

Geometry of the subgaussian body of an isotropic convex body

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Abstract

For a centered convex body $K \subset \mathbb{R}^n$, let $\Psi_2(K)$ denote the symmetric convex body whose support function is given by the ψ_2 -norms of linear functionals on K . The recent solution of Milman's problem on the existence of subgaussian directions by Letwin and Mikulincer naturally motivates the study of the geometry of this body. We prove that $\Psi_2(K)$ has bounded volume ratio with respect to the L_2 -centroid body $Z_2(K)$. In the isotropic case, we also obtain sharp estimates for its mean width and the volume radii of its orthogonal projections, and derive consequences for the existence of subgaussian orthonormal bases. In particular, we construct orthonormal bases with quantitatively controlled subgaussian constants for every isotropic convex body.

1 Introduction

Concentration properties of linear functionals on convex bodies play a central role in asymptotic convex geometry. In particular, directions along which linear functionals exhibit subgaussian behavior have been studied extensively since they provide strong control of moments and tail estimates.

This motivates the following definition of a subgaussian direction. Let K be a centered convex body in $\mathbb{R}^n$, that is, a convex body whose barycenter is the origin. Throughout the paper we assume that convex bodies have volume 1. A direction $v \in S^{n-1}$ is called *subgaussian* for K with constant $b > 0$ if

$$(1.1) \quad \|\langle \cdot, v \rangle\|_{\psi_2} \leq b \|\langle \cdot, v \rangle\|_2.$$

Recall that the ψ_2 -norm of a random variable X on a probability space is defined by

$$\|X\|_{\psi_2} = \inf \{t > 0 : \mathbb{E} \exp((X/t)^2) \leq 2\}.$$

For bounded random variables X ,

$$\|X\|_{\psi_2} \approx \sup_{p \geq 1} \frac{\|X\|_p}{\sqrt{p}} \approx \inf \left\{ \alpha > 0 : \mathbb{P}(|X| > t) \leq 2e^{-t^2/\alpha^2} \text{ for all } t \geq 0 \right\},$$

up to absolute constants. Uniform control of $\|X\|_p/\sqrt{p}$ is equivalent to a ψ_2 -estimate, and hence to subgaussian tail decay (see, for example, [29, Section 2.6]). Throughout the paper, moments and ψ_2 -norms of linear functionals are computed with respect to the uniform probability measure on K .

The following question was posed by V. Milman.

Question. Does there exist an absolute constant $C > 0$ such that every centered convex body admits a subgaussian direction with constant C ?

Klartag [17] proved that every convex body admits a subgaussian direction, obtaining a polylogarithmic estimate for the constant in (1.1). A different proof, yielding a slightly improved bound, was later given in [13]. Subsequently, [14] showed that every convex body admits an $O(\sqrt{\ln(en)})$ -subgaussian direction. Very recently, Letwin and Mikulincer [20] answered Milman's question affirmatively. Their proof introduces several new ideas that play key roles in the present work.

While Milman's problem concerns the existence of a single subgaussian direction, its recent resolution naturally leads to the study of the geometry of the entire collection of such directions. This viewpoint is conveniently encoded in the *subgaussian body* $\Psi_2(K)$, defined as the symmetric convex body with support function

$$h_{\Psi_2(K)}(v) = \|\langle \cdot, v \rangle\|_{\psi_2}.$$

By definition, $\Psi_2(K)$ contains the ellipsoid $cZ_2(K)$, for an absolute constant $c > 0$, where

$$h_{Z_2(K)}(v) = \|\langle \cdot, v \rangle\|_2.$$

It was proved in [14] that

$$\left(\frac{\text{vol}_n(\Psi_2(K))}{\text{vol}_n(Z_2(K))} \right)^{1/n} \leq C \sqrt{\ln(en)},$$

where $C > 0$ is an absolute constant.

Our first result strengthens this estimate by showing that the volume ratio of the subgaussian body with respect to $Z_2(K)$ is bounded by an absolute constant.

Theorem 1.1. *Let K be a centered convex body in $\mathbb{R}^n$. Then*

$$\left(\frac{\text{vol}_n(\Psi_2(K))}{\text{vol}_n(Z_2(K))} \right)^{1/n} \leq C,$$

where $C > 0$ is an absolute constant.

Theorem 1.1 and a standard argument imply the existence of a direction $v \in S^{n-1}$ satisfying $h_{\Psi_2(K)}(v) \leq C h_{Z_2(K)}(v)$; equivalently,

$$\|\langle \cdot, v \rangle\|_{\psi_2} \leq C \|\langle \cdot, v \rangle\|_2.$$

Thus, Theorem 1.1 provides another proof of the affirmative answer to Milman's question. More importantly, however, it yields a substantially stronger geometric statement by controlling the global size of the body $\Psi_2(K)$ rather than merely guaranteeing the existence of a single subgaussian direction.

Letwin and Mikulincer further proved in [20] that every centered convex body K in $\mathbb{R}^n$ admits an orthonormal set $\Theta = \{v_1, \dots, v_m\} \subset S^{n-1}$ with $|\Theta| = m \geq \frac{9n}{10}$ such that

$$c\sqrt{p} \|\langle \cdot, v \rangle\|_2 \leq \|\langle \cdot, v \rangle\|_p \leq C\sqrt{p} \|\langle \cdot, v \rangle\|_2,$$

for every $v \in \Theta$, where the upper estimate holds for all $p \geq 1$, while the lower estimate is valid for $1 \leq p \leq n$. They also conjectured that, if K is isotropic, then one may choose Θ to be a complete orthonormal basis satisfying

$$c\sqrt{p} \leq \|\langle \cdot, v \rangle\|_p \leq C\sqrt{p},$$

where again the upper estimate holds for all $p \geq 1$, while the lower estimate holds for $1 \leq p \leq n$. In this case, $\|\langle \cdot, v \rangle\|_2 = L_K \approx 1$ for every $v \in S^{n-1}$ (see Section 2 for background information on isotropic convex

bodies and the isotropic constant). Note that the upper estimate is equivalent to

$$\|\langle \cdot, v \rangle\|_{\psi_2} \leq C.$$

Our next theorem provides a quantitative partial answer to this conjecture.

Theorem 1.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then there exists an orthonormal basis $\{v_1, \dots, v_n\}$ such that*

$$\|\langle \cdot, v_k \rangle\|_{\psi_2} \leq C \sqrt{\frac{n}{n-k+1}}, \quad 1 \leq k \leq n,$$

where $C > 0$ is an absolute constant.

The proof of Theorem 1.2 is based on the projection estimate below, which is also of independent interest.

Theorem 1.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leq m \leq n$ and any $F \in G_{n,m}$,*

$$\text{vrad}(P_F(\Psi_2(K))) \leq C \sqrt{n/m},$$

where $C > 0$ is an absolute constant.

The case $m = 1$ shows that $\Psi_2(K) \subseteq C\sqrt{n}B_2^n$. The estimates of Theorem 1.3 are sharp. Indeed, let

$$K_1 = \text{vol}_n(B_1^n)^{-1/n} B_1^n,$$

the isotropic cross-polytope. In this case, as shown in [7],

$$\Psi_2(K_1) \approx \sqrt{n} B_1^n.$$

If F_k is a coordinate k -dimensional subspace, then

$$(1.2) \quad \begin{aligned} \max\{\text{vrad}(P_F(\Psi_2(K_1))) : F \in G_{n,k}\} &\geq \text{vrad}(P_{F_k}(\Psi_2(K_1))) \\ &\approx \sqrt{n} \text{vrad}(B_1^k) \approx \sqrt{\frac{n}{k}}. \end{aligned}$$

In fact, Ivanov [16] recently confirmed the conjecture that among all k -dimensional projections of the cross-polytope, the coordinate ones have maximal volume. Consequently, the first inequality in (1.2) is actually an equality.

Using the recent optimal M -estimate of Bizeul [6] for isotropic convex bodies, we obtain a sharp upper bound for the mean width of the subgaussian body.

Theorem 1.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$w(\Psi_2(K)) \leq C \sqrt{\ln(en)},$$

where $C > 0$ is an absolute constant.

Brazitikos and Hioni [9] had previously obtained the upper bound $w(\Psi_2(K)) \leq C(\ln(en))^2$. Note that

$$w(\Psi_2(K_1)) \approx w(\sqrt{n}B_1^n) \approx \sqrt{\ln(en)}$$

for the isotropic cross-polytope K_1 , hence the estimate of Theorem 1.4 is asymptotically sharp.

A further simple observation is that

$$\Psi_2(K) \supseteq c Z_2(K) = c L_K B_2^n$$

for every isotropic convex body K in $\mathbb{R}^n$, which implies that

$$M(\Psi_2(K)) = \int_{S^{n-1}} \|\xi\|_{\Psi_2(K)} d\sigma(\xi) \leq C.$$

Combining this estimate with Theorem 1.4, we obtain

$$M(\Psi_2(K)) w(\Psi_2(K)) \leq C \sqrt{\ln(en)}.$$

Thus, in the terminology of [1, Chapter 6], the subgaussian body of an isotropic convex body is in the ℓ -position.

