The isotropic constant in the theory of high-dimensional convex bodies

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Abstract

The study of the geometry of log-concave probability measures, in parallel with the study of distribution of volume on high-dimensional convex bodies, has led to spectacular results culminating in the final affirmative answer to Bourgain's slicing problem through the recent works of Q. Guan and B. Klartag-J. Lehec. In this article we present the main tools and ideas behind these developments, we survey some of their consequences to the asymptotic theory of convex bodies, and discuss a few remaining open questions regarding the geometry of isotropic convex bodies.

1 Introduction

Bourgain's *slicing problem* asks if there exists an absolute constant $c > 0$ such that

$$\max\{\mathrm{vol}_{n-1}(K \cap \xi^{\perp}) : \xi \in S^{n-1}\} \geqslant c$$

for every convex body K of volume 1 in $\mathbb{R}^n$ that has barycenter at the origin. It turns out that a natural framework for the study of this problem is the isotropic position of a convex body. A convex body K in $\mathbb{R}^n$ is called isotropic if $\mathrm{vol}_n(K) = 1$, its barycenter is at the origin and its inertia matrix is a multiple of the identity, that is, there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \xi \rangle^2 dx = L_K^2$$

for every $\xi \in S^{n-1}$. The number L_K is then called the isotropic constant of K. The affine class of any convex body K contains a unique, up to orthogonal transformations, isotropic convex body; this is the isotropic position of K. It turns out that an affirmative answer to the slicing problem is equivalent to the following statement:

There exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\} \leqslant C. \tag{1.1}$$

The notion of the isotropic constant can be reintroduced in the more general setting of finite log-concave measures, and a more general question can be posed in a way that is equivalent to the above when we consider uniform measures on convex bodies. We say that a finite log-concave measure μ in $\mathbb{R}^n$ is isotropic if μ is a probability measure, its barycenter is at the origin and the covariance matrix $\mathrm{Cov}(\mu)$ of μ is the identity matrix I_n. The isotropic constant of μ is defined in an appropriate way, and a theorem of K. Ball shows that, in fact, for some absolute constant $c > 1$,

$$L_n \leqslant \tilde{L}_n := \sup\{L_\mu : \mu \text{ is isotropic in } \mathbb{R}^n\} \leqslant cL_n.$$

Around 1985-6 (published in 1990), Bourgain [26] obtained the upper bound $L_n \leqslant c\sqrt[4]{n}\ln n$ and, in 2006, this estimate was improved by Klartag [69] to $L_n \leqslant c\sqrt[4]{n}$. Actually, Klartag obtained a solution to the "isomorphic slicing problem", by showing that, for every convex body K in $\mathbb{R}^n$ and any $\varepsilon \in (0,1)$, one can find a convex body $T \subset \mathbb{R}^n$ with barycenter at the origin and a point $x \in \mathbb{R}^n$ such that $(1+\varepsilon)^{-1}T \subseteq K + x \subseteq (1+\varepsilon)T$ and $L_T \leqslant C/\sqrt{\varepsilon}$ for some absolute constant $C > 0$. A few years later, Dafnis and Paouris [38] developed an approach that was based on small ball probability estimates and led to the (slightly) weaker estimate $L_n \leqslant c\sqrt[4]{n}(\ln n)^2$.

A second main question in this area is the *Kannan-Lovász-Simonovits conjecture* about the isoperimetric constant χ_μ of an isotropic log-concave probability measure μ, defined as the largest constant $\chi \geqslant 0$ such that

$$\mu^+(A) \geqslant \chi \min\{\mu(A), 1-\mu(A)\}$$

for every Borel subset A of $\mathbb{R}^n$, where $\mu^+(A)$ is the Minkowski content of A. If we set $\psi_\mu = 1/\chi_\mu$ then the "KLS conjecture" is the question if there exists an absolute constant $C > 0$ such that

$$\psi_n := \sup\{\psi_\mu : \mu \text{ is isotropic log-concave measure on } \mathbb{R}^n\} \leqslant C.$$

An equivalent way to formulate the KLS conjecture is to ask that the Poincaré inequality holds for every isotropic log-concave probability measure μ on $\mathbb{R}^n$ with a constant that does not depend on the measure or the dimension n.

It is now known that Bourgain's slicing problem has an affirmative answer, and it is also known that $\psi_n \leqslant c\sqrt{\ln n}$. Moreover, an affirmative answer has been given for the *thin-shell conjecture*, which asks if there exists an absolute constant $C > 0$ such that, for any $n \geqslant 1$ and any isotropic log-concave probability measure μ on $\mathbb{R}^n$, one has

$$\mathbb{E}_\mu(|x| - \sqrt{n})^2 \leqslant C^2.$$

This statement implies that most of the mass of μ is concentrated on a thin spherical cell whose width t is much smaller than the central radius $\sqrt{n}$. It also implies that high-dimensional log-concave distributions have approximately Gaussian marginals. If we define $\sigma_\mu^2 = \frac{1}{n}\text{Var}_\mu(|x|^2)$ then the thin-shell conjecture is equivalent to the question if

$$\sigma_n := \sup\{\sigma_\mu : \mu \text{ is isotropic log-concave measure on } \mathbb{R}^n\} \leqslant C.$$

Note that $\sigma_\mu \leqslant C\psi_\mu$ for every Borel probability measure μ on $\mathbb{R}^n$, and hence $\sigma_n \leqslant C\psi_n$. On, the other hand, Eldan and Klartag [43] showed that there exists an absolute constant $C > 0$ such that, for every $n \geqslant 1$,

$$L_n \leqslant C\sigma_n,$$

therefore the KLS conjecture implies that L_n is bounded; in fact, any upper bound for ψ_n is an upper bound for L_n, up to an absolute constant.

The key for the developments that we shall discuss in this article is Eldan's stochastic localization, a technique invented by R. Eldan in his PhD thesis. Stochastic localization allows us to decompose a probability measure on a high-dimensional space into a mixture of simpler measures that are "localized" in the sense that they are concentrated on smaller random subsets of the original space. Using this decomposition and stochastic analysis, one can provide sharp estimates for the heat evolution of a probability measure on $\mathbb{R}^n$, under the log-concavity assumption. Eldan [42] showed that there exists an absolute constant $C > 0$ such that

$$\psi_n^2 \leqslant C \ln n \sum_{k=1}^{n} \frac{\sigma_k^2}{k},$$

which means that the thin-shell conjecture implies the KLS conjecture up to a polylogarithmic in the dimension factor. Combining this result with work of Guédon and E. Milman [63], one obtains the bound $\psi_n \leqslant Cn^{1/3}\ln n$. Later, Lee and Vempala [89] developed a variant of Eldan's stochastic localization and obtained the bound $\psi_n \leqslant c\sqrt[4]{n}$. In a breakthrough work, Chen [35] proved that one has $\psi_n \leqslant \exp(C\sqrt{\ln n(\ln\ln n)})$, and hence $\psi_n \leqslant n^\varepsilon$ for any $\varepsilon > 0$ and all large enough n. This development was the starting point for a series of important works, that led to the estimate $\psi_n \leqslant c\sqrt{\ln n}$ by Klartag [74]. This is currently the best known result on the KLS conjecture.

A recent technical breakthrough by Guan [62], who also obtained the bound $\sigma_n = O(\ln\ln n)$, was used by Klartag and Lehec [78] who presented a proof of the conjecture (1.1). Soon afterwards, one more proof of the isotropic constant conjecture was offered by Bizeul [15]. Even more recently, Klartag and Lehec [79] confirmed the thin-shell conjecture.

In this article we first introduce Bourgain's slicing problem and discuss its connection with the asymptotic versions of several classical problems from convex geometry. Then, we survey some of the consequences of the affirmative answer to the slicing problem for the asymptotic theory of high-dimensional convex bodies. The second part of the article presents the tools and ideas behind the developments on the KLS conjecture and the solution of the slicing problem. Finally, we discuss a few remaining open questions regarding the geometry of isotropic convex bodies.

We refer to Schneider's monograph [122] for the classical theory of convex bodies and to the books [2] and [3] for basic facts from asymptotic convex geometry. We also refer to [30] for more information on isotropic convex bodies and log-concave probability measures.

2 Isotropic position and the slicing problem

We start with basic notation and definitions from convex geometry. We work in $\mathbb{R}^n$, which is equipped with the standard inner product $\langle\cdot,\cdot\rangle$. We denote the corresponding Euclidean norm by $|\cdot|$, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume in $\mathbb{R}^n$ is denoted by vol_n. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We also denote the Haar measure on $O(n)$ by ν. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. For any integer $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$ we denote the orthogonal projection from $\mathbb{R}^n$ onto F by P_F. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \approx b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. Also if $K, C \subseteq \mathbb{R}^n$ we will write $K \approx C$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq C \subseteq c_2 K$.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric if $K = -K$, and that K is centered if its barycenter $\mathrm{bar}(K)$ is at the origin, i.e. if

$$\int_K \langle x, \xi\rangle \, dx = 0$$

for every $\xi \in S^{n-1}$. If K is a centered convex body in $\mathbb{R}^n$ then $\{T(K) : T \in GL_n\}$ is the family of positions of K.

The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ of a convex body K with $0 \in \mathrm{int}(K)$ is the function $\rho_K(x) = \max\{t > 0 : tx \in K\}$, and the support function of K is defined for every $y \in \mathbb{R}^n$ by $h_K(y) = \max\{\langle x, y\rangle : x \in K\}$. The mean width of K is the expectation

$$w(K) = \int_{S^{n-1}} h_K(\xi)\, d\sigma(\xi)$$

of h_K over the sphere. The radius of K is

$$R(K) = \max\{|x| : x \in K\}$$

and the volume radius of K is the quantity

$$\mathrm{vrad}(K) = \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B_2^n)}\right)^{1/n}.$$

The polar body K° of a convex body K with $0 \in \mathrm{int}(K)$ is the convex body

(2.1) $$K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } y \in K\}.$$

For every convex body $K \subseteq \mathbb{R}^n$ we write $\overline{K}$ for the multiple of K that has volume 1; in other words, $\overline{K} := \mathrm{vol}_n(K)^{-1/n}K$.

A closed bounded set $K \neq \{0\}$ in $\mathbb{R}^n$ is called a star body if for every $x \in K \setminus \{0\}$ we have that the interval $[0, x)$ is contained in the interior of K (thus, every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin), and the Minkowski functional of K defined by $p_K(x) = \min\{t \geqslant 0 : \ x \in tK\}$ is a continuous function on $\mathbb{R}^n$.

§ 2.1. Isotropic convex bodies. A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume $\mathrm{vol}_n(K) = 1$, it is centered, and there is a constant $\alpha > 0$ such that

(2.2) $$\int_K \langle x, y\rangle^2 dx = \alpha^2 |y|^2$$

for all $y \in \mathbb{R}^n$. It is useful to note that the isotropic condition (2.2) is equivalent to the fact that

(2.3) $$\int_K x_i x_j dx = \alpha^2 \delta_{ij}$$

for every $i, j = 1, \ldots, n$, where $x_j = \langle x, e_j\rangle$ are the coordinates of x with respect to any given orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. This is in turn equivalent to the fact that for every linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ (we write $T \in L(\mathbb{R}^n)$),

(2.4) $$\int_K \langle x, Tx\rangle dx = \alpha^2 \mathrm{tr}(T).$$

Another consequence of the isotropic condition (2.2) is that

$$\int_K |x|^2 dx = \sum_{i=1}^n \int_K \langle x, e_i\rangle^2 dx = n\alpha^2.$$

Also, it is easily checked that if K is an isotropic convex body in $\mathbb{R}^n$ then $U(K)$ is also isotropic for every $U \in O(n)$.

The next proposition shows that every centered convex body has a linear image which satisfies the isotropic condition. Moreover, this isotropic position of K is unique up to an orthogonal transformation.

Proposition 2.1. *Let K be a centered convex body in $\mathbb{R}^n$. There exists $T \in GL_n$ such that $T(K)$ is isotropic.*

To see this, note that the operator $M_K \in L(\mathbb{R}^n)$ defined by $M_K(y) = \int_K \langle x, y\rangle x dx$ is symmetric and positive definite; therefore, it has a symmetric and positive definite square root S. Consider the linear image $\tilde{K} = S^{-1}(K)$ of K. Then, for every $y \in \mathbb{R}^n$ we have

$$\begin{aligned}\int_{\tilde{K}} \langle x, y\rangle^2 dx &= |\det S|^{-1} \int_K \langle S^{-1}x, y\rangle^2 dx = |\det S|^{-1} \int_K \langle x, S^{-1}y\rangle^2 dx \\ &= |\det S|^{-1} \Big\langle \int_K \langle x, S^{-1}y\rangle x dx, S^{-1}y \Big\rangle = |\det S|^{-1} \langle M_K S^{-1}y, S^{-1}y\rangle = |\det S|^{-1}|y|^2.\end{aligned}$$

Normalizing the volume of $\tilde{K}$ we obtain an isotropic convex body.

We can now show that the isotropic position of a convex body is uniquely determined up to orthogonal transformations, and arises as a solution of a minimization problem. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Define

(2.5) $$\Delta(K) = \inf\left\{ \int_{TK} |x|^2 dx : T \in SL_n \right\}.$$

We shall show that a position K_1 of K, of volume 1, is isotropic if and only if

$$\int_{K_1} |x|^2 dx = \Delta(K). \tag{2.6}$$

Fix an isotropic position K_1 of K. We know that there exists $\alpha > 0$ such that

$$\int_{K_1} \langle x, Tx\rangle dx = \alpha^2 \mathrm{tr}(T)$$

for every $T \in L(\mathbb{R}^n)$. Then, for every $T \in SL_n$ we have

$$\int_{TK_1} |x|^2 dx = \int_{K_1} |Tx|^2 dx = \int_{K_1} \langle x, T^*Tx\rangle dx = \alpha^2 \mathrm{tr}(T^*T) \geqslant n\alpha^2 = \int_{K_1} |x|^2 dx, \tag{2.7}$$

where we have used the arithmetic-geometric means inequality in the form $\mathrm{tr}(T^*T) \geqslant n(\det(T^*T))^{1/n}$. This shows that K_1 satisfies (2.6). In particular, the infimum in (2.5) is a minimum.

