} \) for the unit sphere, and \( \sigma \) for the rotationally invariant probability measure on \( S^{n-1} \). Volume is denoted by \( |\cdot| \). The letters \( c_i, C_i \) denote absolute positive constants whose value may change from line to line. Whenever we write \( a \sim b \), we mean that there exist absolute constants \( c_1, c_2 > 0 \) such that \( c_1 a \leq b \leq c_2 a \). We will often use the fact that \( |B_2^n|^{1/n} \sim 1/\sqrt{n} \); to see this, recall that \( |B_2^n| = \pi^{n/2}/\Gamma\left(\frac{n}{2} + 1\right) \) and use Stirling’s formula.

A convex body in \( \mathbb{R}^n \) is a compact convex set \( A \subset \mathbb{R}^n \) with non-empty interior. We say that \( A \) is symmetric if \( x \in A \) implies that \( -x \in A \). We say that \( A \) is unconditional with respect to an orthonormal basis of \( \mathbb{R}^n \) if \( x = (x_1, \ldots, x_n) \in A \) implies that \( (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \in A \) for every choice of signs \( \varepsilon_i \in \{-1, 1\} \). The volume radius of \( A \) is the quantity \( \text{vrad}(A) = \left(\frac{|A|}{|B_2^n|}\right)^{1/n} \). Integration in polar coordinates shows that if the origin is an interior point of \( A \) then the volume radius of \( A \) can be expressed as

\[
\text{vrad}(A) = \left(\int_{S^{n-1}} \|\theta\|^n_A d\sigma(\theta)\right)^{1/n},
\]

where \( \|\theta\|^n_A = \min\{t > 0 : \theta \in tA\} \). The support function of \( A \) is defined by \( h_A(y) := \max\{\langle x, y \rangle : x \in A\} \), and the mean width of \( A \) is the average

\[
w(A) := \int_{S^{n-1}} h_A(\theta) d\sigma(\theta) \tag{2.1}
\]

of \( h_A \) on \( S^{n-1} \). The radius \( R(A) \) of \( A \) is the smallest \( R > 0 \) such that \( A \subseteq RB_2^n \).

For notational convenience we write \( \overline{A} \) for the homothetic image of volume 1 of a convex body \( A \subset \mathbb{R}^n \), i.e. \( \overline{A} := |A|^{-1/n}A \).

The polar body \( A^\circ \) of a symmetric convex body \( A \) in \( \mathbb{R}^n \) is defined by

\[
A^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in A\}. \tag{2.2}
\]

The Blaschke-Santaló inequality states that \( |A||A^\circ| \leq |B_2^n|^2 \), with equality if and only if \( A \) is an ellipsoid. The reverse Santaló inequality of Bourgain and V. Milman [5] states that there exists an absolute constant \( c > 0 \) such that, conversely,

\[
(|A||A^\circ|)^{1/n} \geq c/n. \tag{2.3}
\]

A convex body \( K \) in \( \mathbb{R}^n \) is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and if its inertia matrix is a multiple of the identity matrix: there exists a constant \( L_K > 0 \) such that

\[
\int_K \langle x, \theta \rangle^2 dx = L_K^2 \tag{2.4}
\]
for every $\theta$ in the Euclidean unit sphere $S^{n-1}$. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

\[ L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leq C \]  

for all $n \geq 1$. Bourgain proved in [4] that $L_n \leq c\sqrt[4]{n}\log n$, while Klartag [10] obtained the bound $L_n \leq c\sqrt{n}$. A second proof of Klartag’s bound appears in [12].

Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geq 1$ we consider the $q$-th moment of the Euclidean norm

\[ \Psi_2(K) := \left( \int_K \|x\|_2^q dx \right)^{1/q} \]  

We refer the reader to the article of V. Milman and Pajor [16] and to the book [6] for an updated exposition of isotropic log-concave measures and more information on the hyperplane conjecture.