We next return to the conjecture of Letwin and Mikulincer. We obtain a complementary result by showing that one can always find a complete orthonormal basis consisting entirely of subgaussian directions, with subgaussian constants growing at most logarithmically in the dimension. More precisely, we prove the following theorem.

Theorem 1.5. *Let K be an isotropic convex body in $\mathbb{R}^n$. There exists an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ such that*

$$\|\langle \cdot, v_i \rangle\|_{\psi_2} \leq C \sqrt{\ln(en)}$$

for all $i = 1, \dots, n$. In fact,

$$\int_{O(n)} \max_{1 \leq i \leq n} \|\langle \cdot, U(e_i) \rangle\|_{\psi_2} d\nu(U) \leq C_1 \sqrt{\ln(en)},$$

where ν is the Haar probability measure on $O(n)$ and $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$.

Theorem 1.5 should be viewed as complementary to Theorem 1.2. The latter provides a basis in which the subgaussian constants are uniformly bounded for most vectors and deteriorate only near the end of the basis, whereas Theorem 1.5 yields a complete orthonormal basis whose subgaussian constants are bounded logarithmically in the dimension.

We conclude by identifying an important class of convex bodies for which the preceding methods yield optimal results. In particular, we prove that the Letwin–Mikulincer conjecture holds for every unconditional isotropic convex body. The proof combines our general approach with the sharp ψ_2 -estimates of Bobkov and Nazarov.

The paper is organized as follows. In Section 2 we collect the necessary background and auxiliary results. Section 3 is devoted to the proofs of our main results: the volume estimate for the subgaussian body, the projection theorem and its consequences, the construction of orthonormal bases of subgaussian directions, and the unconditional case. Section 4 contains further remarks and questions.

2 Background and auxiliary facts

We work in $\mathbb{R}^n$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$. The associated Euclidean norm is denoted by $|\cdot|$, and B_2^n and S^{n-1} denote the Euclidean unit ball and unit sphere, respectively. Lebesgue measure on $\mathbb{R}^n$ is denoted by vol_n , and we write $\omega_n = \text{vol}_n(B_2^n)$. The rotationally invariant probability measure on

S^{n-1} is denoted by σ , and ν is the Haar probability measure on $O(n)$. The Grassmannian of k -dimensional subspaces of $\mathbb{R}^n$ is denoted by $G_{n,k}$ and is equipped with the Haar probability measure $\nu_{n,k}$.

Throughout the paper, the symbols $C, c, c', c_1, c_2, \dots$ denote absolute positive constants whose values may change from line to line. Whenever we write $a \approx b$ for positive quantities a, b , we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. For convex bodies, $A \approx B$ means that $c_1 A \subseteq B \subseteq c_2 A$.

For background on isotropic convex bodies and log-concave measures, we refer the reader to [10]; for general background on asymptotic geometric analysis, see [1, 2].

A convex body in $\mathbb{R}^n$ is a compact convex subset K with nonempty interior. We say that K is symmetric if $K = -K$, and centered if its barycenter

$$\text{bar}(K) = \frac{1}{\text{vol}_n(K)} \int_K x \, dx$$

is the origin.

Let K be a convex body with $0 \in \text{int}(K)$. Its radial function is defined by $\rho_K(u) = \max\{t > 0 : tu \in K\}$ for $u \in S^{n-1}$, and its support function is $h_K(y) = \max\{\langle x, y \rangle : x \in K\}$ for $y \in \mathbb{R}^n$. The mean width of K is the average of its support function on S^{n-1} :

$$w(K) = \int_{S^{n-1}} h_K(\xi) \, d\sigma(\xi).$$

We note that if $T \in GL_n$, then

$$\frac{1}{\|T^{-1}\|_{\text{op}}} w(K) \leq w(TK) \leq \|T\|_{\text{op}} w(K).$$

The Minkowski functional of K is $p_K(x) = \inf\{t > 0 : x \in tK\}$. The mean norm of K is the average of its Minkowski functional on S^{n-1} :

$$M(K) = \int_{S^{n-1}} p_K(\xi) \, d\sigma(\xi).$$

The outer radius, inradius and volume radius of K are respectively $R(K) = \max\{|x| : x \in K\}$, $r(K) = \max\{r > 0 : rB_2^n \subseteq K\}$, and

$$\text{vrad}(K) = \left(\frac{\text{vol}_n(K)}{\text{vol}_n(B_2^n)} \right)^{1/n}.$$

The polar body of K is

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

For every convex body $K \subseteq \mathbb{R}^n$ we write $\bar{K}$ for the multiple of K that has volume 1; in other words, $\bar{K} := \text{vol}_n(K)^{-1/n} K$.

An absolutely continuous probability measure μ on $\mathbb{R}^n$ is called log-concave if its density with respect to Lebesgue measure is of the form $f_\mu = e^{-\varphi}$, where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex. The uniform probability measure on a convex body is log-concave. The barycenter of a probability measure μ is

$$\text{bar}(\mu) := \int_{\mathbb{R}^n} x f_\mu(x) \, dx,$$

and its isotropic constant is the quantity

$$L_\mu := \|f_\mu\|_\infty^{\frac{1}{2n}} \det(\text{Cov}(\mu))^{\frac{1}{2n}},$$

where

$$\text{Cov}(\mu) := \int x \otimes x f_\mu(x) dx - \left(\int x f_\mu(x) dx \right) \otimes \left(\int x f_\mu(x) dx \right)$$

is the covariance matrix of μ . A log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if $\text{bar}(\mu) = 0$ and $\text{Cov}(\mu) = I_n$. For an isotropic log-concave probability measure, $L_\mu = \|f_\mu\|_\infty^{1/n}$ since $\det(\text{Cov}(\mu)) = 1$.

A convex body $K \subset \mathbb{R}^n$ is called isotropic if it has volume 1, its barycenter is at the origin, and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \xi \rangle^2 dx = L_K^2 \quad \text{for all } \xi \in S^{n-1}.$$

Every convex body K admits an isotropic affine image, unique up to orthogonal transformations (see [10]). The constant L_K is an affine invariant of K .

A central problem in asymptotic convex geometry, posed by Bourgain [8], is the hyperplane conjecture: whether there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leq C$$

for every $n \geq 1$. In fact, Ball [4] showed that, in every dimension,

$$\sup_{\mu} L_\mu \leq C \sup_K L_K,$$

where the suprema are taken over all log-concave probability measures μ and all convex bodies $K \subseteq \mathbb{R}^n$, respectively.

The hyperplane conjecture was recently resolved affirmatively by Klartag and Lehec [18], following major progress by Guan [15]; an alternative proof was subsequently obtained by Bizeul [5]. The latter approach relies on the following optimal small-ball estimate: if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$, then for every $0 < \varepsilon \leq c_0$ and every $y \in \mathbb{R}^n$,

$$(2.1) \quad \mu(\{x \in \mathbb{R}^n : |x - y|^2 \leq \varepsilon n\}) \leq \varepsilon^{c_0 n},$$

where $c_0 > 0$ is an absolute constant.

E. Milman proved in [22] that if K is an isotropic convex body in $\mathbb{R}^n$, then

$$w(K) \leq C \sqrt{n} (\ln(en))^2 L_K \leq c_1 \sqrt{n} (\ln(en))^2,$$

where the second inequality uses the boundedness of L_n . Very recently, Bizeul [6] obtained the optimal upper bound

$$(2.2) \quad w(K) \leq C \sqrt{n \ln(en)}$$

for any isotropic convex body K in $\mathbb{R}^n$.

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For $p \geq 1$, the L_p -centroid body $Z_p(\mu)$ of μ is the symmetric convex body whose support function is

$$h_{Z_p(\mu)}(y) := \|\langle \cdot, y \rangle\|_{L_p(\mu)} = \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^p f_\mu(x) dx \right)^{1/p}, \quad y \in \mathbb{R}^n.$$

The centroid bodies satisfy the covariance property $Z_p(T_*\mu) = T(Z_p(\mu))$ for every $T \in GL_n$, where

$(T_*\mu)(A) = \mu(T^{-1}(A))$. If μ is isotropic then $Z_2(\mu) = B_2^n$ by the normalization $\text{Cov}(\mu) = I_n$. Moreover, for every $1 \leq p < q$,

$$(2.3) \quad Z_p(\mu) \subseteq Z_q(\mu) \subseteq \frac{cq}{p} Z_p(\mu),$$

where $c > 0$ is an absolute constant. The asymptotic theory of centroid bodies was developed by Paouris in [24, 25].