Note also that if we have equality in (2.7) then $T^*T = I_n$, the identity operator, and hence $T \in O(n)$. This shows that any other position $\tilde{K}$ of K which satisfies (2.6) is an orthogonal image of K_1, therefore it is isotropic. Finally, if K_2 is some other isotropic position of K then the first part of the proof shows that K_2 satisfies (2.6). By the previous step, we must have $K_2 = U(K_1)$ for some $U \in O(n)$.

Based on the above we may define the isotropic constant of any convex body K in $\mathbb{R}^n$ by

$$L_K^2 = \frac{1}{n} \min\left\{ \frac{1}{\mathrm{vol}_n(T\tilde{K})^{1+\frac{2}{n}}} \int_{T\tilde{K}} |x|^2 dx \;\middle|\; T \in GL_n \right\},$$

where $\tilde{K} = K - \mathrm{bar}(K)$ is the centered translate of K. Note that L_K depends only on the affine class of K. Note also that if K is isotropic then for all $\xi \in S^{n-1}$ we have

$$\int_K \langle x, \xi\rangle^2 dx = L_K^2.$$

The isotropic constant conjecture, which is now a theorem is the following statement.

Theorem 2.2 (isotropic constant problem)**.** *There exists an absolute constant $C > 0$ such that for any $n \geqslant 1$ and any convex body K in $\mathbb{R}^n$ we have*

$$L_K \leqslant C.$$

As we will see, the isotropic constant problem is equivalent to Bourgain's slicing problem.

§ 2.2. Bourgain's slicing problem. The isotropic constant problem was stated explicitly as a question in the article of V. Milman and Pajor [106] and in the PhD Thesis of K. Ball [6]. The question appears for the first time in the work of Bourgain [23] on high-dimensional maximal functions associated with arbitrary convex bodies. Bourgain was interested in bounds for the L_p-norm of the maximal function

$$\mathcal{M}_K f(x) = \sup\left\{ \frac{1}{\mathrm{vol}_n(tK)} \int_{tK} |f(x+y)|\, dy \mid t > 0 \right\}$$

of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where K is a symmetric convex body in $\mathbb{R}^n$. Let $C_p(K)$ denote the best constant such that

$$\|\mathcal{M}_K f\|_p \leqslant C_p(K)\|f\|_p$$

and $C_{1,1}(K)$ the best constant so that the weak type inequality

$$\|\mathcal{M}_K f\|_{1,\infty} \leqslant C_{1,1}(K)\|f\|_1$$

is satisfied. Stein proved in [125] that if $K = B_2^n$ is the Euclidean unit ball then $C_p(B_2^n)$ is bounded independently of the dimension for all $p > 1$. Bourgain showed that there exists an absolute constant $C > 0$ (independent of n and K) such that

$$\|\mathcal{M}_K\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leqslant C.$$

Around the same time, the result for $\|\mathcal{M}_K\|_{2\to 2}$ was generalized to all $p > 3/2$ by Bourgain [24] and, independently, by Carbery [33]. By the definition of $\mathcal{M}_K$ it is clear that in order to obtain a uniform bound on $\|\mathcal{M}_K\|_{2\to 2}$ one can start with any suitable position $T(K)$ of K. Bourgain used the isotropic position; the property that played an important role in his argument was that when K is isotropic then $L_K|K \cap \xi^\perp| \approx 1$ for all $\xi \in S^{n-1}$. We shall explain this fact in this subsection. Bourgain [23] mentions the fact that $L_K \geqslant c$ and asks whether a reverse inequality might hold true.

The slicing problem, which is now a theorem, can be formulated as follows.

Theorem 2.3 (slicing problem)**.** *There exists an absolute constant $c > 0$ such that*

$$\max\{\mathrm{vol}_{n-1}(K \cap \xi^\perp) : \xi \in S^{n-1}\} \geqslant c \tag{2.8}$$

for every centered convex body K of volume 1 in $\mathbb{R}^n$.

To see the connection of Bourgain's slicing problem with the isotropic constant problem, we shall exploit the fact that the moments of inertia of a centered convex body are closely related with the volume of its hyperplane sections that pass through the origin. In the isotropic case this relation takes the following form.

Theorem 2.4. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $\xi \in S^{n-1}$ we have*

$$\frac{c_1}{\mathrm{vol}_{n-1}(K \cap \xi^\perp)} \leqslant \left(\int_K \langle x, \theta\rangle^2 dx\right)^{1/2} \leqslant \frac{c_2}{\mathrm{vol}_{n-1}(K \cap \xi^\perp)}, \tag{2.9}$$

where $c_1, c_2 > 0$ are absolute constants. In particular, if K is isotropic then for every $\xi \in S^{n-1}$ we have

$$\frac{c_1}{L_K} \leqslant \mathrm{vol}_{n-1}(K \cap \xi^\perp) \leqslant \frac{c_2}{L_K}. \tag{2.10}$$

For the proof, given $\xi \in S^{n-1}$ we consider the function $f(t) = f_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{x : \langle x, \xi\rangle = t\})$, $t \in \mathbb{R}$. It is a consequence of the Brunn-Minkowski inequality (see [2, Section 1.2]) that $\ln f$ is a concave function on its support.

We restrict our attention to the symmetric case. Then, f is even and $\|f\|_\infty = f(0)$. For the proof in the general case, which is more or less the same, we need an additional fact (that can be found in Fradelizi [47]) which shows that hyperplane sections through the center of mass are, up to an absolute constant, maximal: If K is a centered convex body of volume 1 in $\mathbb{R}^n$ then, for every $\xi \in S^{n-1}$,

$$\|f\|_\infty \leqslant e\, f(0) = e\, \mathrm{vol}_{n-1}(K \cap \xi^\perp).$$

Proof of Theorem 2.4 (symmetric case). Let $f := f_{K,\xi}$. To prove the left-hand side of (2.9) we set $b = \int_0^{+\infty} f(t)dt = \frac{1}{2}$ and define

$$g(t) = \|f\|_\infty \mathbb{1}_{[0,b/\|f\|_\infty]}(t).$$

Since $g \geqslant f$ on the support of g, we have

$$\int_0^s f(t)dt \leqslant \int_0^s g(t)dt$$

for every $0 \leqslant s \leqslant b/\|f\|_\infty$. The integrals of f and g on $[0, +\infty)$ are both equal to b. It follows that

$$\int_s^\infty g(t)dt \leqslant \int_s^\infty f(t)dt$$

for every $s \geqslant 0$. Then,

$$\begin{aligned}\int_0^\infty t^2 f(t)dt &= \int_0^\infty \int_0^t 2sf(t)dsdt = \int_0^\infty 2s\left(\int_s^\infty f(t)dt\right)ds\\ &\geqslant \int_0^\infty 2s\left(\int_s^\infty g(t)dt\right)ds = \int_0^\infty t^2 g(t)dt\\ &= \int_0^{b/\|f\|_\infty} t^2\|f\|_\infty dt = \frac{b^3}{3\|f\|_\infty^2}.\end{aligned}$$

This shows that

$$\int_K \langle x,\xi\rangle^2 dx = 2\int_0^\infty t^2 f(t)\,dt \geqslant \frac{2b^3}{3\|f\|_\infty^2} = \frac{1}{12f(0)^2}.$$

To prove the right-hand side inequality of (2.9) we distinguish two cases. Assume first that there exists $s>0$ such that $f(s)=\frac{1}{2}f(0)$. Then,

$$\frac{1}{2} = \int_0^\infty f(t)dt \geqslant \int_0^s f(t)dt \geqslant sf(s) = \frac{1}{2}sf(0),$$

because, since $\ln f$ is even and concave, we have that $f(t)\geqslant f(s)$ on $[0,s]$. On the other hand, if $t>s$, then

$$f(s) \geqslant \big(f(0)\big)^{1-\frac{s}{t}}\big(f(t)\big)^{\frac{s}{t}},$$

which implies that $f(t)\leqslant f(0)2^{-t/s}$. We now write

$$\begin{aligned}\int_0^\infty t^2 f(t)dt &= \int_0^s t^2 f(t)dt + \int_s^\infty t^2 f(t)dt \leqslant f(0)\int_0^s t^2 dt + \int_s^\infty t^2 f(0)2^{-t/s}dt\\ &= f(0)\left(\frac{s^3}{3} + s^3\int_1^\infty u^2 2^{-u}du\right) \leqslant c_0 f(0)s^3 \leqslant c_0/\big(f(0)\big)^2.\end{aligned}$$

Now, assume that, for every $s>0$ on the support of f, we have $f(s)>\frac{1}{2}f(0)$. Then, the role of s is played by $s_0=\sup\{s>0: f(s)>0\}$. We have $\frac{1}{2}\geqslant\frac{1}{2}f(0)s_0$ and

$$\int_{-\infty}^\infty t^2 f(t)dt = 2\int_0^\infty t^2 f(t)dt = 2\int_0^{s_0} t^2 f(t)dt \leqslant \frac{2f(0)s_0^3}{3} \leqslant \frac{2}{3\big(f(0)\big)^2}.$$

Thus, we get the same estimate as before, without using the fact that $\ln f$ is concave. □

Theorem 2.4 reveals a close connection between the isotropic constant problem and the slicing problem. In fact, one direction is very simple by the previous discussion; assume that the slicing problem has an affirmative answer. If K is isotropic then Theorem 2.4 shows that *all* sections $K\cap\xi^\perp$ have volume bounded from above by c_2/L_K. Since (2.8) must be true for at least one $\xi\in S^{n-1}$, we get $L_K\leqslant c_2/c$.

Conversely, we will show in § 2.5 that if there exists an absolute bound C for the isotropic constant, then the slicing problem has an affirmative answer. We shall first discuss the connection of the isotropic constant problem and of the slicing problem with two other classical problems from convex geometry and then establish the equivalence of all four them.

§ 2.3. Busemann-Petty problem. The Busemann-Petty problem was posed in [32], first in a list of ten problems concerning central sections of symmetric convex bodies in $\mathbb{R}^n$ and coming from questions in Minkowski geometry. It was originally formulated as follows:

Assume that K_1 and K_2 are symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$\mathrm{vol}_{n-1}(K_1\cap\xi^\perp) \leqslant \mathrm{vol}_{n-1}(K_2\cap\xi^\perp)$$

for all $\xi\in S^{n-1}$. Does it follow that $\mathrm{vol}_n(K_1)\leqslant\mathrm{vol}_n(K_2)$?

The first breakthrough on the Busemann-Petty problem came in 1975. Larman and Rogers [85] chose $K_2 = B_2^n$ and proved that if $n \geqslant 12$ then there exist symmetric convex bodies K_1 which are arbitrarily small perturbations of B_2^n such that the pair (K_1, B_2^n) provides a negative answer to the problem. The proof is probabilistic in nature.

Ball [7] proved that if $Q_n = [-1/2, 1/2]^n$ is the cube of volume 1 in $\mathbb{R}^n$ then $\mathrm{vol}_{n-1}(Q_n \cap \xi^\perp) \leqslant \sqrt{2}$ for all $\xi \in S^{n-1}$. Then, he observed in [9] that, if $n \geqslant 10$, and if $K_1 = Q_n$ and K_2 is the Euclidean ball of volume 1, then

$$\mathrm{vol}_{n-1}(Q_n \cap \xi^\perp) \leqslant \sqrt{2} < \omega_{n-1}\omega_n^{-\frac{n-1}{n}} = \mathrm{vol}_{n-1}(K_2 \cap \xi^\perp)$$

for all $\xi \in S^{n-1}$. Ball's counterexample is essentially different from the one of Larman and Rogers: the cube is far from being a perturbation of a ball. After Ball's example, counterexamples to the Busemann-Petty problem were given independently by Giannopoulos [54] and Bourgain [25] for $n \geqslant 7$, and by Papadimitrakis [115] in dimensions $n = 6$ and $n = 5$.

The key notion that led to the final solution to the Busemann-Petty problem is Lutwak's definition of an intersection body. The intersection body IK of a symmetric convex body K in $\mathbb{R}^n$ is the symmetric convex body whose radial function is defined by

$$\rho_{IK}(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp), \quad \xi \in S^{n-1}.$$

Lutwak observed in [96] that intersection bodies are closely connected with the Busemann-Petty problem. Using simple facts from the theory of dual mixed volumes one can see that if K_1 is an intersection body then the answer to the Busemann-Petty problem is affirmative for K_1 and any symmetric convex body K_2. However, if K_1 is a symmetric convex body that is not an intersection body, then K_1 and a perturbation K_2 of K_1 provide a counterexample to the question . Therefore, the Busemann-Petty problem has an affirmative answer in $\mathbb{R}^n$ if and only if every symmetric convex body in $\mathbb{R}^n$ is an intersection body. Using this reduction of the problem, Gardner [49], [50] and Zhang [132] gave a negative answer to the problem for $n \geqslant 5$ by providing examples of non-intersection bodies in $\mathbb{R}^5$. Around the same time, Gardner [51] proved that every symmetric convex body in $\mathbb{R}^3$ is an intersection body, and hence the Busemann-Petty problem has an affirmative answer in dimension $n = 3$. For a few years it was believed that the problem has a negative answer in the remaining case $n = 4$. Through the work of Koldobsky who established a Fourier analytic characterization of intersection bodies in [82] it was understood that the case $n = 4$ was still open. Zhang [134] proved that the answer in $\mathbb{R}^4$ is affirmative, and around the same time, using the Fourier analytic approach, Gardner, Koldobsky and Schlumprecht [53] gave a unified solution to the problem in all dimensions. Thus, the answer to the Busemann-Petty problem is positive if $n \leqslant 4$ and negative for all higher dimensions (see the books of Gardner [52] and Koldobsky [83] for a complete discussion of the problem and its history).