### 3 Proof of the results

Recall that $\Psi_2(K)$ is the symmetric convex body with support function $h_{\Psi_2(K)}(y) = \|\langle \cdot, y \rangle\|_{L_\psi(K)}$. One also has

\[ h_{\Psi_2(K)}(y) \simeq \sup_{q \geq 2} \frac{h_{\Psi_2(K)}(y)}{\sqrt{q}} \simeq \sup_{2 \leq q \leq n} \frac{h_{\Psi_2(K)}(y)}{\sqrt{q}} \]
because \( h_{Z_n(K)}(y) \simeq h_{Z_n(K)}(y) \) for all \( q \geq n \). Taking into account the fact that if \( 2^s < q < 2^{s+1} \) then
\[
\frac{h_{Z_n(K)}(y)}{\sqrt{q}} \leq \frac{h_{Z_{q+1}(K)}(y)}{2^{s/2}} \leq \sqrt{2} \frac{h_{Z_{q+1}(K)}(y)}{2^{(s+1)/2}},
\]
we can further simplify and write
\[
h_{\Psi_2(K)}(y) \simeq \max_{1 \leq s \leq m} \frac{h_{Z_{q+1}(K)}(y)}{2^{s/2}} \quad (3.1)
\]
where \( m = \lfloor \log_2 n \rfloor \) (for all the above see [6, Chapter 5]).

**Proof of Proposition 1.1.** Since \( \Psi_2(T(K)) = T(\Psi_2(K)) \) for every \( T \in SL(n) \), we may find \( T \in SL(n) \) such that \( K_1 = T(K) \) has the property that \( \Psi_2(K_1) \) has minimal mean width among all its linear images of the same volume. It is well known (see [21] or [6, Chapter 1]) that in this case one has the estimate
\[
w(\Psi_2(K_1)) \leq C_1(\log n) [vrad(\Psi_2(K_1))],
\]
where \( C_1 > 0 \) is an absolute constant. On the other hand, we may write
\[
\int_{S^{n-1}} \frac{h_{\Psi_2(K_1)}(\theta)}{h_{Z_2(K_1)}(\theta)} d\sigma(\theta) \leq \left( \int_{S^{n-1}} \frac{h_{\Psi_2(K_1)}^2(\theta)}{h_{Z_2(K_1)}^2(\theta)} d\sigma(\theta) \right)^{\frac{1}{2}} \left( \int_{S^{n-1}} \frac{h_{Z_2(K_1)}^{-2}(\theta)}{h_{Z_2(K_1)}^2(\theta)} d\sigma(\theta) \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_{S^{n-1}} \frac{h_{\Psi_2(K_1)}^2(\theta)}{h_{Z_2(K_1)}^2(\theta)} d\sigma(\theta) \right)^{\frac{1}{2}} \left( \int_{S^{n-1}} \frac{h_{Z_2(K_1)}^{-2}(\theta)}{h_{Z_2(K_1)}^2(\theta)} d\sigma(\theta) \right)^{\frac{1}{2}} \leq C_2 w(\Psi_2(K_1)) vrad(Z_2^2(K_1)) = C_2 w(\Psi_2(K_1)) vrad(Z_2^2(K_1)) = C_2 w(\Psi_2(K_1)) vrad(Z_2^2(K_1)),
\]
where we have used Cauchy-Schwarz inequality, Hölder’s inequality, the equality \( vrad(Z_2^2(K_1)) vrad(Z_2^2(K_1)) = 1 \) which holds true because \( Z_2(K_1) \) is an ellipsoid, and the equivalence of the \( L_1 \) and the \( L_2 \) norm of the function \( h_{\Psi_2(K_1)} \) on \( S^{n-1} \) (this is a well-known Kahane-Khintchine type inequality; in fact, one can view it as a special case of the stronger inequality (3.2), due to Litvak, V. Milman and Schechtman [13]).

Combining the previous estimates we conclude that
\[
\int_{S^{n-1}} h_{\Psi_2(K_1)}(\theta) d\sigma(\theta) \leq C_3 \log n \left( \frac{[\Psi_2(K_1)]}{[Z_2(K_1)]} \right)^{1/n} \leq C_4 (\log n)^{3/2},
\]
where in the last step we have used (1.1). An application of Markov’s inequality shows that for any \( \delta \in (0, 1) \) we may find \( \Theta_\delta \subseteq S^{n-1} \) with measure \( \sigma(\Theta_\delta) \geq 1 - \delta \) such that every \( \theta \in \Theta_\delta \) is a \( \psi_2 \)-direction for \( K_1 \) with constant \( C_4 \delta^{-1}(\log n)^{3/2} \). \( \square \)