If K is a centered convex body and λ_K denotes the uniform probability measure on K , we write $Z_p(K) := Z_p(\lambda_K)$. Note that

$$Z_p(T(K)) = T(Z_p(K)), \quad T \in SL_n.$$

If K is isotropic then the probability measure μ_K , with density $L_K^n \mathbb{1}_{\frac{K}{L_K}}$, is isotropic, and $Z_p(K) = L_K Z_p(\mu_K)$. For every $p \geq n$ and every $\xi \in S^{n-1}$ one has

$$(2.4) \quad \|\langle \cdot, \xi \rangle\|_p \approx \|\langle \cdot, \xi \rangle\|_\infty$$

(see [10, Lemma 3.2.8/Proposition 5.1.2]) and hence $Z_p(K) \supseteq c_2 Z_\infty(K)$, where $c_2 > 0$ is an absolute constant and $Z_\infty(K) = \text{conv}\{K, -K\}$.

Recall that $\Psi_2(K)$ is the symmetric convex body with support function

$$h_{\Psi_2(K)}(\xi) = \|\langle \cdot, \xi \rangle\|_{\psi_2}, \quad \xi \in S^{n-1}.$$

An important consequence for the present paper is the following representation of the ψ_2 -norm:

$$\|\langle \cdot, \xi \rangle\|_{L_{\psi_2}(K)} \approx \sup_{p \geq 1} \frac{\|\langle \cdot, \xi \rangle\|_{L_p(K)}}{\sqrt{p}} \approx \sup_{1 \leq p \leq n} \frac{\|\langle \cdot, \xi \rangle\|_{L_p(K)}}{\sqrt{p}}$$

for every centered convex body K and any $\xi \in S^{n-1}$, where the second equivalence is immediate from the fact that $p \mapsto \|\langle \cdot, \xi \rangle\|_{L_p(K)}/\sqrt{p}$ is decreasing on $[n, \infty)$ by (2.4). This implies

$$(2.5) \quad \Psi_2(K) \approx \text{conv} \left\{ \frac{Z_p(K)}{\sqrt{p}} : 1 \leq p \leq n \right\}.$$

For a subspace $F \in G_{n,k}$, let $P_F : \mathbb{R}^n \rightarrow F$ denote the orthogonal projection. The marginal of a log-concave probability measure μ with respect to F is defined by

$$\pi_F(\mu)(A) := \mu(P_F^{-1}(A))$$

for every Borel subset A of F . Since linear images preserve log-concavity, $\pi_F(\mu)$ is again a log-concave probability measure on F . Moreover, if μ is centered, then $\pi_F(\mu)$ is centered, and if μ is isotropic then $\pi_F(\mu)$ is isotropic.

A basic observation of Paouris [24] (see also [10, Theorem 5.1.12]) is that projections commute with centroid bodies. Namely, for every $1 \leq k \leq n$ and any $F \in G_{n,k}$, and every $p \geq 1$,

$$(2.6) \quad P_F(Z_p(\mu)) = Z_p(\pi_F(\mu)).$$

For $q \neq 0$ and $q > -n$, define the q -th moment of the Euclidean norm by

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} |x|^q d\mu(x) \right)^{1/q}.$$

Combining the small-ball estimate (2.1) of Bizeul [5] with the standard representation of negative moments through small-ball probabilities, one obtains

$$I_{-(n-1)}(\mu) \approx I_2(\mu)$$

for every isotropic log-concave probability measure μ on $\mathbb{R}^n$ (see [11] and [12, Theorem 6.9]).

The above equivalence of negative moments was used by Letwin and Mikulincer [20, Proposition 2.4] to obtain the following result.

Proposition 2.1. *There exist absolute constants $c_0 \in (0, 1/8]$ and $c, C > 0$ such that the following holds. If $K \subset \mathbb{R}^n$ is an isotropic convex body, then for every $1 \leq p \leq 2c_0n$,*

$$w_{-p}(Z_p(K)) \approx w_{-2p}(Z_p(K)) \approx \sqrt{p} L_K.$$

Here, for a convex body L and $q \neq 0$, the q -th mean width is defined by

$$(2.7) \quad w_q(L) = \left(\int_{S^{n-1}} h_L(\xi)^q d\sigma(\xi) \right)^{1/q}.$$

In particular, this gives precise control of the negative moments of the support function of $Z_p(K)$.

The previous facts allow us to convert information on the family of centroid bodies $Z_p(K)$ into information on the subgaussian body $\Psi_2(K)$. The following construction provides the main mechanism for doing so. Let K be an isotropic convex body in $\mathbb{R}^n$ and fix $1 \leq p \leq c_0n$, where c_0 is the constant from Proposition 2.1. Define

$$(2.8) \quad A_p(K) = \{y \in \mathbb{R}^n : \|\langle \cdot, y \rangle\|_p \leq C_0 \sqrt{np} L_K\}.$$

Equivalently,

$$A_p := A_p(K) = C_0 \sqrt{np} L_K Z_p(K)^\circ.$$

In view of (2.5), one of the main new ideas of Letwin and Mikulincer [20] is to estimate the Gaussian measure of finite intersections of the sets A_p from below. Using Proposition 2.1 they obtain the next lower bound for each individual A_p .

Lemma 2.2. *For every $1 \leq p \leq 2c_0n$, if the constant C_0 in (2.8) is chosen sufficiently large, then*

$$\gamma_n(A_p) \geq e^{-C_1 p},$$

where $C_1 > 0$ is an absolute constant.

Then, Letwin and Mikulincer combine Lemma 2.2 with the Gaussian correlation inequality.

Theorem 2.3 (Gaussian correlation inequality). *Let γ_n be the standard Gaussian measure on $\mathbb{R}^n$. If A, B are symmetric convex Borel sets in $\mathbb{R}^n$, then*

$$\gamma_n(A \cap B) \geq \gamma_n(A) \gamma_n(B).$$

By induction, for every finite family of symmetric convex Borel sets $A_1, \dots, A_m$,

$$\gamma_n \left(\bigcap_{j=1}^m A_j \right) \geq \prod_{j=1}^m \gamma_n(A_j).$$

The Gaussian correlation inequality was proved by Royen [27]; see also the exposition of Latała and Matlak [19]. We will use this inequality together with Lemma 2.2 to obtain simultaneous control of the sets A_p , which is the key step in constructing large families of subgaussian directions.

3 Geometry of the subgaussian body

In this section, we prove our main results concerning the geometry of the subgaussian body $\Psi_2(K)$ associated with a centered convex body K in $\mathbb{R}^n$.

§ 3.1. Volume estimates. We begin with the proof of Theorem 1.1, which asserts that $\Psi_2(K)$ has uniformly bounded volume ratio.

Proof of Theorem 1.1. The subgaussian body $\Psi_2(K)$ satisfies the following affine invariance property:

$$\Psi_2(T(K)) = T(\Psi_2(K)), \quad T \in SL_n.$$

Since the L_p -centroid bodies $Z_p(K)$ (and hence $Z_2(K)$) satisfy the same property, the volume ratio is affine invariant. Consequently, we may assume that K is isotropic.

Our starting point is (2.5):

$$\Psi_2(K) \approx \text{conv} \left\{ \frac{Z_p(K)}{\sqrt{p}} : 1 \leq p \leq n \right\}.$$

By (2.3) we have $Z_{2p}(K) \approx Z_p(K)$ for all $p \geq 1$. This implies that the above convex hull may be restricted, up to an absolute constant, to dyadic values of p . Hence,

$$\Psi_2(K) \approx \text{conv} \left\{ \frac{Z_{2^k}(K)}{\sqrt{2^k}} : k = 0, 1, \dots, \lfloor \log_2 n \rfloor \right\}.$$

In fact, using the inclusion $Z_n(K) \subseteq \frac{c}{c_0} Z_{c_0 n}(K)$, where c_0 is the absolute constant in Proposition 2.1, we see that

$$\Psi_2(K) \approx \text{conv} \left\{ \frac{Z_{2^k}(K)}{\sqrt{2^k}} : k = 0, 1, \dots, k_0 \right\}.$$

where $k_0 = \lceil \log_2(c_0 n) \rceil$. Taking polars, we obtain

$$\Psi_2(K)^\circ \approx \bigcap_{k=0}^{k_0} \sqrt{2^k} Z_{2^k}(K)^\circ.$$

By the definition of the sets A_p in (2.8),

$$\sqrt{p} Z_p(K)^\circ = \frac{1}{C_0 \sqrt{n} L_K} A_p.$$

Therefore,

$$\Psi_2(K)^\circ \approx \frac{1}{C_0 \sqrt{n} L_K} \bigcap_{k=0}^{k_0} A_{2^k}.$$

Set

$$A = \bigcap_{k=0}^{k_0} A_{2^k}.$$

Each A_{2^k} is a symmetric convex set. Therefore, the Gaussian correlation inequality (Theorem 2.3) gives

$$\gamma_n(A) = \gamma_n\left(\bigcap_{k=0}^{k_0} A_{2^k}\right) \geq \prod_{k=0}^{k_0} \gamma_n(A_{2^k}).$$