The asymptotic version of the Busemann-Petty problem, which is now a theorem, can be formulated as follows.

Theorem 2.5 (asymptotic Busemann-Petty problem)**.** *There exists an absolute constant $c > 0$ such that if K_1 and K_2 are centered convex bodies in $\mathbb{R}^n$ that satisfy $\mathrm{vol}_{n-1}(K_1 \cap \xi^\perp) \leqslant \mathrm{vol}_{n-1}(K_2 \cap \xi^\perp)$ for all $\xi \in S^{n-1}$ then $\mathrm{vol}_n(K_1)^{\frac{n-1}{n}} \leqslant c\,\mathrm{vol}_n(K_2)^{\frac{n-1}{n}}$.*

In Section 2.5, assuming that the isotropic constant problem has an affirmative answer, we shall give a proof of Theorem 2.5, restricted to the class of symmetric convex bodies, with the help of Busemann's formula. Later on we shall discuss an alternative approach which works for the class of centered convex bodies and also settles the lower-dimensional asymptotic Busemann-Petty problem.

§ 2.4. Sylvester problem Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $x_1, \ldots, x_{n+1}$ be random points which are independently and uniformly distributed in K. Their convex hull $\mathrm{conv}\{x_1, \ldots, x_{n+1}\}$ is a random simplex contained in K. For every $p > 0$ we define

$$m_p(K) = \left(\int_K \cdots \int_K \mathrm{vol}_n(\mathrm{conv}\{x_1, \ldots, x_{n+1}\})^p dx_{n+1} \cdots dx_1 \right)^{1/p}.$$

If we drop the assumption that $|K| = 1$ then we normalize as follows:

$$m_p(K) = \left(\frac{1}{\mathrm{vol}_n(K)^{n+p+1}} \int_K \cdots \int_K \mathrm{vol}_n(\mathrm{conv}\{x_1, \dots, x_{n+1}\})^p dx_{n+1} \cdots dx_1\right)^{1/p}.$$

Then, $m_p(K)$ is invariant under non-degenerate affine transformations: If $T \in GL_n$ and $u \in \mathbb{R}^n$, then $m_p(K) = m_p(TK + u)$ for all $p > 0$. The quantity $m_1(K)$ is the expectation of the normalized volume of a random simplex inside K.

Sylvester's problem is the question to determine the affine classes of convex bodies for which $m_p(K)$ is minimized or maximized. It is known that, for every $p > 0$,

$$m_p(K) \geqslant m_p(B_2^n)$$

with equality if and only if K is an ellipsoid (see e.g. Groemer [60] and [61] for the case $p \geqslant 1$ and Giannopoulos-Tsolomitis [56] for an extension to all $p > 0$). The problem of the maximum is completely open in dimensions $n \geqslant 3$. The simplex conjecture asserts that for every convex body K in $\mathbb{R}^n$,

$$m_1(K) \leqslant m_1(\Delta_n)$$

where Δ_n is a simplex in $\mathbb{R}^n$. This has been verified only in the planar case (see [16] and [17]).

Sylvester's problem is related to the isotropic constant problem: in fact, one can check that the simplex conjecture implies that $L_n \leqslant C$. To see this, we consider the variant of $m_p(K)$

$$S_p(K) = \left(\frac{1}{\mathrm{vol}_n(K)^{n+p}} \int_K \cdots \int_K \mathrm{vol}_n(\mathrm{conv}\{0, x_1, \dots, x_n\})^p dx_n \cdots dx_1\right)^{1/p}.$$

It is not hard to compare the quantities $m_p(K)$ and $S_p(K)$.

Proposition 2.6. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $p \geqslant 1$ we have*

$$S_p(K) \leqslant m_p(K) \leqslant (n+1) S_p(K).$$

Now, note that $\mathrm{vol}_n(\mathrm{conv}\{0, x_1, \dots, x_n\}) = \frac{1}{n!}|\det(x_1 \dots, x_n)|$. The function $f_i : K \to \mathbb{R}$ defined by $x_i \mapsto |\det(x_1, \dots, x_n)|$ for fixed x_j in K, $j \neq i$, is a seminorm, as is the function $g_i : K \to \mathbb{R}$ defined by

$$x_i \mapsto \int_K \cdots \int_K |\det(x_1, \dots, x_n)|\, dx_{i+1} \cdots dx_n$$

for fixed x_j in K when $j < i$. It is a well-known fact, following from Borell's lemma (see Section 4 below) that

$$\|g\|_{L^q(K)} \leqslant \frac{cq}{p} \|g\|_{L^p(K)} \qquad (2.11)$$

for any seminorm g and any $1 \leqslant p < q$. With successive applications of Fubini's theorem, using each time (2.11) with $p = 1$ and $q = 2$, we obtain the following.

Proposition 2.7. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$S_2(K) \leqslant c^n S_1(K),$$

where $c > 0$ is an absolute constant.

Consider the matrix of inertia $(M_K)_{i,j} = \int_K x_i x_j dx$ of K with respect to some fixed orthonormal basis of $\mathbb{R}^n$. The connection of $m_p(K)$ and $S_p(K)$ with the isotropic constant of K becomes clear by the next identity, which is known as Blaschke formula.

Proposition 2.8 (Blaschke formula)**.** *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$S_2^2(K) = \frac{\det(M_K)}{n!}.$$

Proof. By definition,

$$S_2^2(K) = \int_K \cdots \int_K \mathrm{vol}_n(\mathrm{conv}\{0, x_1, \ldots, x_n\})^2 dx_n \cdots dx_1.$$

We write $x_i = (x_{ij}), j = 1, \ldots, n$. Then,

$$(n!)^2 S_2^2(K) = \int_K \cdots \int_K |\det(x_1, \ldots, x_n)|^2 dx_n \cdots dx_1,$$

and expanding the determinant we get

$$\begin{aligned}
(n!)^2 S_2^2(K) &= \int_K \cdots \int_K \Big(\sum_\sigma \varepsilon_\sigma \prod_{i=1}^n x_{i,\sigma(i)}\Big)\Big(\sum_\tau \varepsilon_\tau \prod_{i=1}^n x_{i,\tau(i)}\Big) dx_n \cdots dx_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma,\tau} \varepsilon_\sigma \varepsilon_\tau \prod_{i=1}^n x_{i,\sigma(i)} x_{i,\tau(i)}\right) dx_n \cdots dx_1 \\
&= \int_K \cdots \int_K \left(\sum_{\sigma,\varphi} \varepsilon_\varphi \prod_{i=1}^n x_{i,\sigma(i)} x_{i,\varphi(\sigma(i))}\right) dx_n \cdots dx_1 \\
&= \sum_{\sigma,\varphi} \varepsilon_\varphi \prod_{i=1}^n \left(\int_K x_i x_{\varphi(i)} dx\right) \\
&= n!\ \det(M_K),
\end{aligned}$$

which completes the proof. □

It is not hard to check that if K is a centered convex body of volume 1 in $\mathbb{R}^n$ then for every $T \in SL_n$ we have that $M_{T(K)} = TM_K T^*$, and hence $\det(M_K) = \det(M_{T(K)})$. If we choose T so that $T(K)$ will be isotropic, then $M_{T(K)} = L_K^2 I_n$, and hence $\det(M_K) = \det(M_{T(K)}) = L_K^{2n}$. Therefore, we get the next theorem.

Theorem 2.9. *Let K be a centered convex body in $\mathbb{R}^n$. Then,*

$$L_K^{2n} = n!\ S_2^2(K).$$

Remark 2.10. Theorem 2.9 already gives the simple bound $L_n \leqslant \sqrt{n}$. Indeed, if K is an isotropic convex body in $\mathbb{R}^n$ then, noting that $S_2(K)$ is obviously bounded by 1, we get

$$L_K \leqslant \sqrt[2n]{n!} \leqslant \sqrt{n}.$$

What is more interesting is the next observation.

Fact 2.11. *If the simplex conjecture is true, then $L_K \leqslant C$ for every $n \geqslant 1$ and any convex body K in $\mathbb{R}^n$, where $C > 0$ is an absolute constant.*

Proof. Consider the simplex $\Delta_n = \left\{x \in \mathbb{R}^n : -\frac{1}{n+1} \leqslant x_i \leqslant \frac{n}{n+1}\ ,\ \sum_{i=1}^n x_i \leqslant \frac{1}{n+1}\right\}$. Then, $\Delta_n' = (n!)^{1/n}\Delta_n$ has volume 1 and barycenter at the origin. A simple computation shows that

$$\int_{\Delta_n'} x_i^2 dx < \frac{(n!)^{1+\frac{2}{n}}}{(n+2)!},$$

and since M_K is symmetric and positive definite, Hadamard's inequality gives

$$S_2^2(\Delta_n') = \frac{\det(M_{\Delta_n'})}{n!} \leqslant \frac{1}{n!}\left(\frac{(n!)^{1+\frac{2}{n}}}{(n+2)!}\right)^n \leqslant \frac{1}{n!}.$$

Now, if K is an isotropic convex body in $\mathbb{R}^n$ we have assumed that $m_1(K) \leqslant m_1(\Delta_n')$, and combining Propositions 2.6 and 2.8 with Theorem 2.9 we obtain

$$\begin{aligned} L_K^n &= \sqrt{n!}\, S_2(K) \leqslant \sqrt{n!}\, c^n S_1(K) \leqslant \sqrt{n!}\, c^n m_1(K) \leqslant \sqrt{n!}\, c^n m_1(\Delta_n') \\ &\leqslant \sqrt{n!}\, c^n (n+1) S_1(\Delta_n') \leqslant \sqrt{n!}\, c^n (n+1) S_2(\Delta_n') \leqslant (n+1)c^n. \end{aligned}$$

This shows that $L_K \leqslant 2c$. □

Fact 2.11 has probably been, at least for some people, the strongest evidence that Bourgain's slicing problem should have an affirmative answer. Now that this has been confirmed, we can reverse the argument to obtain an affirmative answer to the so-called asymptotic Sylvester problem.

Theorem 2.12 (asymptotic Sylvester problem)**.** *Let K be a convex body in $\mathbb{R}^n$. Then,*

$$\big(m_1(K)\big)^{1/n} \approx \frac{1}{\sqrt{n}}.$$

In fact, Blaschke formula shows that Theorem 2.12 is equivalent with an affirmative answer to the isotropic constant problem.

§ 2.5. Equivalence of the four problems. We have already seen that Theorem 2.3 implies Theorem 2.2, and that Theorem 2.12 is equivalent to Theorem 2.2.

Next, we show that Theorem 2.2 implies Theorem 2.3. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Consider the ellipsoid $\mathcal{E}_B(K)$ defined by

$$\|y\|_{\mathcal{E}_B(K)}^2 = \langle M_K y, y\rangle = \int_K \langle x, y\rangle^2 dx$$

where M_K is the matrix of inertia of K. The ellipsoid $\mathcal{E}_B(K)$ is the Binet ellipsoid of K. Note that K is in isotropic position if and only if $\mathcal{E}_B(K) = L_K^{-1}B_2^n$. It is not hard to see that the volume of $\mathcal{E}_B(K)$ is invariant under the action of SL_n. From the definition of the Binet ellipsoid we check that if $T \in SL_n$ then $\|y\|_{\mathcal{E}_B(T(K))} = \|T^* y\|_{\mathcal{E}_B(K)}$ for all $y \in \mathbb{R}^n$, therefore

$$\mathcal{E}_B(T(K)) = (T^*)^{-1}(\mathcal{E}_B(K)).$$

It follows that

$$\mathrm{vol}_n(\mathcal{E}_B(T(K))) = \mathrm{vol}_n(\mathcal{E}_B(K))$$

for every $T \in SL_n$. Now, we use the fact that $T_0(K)$ is isotropic for some $T_0 \in SL_n$ and this implies that

$$\mathrm{vol}_n(\mathcal{E}_B(T(K))) = \mathrm{vol}_n(\mathcal{E}_B(T_0(K))) = \mathrm{vol}_n(L_K^{-1}B_2^n) = \omega_n L_K^{-n}.$$

Assume that the isotropic constant conjecture has an affirmative answer and let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Integration in spherical coordinates shows that

$$L_K^{-n} = \frac{\mathrm{vol}_n(\mathcal{E}_B(K))}{\omega_n} = \int_{S^{n-1}} \|\xi\|_{\mathcal{E}_B(K)}^{-n} d\sigma(\xi),$$

and hence there exists a direction $\xi \in S^{n-1}$ such that

$$\int_K \langle x, \xi\rangle^2 dx = \|\xi\|_{\mathcal{E}_B(K)} \leqslant L_K^2 \leqslant C^2.$$

Then, from (2.9) we see that $\mathrm{vol}_{n-1}(K\cap\xi^\perp)\geqslant c_1/C$ and this proves the slicing conjecture.

It remains to show the equivalence of Theorem 2.2 with Theorem 2.5. In the symmetric case, one way to do this is using the Busemann formula.

Theorem 2.13 (Busemann formula)**.** *If K is a convex body in $\mathbb{R}^n$ with $0\in\mathrm{int}(K)$, then*

$$\mathrm{vol}_n(K)^{n-1}=\frac{n!\,\omega_n}{2}\int_{S^{n-1}}\mathrm{vol}_{n-1}(K\cap\xi^\perp)^nS_1(K\cap\xi^\perp)d\sigma(\xi).$$

Theorem 2.13 is an example of the so-called Blaschke-Petkantschin formulas which, roughly speaking, allow us to compute the integral of a function of k-tuples $(x_1,\dots,x_k)$ of points in $\mathbb{R}^n$ by first integrating over all such k-tuples in some $F\in G_{n,k}$ and then averaging with respect to the Haar measure $\nu_{n,k}$ on $G_{n,k}$.