We proceed to the proof of Theorem 1.2 and Theorem 1.3. First, we give a simple argument that leads to the upper bound of Theorem 1.3 for
\[
w(\Psi_2(K)) = \int_{S^{n-1}} \|\langle \cdot, \theta \rangle\|_{L_{\psi_2(K)}} d\sigma(\theta).
\]
Let $K$ be an isotropic convex body in $\mathbb{R}^n$. By (3.1), for any $y \in S^{n-1}$ we have
\[
h_{\Psi^2(K)}(y) \leq C_1 \max_{1 \leq s \leq m} \frac{h_{Z_{2s}^+(K)}(y)}{2^{s/2}} \leq C_1 \sum_{s=1}^m \frac{h_{Z_{2s}^+(K)}(y)}{2^{s/2}}
\]
where $m = \lfloor \log_2 n \rfloor$. It trivially follows that
\[
w(\Psi^2(K)) \leq C_1 \sum_{s=1}^m w(Z_{2s}^+(K)) \frac{2^{s/2}}{s^{2s/2}}
\]
From Theorem 1.4 we know that
\[
w(Z_{2s}^+(K)) \leq C_2 s^{2s/2} \max \left\{ \frac{s^{2s/2}}{\sqrt{n}}, 1 \right\} L_K.
\]
Therefore, denoting by $k$ the largest integer for which $k^2 2^k \leq n$, and using summation by parts in the final step, we see that
\[
w(\Psi^2(K)) \leq C_3 \sum_{s=1}^m s \max \left\{ \frac{s^{2s/2}}{\sqrt{n}}, 1 \right\} L_K \leq C_3 \left( \sum_{s=1}^k s + \frac{1}{\sqrt{n}} \sum_{s=k+1}^m s^2 2^{s/2} \right) L_K
\]
\[
\leq C_4 \left( k^2 + \frac{n^2 2^{m/2}}{\sqrt{n}} \right) L_K \leq C_5 m^2 L_K \leq C (\log n)^2 L_K,
\]
where $C > 0$ is an absolute constant.

A more careful use of the theory of centroid bodies, given below, leads to the probability estimate of Theorem 1.2 and to a second proof of Theorem 1.3.

**Proof of Theorem 1.2 and Theorem 1.3.** We will use the following observations that can be found e.g. in [6, Chapter 5]: given a symmetric convex body $A$ in $\mathbb{R}^n$, if we set $k = k_A(A) = n \left( \frac{w(A)}{R(A)} \right)^2$, then
\[
w_k(A) := \left( \int_{S^{n-1}} h_A^k(\theta) d\sigma(\theta) \right)^{1/k} \leq C_1 w(A) \tag{3.2}
\]
where $C_1 > 0$ is an absolute constant; this is a result of Litvak, V. Milman and Schechtman from [13]. If $A = Z_q(K)$ then the results of Paouris in [19] (or, earlier, in [18]) show that, for all $2 \leq q \leq n$, we have
\[
w(Z_q(K)) \geq c_1 w_q(Z_q(K)) \geq c_2 \sqrt{q/n} I_q(K) \geq c_2 \sqrt{q/n} I_2(K) = c_2 \sqrt{qL_K}.
\]
In fact, E. Milman and Klartag have obtained the stronger bound $vrad(Z_q(K)) \geq c_3 \sqrt{qL_K}$ for all $2 \leq q \leq q_H(K)$, where $q_H(K) \geq c_4 \sqrt{n}$ is a hereditary parameter of $K$ that was introduced and studied in [12] for this purpose. On the other hand, $R(Z_q(K)) \leq C_2 qL_K$, and hence $k_A(Z_q(K)) \geq c_5 n/q$ for all $2 \leq q \leq \sqrt{n}$. In the
range $\sqrt{n} \leq q \leq n$ one has the weaker bound $w(Z_q(K)) \geq \text{vrad}(Z_q(K)) \geq c_0 \sqrt{q}$ which follows from Urysohn’s inequality and a lower bound for $\text{vrad}(Z_q(K))$ for the full range $2 \leq q \leq n$, which is due to Lutwak, Yang and Zhang [14]; this results in the estimate $k_*(Z_q(K)) \geq c_7 n/(q L_K^2)$.

Using Theorem 1.4 and (3.2) we get

$$\left( \int_{S^{n-1}} h_{Z_q(K)}^k(\theta) d\sigma(\theta) \right)^{1/k} \leq C_3 \log(1 + q) \max \left\{ q \log(1 + q) \sqrt{n} / \sqrt{q}, 1 \right\} L_K$$

where $k_* := k_*(Z_q(K))$, and using Markov’s inequality we conclude that, for every $q \leq n$ there exists a subset $\Theta_q$ of $S^{n-1}$ such that $\sigma(S^{n-1} \setminus \Theta_q) \leq \exp(-c_9 n/(q L_K^2))$ and

$$h_{Z_q(K)}(\theta) \leq C_4 \sqrt{q} \log(1 + q) \max \left\{ \sqrt{q} \log(1 + q) \sqrt{n} / \sqrt{q}, 1 \right\} L_K$$

for all $\theta \in \Theta_q$.