Applying Lemma 2.2, we obtain

$$\gamma_n(A) \geq \prod_{k=0}^{k_0} e^{-C_1 2^k} = \exp\left(-C_1 \sum_{k=0}^{k_0} 2^k\right) \geq e^{-C_2 n}.$$

Since

$$\gamma_n(A) = (2\pi)^{-n/2} \int_A e^{-|x|^2/2} dx$$

and $e^{-|x|^2/2} \leq 1$, we have

$$\text{vol}_n(A) \geq \int_A e^{-|x|^2/2} dx = (2\pi)^{n/2} \gamma_n(A) \geq e^{-C_3 n}.$$

Consequently,

$$\text{vol}_n(\Psi_2(K)^\circ) \geq \left(\frac{c}{\sqrt{n}L_K}\right)^n.$$

By the Blaschke–Santaló inequality,

$$\text{vol}_n(\Psi_2(K))\text{vol}_n(\Psi_2(K)^\circ) \leq \omega_n^2,$$

together with $\omega_n^{1/n} \approx \frac{1}{\sqrt{n}}$ and the boundedness of the isotropic constant $L_K \leq C$, we conclude that

$$\text{vrad}(\Psi_2(K)) \leq C.$$

This proves the theorem. $\square$

For a centered log-concave probability measure μ on $\mathbb{R}^n$, we define the symmetric convex body $\Psi_2(\mu)$ by

$$h_{\Psi_2(\mu)}(v) := \sup_{1 \leq q \leq n} \frac{\|\langle \cdot, v \rangle\|_{L_q(\mu)}}{\sqrt{q}}, \quad v \in \mathbb{R}^n.$$

If K is isotropic then the probability measure μ_K , with density $L_K^n \mathbb{1}_{\frac{K}{L_K}}$, satisfies $\Psi_2(K) \approx L_K \Psi_2(\mu_K) \approx \Psi_2(\mu_K)$.

We shall use an observation from [14]. A proof of the next proposition, in this exact formulation, can be found in [10, Proposition 8.1.7] (we also take into account the fact that all isotropic constants are now known to be of order 1 up to an absolute constant).

Proposition 3.1. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then there exists a centered convex body $T = T_\mu$ in $\mathbb{R}^n$ with the following property: For every $1 \leq q \leq n$,*

$$c_1 Z_q(T) \subseteq Z_q(\mu) + \sqrt{q} B_2^n \subseteq c_2 Z_q(T),$$

where $c_1, c_2 > 0$ are absolute constants. In particular, $Z_q(\mu) \subseteq c_2 Z_q(T)$ for every $1 \leq q \leq n$.

The convex body T with these properties can be taken to be

$$T = \overline{K_{n+1}}(\mu * \gamma_n),$$

where γ_n denotes the standard Gaussian measure on $\mathbb{R}^n$. See [10, Chapter 2] for the construction of K. Ball's bodies $K_p(\mu)$.

We now extend the volume estimate for Ψ_2 from convex bodies to log-concave measures.

Theorem 3.2. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Then,*

$$\left(\frac{\text{vol}_n(\Psi_2(\mu))}{\text{vol}_n(Z_2(\mu))} \right)^{1/n} \leq C,$$

where $C > 0$ is an absolute constant.

Proof. By affine invariance, we may assume that μ is isotropic. Let $T = T_\mu$ be the convex body given by Proposition 3.1. From the definition of Ψ_2 and Proposition 3.1, we have

$$(3.1) \quad \Psi_2(\mu) \subseteq c_2 \Psi_2(T).$$

Since μ is isotropic, we have $Z_2(\mu) = B_2^n$. Moreover, $\text{vol}_n(Z_2(T)) = \omega_n L_T^n$ and hence,

$$\text{vol}_n(Z_2(T)) = \text{vol}_n(Z_2(\mu)) L_T^n.$$

Combining the above and applying Theorem 1.1 to the centered convex body T , we obtain

$$\left(\frac{\text{vol}_n(\Psi_2(\mu))}{\text{vol}_n(Z_2(\mu))} \right)^{1/n} \leq c_2 L_T \left(\frac{\text{vol}_n(\Psi_2(T))}{\text{vol}_n(Z_2(T))} \right)^{1/n} \leq c_2 C L_T.$$

Since L_T is bounded by an absolute constant, this completes the proof. $\square$

We now prove Theorem 1.3, which gives an upper bound for the volume of projections of $\Psi_2(K)$ in the isotropic case.

Proof of Theorem 1.3. Let $F \in G_{n,m}$. By the projection property of centroid bodies, (2.6), we have

$$\begin{aligned} P_F(\Psi_2(K)) &\approx \text{conv} \left\{ \frac{P_F(Z_p(K))}{\sqrt{p}} : p = 2^k, k = 0, 1, \dots, \lfloor \log_2 n \rfloor \right\} \\ &= \text{conv} \left\{ \frac{L_K Z_p(\pi_F(\mu_K))}{\sqrt{p}} : p = 2^k, k = 0, 1, \dots, \lfloor \log_2 n \rfloor \right\}. \end{aligned}$$

If $p > m$, the moment comparison inequality (2.3) in dimension m gives

$$Z_p(\pi_F(\mu_K)) \subseteq c \frac{p}{m} Z_m(\pi_F(\mu_K)).$$

Hence,

$$\frac{Z_p(\pi_F(\mu_K))}{\sqrt{p}} \subseteq c \sqrt{\frac{p}{m}} \frac{Z_m(\pi_F(\mu_K))}{\sqrt{m}}.$$

Since $p \leq n$, we obtain

$$\frac{Z_p(\pi_F(\mu_K))}{\sqrt{p}} \subseteq c \sqrt{\frac{n}{m}} \frac{Z_m(\pi_F(\mu_K))}{\sqrt{m}}.$$

Therefore,

$$\begin{aligned} P_F(\Psi_2(K)) &\subseteq c\sqrt{\frac{n}{m}} L_K \operatorname{conv} \left\{ \frac{Z_p(\pi_F(\mu_K))}{\sqrt{p}} : p = 2^k, k = 0, 1, \dots, \lfloor \log_2 m \rfloor \right\} \\ &\subseteq c\sqrt{\frac{n}{m}} L_K \Psi_2(\pi_F(\mu_K)). \end{aligned}$$

The marginal $\pi_F(\mu_K)$ is isotropic on F . Applying Theorem 3.2 in dimension m gives

$$\operatorname{vrad}(\Psi_2(\pi_F(\mu_K))) \leq C.$$

Consequently,

$$\operatorname{vrad}(P_F(\Psi_2(K))) \leq C\sqrt{\frac{n}{m}},$$

which proves the theorem. $\square$

Choosing $m = 1$ in Theorem 1.3 we see that $\operatorname{vol}_1(P_F(\Psi_2(K))) \leq 2C\sqrt{n}$ for every $F \in G_{n,1}$. Since $\Psi_2(K)$ is symmetric, this is equivalent to $h_{\Psi_2(K)}(v) \leq C\sqrt{n}$ for every $v \in S^{n-1}$. Hence, we obtain the next fact.

Corollary 3.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then $R(\Psi_2(K)) \leq C\sqrt{n}$, where $C > 0$ is an absolute constant.*

The same fact can be also seen as follows: since K is isotropic, standard estimates for isotropic convex bodies (that can be found e.g. in [10, Chapter 3]) give $R(K) \leq c_1 n L_K$ and $\|\langle \cdot, v \rangle\|_{\psi_1} \leq c_2 L_K$ for all $v \in S^{n-1}$, where

$$\|\langle \cdot, v \rangle\|_{\psi_1} = \inf \{t > 0 : \mathbb{E} \exp(|\langle \cdot, v \rangle|/t) \leq 2\}.$$

Consequently,

$$\|\langle \cdot, v \rangle\|_{\psi_2} \leq C\sqrt{\|\langle \cdot, v \rangle\|_{\psi_1} \|\langle \cdot, v \rangle\|_{\infty}} \leq C_1 \sqrt{n} L_K \leq C\sqrt{n}.$$

§ 3.2. Construction of subgaussian bases. We next use Theorem 1.3 to construct an orthonormal basis consisting of directions with controlled subgaussian norms, following the approach of [20].