Let K be a symmetric convex body in $\mathbb{R}^n$. Since all central hyperplane sections $K\cap\xi^\perp$ are symmetric, and hence centered, from Theorem 2.9 we know that

$$\big(S_1(K\cap\xi^\perp)\big)^{\frac{1}{n-1}}\approx\big(S_2(K\cap\xi^\perp)\big)^{\frac{1}{n-1}}\approx\frac{L_{K\cap\xi^\perp}}{\sqrt{n}}.$$

Therefore, Theorem 2.13 has the following immediate consequence.

Corollary 2.14. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$\mathrm{vol}_n(K)^{\frac{n-1}{n}}\approx\left(\int_{S^{n-1}}L_{K\cap\xi^\perp}^{n-1}\mathrm{vol}_{n-1}(K\cap\xi^\perp)^nd\sigma(\xi)\right)^{1/n}.$$

Given that the isotropic constant of any convex body is bounded from above and from below by an absolute positive constant, we get:

Corollary 2.15. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$\mathrm{vol}_n(K)^{\frac{n-1}{n}}\approx\left(\int_{S^{n-1}}\mathrm{vol}_{n-1}(K\cap\xi^\perp)^nd\sigma(\xi)\right)^{1/n}.$$

Now, let K_1 and K_2 be two symmetric convex bodies in $\mathbb{R}^n$ that satisfy

$$\mathrm{vol}_{n-1}(K_1\cap\xi^\perp)\leqslant\mathrm{vol}_{n-1}(K_2\cap\xi^\perp)$$

for all $\xi\in S^{n-1}$. Corollary 2.15 shows that

$$\begin{aligned}\mathrm{vol}_n(K_1)^{n-1}&\leqslant c_1^n\int_{S^{n-1}}\mathrm{vol}_{n-1}(K_1\cap\xi^\perp)^nd\sigma(\xi)\leqslant c_1^n\int_{S^{n-1}}\mathrm{vol}_{n-1}(K_2\cap\xi^\perp)^n\sigma(d\xi)\\&\leqslant c_2^n\mathrm{vol}_n(K_2)^{n-1},\end{aligned}$$

which gives

$$\mathrm{vol}_n(K_1)^{\frac{n-1}{n}}\leqslant c_3\mathrm{vol}_n(K_2)^{\frac{n-1}{n}}$$

for an absolute constant $c_3>0$. This shows that Theorem 2.2 implies Theorem 2.5 in the symmetric case.

Conversely, assuming Theorem 2.5 we can give a direct proof of Theorem 2.2. Let K be an isotropic convex body in $\mathbb{R}^n$. Choose $\xi_0\in S^{n-1}$ so that

$$\mathrm{vol}_{n-1}(K\cap\xi_0^\perp)=\max_{\xi\in S^{n-1}}\mathrm{vol}_{n-1}(K\cap\xi^\perp)$$

and $r>0$ so that $\omega_{n-1}r^{n-1}=\mathrm{vol}_{n-1}(K\cap\xi_0^\perp)$. Then,

$$\mathrm{vol}_{n-1}(K\cap\xi^\perp)\leqslant\omega_{n-1}r^{n-1}=\mathrm{vol}_{n-1}((rB_2^n)\cap\xi^\perp)$$

for all $\xi \in S^{n-1}$, therefore

$$\mathrm{vol}_n(K)^{n-1} \leqslant c^n \mathrm{vol}_n(rB_2^n)^{n-1} = \frac{c^n \omega_n^{n-1}}{\omega_{n-1}^n} \mathrm{vol}_{n-1}(K \cap \xi_0^{\perp})^n \leqslant c_1^n \mathrm{vol}_{n-1}(K \cap \xi_0^{\perp})^n,$$

for some absolute constant $c_1 > 0$. Since $\mathrm{vol}_n(K) = 1$, we see that

$$\mathrm{vol}_{n-1}(K \cap \xi_0^{\perp}) \geqslant 1/c_1.$$

On the other hand, K is isotropic and hence we have $\mathrm{vol}_{n-1}(K \cap \xi^{\perp}) \approx 1/L_K$ for every $\xi \in S^{n-1}$. It follows that $L_K \leqslant C$ for some absolute constant $C > 0$.

3 The isotropic position is an M-position

Let K and C be two convex bodies in $\mathbb{R}^n$. The covering number $N(K,C)$ of K by C is the least integer N for which there exist N translates of C whose union covers K:

$$N(K,C) = \min\Big\{N \in \mathbb{N} : \ \exists x_1, \dots, x_N \in \mathbb{R}^n \text{ such that } K \subseteq \bigcup_{j=1}^{N} (x_j + C)\Big\}.$$

The next theorem of V. Milman [105] establishes the existence of the so-called M-position of a convex body.

Theorem 3.1 (V. Milman)**.** *There exists an absolute constant $\beta > 0$ such that every centered convex body K in $\mathbb{R}^n$ has a linear image $\tilde{K}$ which satisfies $\mathrm{vol}_n(\tilde{K}) = \mathrm{vol}_n(B_2^n)$ and*

$$\max\big\{N(\tilde{K}, B_2^n), N(B_2^n, \tilde{K}), N(\tilde{K}^\circ, B_2^n), N(B_2^n, \tilde{K}^\circ)\big\} \leqslant \exp(\beta n). \tag{3.1}$$

We say that a convex body $\tilde{K}$ which has volume $\mathrm{vol}_n(\tilde{K}) = \mathrm{vol}_n(B_2^n)$ and satisfies (3.1) is in M-position with constant β.

Pisier [117] (see also [118]) has proposed a different approach to this result in the symmetric case, which allows one to find a whole family of M-positions and to give more detailed information on the behavior of the corresponding covering numbers. The precise statement is as follows.

Theorem 3.2 (Pisier)**.** *For every $0 < \alpha < 2$ and every symmetric convex body K in $\mathbb{R}^n$ there exists a linear image K_α of K such that*

$$\max\big\{N(K_\alpha, tB_2^n), N(B_2^n, tK_\alpha), N(K_\alpha^\circ, tB_2^n), N(B_2^n, tK_\alpha^\circ)\big\} \leqslant \exp\left(\frac{c(\alpha)n}{t^\alpha}\right)$$

for every $t \geqslant c(\alpha)^{1/\alpha}$, where $c(\alpha)$ depends only on α, and $c(\alpha) = O\big((2-\alpha)^{-\alpha/2}\big)$ as $\alpha \to 2^-$.

As we will see in this section, the fact that $L_n \leqslant C$ implies that the isotropic position of any convex body is an M-position with an absolute constant β. This in turn shows that isotropic convex bodies satisfy the reverse Brunn-Minkowski inequality. It also leads to a simple proof of the reverse Santaló inequality of Bourgain and V. Milman [27].

§ 3.1. Covering estimates. We assume that $L_n \leqslant C$. Our goal is to show that every isotropic convex body is in M-position with an absolute constant β.

Theorem 3.3. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\max\big\{\ln N(K, D_n), \ln N(D_n, K), \ln N(K^\circ, D_n^\circ), \ln N(D_n^\circ, K^\circ)\big\} \leqslant cn, \tag{3.2}$$

where D_n is the centered Euclidean ball of volume 1 in $\mathbb{R}^n$ and $c > 0$ is an absolute constant.

The proof is based on an observation of V. Milman and Pajor in [106]; they gave an estimate for the covering numbers $N(K,tB_2^n)$, $t>0$, where K is a convex body in $\mathbb{R}^n$, in terms of the quantity

$$I_1(K)=\frac{1}{\mathrm{vol}_n(K)^{1+\frac{1}{n}}}\int_K |x|\,dx.$$

Lemma 3.4. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ such that $0\in\mathrm{int}(K)$. For any $t>0$ we have that*

$$\ln N(K,tB_2^n)\leqslant \frac{c_1 n I_1(K)}{t}+\ln 2, \tag{3.3}$$

where $c_1>0$ is an absolute constant.

Proof. Consider the Borel probability measure μ on $\mathbb{R}^n$ defined by

$$\mu(A)=\frac{1}{c_K}\int_A e^{-p_K(x)}dx.$$

where $p_K(x)=\inf\{t>0:x\in tK\}$ is the Minkowski functional of K and $c_K=\int_{\mathbb{R}^n}\exp(-p_K(x))dx$. A simple computation, based on the fact that $\{x\in\mathbb{R}^n:p_K(x)\leqslant t\}=tK$ for any $t>0$, shows that $c_K=n!$.

Let $\{x_1,\dots,x_N\}$ be a subset of K which is maximal with respect to the condition $|x_i-x_j|\geqslant t$ for $i\neq j$. Then $K\subseteq\bigcup_{1\leqslant i\leqslant N}(x_i+tB_2^n)$, and hence $N(K,tB_2^n)\leqslant N$. Let $\alpha>0$. Note that if we set $y_i=(2\alpha/t)x_i$, by the subadditivity and positive homogeneity of p_K and the fact that $p_K(x_i)\leqslant 2$, we have

$$\mu(y_i+\alpha B_2^n)\geqslant\frac{1}{c_K}\int_{\alpha B_2^n}e^{-p_K(x)}e^{-p_K(y_i)}dx\geqslant e^{-2\alpha/t}\mu(\alpha B_2^n).$$

The bodies $y_i+\alpha B_2^n$ have disjoint interiors, therefore $Ne^{-2\alpha/t}\mu(aB_2^n)\leqslant 1$. This shows that

$$N(K,tB_2^n)\leqslant e^{2\alpha/t}(\mu(\alpha B_2^n))^{-1}.$$

Now, we choose $\alpha>0$ so that $\mu(\alpha B_2^n)\geqslant 1/2$. A simple computation shows that

$$\gamma:=\int_{\mathbb{R}^n}|x|\,d\mu(x)=(n+1)I_1(K). \tag{3.4}$$

By Markov's inequality, $\mu(2\gamma B_2^n)\geqslant 1/2$, so if we choose $\alpha=2\gamma$, we get

$$N(K,tB_2^n)\leqslant 2\exp\big(4(n+1)I_1(K)/t\big)$$

for every $t>0$. $\square$

If K is an isotropic convex body then $I_1(K)\leqslant\sqrt{n}L_K$. Let $D_n=\overline{B}_2^n$ be the centered Euclidean ball of volume 1. Since $D_n\approx\sqrt{n}B_2^n$, Lemma 3.4 shows that

$$\ln N(K,tD_n)\leqslant\frac{c_2 nL_K}{t}\leqslant\frac{c_3 n}{t} \tag{3.5}$$

for any $t>0$ (note that if t is large then this estimate is trivially true, since every isotropic body K satisfies the inclusion $K\subseteq cnL_KB_2^n$ for some absolute constant $c>0$).

Knowing that, for any set S,

$$N(S-S,2D_n)=N(S-S,D_n-D_n)\leqslant N(S,D_n)^2,$$

we can use (3.5) to also get an upper bound for the covering numbers of the difference body of an isotropic convex body K by D_n:

$$\ln N(K-K,tD_n)\leqslant\frac{2c_3n}{t}.$$

The next lemma allows us to bound the dual covering numbers $N(B_2^n,tK^\circ)$.

Lemma 3.5. *Let K be a convex body in $\mathbb{R}^n$ which contains 0 in its interior. For every $t>0$ we set $A(t):=t\ln N(K,tB_2^n)$ and $B(t):=t\ln N(B_2^n,tK^\circ)$. Then, one has*

$$\sup_{t>0} B(t)\leqslant 16\sup_{t>0} A(t). \tag{3.6}$$

In particular, if K is isotropic (or a translate of an isotropic convex body which still contains 0 in its interior), then

$$\ln N(B_2^n,tK^\circ)\leqslant \ln N\big(B_2^n,t(K-K)^\circ\big)\leqslant \frac{c_2n^{3/2}L_K}{t}, \tag{3.7}$$

where $c_2>0$ is an absolute constant.

Proof. We use a well-known idea from [128] (see also [88, Section 3.3]). For any $t>0$ we have $(t^2K^\circ)\cap(4K)\subseteq 2tB_2^n$. Passing to the polar bodies we see that

$$B_2^n\subseteq \mathrm{conv}\left(\frac{t}{2}K^\circ,\frac{2}{t}K\right)\subseteq \frac{t}{2}K^\circ+\frac{2}{t}K.$$

We write

$$\begin{aligned}N(B_2^n,tK^\circ)&\leqslant N\left(\frac{t}{2}K^\circ+\frac{2}{t}K,tK^\circ\right)\leqslant N\left(\frac{2}{t}K,\frac{t}{2}K^\circ\right)\leqslant N\left(\frac{2}{t}K,\frac{1}{4}B_2^n\right)N\left(\frac{1}{4}B_2^n,\frac{t}{2}K^\circ\right)\\ &=N\left(K,\frac{t}{8}B_2^n\right)N(B_2^n,2tK^\circ).\end{aligned}$$

Taking logarithms we get $B(t)\leqslant 8A(t/8)+\frac{1}{2}B(2t)$ for all $t>0$. This implies that $B:=\sup_{t>0}B(t)\leqslant 16A$, and the result follows. $\square$

Since $D_n^\circ\approx(1/\sqrt{n})B_2^n$, (3.5) and (3.7) immediately imply the following.

Proposition 3.6. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\max\{\ln N(K,tD_n),\ln N(D_n^\circ,tK^\circ)\}\leqslant \frac{cn}{t} \tag{3.8}$$

for all $t>0$, where $c>0$ is an absolute constant.

We shall also use the next covering lemma, which provides some standard entropy estimates that are valid for arbitrary convex bodies in $\mathbb{R}^n$.