We fix $a > 1$ and define $q_0 = \frac{c_9 n}{2a L_K \log n}$. Then, for every $q \leq q_0$ we have $\sigma(S^{n-1} \setminus \Theta_q) \leq \frac{1}{n^{2a}}$. It follows that

$$\sigma \left( S^{n-1} \setminus \bigcap_{s=1}^{[\log_2 q_0]} \Theta_{2^s} \right) \leq \frac{c_9 \log n}{n^{2a}} \leq \frac{1}{n^{2a}}.$$ 

If $\Theta := \bigcap_{s=1}^{[\log_2 q_0]} \Theta_{2^s}$ then, for every $\theta \in \Theta$ and every $q \leq q_0$ we have

$$\frac{h_{Z_q(K)}(\theta)}{\sqrt{q}} \leq C_4 \log(1 + q) \max \left\{ \sqrt{q} \log(1 + q) \sqrt{n} / \sqrt{q}, 1 \right\} L_K \tag{3.3}$$

while for $q_0 \leq q \leq n$ we use (2.7) to write

$$\frac{h_{Z_q(K)}(\theta)}{\sqrt{q}} \leq C_6 q \frac{h_{Z_q(K)}(\theta)}{q_0} \sqrt{\frac{q_0}{q}} = C_6 \sqrt{q/q_0} \frac{h_{Z_q(K)}(\theta)}{q_0 \sqrt{q}} \leq C_6 \log(1 + q_0) \log(1 + q_0) \max \left\{ \log(1 + q_0) \sqrt{a L_K \log n} / \sqrt{q_0}, 1 \right\} L_K$$

$$\leq C_7 \sqrt{\log(\log n)^{3/2}} \max \left\{ \sqrt{\log n} / \sqrt{a L_K}, 1 \right\} L_K.$$

Combining this estimate with (3.3) we see that

$$\|\langle \cdot, \theta \rangle\|_{L_{\psi_2}(K)} \leq C_8 \sqrt{\log n} \max \left\{ \sqrt{\log n} / \sqrt{a L_K}, 1 \right\} L_K^2$$

with probability greater that $1 - n^{-a}$. 

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Now, we use Theorem 3.1 from [8]: If $K$ is an isotropic convex body in $\mathbb{R}^n$, there exists an isotropic convex body $K_1$ such that $L_{K_1} \leq C_0$, where $C_0 > 0$ is an absolute constant, and

$$\Psi_2(K) \leq c_{10} L_K \Psi_2(K_1).$$

Our previous reasoning, applied to $K_1$, shows that, with probability greater than $1 - n^{-a}$,

$$\|\langle \cdot, \theta \rangle\|_{L^2(\Psi_2)} \leq c_{10} L_K \|\langle \cdot, \theta \rangle\|_{L^2(\Psi_2)} \leq C_8 c_{10} L_K \sqrt{a} \left( \frac{\sqrt{\log n}}{\sqrt{a}} \right)^{3/2} \max \left\{ \frac{\sqrt{\log n}}{\sqrt{a}}, 1 \right\}.$$  

This proves Theorem 1.2. In particular, we have

$$\|\langle \cdot, \theta \rangle\|_{L^2(\Psi_2)} \leq C_{10} (\log n)^2 L_K$$

with probability greater than $1 - \frac{1}{n}$. Since $\|\langle \cdot, \theta \rangle\|_{L^2(\Psi_2)} \leq C_{11} \sqrt{a} L_K$ for all $\theta \in S^{n-1}$ (see e.g. [6]) this gives one more proof of Theorem 1.3.

**Remark.** Let $K$ be an isotropic convex body in $\mathbb{R}^n$. The function $\psi_K : [1, \infty) \to \mathbb{R}$ with

$$\psi_K(t) := \sigma \left( \{ \theta \in S^{n-1} : \|\langle \cdot, \theta \rangle\|_{L^2(\Psi_2)} \leq c t \sqrt{\log n L_K} \} \right)$$

was introduced in [9], where it was shown that for every $t \geq 1$ one has

$$\psi_K(t) \geq \exp(-ct^2),$$

where $c > 0$ is an absolute constant. Theorem 1.3 provides much stronger information; it implies that $\psi_K(t) \geq 1/2$ for $t \approx (\log n)^{3/2}$.

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