Proof of Theorem 1.2. Since $Z_2(K) = L_K B_2^n \approx B_2^n$, by Theorem 1.1 we have that $r := \operatorname{vrad}(\Psi_2(K)) \leq C$ for an absolute constant $C > 0$. Note that

$$(3.2) \quad \min\{h_{\Psi_2(K)}(v) : v \in S^{n-1}\} \leq r.$$

Otherwise, there exists $s > r$ such that $h_{\Psi_2(K)}(v) \geq s = h_{sB_2^n}(v)$ for all $v \in S^{n-1}$, which implies $\Psi_2(K) \supseteq sB_2^n$ and hence $\operatorname{vrad}(\Psi_2(K)) \geq s > r$, a contradiction. From (3.2) we see that there exists $v_1 \in S^{n-1}$ such that

$$\|\langle \cdot, v_1 \rangle\|_{\psi_2} \leq r = \operatorname{vrad}(\Psi_2(K)) \leq C.$$

We proceed inductively. Assume that $v_1, \dots, v_{k-1}$ have been chosen so that they form an orthonormal sequence and satisfy

$$\|\langle \cdot, v_i \rangle\|_{\psi_2} \leq C\sqrt{\frac{n}{n-i+1}}, \quad i = 1, \dots, k-1.$$

Let $F_k = (\operatorname{span}\{v_1, \dots, v_{k-1}\})^\perp$. Then $F_k \in G_{n, n-k+1}$. By Theorem 1.3,

$$\operatorname{vrad}(P_{F_k}(\Psi_2(K))) \leq C\sqrt{\frac{n}{n-k+1}}.$$

Since $P_{F_k}(\Psi_2(K))$ is a convex body in F_k , there exists a vector $v_k \in S^{n-1} \cap F_k$ such that

$$h_{P_{F_k}(\Psi_2(K))}(v_k) \leq C \sqrt{\frac{n}{n-k+1}}.$$

Because $v_k \in F_k$,

$$h_{P_{F_k}(\Psi_2(K))}(v_k) = h_{\Psi_2(K)}(v_k),$$

and therefore

$$\|\langle \cdot, v_k \rangle\|_{\psi_2} \leq C \sqrt{\frac{n}{n-k+1}}.$$

By construction, the vectors remain orthonormal. Iterating the argument gives an orthonormal basis $\{v_1, \dots, v_n\}$ with the desired estimates. $\square$

§ 3.3. Mean width and mean norm. Recall from (2.2) that $w(K) \leq C\sqrt{n \ln(en)}$ for every isotropic convex body K in $\mathbb{R}^n$. We begin with a version of this result for isotropic log-concave probability measures.

Lemma 3.4. *Let ν be an isotropic log-concave probability measure on $\mathbb{R}^m$. Then*

$$(3.3) \quad w(Z_m(\nu)) \leq C\sqrt{m \ln(em)}.$$

Proof. Let f_ν denote the density of ν . We shall use a number of known facts about Ball's bodies $K_p(\nu)$ associated with ν . Recall that, for any $p > 0$, the convex body $K_p(\nu)$ has radial function

$$\rho_{K_p(\nu)}(x) = \left(\frac{1}{f_\nu(0)} \int_0^\infty pr^{p-1} f_\nu(rx) dr \right)^{1/p}$$

for $x \neq 0$. We refer the reader to [10] for more details about the family $\{K_p(\nu)\}_{p>0}$. From [10, Proposition 2.5.8] we have

$$f_\nu(0) \text{vol}_m(K_{m+1}(\nu)) \approx 1.$$

Then, [10, Proposition 2.5.12] shows that if we set

$$a := \text{vol}_m(K_{m+1}(\nu))^{-1/m} \approx f_\nu(0)^{1/m} \approx L_\nu \approx 1$$

then $M = aK_{m+1}(\nu)$ has volume one and is almost isotropic. This means (see [10, Definition 2.5.11]) that if $T \in SL_m$ is chosen so that $T(M)$ is isotropic then $T(B_2^m) \approx B_2^m$. By Bizeul's estimate,

$$(3.4) \quad w(K_{m+1}(\nu)) \approx w(T(M)) \leq C\sqrt{m \ln(em)}.$$

On the other hand, [10, Theorem 5.1.7] and the fact that $f_\nu(0)^{1/m} \approx L_\nu \approx 1$ imply

$$Z_m(\nu) \approx Z_m(M).$$

Since $Z_m(M) \subseteq \text{conv}\{M, -M\}$, we conclude that $w(Z_m(\nu)) \leq C_1 w(K_{m+1}(\nu))$, and the lemma follows from (3.4). $\square$

Using Lemma 3.4 we obtain the following consequence of Bizeul's sharp bound.

Lemma 3.5. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leq q \leq n$,*

$$w(Z_q(K)) \leq C\sqrt{q \ln(eq)}.$$

Proof. Without loss of generality, we may assume that q is an integer. Let μ be the isotropic measure with density $L_K^n \mathbf{1}_{\frac{K}{L_K}}$. For any $F \in G_{n,q}$ we have that $\pi_F(\mu)$ is isotropic on the q -dimensional space F . Since, $P_F(Z_q(\mu)) = Z_q(\pi_F(\mu))$, applying (3.3) for $\pi_F(\mu)$ we get

$$w_F(P_F(Z_q(\mu))) = w_F(Z_q(\pi_F(\mu))) \leq C\sqrt{q \ln(eq)}.$$

Now, we use the identity

$$\begin{aligned} w(A) &= \int_{G_{n,q}} \int_{S_F} h_A(\xi) d\sigma_F(\xi) d\nu_{n,q}(F) = \int_{G_{n,q}} \int_{S_F} h_{P_F(A)}(\xi) d\sigma_F(\xi) d\nu_{n,q}(F) \\ &= \int_{G_{n,q}} w_F(P_F(A)) d\nu_{n,q}(F) \end{aligned}$$

to write

$$w(Z_q(\mu)) = \int_{G_{n,q}} w_F(P_F(Z_q(\mu))) d\nu_{n,q}(F) \leq C\sqrt{q \ln(eq)}$$

completing the proof. $\square$

Theorem 3.6. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$w(\Psi_2(K)) \leq C\sqrt{\ln(en)},$$

where $C > 0$ is an absolute constant.

Proof. We start from the formula

$$\Psi_2(K) \approx \text{conv} \left\{ \frac{Z_{2^k}(K)}{\sqrt{2^k}} : k = 0, 1, \dots, \lfloor \log_2 n \rfloor \right\}.$$

This implies that for any $s \geq 1$ we have

$$h_{\Psi_2(K)}(v) \approx \max_{1 \leq k \leq \lfloor \log_2 n \rfloor} \frac{h_{Z_{2^k}}(v)}{\sqrt{2^k}} \leq c_5 \left(\sum_{k=0}^{\lfloor \log_2 n \rfloor} \frac{h_{Z_{2^k}}^s(v)}{(\sqrt{2^k})^s} \right)^{1/s}.$$

We set $q_k = 2^k$. Integrating on S^{n-1} and using Hölder's inequality, we obtain

$$(3.5) \quad w(\Psi_2(K)) \leq c_5 \left(\sum_{k=0}^{\lfloor \log_2 n \rfloor} \frac{1}{(\sqrt{q_k})^s} \int_{S^{n-1}} h_{Z_{q_k}}^s(v) d\sigma(v) \right)^{1/s} = c_5 \left(\sum_{k=0}^{\lfloor \log_2 n \rfloor} \left(\frac{w_s(Z_{q_k}(K))}{\sqrt{q_k}} \right)^s \right)^{1/s},$$

where the s -mean width $w_s(C)$ of a symmetric convex body C in $\mathbb{R}^n$ is defined in (2.7). These parameters were studied by Litvak, V. Milman and Schechtman who showed in [21] that

$$(3.6) \quad w_s(C) \approx \max \left\{ w(C), \frac{R(C)\sqrt{s}}{\sqrt{n}} \right\}$$

for all $1 \leq s \leq n$.

We choose $s = \ln(en)$. We shall use the fact that for every k we have $R(Z_{q_k}(K)) \leq c_6 q_k$ (see [10,

Chapter 5]). From (3.6) we get

$$(3.7) \quad \frac{w_s(Z_{q_k}(K))}{\sqrt{q_k}} \leq c_8 \max \left\{ \frac{w(Z_{q_k}(K))}{\sqrt{q_k}}, \frac{\sqrt{q_k s}}{\sqrt{n}} \right\} \leq c_9 \max \left\{ \sqrt{\ln(eq_k)}, \sqrt{s} \right\} \leq c_{10} \sqrt{\ln(en)}$$

for every $k = 0, 1, \dots, \lfloor \log_2 n \rfloor$. Since there are only $O(\ln(en))$ dyadic indices and $s \approx \ln(en)$, the factor $\ln^{1/s}(en)$ is bounded by an absolute constant. Going back to (3.5) we conclude the proof. $\square$

We also observe that $\Psi_2(K)$ has bounded M -parameter. From $\Psi_2(K) \supseteq c Z_2(K)$ we readily see that, for every isotropic convex body K in $\mathbb{R}^n$,

$$M(\Psi_2(K)) = \int_{S^{n-1}} \|\xi\|_{\Psi_2(K)} d\sigma(\xi) \leq C.$$

Combining this estimate with Theorem 3.6, we obtain

$$(3.8) \quad M(\Psi_2(K)) w(\Psi_2(K)) \leq C \sqrt{\ln(en)},$$

which means that the subgaussian body of an isotropic convex body is essentially in the ℓ -position.

Remark 3.7. Similar arguments extend Lemma 3.5 and Theorem 3.6 to isotropic log-concave probability measures.