Lemma 3.7. *Let K and L be convex bodies in $\mathbb{R}^n$. If L is symmetric, then*

$$N(K,L)\leqslant \frac{\mathrm{vol}_n(K+L/2)}{\mathrm{vol}_n(L/2)}\leqslant 2^n\frac{\mathrm{vol}_n(K+L)}{\mathrm{vol}_n(L)}. \tag{3.9}$$

If L is arbitrary, then

$$N(K,L)\leqslant 4^n\frac{\mathrm{vol}_n(K+L)}{\mathrm{vol}_n(L)}. \tag{3.10}$$

Moreover,

$$\frac{\mathrm{vol}_n(K+L)}{\mathrm{vol}_n(L)}\leqslant 2^nN(K,L). \tag{3.11}$$

Proof. The proof of (3.11) is an immediate consequence of the definitions. To prove (3.9), note that if N is a maximal subset of K with respect to the property

$$x, y \in N \text{ and } x \neq y \implies \|x - y\|_L \geqslant 1, \tag{3.12}$$

then $K \subseteq \bigcup_{x \in N}(x + L)$, while every two sets $x + L/2, y + L/2$ $(x, y \in N)$ have disjoint interiors when $x \neq y$.

Finally, when L is not necessarily symmetric, we recall that $N(K + x, L + y) = N(K, L)$ for every $x, y \in \mathbb{R}^n$, and also that the ratio $\mathrm{vol}_n(K + L)/\mathrm{vol}_n(L)$ obviously remains unaltered if we translate K or L. Hence, we can assume that L is centered, in which case it follows from [107, Corollary 3] that

$$\mathrm{vol}_n(L \cap (-L)) \geqslant 2^{-n}\mathrm{vol}_n(L). \tag{3.13}$$

But then, from (3.9) we get that

$$N(K, L) \leqslant N(K, L \cap (-L)) \leqslant 2^n \frac{\mathrm{vol}_n(K + (L \cap (-L)))}{\mathrm{vol}_n(L \cap (-L))} \leqslant 4^n \frac{\mathrm{vol}_n(K + L)}{\mathrm{vol}_n(L)}, \tag{3.14}$$

and this implies (3.10). □

Corollary 3.8. *Let K and L be two convex bodies in $\mathbb{R}^n$. Then,*

$$N(K, L)^{1/n} \approx \frac{\mathrm{vol}_n(K + L)^{1/n}}{\mathrm{vol}_n(L)^{1/n}}. \tag{3.15}$$

Consequently, if K and L have the same volume, then

$$N(K, L)^{1/n} \leqslant 8N(L, K)^{1/n}. \tag{3.16}$$

Let K be a convex body in $\mathbb{R}^n$. The function $z \mapsto \mathrm{vol}_n(K)\,\mathrm{vol}_n((K - z)^\circ)$, defined on $\mathrm{int}(K)$, is strictly convex and has a unique point of minimum, the Santaló point $s(K)$ of K. The Blaschke-Santaló inequality states that $\mathrm{vol}_n(K)\,\mathrm{vol}_n((K - s(K))^\circ) \leqslant \omega_n^2$. One can also prove that if $\mathrm{bar}(K) = 0$ then $s(K^\circ) = 0$. Since $(K^\circ)^\circ = K$, we see that if $\mathrm{bar}(K) = 0$ then

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) = \mathrm{vol}_n((K^\circ - s(K^\circ))^\circ)\,\mathrm{vol}_n(K^\circ) \leqslant \omega_n^2.$$

Therefore, we get:

Theorem 3.9 (Blaschke-Santaló)**.** *Let K be a convex body in $\mathbb{R}^n$ with either $\mathrm{bar}(K) = 0$ or $s(K) = 0$. Then,*

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) \leqslant \omega_n^2 = \mathrm{vol}_n(B_2^n)^2.$$

Combining Proposition 3.6 with the Blaschke-Santaló inequality and Corollary 3.8, we can now prove Theorem 3.3.

Proof of Theorem 3.3. From Proposition 3.6 we already know that

$$\max\left\{N(K, D_n), N(D_n^\circ, K^\circ)\right\} \leqslant \exp(cn). \tag{3.17}$$

For the other two covering numbers we note that $N(D_n, K) \leqslant 8^n N(K, D_n)$ by Lemma 3.7, which means that $\ln N(D_n, K) \leqslant (c + \ln 8)n$. Similarly,

$$N(K^\circ, D_n^\circ) \leqslant 2^n \frac{\mathrm{vol}_n(K^\circ + D_n^\circ)}{\mathrm{vol}_n(D_n^\circ)} \leqslant 2^n \frac{\mathrm{vol}_n(K^\circ + D_n^\circ)}{\mathrm{vol}_n(K^\circ)} \leqslant 4^n N(D_n^\circ, K^\circ), \tag{3.18}$$

which means that $\ln N(K^\circ, D_n^\circ) \leqslant (c + \ln 4)n$, where we have also used the fact that $\mathrm{vol}_n(K) = \mathrm{vol}_n(D_n)$ and hence $\mathrm{vol}_n(K^\circ) \leqslant \mathrm{vol}_n(D_n^\circ)$ from the Blaschke-Santaló inequality. This completes the proof. □

§ 3.2. Bourgain-Milman inequality. The Bourgain-Milman inequality [27], also known as reverse Santaló inequality, states that there exists an absolute constant $0 < c < 1$ with the following property: for every $n \geqslant 1$ and any convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$,

$$s(K) = \mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) \geqslant c^n \omega_n^2 = c^n \mathrm{vol}_n(B_2^n)^2. \tag{3.19}$$

The existence of an M-position for any convex body and the asymptotic form of the Santaló inequality and its inverse are interconnected results. We illustrate this by giving a very simple proof of the Bourgain-Milman inequality starting from the fact that every isotropic convex body is in M-position.

The discussion about the Blaschke-Santaló inequality in the previous subsection shows that if $0 \in \mathrm{int}(K)$ then

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) \geqslant \mathrm{vol}_n(K - s(K))\,\mathrm{vol}_n((K - s(K))^\circ)$$

therefore we may replace K by $K_1 := K - s(K)$. Then, we know that $\mathrm{bar}(K_1^\circ) = 0$. It is also easily checked that

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) = \mathrm{vol}_n(T(K))\,\mathrm{vol}_n((T(K))^\circ)$$

for every $T \in GL_n$, and hence we may assume that K_1° is in isotropic position. Then, from Theorem 3.3 we know that

$$\max\big\{\ln N(K_1^\circ, D_n), \ln N(D_n, K_1^\circ), \ln N(K_1, D_n^\circ), \ln N(D_n^\circ, K_1)\big\} \leqslant cn,$$

where D_n is the centered Euclidean ball of volume 1 in $\mathbb{R}^n$ and $c > 0$ is an absolute constant. It follows that

$$\begin{aligned}\mathrm{vol}_n(D_n) &\leqslant N(D_n, K_1^\circ)\,\mathrm{vol}_n(K_1^\circ) \leqslant e^{cn}\mathrm{vol}_n(K_1^\circ),\\ \mathrm{vol}_n(D_n^\circ) &\leqslant N(D_n^\circ, K_1)\,\mathrm{vol}_n(K_1) \leqslant e^{cn}\mathrm{vol}_n(K_1),\end{aligned}$$

and hence

$$\omega_n^2 = \mathrm{vol}_n(D_n)\,\mathrm{vol}_n(D_n^\circ) \leqslant e^{2cn}\mathrm{vol}_n(K_1)\,\mathrm{vol}_n(K_1^\circ)$$

which proves (3.19).

§ 3.3. Reverse Brunn–Minkowski inequality. As a consequence of Theorem 3.3 and Corollary 3.8, we get the "reverse" Brunn-Minkowski inequality.

Theorem 3.10. *Let $K_1, \dots, K_m$ be isotropic convex bodies in $\mathbb{R}^n$. Then, for any $\lambda_1, \dots, \lambda_m > 0$,*

$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m)^{1/n} \leqslant c_1 m \sum_{j=1}^m \lambda_j \mathrm{vol}_n(K_j)^{1/n} \tag{3.20}$$

where $c_1 > 0$ is an absolute constant. Moreover, if K_1 and K_2 are isotropic convex bodies in $\mathbb{R}^n$ then, for any $\lambda_1, \lambda_2 > 0$,

$$\mathrm{vol}_n(\lambda_1 K_1^\circ + \lambda_2 K_2^\circ)^{1/n} \leqslant c_2(\lambda_1 \mathrm{vol}_n(K_1^\circ)^{1/n} + \lambda_2 \mathrm{vol}_n(K_2^\circ)^{1/n}). \tag{3.21}$$

where $c_2 > 0$ is an absolute constant.

Proof. From Proposition 3.6 we know that

$$N(\lambda_1 K_1 + \cdots + \lambda_m K_m, t(\lambda_1 + \cdots + \lambda_m)D_n) \leqslant \prod_{j=1}^m N(K_j, tD_n) \leqslant \exp(cmn/t),$$

and hence

$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m)^{1/n} \leqslant e^{cm/t}\,(t(\lambda_1 + \cdots + \lambda_m))$$

for every $t > 0$. Choosing $t = cm$ we see that

$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m)^{1/n} \leqslant c_1 m \sum_{j=1}^{m} \lambda_j = c_1 m \sum_{j=1}^{m} \lambda_j \mathrm{vol}_n(K_j)^{1/n}$$

with $c_1 = ec$. For the second claim, note that Theorem 3.3 implies that

$$N(\lambda_1 K_1^\circ + \lambda_2 K_2^\circ, t(\lambda_1 + \lambda_2) D_n^\circ) \leqslant \exp(2cn/t)$$

for every $t > 0$, and apply the same reasoning as before to write

$$\mathrm{vol}_n(\lambda_1 K_1^\circ + \lambda_2 K_2^\circ)^{1/n} \leqslant 2c(\lambda_1 + \lambda_2)\mathrm{vol}_n(D_n^\circ)^{1/n}.$$

Since K_1 and K_2 are isotropic, the argument of the previous subsection shows that $\mathrm{vol}_n(D_n^\circ) \leqslant e^{cn}\mathrm{vol}_n(K_i^\circ)$ for $i = 1, 2$. In this way we obtain (3.21) with $c_2 = 2ce^c$. □

4 Log-concave probability measures

A Borel measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(H) < 1$ for every hyperplane H in $\mathbb{R}^n$ and

$$\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for any pair of compact sets A, B in $\mathbb{R}^n$ and any $\lambda \in (0,1)$. Borell [21] has proved that, under these assumptions, μ has a log-concave density f_μ. Recall that a function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set in $\mathbb{R}^n$ and the restriction of $\ln f$ to it is concave. The Brunn-Minkowski inequality implies that if K is a convex body in $\mathbb{R}^n$ then the indicator function $\mathbb{1}_K$ of K is the density of a log-concave measure, the Lebesgue measure on K.

We say that μ is symmetric if $\mu(-B) = \mu(B)$ for every Borel subset B of $\mathbb{R}^n$ and that μ is centered if the barycenter $\mathrm{bar}(\mu) = \int_{\mathbb{R}^n} x \, d\mu(x)$ of μ is at the origin, i.e.

$$\int_{\mathbb{R}^n} \langle x, \xi \rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x, \xi \rangle f_\mu(x) dx = 0$$

for all $\xi \in S^{n-1}$. If μ is symmetric then f_μ is even and it follows that $\|f_\mu\|_\infty = f_\mu(0)$. On the other hand, Fradelizi [47] has shown that if μ is a centered log-concave probability measure then

$$\|f_\mu\|_\infty \leqslant e^n f_\mu(0). \tag{4.1}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. For any $\alpha \geqslant 1$ we define the ψ_α-norm of f as follows:

$$\|f\|_{\psi_\alpha} := \inf \left\{ t > 0 : \int_\Omega \exp\left((|f|/t)^\alpha \right) \, d\mu \leqslant 2 \right\},$$

provided that the set on the right-hand side is non-empty. Note that the ψ_α-norm is exactly the Orlicz norm corresponding to the function $t \in \mathbb{R} \to e^{|t|^\alpha} - 1$. An equivalent expression for the ψ_α-norm in terms of the L_p-norms is that

$$\|f\|_{\psi_\alpha} \approx \sup_{p \geqslant \alpha} \frac{\|f\|_{L_p(\mu)}}{p^{1/\alpha}},$$

up to some absolute constants.

A well-known lemma of Borell (see [30, Lemma 2.4.5] for a proof) asserts that if μ is a log-concave probability measure on $\mathbb{R}^n$ then, for any symmetric convex set A in $\mathbb{R}^n$ with $\mu(A) = \alpha \in (0,1)$ and any $t > 1$ we have

$$1 - \mu(tA) \leqslant \alpha \left(\frac{1-\alpha}{\alpha} \right)^{\frac{t+1}{2}}. \tag{4.2}$$

Using Borell's lemma one can show that there exists an absolute constant $c_0>0$ such that for every log-concave probability measure μ on $\mathbb{R}^n$, for any seminorm $f:\mathbb{R}^n\to\mathbb{R}$ and any $q>p\geqslant 1$, we have

$$\left(\int_{\mathbb{R}^n}|f|^p\,d\mu\right)^{1/p}\leqslant\left(\int_{\mathbb{R}^n}|f|^q\,d\mu\right)^{1/q}\leqslant c_0\frac{q}{p}\left(\int_{\mathbb{R}^n}|f|^p\,d\mu\right)^{1/p}\tag{4.3}$$

(a proof is given in [30, Theorem 2.4.6]). In particular, $\|f\|_{\psi_1}\leqslant c_1\|f\|_1$, where $c_1>0$ is an absolute constant.