Proposition 3.8. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then*

$$(3.9) \quad w(Z_q(\mu)) \leq C \sqrt{q \ln(eq)}, \quad 1 \leq q \leq n$$

and

$$w(\Psi_2(\mu)) \leq C \sqrt{\ln(en)},$$

where $C > 0$ is an absolute constant.

§ 3.4. Large sets of subgaussian directions. In this subsection we study the distribution of the ψ_2 -norm over directions. Let K be an isotropic convex body in $\mathbb{R}^n$. Brazitikos and Hioni [9] obtained logarithmic estimates for the ψ_2 -norm of most directions. More precisely, they proved that, for every $a > 1$, there exists a set $V_a \subseteq S^{n-1}$ with $\sigma(V_a) \geq 1 - n^{-a}$ such that

$$\|\langle \cdot, v \rangle\|_{\psi_2} \leq C \ln^{3/2}(en) \max\{\sqrt{\ln(en)}, \sqrt{a}\}$$

for every $v \in V_a$.

In view of the bound $w(\Psi_2(K)) \leq C \sqrt{\ln(en)}$ of Theorem 3.6, the next theorem yields that a random orthonormal basis of $\mathbb{R}^n$ consists of ψ_2 -directions with better behavior. Theorem 1.5 is an immediate consequence of Proposition 3.11 below, applied e.g. with $\delta = 1/2$.

Theorem 3.9. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $t > 0$ there exists a subset $V_t \subseteq S^{n-1}$ with*

$$\sigma(V_t) \geq 1 - \exp(-t^2/2)$$

such that, for every $v \in V_t$,

$$(3.10) \quad \|\langle \cdot, v \rangle\|_{\psi_2} = h_{\Psi_2(K)}(v) \leq w(\Psi_2(K)) + Ct \leq C(\sqrt{\ln(en)} + t),$$

where $c, C > 0$ are absolute constants.

The proof of Theorem 3.9 relies on the classical deviation inequality for Lipschitz functions $f : S^{n-1} \rightarrow \mathbb{R}$. (see [3, Proposition 5.20] for the exact formulation below), which is a consequence of the spherical isoperimetric inequality.

Theorem 3.10. *Let $f : S^{n-1} \rightarrow \mathbb{R}$ be a Lipschitz continuous function with constant b . Then, for every $s > 0$,*

$$\sigma(\{v \in S^{n-1} : f(v) \geq \mathbb{E}(f) + bs\}) \leq \exp(-s^2 n/2),$$

where $\mathbb{E}(f)$ is the expectation of f .

Proof of Theorem 3.9. Corollary 3.3 implies that $f(v) = \|\langle \cdot, v \rangle\|_{\psi_2}$ is Lipschitz continuous on S^{n-1} with constant $C\sqrt{n}$. Note that $\mathbb{E}(f) = w(\Psi_2(K))$. Therefore, Theorem 3.10 implies that

$$\sigma(\{v \in S^{n-1} : \|\langle \cdot, v \rangle\|_{\psi_2} \geq w(\Psi_2(K)) + Ct\}) \leq \exp(-t^2/2)$$

for every $t > 0$.

Define

$$V_t = \{v \in S^{n-1} : \|\langle \cdot, v \rangle\|_{\psi_2} \leq w(\Psi_2(K)) + Ct\}.$$

Then, $\sigma(V_t) \geq 1 - \exp(-t^2/2)$, which is precisely (3.10). $\square$

The next proposition shows that a random orthonormal basis of $\mathbb{R}^n$ consists entirely of subgaussian directions, with subgaussian constants growing at most logarithmically in the dimension.

Proposition 3.11. *There exists an absolute constant $C_1 > 0$ such that, for every isotropic convex body K in $\mathbb{R}^n$,*

$$\mathbb{E} \max_{1 \leq i \leq n} \|\langle \cdot, U(e_i) \rangle\|_{\psi_2} \leq C_1 \sqrt{\ln(en)}.$$

Moreover, for any $\delta \in (0, 1)$, a random orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ satisfies with probability at least $1 - \delta$

$$(3.11) \quad \|\langle \cdot, v_i \rangle\|_{\psi_2} \leq w(\Psi_2(K)) + C_1 \sqrt{\ln(2n/\delta)} \leq C_2 \sqrt{\ln(2n/\delta)}, \quad i = 1, \dots, n.$$

where $C_2 > 0$ is an absolute constant.

Proof. Let $U \in O(n)$ and define $f_i(U) = \|\langle \cdot, U(e_i) \rangle\|_{\psi_2}$. Then

$$\begin{aligned} \mathbb{E} \max_{1 \leq i \leq n} \|\langle \cdot, U(e_i) \rangle\|_{\psi_2} &= \int_0^\infty \nu(\{U \in O(n) : \max_{1 \leq i \leq n} f_i(U) \geq t\}) dt \\ &\leq t_0 + n \int_{t_0}^\infty \nu(\{U \in O(n) : f_1(U) \geq t\}) dt \\ &= t_0 + n \int_{t_0}^\infty \sigma(\{v \in S^{n-1} : h_{\Psi_2(K)}(v) \geq t\}) dt, \end{aligned}$$

where t_0 will be chosen later. Assume that

$$t_0 \geq 2C\sqrt{2\ln(en)} + w(\Psi_2(K)),$$

where $C > 0$ is the absolute constant in Theorem 3.9. By the change of variables $t = w(\Psi_2(K)) + Cs$ we obtain

$$\int_{t_0}^\infty \sigma(\{v \in S^{n-1} : h_{\Psi_2(K)}(v) \geq t\}) dt = C \int_{\frac{1}{C}(t_0 - w(\Psi_2(K)))}^\infty \sigma(\{v \in S^{n-1} : h_{\Psi_2(K)}(v) \geq w(\Psi_2(K)) + Cs\}) ds.$$

Then, for every $s > 0$ there exists $V_s \subset S^{n-1}$ such that $\sigma(V_s) \geq 1 - \exp(-s^2/2)$ and for every $v \in V_s$ we have that $h_{\Psi_2(K)}(v) \leq w(\Psi_2(K)) + Cs$. Thus

$$\sigma(\{v \in S^{n-1} : h_{\Psi_2(K)}(v) \geq w(\Psi_2(K)) + Cs\}) \leq \sigma(S^{n-1} \setminus V_s) \leq \exp(-s^2/2)$$

Using the change of variables $t = 2s\sqrt{2\ln(en)}$ we obtain

$$\begin{aligned} \int_{\frac{1}{C}(t_0 - w(\Psi_2(K)))}^{\infty} \exp(-t^2/2) dt &\leq \int_{2\sqrt{2\ln(en)}}^{\infty} \exp(-t^2/2) dt \\ &= 2\sqrt{2\ln(en)} \int_1^{\infty} \exp(-4s^2 \ln(en)) ds \\ &\leq 2\sqrt{2\ln(en)} \int_1^{\infty} \exp(-4s \ln(en)) ds \\ &\leq \frac{2}{e^3} \sqrt{2\ln(en)} n^{-3} \int_1^{\infty} e^{-s} ds = \frac{2}{e^4} n^{-3} \sqrt{2\ln(en)}, \end{aligned}$$

where we used the inequality $\exp(-4s \ln(en)) \leq e^{-s} \exp(-3 \ln(en))$ which holds for every $s \geq 1$. Finally, since $w(\Psi_2(K)) \leq C\sqrt{\ln(en)}$, we may choose $t_0 = C_0\sqrt{\ln(en)}$ where $C_0 > 0$ is a sufficiently large absolute constant. Therefore, combining the above estimates, we obtain

$$\mathbb{E} \max_{1 \leq i \leq n} \|\langle \cdot, U(e_i) \rangle\|_{\psi_2} \leq C_1 \sqrt{\ln(en)}.$$

For the second claim, let $t > 0$ and consider the set V_t from Theorem 3.9. Fix an orthonormal basis $\{e_1, \dots, e_n\}$ and observe that, for every $i = 1, \dots, n$,

$$\nu(\{U \in O(n) : U(e_i) \notin V_t\}) \leq \exp(-t^2/2).$$

If $n \exp(-t^2/2) < \delta$ then $\{U(e_1), \dots, U(e_n)\} \subset V_t$ with probability greater than $1 - \delta$. This shows that

$$\nu(\{U \in O(n) : \max_{1 \leq i \leq n} \|\langle \cdot, U(e_i) \rangle\|_{\psi_2} \leq w(\Psi_2(K)) + (C/\sqrt{c})\sqrt{\ln(2n/\delta)}\}) \geq 1 - \delta.$$

In other words, a random orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ satisfies (3.11) with probability at least $1 - \delta$. $\square$

We can also show that if K is isotropic then, for any $1 \leq k \leq n-1$, a random $F \in G_{n,k}$ is “subgaussian with constant $\sqrt{k}$ for K ” in the sense that $\|\langle \cdot, v \rangle\|_{\psi_2} \leq C \max\{w(\Psi_2(K)), \sqrt{k}\}$ for all $v \in S^{n-1} \cap F$.

Proposition 3.12. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leq k \leq n-1$ there exists a subset $\Gamma_k \subseteq G_{n,k}$ with $\nu_{n,k}(\Gamma_k) \geq 1 - e^{-ck}$ such that for every $F \in \Gamma_k$ and every $v \in S^{n-1} \cap F$,*

$$\|\langle \cdot, v \rangle\|_{\psi_2} \leq C \max\{w(\Psi_2(K)), \sqrt{k}\} \leq C_1 \max\{\sqrt{\ln(en)}, \sqrt{k}\}.$$

The proof of Proposition 3.12 relies on a result of V. Milman [23]; see also [1, Proposition 5.7.1], which gives a bound for the diameter of random low-dimensional projections of a symmetric convex body.

Proposition 3.13. *Let K be a symmetric convex body in $\mathbb{R}^n$ satisfying $K \subseteq RB_2^n$. For every $1 \leq k \leq n-1$, there exists a subset $\Gamma_k \subseteq G_{n,k}$ with $\nu_{n,k}(\Gamma_k) \geq 1 - e^{-ck}$ such that for every $F \in \Gamma_k$,*

$$P_F(K) \subseteq C \max\{w(K), R\sqrt{k/n}\} B_2^n \cap F,$$

where $c > 0$ and $C > 0$ are absolute constants.

Proof of Proposition 3.12. Recall from the proof of Theorem 3.9 that

$$R(\Psi_2(K)) \leq C\sqrt{n}.$$

Given k , we apply Proposition 3.13 to the symmetric convex body $\Psi_2(K)$ to find a subset $\Gamma_k \subseteq G_{n,k}$ with

$$\nu_{n,k}(\Gamma_k) \geq 1 - e^{-ck}$$

such that for every $F \in \Gamma_k$,

$$P_F(\Psi_2(K)) \subseteq C \max \left\{ w(\Psi_2(K)), R(\Psi_2(K))\sqrt{k/n} \right\} B_2^n \cap F.$$

Using the estimate for the radius of $\Psi_2(K)$, we have

$$R(\Psi_2(K))\sqrt{k/n} \leq C_1\sqrt{k}.$$

Therefore,

$$P_F(\Psi_2(K)) \subseteq C \max \{ w(\Psi_2(K)), \sqrt{k} \} B_2^n \cap F$$

for every $F \in \Gamma_k$. Equivalently, for every $F \in \Gamma_k$ and any $v \in S^{n-1} \cap F$,

$$h_{P_F(\Psi_2(K))}(v) \leq C \max \{ w(\Psi_2(K)), \sqrt{k} \}.$$

Since $h_{P_F(\Psi_2(K))}(v) = h_{\Psi_2(K)}(v) = \|\langle \cdot, v \rangle\|_{\psi_2}$, the result follows. $\square$

§ 3.5. The case of unconditional bodies. Recall that a convex body K in $\mathbb{R}^n$ is called unconditional if $x = \sum_{i=1}^n x_i e_i \in K$ implies that $\sum_{i=1}^n \varepsilon_i x_i e_i \in K$ for every choice of signs $\varepsilon_i \in \{-1, 1\}$, where $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. We conclude this article by showing that unconditionality yields a particularly strong form of the previous results: one can construct an explicit orthonormal basis all of whose vectors are uniformly subgaussian directions.

We begin with the model example of the unit ball of ℓ_1^n equipped with the uniform probability measure μ_n , whose density is

$$\frac{d\mu_n}{dx} = \frac{n!}{2^n} \mathbb{1}_{B_1^n}.$$

A direct computation (see [7]) shows that, for every $v \in \mathbb{R}^n$,

$$(3.12) \quad c_1 \|v\|_\infty \leq \sqrt{n} \|\langle \cdot, v \rangle\|_{\psi_2(\mu_n)} \leq c_2 \|v\|_\infty,$$

where one may take $c_1 = 1/\sqrt{6}$ and $c_2 = 2\sqrt{2}$.

Bobkov and Nazarov [7] proved a comparison theorem implying, in particular, that the upper estimate in (3.12) extends to arbitrary isotropic unconditional convex bodies.

Theorem 3.14 (Bobkov–Nazarov). *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then, for every $v \in \mathbb{R}^n$,*

$$\|\langle \cdot, v \rangle\|_{\psi_2} \leq C\sqrt{n} \|v\|_\infty.$$

The next elementary lemma provides a convenient orthonormal basis adapted to the ℓ_∞ -norm.

Lemma 3.15. *For every $n \geq 1$, there exists an orthonormal basis $\{v_0, \dots, v_{n-1}\}$ of $\mathbb{R}^n$ such that*

$$\|v_i\|_\infty \leq \frac{\sqrt{2}}{\sqrt{n}}, \quad i = 0, \dots, n-1.$$

Proof. Let $\{u_0, \dots, u_{n-1}\}$ denote the standard discrete Fourier basis of $\mathbb{C}^n$, given by

$$(u_k)_m = \frac{1}{\sqrt{n}} e^{2\pi i k m / n}, \quad 0 \leq k, m \leq n-1.$$

For $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, define

$$v_{2k-1} = \sqrt{2} \operatorname{Re}(u_k), \quad v_{2k} = \sqrt{2} \operatorname{Im}(u_k),$$

and set

$$v_0 = u_0.$$

If n is even, define in addition

$$v_{n-1} = u_{n/2} = \frac{1}{\sqrt{n}} (1, -1, 1, -1, \dots, -1).$$

The standard orthogonality relations for the discrete Fourier basis imply that these vectors form an orthonormal basis of $\mathbb{R}^n$.

Moreover,

$$(v_{2k-1})_m = \sqrt{\frac{2}{n}} \cos\left(\frac{2\pi k m}{n}\right), \quad (v_{2k})_m = \sqrt{\frac{2}{n}} \sin\left(\frac{2\pi k m}{n}\right),$$

so every coordinate of v_{2k-1} and v_{2k} has absolute value at most $\sqrt{2/n}$. Since

$$\|v_0\|_\infty = \frac{1}{\sqrt{n}},$$

and the same estimate holds for v_{n-1} when n is even, every basis vector satisfies

$$\|v_i\|_\infty \leq \frac{\sqrt{2}}{\sqrt{n}},$$

as claimed. □

Combining the previous two results immediately yields the following consequence.

Corollary 3.16. *Let K be an isotropic unconditional convex body in $\mathbb{R}^n$. Then there exists an explicit orthonormal basis $\{v_0, \dots, v_{n-1}\}$ of $\mathbb{R}^n$ such that*

$$\|\langle \cdot, v_i \rangle\|_{\psi_2} \leq C, \quad i = 0, \dots, n-1.$$

Proof. Let $\{v_0, \dots, v_{n-1}\}$ be the basis from Lemma 3.15. By Theorem 3.14,

$$\|\langle \cdot, v_i \rangle\|_{\psi_2} \leq C_1 \sqrt{n} \|v_i\|_\infty \leq C,$$

which proves the claim. □

Thus, for isotropic unconditional convex bodies, one obtains an explicit orthonormal basis of uniformly subgaussian directions with no logarithmic loss. This contrasts with the general isotropic case, where the methods developed in this work yield bounds depending on $w(\Psi_2(K))$ and $\sqrt{\ln(en)}$.

4 Further remarks and questions

§ 4.1. Covering numbers. Recall that if A and B are two convex bodies in $\mathbb{R}^n$, then the covering number $N(A, B)$ of A by B is the least integer N for which there exist N translates of B whose union covers A . Sudakov's inequality (see [1, Chapter 4]) asserts that if K is a symmetric convex body in $\mathbb{R}^n$, then for every $t > 0$,

$$\ln N(K, tB_2^n) \leq n \frac{w(K)^2}{t^2},$$

where $c > 0$ is an absolute constant. Theorem 1.4 shows that if K is isotropic, then

$$\ln N(\Psi_2(K), tB_2^n) \leq cn \frac{\ln(en)}{t^2}$$

for all $t > 0$. Note that the interesting range for t is $[c, C\sqrt{n}]$ because $cB_2^n \subseteq \Psi_2(K) \subseteq C\sqrt{n} B_2^n$.

Schütt [28] has determined the covering numbers $N(\sqrt{n}B_1^n, tB_2^n)$. He showed that

$$(4.1) \quad \ln N(\sqrt{n}B_1^n, tB_2^n) \leq cn \frac{\ln(2t)}{t^2}, \quad 1 \leq t \leq \sqrt{n}/2.$$

Since the subgaussian body of the isotropic cross-polytope is $\approx \sqrt{n}B_1^n$, this estimate suggests the question to determine if

$$(4.2) \quad \ln N(\Psi_2(K), tB_2^n) \leq cn \frac{\ln(c_0 t)}{t^2}, \quad 1 \leq t \leq C\sqrt{n},$$

for every isotropic convex body K in $\mathbb{R}^n$.