§ 4.1. Isotropic log-concave probability measures. For any log-concave probability measure μ on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu:=\left(\frac{\sup_{x\in\mathbb{R}^n}f_\mu(x)}{\int_{\mathbb{R}^n}f_\mu(x)dx}\right)^{\frac{1}{n}}\big(\det\mathrm{Cov}(\mu)\big)^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{i,j}:=\frac{\int_{\mathbb{R}^n}x_ix_jf_\mu(x)\,dx}{\int_{\mathbb{R}^n}f_\mu(x)\,dx}-\frac{\int_{\mathbb{R}^n}x_if_\mu(x)\,dx}{\int_{\mathbb{R}^n}f_\mu(x)\,dx}\frac{\int_{\mathbb{R}^n}x_jf_\mu(x)\,dx}{\int_{\mathbb{R}^n}f_\mu(x)\,dx}.$$

A log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if it is centered and $\mathrm{Cov}(\mu)=I_n$, where I_n is the identity $n\times n$ matrix. By the definition of the isotropic constant, if μ is isotropic then $L_\mu=\|f_\mu\|_\infty^{1/n}$. Note that a convex body K of volume 1 is isotropic if and only if the log-concave probability measure μ_K with density $L_K^n\mathbb{1}_{K/L_K}$ is isotropic.

Let μ and ν be two log-concave probability measures on $\mathbb{R}^n$. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a measurable function which is defined μ-almost everywhere and satisfies

$$\nu(B)=\mu(T^{-1}(B))$$

for every Borel subset B of $\mathbb{R}^n$. We then say that T pushes forward μ to ν and write $T_*\mu=\nu$. It is not hard to check that for every log-concave probability measure μ on $\mathbb{R}^n$ there exists an invertible affine transformation T such that the log-concave probability measure $T_*\mu$ is isotropic, and $L_{T_*\mu}=L_\mu$.

One can prove that the isotropic constants of all log-concave measures are uniformly bounded from below by a constant $c>0$ which is independent of the dimension. If μ is an isotropic log-concave probability measure, then

$$L_\mu=\|f_\mu\|_\infty^{1/n}\approx[f_\mu(0)]^{1/n}\geqslant c,\tag{4.4}$$

where $c>0$ is an absolute constant (see [30, Proposition 2.3.12]). The isotropic constant problem for log-concave measures, which is now a theorem, can now be stated as follows:

> *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then, $L_\mu=\|f_\mu\|_\infty^{1/n}\leqslant C$, where $C>0$ is an absolute constant.*

We can define $\tilde{L}_n:=\max\{L_\mu:\mu$ is an isotropic log-concave probability measure on $\mathbb{R}^n\}$. Since $L_K=L_{\mu_K}\leqslant\tilde{L}_n$ for every centered convex body K in $\mathbb{R}^n$, it is clear that $L_n\leqslant\tilde{L}_n$.

§ 4.2. K. Ball's bodies. Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. An important family of convex bodies associated with μ was introduced by K. Ball, who also established their convexity in [8]: for every $p>0$, we define

$$K_p(\mu)=\left\{x\in\mathbb{R}^n:\int_0^\infty r^{p-1}f_\mu(rx)\,dr\geqslant\frac{f_\mu(0)}{p}\right\}.$$

From the definition it follows that the radial function of $K_p(\mu)$ is given by

$$\varrho_{K_p(\mu)}(x)=\left(\frac{1}{f_\mu(0)}\int_0^\infty pr^{p-1}f_\mu(rx)\,dr\right)^{1/p}\tag{4.5}$$

for $x\neq 0$. It is easily checked that if K is a convex body in $\mathbb{R}^n$ with $0\in K$ then $K_p(\mathbb{1}_K)=K$ for all $p>0$. A very useful observation is that

$$f_\mu(0)\mathrm{vol}_n(K_n(\mu))=1.$$

To see this, we write

$$\begin{aligned}\mathrm{vol}_n(K_n(\mu)) &= \int_{K_n(\mu)}\mathbf{1}\,dx = n\omega_n\int_{S^{n-1}}\int_0^{\rho_{K_n(\mu)}(\xi)} r^{n-1}drd\sigma(\xi)\\ &= \frac{n\omega_n}{f_\mu(0)}\int_{S^{n-1}}\int_0^\infty r^{n-1}f_\mu(r\xi)drd\sigma(\xi)=\frac{1}{f_\mu(0)}\int_{\mathbb{R}^n}f_\mu(x)\,dx=\frac{1}{f_\mu(0)}\end{aligned}$$

using (4.5) and integration in spherical coordinates.

Another important observation is that, for every $0<p\leqslant q$ we have that

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}}K_q(\mu)\subseteq K_p(\mu)\subseteq e^{\frac{n}{p}-\frac{n}{q}}K_q(\mu). \tag{4.6}$$

A proof is given in [30, Proposition 2.5.7]. As a consequence we obtain an approximate formula for the volume of $K_{n+p}(\mu)$. When $p>0$, we have

$$e^{-1}\leqslant f_\mu(0)^{\frac{1}{n}+\frac{1}{p}}\mathrm{vol}_n(K_{n+p}(\mu))^{\frac{1}{n}+\frac{1}{p}}\leqslant e\frac{n+p}{n}, \tag{4.7}$$

while for $-n<p<0$ we have

$$e^{-1}\leqslant f_\mu(0)^{\frac{1}{-p}-\frac{1}{n}}\mathrm{vol}_n(K_{n+p}(\mu))^{\frac{1}{-p}-\frac{1}{n}}\leqslant e. \tag{4.8}$$

If μ is an even log-concave probability measure on $\mathbb{R}^n$ then $T=K_{n+2}(\mu)$ is a symmetric convex body that satisfies

$$c_1L_\mu\leqslant L_T\leqslant c_2L_\mu,$$

where $c_1,c_2>0$ are absolute constants (see [30, Proposition 2.5.9]). Furthermore, if μ is isotropic, then $\overline{T}=\mathrm{vol}_n(T)^{-1/n}T$ is an isotropic convex body. In the case where the measure μ is centered, but not necessarily symmetric, we prefer to work with the convex body $K_{n+1}(\mu)$ instead of $K_{n+2}(\mu)$. The reason is that $T=K_{n+1}(\mu)$ is a centered convex body in $\mathbb{R}^n$ and we still have that

$$c_1L_\mu\leqslant L_T\leqslant c_2L_\mu,$$

where $c_1,c_2>0$ are absolute constants.

We briefly sketch the proof (for more details see [30, Proposition 2.5.12]). First we check that T is centered and satisfies

$$\int_T|\langle x,\xi\rangle|\,dx=\frac{1}{f_\mu(0)}\int|\langle x,\xi\rangle|f_\mu(x)\,dx$$

for all $\xi\in S^{n-1}$. Borell's lemma implies that for every $y\in\mathbb{R}^n$

$$\begin{aligned}\left(\frac{1}{\mathrm{vol}_n(T)}\int_T\langle x,y\rangle^2\,dx\right)^{1/2} &\approx \frac{1}{\mathrm{vol}_n(T)}\int_T|\langle x,y\rangle|\,dx=\frac{1}{f_\mu(0)\mathrm{vol}_n(T)}\int|\langle x,y\rangle|f_\mu(x)\,dx\\ &\approx \frac{1}{f_\mu(0)\mathrm{vol}_n(T)}\left(\int\langle x,y\rangle^2f_\mu(x)\,dx\right)^{1/2},\end{aligned}$$

which, combined with the fact that T and f_μ are both centered, implies that there exist absolute constants $c_1,c_2>0$ such that as positive definite matrices

$$c_1\mathrm{Cov}(\mathbb{1}_T)\leqslant(\mathrm{vol}_n(T)f_\mu(0))^{-2}\mathrm{Cov}(\mu)\leqslant c_2\mathrm{Cov}(\mathbb{1}_T).$$

Therefore,

$$\big(\det \mathrm{Cov}(\mathbb{1}_T)\big)^{1/n} \approx (\mathrm{vol}_n(T) f_\mu(0))^{-2}\big(\det \mathrm{Cov}(\mu)\big)^{1/n}. \tag{4.9}$$

From the definition of the isotropic constant it follows that

$$\begin{aligned} L_T &= \frac{1}{\mathrm{vol}_n(T)^{1/n}}\big(\det \mathrm{Cov}(\mathbb{1}_T)\big)^{\frac{1}{2n}} \approx \mathrm{vol}_n(T)^{-1/n}(f_\mu(0)\mathrm{vol}_n(T))^{-1}\big(\det \mathrm{Cov}(\mu)\big)^{\frac{1}{2n}} \\ &\approx (f_\mu(0)\mathrm{vol}_n(T))^{-1-\frac{1}{n}} L_\mu, \end{aligned}$$

where we have also used the fact that $\|f_\mu\|_\infty^{1/n} \approx f_\mu(0)^{1/n}$ by Fradelizi's inequality (4.1). Finally, applying (4.7) with $p = 1$ we get that

$$e^{-1} \leqslant (f_\mu(0)\mathrm{vol}_n(T))^{1+\frac{1}{n}} \leqslant e\frac{n+1}{n} \leqslant 2e, \tag{4.10}$$

and this completes the proof. This discussion shows that

$$\tilde{L}_n \approx L_n.$$

In particular, knowing now that $L_n \leqslant C$ we get:

Theorem 4.1. *There exists an absolute constant $C > 0$ such that $L_\mu \leqslant C$ for every isotropic log-concave probability measure on $\mathbb{R}^n$.*

§ 4.3. L_p-centroid bodies. Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For any $p \geqslant 1$ we define the L_p-centroid body $Z_p(\mu)$ of μ as the convex body whose support function is

$$h_{Z_p(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^p f_\mu(x) dx\right)^{1/p}, \qquad y \in \mathbb{R}^n.$$

The convex bodies $Z_p(\mu)$ are always symmetric, and $Z_p(T_*\mu) = T(Z_p(\mu))$ for every $T \in GL_n$ and $p \geqslant 1$. Note that if μ is isotropic then $Z_2(\mu) = B_2^n$. Also, from (4.3) it follows that, for every $1 \leqslant p < q$,

$$Z_p(\mu) \subseteq Z_q(\mu) \subseteq \frac{cq}{p} Z_p(\mu), \tag{4.11}$$

where $c > 0$ is an absolute constant.

If $\mu = \lambda_K$ is the Lenesgue measure on a centered convex body K of volume 1 in $\mathbb{R}^n$ then we denote $Z_p(K) := Z_p(\lambda_K)$. In this case we have some additional results. For every $p \geqslant n$ we have that

$$Z_p(K) \supseteq c_2 Z_\infty(K) \tag{4.12}$$

where $c_2 > 0$ is an absolute constant and $Z_\infty(K) = \mathrm{conv}\{K, -K\}$. This is a consequence of the inequality

$$\int_K |\langle x, \xi\rangle|^p dx \geqslant \frac{\Gamma(p+1)\Gamma(n)}{2e\Gamma(p+n+1)} \max\left\{h_K^p(\xi), h_K^p(-\xi)\right\},$$

which holds true for all $\xi \in S^{n-1}$ and $p \geqslant 1$. Therefore, if $p \geqslant n$ we see that

$$\|\langle \cdot, \xi\rangle\|_{L_p(K)} \approx \max\{h_K(\xi), h_K(-\xi)\},$$

and hence $Z_p(K) \supseteq cZ_\infty(K)$. In particular, the Rogers-Shephard inequality $\mathrm{vol}_n(K-K) \leqslant \binom{2n}{n}\mathrm{vol}_n(K)$ for the difference body $K - K$ of K (see [2, Theorem 1.5.2]) implies that

$$c \leqslant \mathrm{vol}_n(Z_n(K))^{1/n} \leqslant \mathrm{vol}_n(K-K)^{1/n} \leqslant 4 \tag{4.13}$$

for some absolute constant $c > 0$.

A nonsymmetric variant of the L_p-centroid bodies of μ is defined as follows. For every $p \geqslant 1$ we consider the convex body $Z_p^+(\mu)$ with support function

$$h_{Z_p^+(\mu)}(y) = \left(\int_{\mathbb{R}^n} \langle x, y\rangle_+^p f_\mu(x)dx\right)^{1/p}, \qquad y \in \mathbb{R}^n,$$

where $a_+ = \max\{a, 0\}$. When f_μ is even, we have that $Z_p^+(\mu) = 2^{-1/p} Z_p(\mu)$. In any case, it is clear that $Z_p^+(\mu) \subseteq Z_p(\mu)$. One can also check that if $1 \leqslant p < q$ then

$$\left(\frac{4}{e}\right)^{\frac{1}{p}-\frac{1}{q}} Z_p^+(\mu) \subseteq Z_q^+(\mu) \subseteq c_1 \left(\frac{4(e-1)}{e}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{q}{p} Z_p^+(\mu). \tag{4.14}$$

(for a proof see [63], where the family of bodies $\tilde{Z}_p^+(\mu) = 2^{1/p} Z_p^+(\mu)$ is considered).

The L_p-centroid bodies were introduced, under a different normalization, by Lutwak and Zhang in [97], while in [112] for the first time, and in [113] later on, Paouris used geometric properties of them to acquire detailed information about the distribution of the Euclidean norm with respect to μ. A basic observation of Paouris is the next asymptotic formula that appears in [113].

Theorem 4.2 (Paouris)**.** *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Then,*

$$c_1 \leqslant \big(f_\mu(0)\, \mathrm{vol}_n(Z_n(\mu))\big)^{1/n} \leqslant c_2, \tag{4.15}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. Direct computation shows that, for every $p \geqslant 1$,

$$\mathrm{vol}_n(K_{n+p}(\mu))^{1+\frac{p}{n}} \int_{\overline{K_{n+p}(\mu)}} |\langle x, \xi\rangle|^p \, dx = \int_{K_{n+p}(\mu)} |\langle x, \xi\rangle|^p \, dx = \frac{1}{f_\mu(0)} \int_{\mathbb{R}^n} |\langle x, \xi\rangle|^p \, f_\mu(x)\, dx$$

for all $\xi \in S^{n-1}$, or equivalently

$$Z_p(\overline{K_{n+p}}(\mu))\mathrm{vol}_n(K_{n+p}(\mu))^{\frac{1}{p}+\frac{1}{n}} f_\mu(0)^{\frac{1}{p}} = Z_p(\mu). \tag{4.16}$$

Now, let $1 \leqslant p \leqslant n$. Using also (4.7) we see that

$$\frac{1}{e} Z_p(\overline{K_{n+p}}(\mu)) \subseteq f_\mu(0)^{1/n} Z_p(\mu) \subseteq e\frac{n+p}{n} Z_p(\overline{K_{n+p}}(\mu)) \subseteq 2e Z_p(\overline{K_{n+p}}(\mu)). \tag{4.17}$$

On the other hand, using (4.6) we can check that

$$h_{Z_p(\overline{K_{n+p}}(\mu))}(\xi) \approx h_{Z_p(\overline{K_{n+1}}(\mu))}(\xi)$$

for every $\xi \in S^{n-1}$. This shows that $Z_p(\overline{K_{n+p}}(\mu)) \approx Z_p(\overline{K_{n+1}}(\mu))$. Therefore, for all $1 \leqslant p \leqslant n$ we get

$$c_1 f_\mu(0)^{1/n} Z_p(\mu) \subseteq Z_p(\overline{K_{n+1}}(\mu)) \subseteq c_2 f_\mu(0)^{1/n} Z_p(\mu) \tag{4.18}$$

where $c_1, c_2 > 0$ are absolute constants.