Let us note that, recently, Paouris and Pathak [26] proved that every convex body K in $\mathbb{R}^n$ has an affine image TK with $\text{vol}_n(TK) = \text{vol}_n(\sqrt{n}B_1^n)$ that satisfies (4.1).

§ 4.2. Distribution of subgaussian directions. For every $t \geq 1$, we define

$$S_t(K) := \{v \in S^{n-1} : \|\langle \cdot, v \rangle\|_{\psi_2} \leq t\}.$$

Consider the function

$$\psi_K(t) = \sigma(S_t(K)).$$

Set $r := r(\Psi_2(K))$ and $w := w(\Psi_2(K))$. An interesting question is to give a lower bound for $\psi_K(t)$, especially when $r \ll w$ and $t \in (r, w)$.

From the proof of Theorem 3.9, we see that if $t > w(\Psi_2(K))$, then

$$\sigma(\{v \in S^{n-1} : \|\langle \cdot, v \rangle\|_{\psi_2} > t\}) \leq \exp\left(-\frac{(t - w(\Psi_2(K)))^2}{2C^2}\right).$$

Hence,

$$\psi_K(t) \geq 1 - \exp(-c_1(t - w(\Psi_2(K)))^2),$$

where $c_1 > 0$ is an absolute constant. In particular, if $t \geq 2w(\Psi_2(K))$, then $t - w(\Psi_2(K)) \geq t/2$, which

implies

$$\psi_K(t) \geq 1 - \exp(-c_2 t^2),$$

where $c_2 = c_1/4$.

A subtle question is to study $\psi_K(t)$ in the case $t < w(\Psi_2(K))$. It is instructive to examine the asymptotics for the isotropic cross-polytope $K_1 = \text{vol}_n(B_1^n)^{-1/n} B_1^n$. Then, $\|\langle \cdot, v \rangle\|_{\psi_2(K_1)} \approx \sqrt{n} \|v\|_\infty$. Let

$$M_n = \sqrt{n} \|V\|_\infty,$$

where V is uniformly distributed on the Euclidean sphere S^{n-1} , and define

$$\psi(t) = \mathbb{P}(M_n \leq t).$$

One can check that

$$|\mathbb{E} M_n - \sqrt{2 \ln n}| \leq C \left(\frac{\ln \ln n}{\sqrt{\ln n}} \right),$$

where $C > 0$ is an absolute constant.

Let $G = (g_1, \dots, g_n)$ be a standard Gaussian random vector. Then $V = G/|G|$ is uniformly distributed on the sphere. Hence $M_n = \frac{\sqrt{n} \max_i |g_i|}{|G|}$. Direct computation shows that if $t \gg 1$, then

$$\psi(t) \geq \left(1 - \frac{c}{t} e^{-t^2/2}\right)^n,$$

and hence

$$\psi(t) \geq \exp\left(-\frac{cn}{t} e^{-t^2/2}\right).$$

Writing $t = \delta \sqrt{2 \ln n}$, where $\frac{1}{\sqrt{2 \ln n}} < \delta < 1$, we obtain

$$\psi(\delta \sqrt{2 \ln n}) \geq \exp\left(-\frac{c}{\delta \sqrt{2 \ln n}} n^{1-\delta^2}\right).$$

This extremal example raises the question whether, for any isotropic convex body K , one can obtain an analogous lower bound for $\psi_K(t)$ in the regime $r(\Psi_2(K)) < t < w(\Psi_2(K))$ under the assumption that $r \ll w$.

Note that if K is the isotropic Euclidean ball D_n then $r = w$ and the question becomes trivial; for every $t < w$ we have $S_t(D_n) = \emptyset$.

§ 4.3. Gaussian width. In this paragraph we make some observations about the Gaussian mean width of $S_t(K)$ for values of t that are below $w(\Psi_2(K))$.

If S is a closed subset of S^{n-1} , we set

$$w_G(S) := \mathbb{E} \left(\max_{x \in S} \sum_{j=1}^n x_j g_j \right),$$

where g_j , $1 \leq j \leq n$, are independent standard Gaussian random variables.

Define

$$a_k := \mathbb{E} \left[\left(\sum_{i=1}^k g_i^2 \right)^{1/2} \right] = \sqrt{2} \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} = \sqrt{k} \left(1 - \frac{1}{4k} + \frac{1}{32k^2} + O(k^{-3}) \right)$$

for $k = 1, \dots, n$. Gordon's escape theorem states that if

$$a_k > w_G(S)$$

for some k , then there exists a subspace $F \in G_{n,n-k}$ such that $F \cap S = \emptyset$.

Proposition 4.1. *There exists an absolute constant $C_0 > 0$ such that the following holds. Let K be an isotropic convex body in $\mathbb{R}^n$ and define $S_t := S_t(K) := \{v \in S^{n-1} : \|\langle \cdot, v \rangle\|_{\psi_2} \leq t\}$, $t > 0$. Assume that $t \geq 2C_0$. If*

$$k \geq k_t := \frac{4}{3} \left\lceil C_0^2 \frac{n}{t^2} \right\rceil,$$

then, for every $E \in G_{n,k}$,

$$w_G(S_t \cap E) \geq \frac{1}{2} \sqrt{k}.$$

In particular,

$$w_G(S_t) \geq \frac{1}{2} \sqrt{n}.$$

Proof. We know that for every $1 \leq m \leq n$ and any $F \in G_{n,m}$,

$$\text{vrad}(P_F(\Psi_2(K))) \leq C_0 \sqrt{\frac{n}{m}},$$

where $C_0 > 0$ is an absolute constant. Let $t \geq 2C_0$ and define m to be the smallest positive integer such that

$$C_0 \sqrt{\frac{n}{m}} \leq t.$$

More precisely, set

$$m = m(t) = \left\lceil C_0^2 \frac{n}{t^2} \right\rceil.$$

Then $S_t \cap F \neq \emptyset$ for every $F \in G_{n,m}$.

Let k be large enough so that $k - m \geq \frac{k}{4}$. Equivalently,

$$k \geq k_t := \frac{4}{3} \left\lceil C_0^2 \frac{n}{t^2} \right\rceil.$$

Consider any $E \in G_{n,k}$. Then $(S_t \cap E) \cap F \neq \emptyset$ for every m -dimensional subspace F of E . Therefore, Gordon's theorem yields

$$w_G(S_t \cap E) \geq a_{k-m} \geq \sqrt{k-m} \geq \frac{1}{2} \sqrt{k}.$$

This proves the proposition. □

Combining this information with Dudley's bound (see [29, Exercise 8.6]), we obtain the next fact.

Proposition 4.2. *With the notation of Proposition 4.1, assume that $t \geq 2C_0$ and*

$$k \geq k_t := \frac{4}{3} \left\lceil C_0^2 \frac{n}{t^2} \right\rceil.$$

Then, for every $E \in G_{n,k}$,

$$N(S_t \cap E, \epsilon_0 B_2^n) \geq \exp(c_0 k),$$

where ϵ_0, c_0 are absolute positive constants. In particular, for every $E \in G_{n,k}$ we may find a set $D_E \subseteq S_t \cap E$ such that

$$|D_E| \geq \exp(c_0 k) \quad \text{and} \quad |v - w| \geq \epsilon_0$$

for any pair of distinct $v, w \in D_E$.

Proof. Let $k \geq k_t$ and consider any $E \in G_{n,k}$. We use the fact that

$$w_G(S_t \cap E) \leq C_2 \int_a^b \sqrt{\ln N(S_t \cap E, \epsilon B_2^n)} d\epsilon,$$

where

$$a = \frac{c_1 w_G(S_t \cap E)}{\sqrt{k}} \quad \text{and} \quad b = \text{diam}(S_t \cap E).$$

Since $w_G(S_t \cap E) \geq \frac{1}{2}\sqrt{k}$ and $\text{diam}(S_t \cap E) = 2$, we get

$$a \geq \epsilon_0 := \frac{c_1}{2} \quad \text{and} \quad b = 2.$$

It follows that

$$\frac{1}{2}\sqrt{k} \leq w_G(S_t \cap E) \leq 2C_2 \sqrt{\ln N(S_t \cap E, \epsilon_0 B_2^n)}.$$

This shows that

$$N(S_t \cap E, \epsilon_0 B_2^n) \geq \exp(c_0 k),$$

with $c_0 = \frac{1}{16C_2^2}$. The second claim follows from the fact that

$$N(A, B) \leq M(A, B)$$

for any bounded $A \subset \mathbb{R}^n$ and any symmetric convex body B , where $M(A, B)$ is the largest cardinality of a set $D \subseteq A$ such that $v - w \notin B$ for distinct $v, w \in D$. $\square$

Thus, even when the set of subgaussian directions is potentially small in spherical measure, it has substantial Gaussian width and contains exponentially large separated subsets in every sufficiently large-dimensional subspace.

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