Now recall that, since μ is centered, the body $\overline{K_{n+1}}(\mu)$ is also centered. Applying (4.13) for the body $\overline{K_{n+1}}(\mu)$ we see that

$$\mathrm{vol}_n(Z_n(\overline{K_{n+1}}(\mu)))^{1/n} \approx 1$$

and hence, by (4.18),

$$\big(f_\mu(0)\, \mathrm{vol}_n(Z_n(\mu))\big)^{1/n} \approx \mathrm{vol}_n(Z_n(\overline{K_{n+1}}(\mu)))^{1/n} \approx 1.$$

This completes the proof. $\square$

§ 4.4. Marginals and projections. Let μ be a log-concave probability measure on $\mathbb{R}^n$. For any $1 \leqslant k \leqslant n-1$ and $F \in G_{n,k}$ we define the marginal of μ with respect to F setting

$$\pi_F(\mu)(A) := \mu(P_F^{-1}(A))$$

for every Borel subset A of F. One can check that $\pi_F(\mu)$ is a log-concave probability measure on F, whose log-concave density $f_{\pi_F(\mu)}$ agrees almost everywhere with the function $\pi_F(f_\mu)$ defined by

$$\pi_F(f_\mu)(x) = \int_{x+F^\perp} f_\mu(y)dy. \tag{4.19}$$

Then, for every measurable function $g : F \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} g(P_F x) f_\mu(x)\, dx = \int_F g(x)\pi_F(f_\mu)(x)\, dx.$$

It follows that if f_μ is centered then, for every $F \in G_{n,k}$ we have that $\pi_F(f_\mu)$ is also centered, and if f is isotropic then $\pi_F(f_\mu)$ is also isotropic.

A basic observation of Paouris from [112] is that any projection of the L_p-centroid body of μ coincides with the L_p-centroid body of the corresponding marginal of μ.

Theorem 4.3. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For every $1 \leqslant k \leqslant n$ and any $F \in G_{n,k}$ and $p \geqslant 1$, we have that*

$$P_F(Z_p(\mu)) = Z_p(\pi_F(\mu)). \tag{4.20}$$

The proof is a direct application of Fubini's theorem.

Applying Theorem 4.2 to $\pi_F(\mu)$ and taking into account (4.20) we obtain the following result.

Theorem 4.4. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. Then, for every $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$ we have*

$$c_1 \leqslant \left(\pi_F(f_\mu)(0)\, \mathrm{vol}_k(P_F(Z_k(f)))\right)^{1/k} \leqslant c_2, \tag{4.21}$$

where $c_1, c_2 > 0$ are absolute constants.

5 Geometry of isotropic convex bodies

In this section we still assume that $L_n \leqslant C$. We shall discuss a few important consequences of this fact on the geometry of high-dimensional isotropic convex bodies.

§ 5.1. The ellipsoid intersection conjecture. We start with the positive answer to the following conjecture.

Theorem 5.1 (ellipsoid intersection conjecture)**.** *For every centered convex body K of volume 1 in $\mathbb{R}^n$ there exists a centered ellipsoid $\mathcal{E}$ of volume $\mathrm{vol}_n(\mathcal{E}) = \mathrm{vol}_n(K) = 1$ such that*

$$\mathrm{vol}_n(K \cap c_1\mathcal{E}) \geqslant \frac{1}{2}\mathrm{vol}_n(K)$$

where $c_1 > 0$ is an absolute constant.

For the proof we may assume that K is isotropic. Then, $\int_K |x|^2 dx = nL_K^2 \leqslant Cn$ for some absolute constant $C > 0$. From Markov's inequality we immediately see that

$$\mathrm{vol}_n(K \setminus \sqrt{2Cn}B_2^n) = \mathrm{vol}_n(\{x \in K : |x| > \sqrt{2Cn}\}) \leqslant \frac{1}{2Cn} \int_K |x|^2 dx \leqslant \frac{1}{2}.$$

It follows that $\mathrm{vol}_n(K \cap \sqrt{2C}\sqrt{n}B_2^n) \geqslant \frac{1}{2}\mathrm{vol}_n(K)$ and since $D_n \approx \sqrt{n}B_2^n$ it is simple to check that

$$\mathrm{vol}_n(K \cap c_1 D_n) \geqslant \frac{1}{2}\mathrm{vol}_n(K)$$

for some absolute constant $c_1 > 0$.

Let us note that the ellipsoid intersection conjecture is in fact equivalent to the isotropic constant conjecture.

§ 5.2. Volume of sections and projections. The next theorem gives a very useful estimate for the volume of sections of an isotropic convex body through its barycenter.

Theorem 5.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, for every $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$,*

$$\mathrm{vol}_{n-k}(K \cap F^{\perp})^{1/k} \approx 1. \tag{5.1}$$

Proof. We denote by μ_K the isotropic log-concave probability measure with density $L_K^n \mathbb{1}_{\frac{K}{L_K}}$ and write f_K for the density of μ_K. Fix $1 \leqslant k \leqslant n-1$ and $F \in G_{n,k}$. We know that $\pi_F(\mu_K)$ is isotropic. Using (4.18) with $p = 2$ we get:

$$\begin{aligned} L_{\overline{K_{k+1}}(\pi_F(\mu_K))} &= \left(\frac{\mathrm{vol}_k(Z_2(\overline{K_{k+1}}(\pi_F(\mu_K))))}{\mathrm{vol}_k(B_F)}\right)^{1/k} \approx \pi_F(f_K)(0)^{1/k}\left(\frac{\mathrm{vol}_k(Z_2(\pi_F(\mu_K)))}{\mathrm{vol}_k(B_F)}\right)^{1/k} \\ &= \pi_F(f_K)(0)^{1/k}\left(\frac{\mathrm{vol}_k(P_F(Z_2(\mu_K)))}{\mathrm{vol}_k(B_F)}\right)^{1/k}, \end{aligned}$$

where we have also used the identity $Z_2(\pi_F(\mu_K)) = P_F(Z_2(\mu_K))$ from Theorem 4.3. Since K is isotropic, we get

$$Z_2(\mu_K) = L_K^{-1} Z_2(K) = B_2^n \;\text{ and hence }\; P_F(Z_2(\mu_K)) = B_F.$$

Moreover, we have

$$\pi_F(f_K)(0) = \int_{F^{\perp}} f_K(y)\,dy = L_K^n \mathrm{vol}_{n-k}\left(\tfrac{1}{L_K}K \cap F^{\perp}\right) = L_K^k \mathrm{vol}_{n-k}(K \cap F^{\perp}).$$

Combining the above we conclude that $L_{\overline{K_{k+1}}(\pi_F(\mu_K))} \approx L_K \mathrm{vol}_{n-k}(K \cap F^{\perp})^{1/k}$, which is assertion of the theorem. $\square$

The next inequality estimates the product of the volumes of a projection of a convex body K in $\mathbb{R}^n$ and of the section of K with the orthogonal subspace (see [119] for the first and [124] for the second claim).

Theorem 5.3 (Rogers-Shephard/Spingarn)**.** *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. Then, for any $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$ we have that*

$$\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K \cap F^{\perp}) \leqslant \binom{n}{k}\mathrm{vol}_n(K).$$

If $\mathrm{bar}(K) = 0$ *then we also have that*

$$\mathrm{vol}_n(K) \leqslant \mathrm{vol}_k(P_F(K))\,\mathrm{vol}_{n-k}(K \cap F^{\perp}).$$

In the case where K is isotropic, these estimates lead to the next fact.

Theorem 5.4. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then, for any $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$ we have that*

$$c_1 \leqslant \mathrm{vol}_k(P_F(K))^{1/k} \leqslant c_2 \frac{n}{k}$$

where $c_1, c_2 > 0$ are absolute constants.

This follows from Theorem 5.3 and from the fact that if K is isotropic then $\mathrm{vol}_{n-k}(K \cap F^{\perp})^{1/k} \approx 1$ by Theorem 5.2.

§ 5.3. Volume of the centroid bodies. A lower bound for the volume of the L_p-centroid bodies $Z_p(K)$ of a star body K in $\mathbb{R}^n$ has been given by Lutwak, Yang and Zhang in [98], and later Haberl and Schuster in [64] obtained a similar lower bound for the nonsymmetric L_p-centroid bodies $Z_p^+(K)$ of K. If K is a star body, with respect to the origin, in $\mathbb{R}^n$ then, for every $1 \leqslant p < \infty$, the body $M_p^+(K)$ is defined through its support function

$$h_{M_p^+(K)}(y) = \left(c_{n,p}(n+p) \int_K \langle x, y\rangle_+^p \, dx \right)^{1/p},$$

where

$$c_{n,p} = \frac{\Gamma\left(\frac{n+p}{2}\right)}{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{p+1}{2}\right)}.$$

The normalization of $M_p^+(K)$ is chosen so that $M_p^+(B_2^n) = B_2^n$ for every p. Haberl and Schuster [64, Theorem 6.4] proved that if K is a star body, with respect to the origin, in $\mathbb{R}^n$ then, for every $p \geqslant 1$,

$$\mathrm{vol}_n(K)^{-\frac{n}{p}-1} \mathrm{vol}_n(M_p^+(K)) \geqslant \mathrm{vol}_n(B_2^n)^{-\frac{n}{p}}$$

with equality if and only if K is a centered ellipsoid in $\mathbb{R}^n$. Since $M_p^+(K) = (c_{n,p}(n+p))^{1/p} Z_p^+(K)$, we conclude that if $\mathrm{vol}_n(K) = 1$ then

$$\mathrm{vol}_n(Z_p^+(K))^{1/n} = (c_{n,p}(n+p))^{-1/p} \mathrm{vol}_n(M_p^+(K))^{1/n} \geqslant \left(\frac{1}{c_{n,p}(n+p)\omega_n} \right)^{1/p}.$$

Taking into account the value of the constant $c_{n,p}$ we can formulate this result as follows.

Proposition 5.5 (Lutwak-Yang-Zhang/Haberl-Schuster)**.** *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$\mathrm{vol}_n(Z_p(K))^{1/n} \geqslant \mathrm{vol}_n(Z_p(\overline{B}_2^n))^{1/n} \geqslant c\sqrt{p/n}$$

and

$$\mathrm{vol}_n(Z_p^+(K))^{1/n} \geqslant \mathrm{vol}_n(Z_p^+(\overline{B}_2^n))^{1/n} \geqslant c\sqrt{p/n}$$

for every $1 \leqslant p \leqslant n$, where $c > 0$ is an absolute constant.

Paouris [112] showed that a reverse inequality holds true (up to the isotropic constant).

Theorem 5.6 (Paouris)**.** *If μ is an isotropic log-concave measure on $\mathbb{R}^n$, then for every $1 \leqslant p \leqslant n$ we have that*

$$\mathrm{vol}_n(Z_p(\mu))^{1/n} \leqslant c\sqrt{p/n}. \tag{5.2}$$

Moreover, if K is a centered convex body of volume 1 in $\mathbb{R}^n$, then for every $1 \leqslant p \leqslant n$ we have that

$$\mathrm{vol}_n(Z_p(K))^{1/n} \leqslant c\sqrt{p/n}\, L_K \leqslant c_1\sqrt{p/n}, \tag{5.3}$$

where $c, c_1 > 0$ are absolute constants.

For the proof we shall use a number of classical facts from the Brunn-Minkowski theory (see [122] for a complete exposition). Steiner's formula asserts that for every convex body C in $\mathbb{R}^n$ we have

$$\mathrm{vol}_n(C + tB_2^n) = \sum_{k=0}^{n} \binom{n}{k} W_k(C) t^k$$

for all $t > 0$, where $W_k(C) = V_{n-k}(C) = V(C; n-k, B_2^n; k)$ is the k-th quermassintegral of C. Also, the Aleksandrov-Fenchel inequality implies the log-concavity of the sequence $(W_0(C), \dots, W_n(C))$, and in particular we have that

$$\left(\frac{W_{n-i}(C)}{\omega_n}\right)^{1/i} \geqslant \left(\frac{W_{n-j}(C)}{\omega_n}\right)^{1/j}, \tag{5.4}$$

for all $1 \leqslant i < j \leqslant n$. We shall also use Kubota's integral formula:

$$W_{n-m}(C) = \frac{\omega_n}{\omega_m} \int_{G_{n,m}} \mathrm{vol}_m(P_F(C))\, d\nu_{n,m}(F), \quad 1 \leqslant m \leqslant n. \tag{5.5}$$

Proof of Theorem 5.6. It is enough to prove (5.2) for integer values of $1 \leqslant p \leqslant n-1$. Observe that for any $F \in G_{n,p}$ we have

$$\mathrm{vol}_p(P_F(Z_p(\mu)))^{1/p} = \mathrm{vol}_p(Z_p(\pi_F(\mu)))^{1/p} \leqslant \frac{c_1}{\left(f_{\pi_F(\mu)}(0)\right)^{1/p}} \leqslant c_2,$$

where we have used Theorem 4.3, Theorem 4.2 and (4.4) for the isotropic density $f_{\pi_F(\mu)} = \pi_F(f_\mu)$. Applying (5.5) we get

$$W_{n-p}(Z_p(\mu)) \leqslant \frac{\omega_n}{\omega_p} c_2^p.$$

Now, we apply (5.4) for the convex body $C = Z_p(\mu)$ with $j = n$ and $i = p$; this gives

$$W_{n-p}^{1/p}(Z_p(\mu)) \geqslant \mathrm{vol}_n(Z_p(\mu))^{1/n} \omega_n^{1/p - 1/n}.$$

Combining the above, we get

$$\mathrm{vol}_n(Z_p(\mu))^{1/n} \leqslant \frac{\omega_n^{1/n}}{\omega_p^{1/p}} c_2.$$

Since $\omega_k^{1/k} \approx 1/\sqrt{k}$, we obtain (5.2). For the second assertion of the theorem we may assume that K is isotropic (because the volume of $Z_p(T(K))$ is the same for all $T \in SL_n$). Consider the measure μ with density $f_\mu = L_K^n \mathbb{1}_{\frac{K}{L_K}}$. Then, μ is isotropic and we easily check that $Z_p(\mu) = L_K^{-1} Z_p(K)$. Thus, the result follows immediately from (5.2) and the fact that the isotropic constant conjecture is true. $\square$

§ 5.4. Lower dimensional Busemann-Petty problem. In this subsection we discuss a natural generalization of the Busemann-Petty problem, for lower dimensional sections. Let $1 \leqslant k \leqslant n-1$ and let $\beta_{n,k}$ be the smallest constant $\beta > 0$ with the property that for every pair of centered convex bodies K and C in $\mathbb{R}^n$ that satisfy

$$\mathrm{vol}_{n-k}(K \cap F) \leqslant \mathrm{vol}_{n-k}(C \cap F)$$

for all $F \in G_{n,n-k}$, one has

$$\mathrm{vol}_n(K)^{\frac{n-k}{n}} \leqslant \beta^k \, \mathrm{vol}_n(C)^{\frac{n-k}{n}}.$$

It is known that $\beta_{n,k} > 1$ if $k < n-3$, while for $k = n-2$ and $k = n-3$ (two- and three-dimensional sections) it is not known whether $\beta_{n,k}$ has to be strictly greater than 1. The asymptotic lower dimensional Busemann-Petty problem asks if the constants $\beta_{n,k}$ are uniformly bounded. The next theorem provides an affirmative answer.

Theorem 5.7. *There exists an absolute constant $C > 0$ such that $\beta_{n,k} \leqslant C$ for all n and k.*

For the proof we introduce the dual affine quermassintegrals of a convex body. Let $1 \leqslant k \leqslant n-1$. For every convex body K, or more generally for any bounded Borel set, in $\mathbb{R}^n$, the k-th dual affine quermassintegral of K is defined by

$$\Psi_k(K) = \mathrm{vol}_n(K)^{-\frac{n-k}{kn}} \left(\int_{G_{n,k}} \mathrm{vol}_{n-k}(K \cap F^\perp)^n d\nu_{n,k}(F) \right)^{\frac{1}{kn}}.$$

Grinberg proved in [59] that this is an affinely invariant quantity and that

$$\Psi_k(K) \leqslant \Psi_k(B_2^n) \leqslant \sqrt{e} \tag{5.6}$$

for every bounded Borel set K of positive volume in $\mathbb{R}^n$.

Now, let K be a centered convex body in $\mathbb{R}^n$. Since $\Psi_k(K) = \Psi_k(T(K))$ for every $T \in GL_n$, we may assume that K is isotropic. From Theorem 5.2 we know that $\mathrm{vol}_{n-k}(K \cap F^\perp)^{1/k} \approx 1$ for every $F \in G_{n,k}$, and hence

$$\Psi_k(K) = \left(\int_{G_{n,k}} \mathrm{vol}_{n-k}(K \cap F^\perp)^n d\nu_{n,k}(F) \right)^{\frac{1}{kn}} \approx 1.$$

In other words, for every centered convex body K in $\mathbb{R}^n$ and any $1 \leqslant k \leqslant n-1$ we have that

$$c_1 \leqslant \Psi_k(K) \leqslant c_2 \tag{5.7}$$

where $c_1, c_2 > 0$ are absolute constants.

Proof of Theorem 5.7. Let K, C be two centered convex bodies in $\mathbb{R}^n$. Assume that for some $1 \leqslant k \leqslant n-1$ we have that

$$\mathrm{vol}_{n-k}(K \cap F) \leqslant \mathrm{vol}_{n-k}(C \cap F)$$

for all $F \in G_{n,n-k}$. Then,

$$\begin{aligned} \Psi_k(K)\,\mathrm{vol}_n(K)^{\frac{n-k}{kn}} &= \left(\int_{G_{n,k}} \mathrm{vol}_{n-k}(K \cap F^\perp)^n d\nu_{n,k}(F) \right)^{\frac{1}{kn}} \\ &\leqslant \left(\int_{G_{n,k}} \mathrm{vol}_{n-k}(C \cap F^\perp)^n d\nu_{n,k}(F) \right)^{\frac{1}{kn}} = \Psi_k(C)\,\mathrm{vol}_n(C)^{\frac{n-k}{kn}}. \end{aligned}$$

Taking into account (5.7) we get

$$\mathrm{vol}_n(K)^{\frac{n-k}{n}} \leqslant \left(\frac{\Psi_k(C)}{\Psi_k(K)} \right)^k \mathrm{vol}_n(C)^{\frac{n-k}{n}} \leqslant \left(\frac{c_2}{c_1} \right)^k \mathrm{vol}_n(C)^{\frac{n-k}{n}}$$

and the result follows. □

Note. In particular, the case $k = 1$ of Theorem 5.7 shows that the isotropic constant conjecture implies the asymptotic Busemann-Petty conjecture for the class of centered convex bodies, thus completing the discussion in §2.5.

The affine quermassintegrals of a convex body K in $\mathbb{R}^n$ were introduced by Lutwak in [95]. We shall discuss an appropriately normalized variant that was considered by Dafnis and Paouris in [39]. For every convex body K in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n$, we define the *normalized k-th affine quermassintegral* of K by

$$\Phi_k(K) := \mathrm{vol}_n(K)^{-\frac{1}{n}} \left(\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))^{-n} \, d\nu_{n,k}(F) \right)^{-\frac{1}{kn}}.$$

Grinberg proved in [59] that these quantities are invariant under volume preserving affine transformations. In this language, Lutwak conjectured in [96] that the affine quermassintegrals satisfy the inequalities

$$\Phi_k(K) \geqslant \Phi_k(B_2^n). \tag{5.8}$$

Dafnis and Paouris studied in [39] an isomorphic variant of Lutwak's conjecture, which asks if there exist absolute constants $c_1, c_2 > 0$ such that for every convex body K in $\mathbb{R}^n$ and any $1 \leqslant k \leqslant n-1$,

$$c_1\sqrt{n/k} \leqslant \Phi_k(K) \leqslant c_2\sqrt{n/k}. \tag{5.9}$$

Note that in the case $k = 1$, (5.9) follows by the Blaschke-Santaló and the Bourgain-Milman inequality, while in the case $k = n-1$ the conjectured rate of growth for $\Phi_{n-1}(K)$ is again true, by the Petty projection inequality and its reverse, proved by Zhang [133].

The left-hand side of (5.9) was proved by Paouris and Pivovarov in [114]; it confirms Lutwak's conjecture in an isomorphic sense. The proof relies on a duality argument, that employs the Blaschke Santaló inequality and the Bourgain-Milman inequality, combined with Grinberg's inequality (5.6). Subsequently, E. Milman and Yehudayoff [104] established the sharp lower bound $\Phi_k(B_2^n) \leqslant \Phi_k(K)$ and verified Lutwak's conjecture, including a characterization of the equality cases, for all $1 \leqslant k \leqslant n-1$: ellipsoids are the only local minimizers with respect to the Hausdorff metric.

Regarding the upper bound in (5.9), an almost optimal estimate (up to a $\ln n$-term) was given by Dafnis and Paouris in [39]. Let us briefly recall their argument: The Aleksandrov inequalities (see [122, Section 6.4]) imply that if K is a convex body in $\mathbb{R}^n$ then the sequence

$$Q_k(K) = \left(\frac{1}{\omega_k}\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))\, d\nu_{n,k}(F)\right)^{1/k} \tag{5.10}$$

is decreasing in k. In particular, for any $1 \leqslant k \leqslant n-1$ we have $Q_k(K) \leqslant Q_1(K)$, which may be written in the equivalent form

$$\left(\frac{1}{\omega_k}\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))\, d\nu_{n,k}(F)\right)^{\frac{1}{k}} \leqslant w(K), \tag{5.11}$$

where $w(K)$ is the mean width of K. Then, by Hölder's inequality,

$$\left(\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))^{-n}\, d\nu_{n,k}(F)\right)^{-\frac{1}{kn}} \leqslant \left(\int_{G_{n,k}} \mathrm{vol}_k(P_F(K))\, d\nu_{n,k}(F)\right)^{\frac{1}{k}} \leqslant \omega_k^{1/k} w(K).$$

Since the term on the left-hand side of this inequality is invariant under volume preserving affine transformations, we may assume that K has minimal mean width, and it is known that in this case we have $w(K) \leqslant c\sqrt{n}\ln n\, \mathrm{vol}_n(K)^{1/n}$ for some absolute constant $c > 0$ (see [2, Chapter 6]). Combining the above we get

$$\Phi_k(K) \leqslant c_2\sqrt{n/k}\ln n. \tag{5.12}$$

It was also shown in [39] that

$$\Phi_k(K) \leqslant c_3(n/k)^{3/2}\sqrt{\ln(en/k)}.$$

In other words, if k is proportional to n then the upper bound for $\Phi_k(K)$ is of the order of 1. The main question that remains open is whether the $\ln n$-term in (5.12) can actually be dropped.

§ 5.5. The deviation inequality of Paouris. Our goal in this subsection is to briefly explain the proof of a very useful deviation inequality of Paouris from [112].

Theorem 5.8 (Paouris)**.** *Let μ be an isotropic log-concave probability measure in $\mathbb{R}^n$. Then,*

$$\mu(\{x \in \mathbb{R}^n : |x| \geqslant ct\sqrt{n}\}) \leqslant \exp\left(-t\sqrt{n}\right) \tag{5.13}$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant.

The proof of Theorem 5.8 is reduced to the behavior of the moments of the function $x \mapsto |x|$. For every $p \geqslant 1$ we define

$$I_p(\mu) = \left(\int_{\mathbb{R}^n} |x|^p d\mu(x)\right)^{1/p}.$$

From (4.3) we see that for all $y \in \mathbb{R}^n$ and $p, q \geqslant 1$ we have

$$I_{pq}(K) \leqslant c_1 p I_q(K).$$

In particular, we have

$$I_p(\mu) \leqslant c_2 p I_2(\mu) \tag{5.14}$$

for all $p \geqslant 2$. Paouris proved the following.

Theorem 5.9 (Paouris)**.** *There exist absolute constants $c_1, c_2 > 0$ such that if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ then*

$$I_p(\mu) \leqslant c_1 I_2(\mu) \tag{5.15}$$

for all $p \leqslant c_2\sqrt{n}$.

Assuming that we have proved Theorem 5.9, we obtain Theorem 5.8 as follows: we consider an isotropic log-concave probability measure μ in $\mathbb{R}^n$. From Markov's inequality, for every $p \geqslant 2$ we have

$$\mu(\{|x| \geqslant e^3 I_p(\mu)\}) \leqslant e^{-3p}.$$

Then, Borell's lemma gives

$$\mu(\{|x| \geqslant e^3 I_p(\mu)s\}) \leqslant (1 - e^{-3p})\left(\frac{e^{-3p}}{1 - e^{-3p}}\right)^{(s+1)/2} \leqslant e^{-ps}$$

for every $s \geqslant 1$. Choosing $p = c_2\sqrt{n}$, and using (5.15), we see that

$$\mu(\{|x| \geqslant c_1 e^3 I_2(\mu)s\}) \leqslant \exp(-c_2\sqrt{n}s)$$

for all $s \geqslant 1$. Since μ is isotropic, we have $I_2(\mu) = \sqrt{n}$. This proves Theorem 5.8.

We pass to the proof of Theorem 5.9. We will actually prove a stronger statement.

Theorem 5.10. *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For every $p \geqslant 1$,*

$$I_p(\mu) \leqslant C\left(I_2(\mu) + R(Z_p(\mu))\right), \tag{5.16}$$

where $R(Z_p(\mu)) = \max\{|x| : x \in Z_p(\mu)\}$ is the radius of $Z_p(\mu)$.

Note that if μ is isotropic then $R(Z_p(\mu)) \leqslant cp$, and hence the right-hand side of (5.16) is bounded by $c_1 \max\{I_2(\mu), p\}$. Since $I_2(\mu) = \sqrt{n}$, for all $p \leqslant \sqrt{n}$ we get

$$I_p(\mu) \leqslant c_1 \max\{I_2(\mu), p\} = c_1 I_2(\mu),$$

which is exactly the statement of Theorem 5.9.

The proof of Theorem 5.10 requires a number of basic results from the asymptotic theory of convex bodies. We write $k_*(C)$ for the largest integer $k \leqslant n$ which satisfies

$$\nu_{n,k}\left(\left\{F \in G_{n,k} : \frac{w(C)}{2}|x| \leqslant h_C(x) \leqslant 2w(C)|x|,\ x \in F\right\}\right) \geqslant \frac{n}{n+k}.$$

In the terminology of [2] this is the so-called Dvoretzky dimension of C°, i.e. the largest k for which the majority of k-dimensional sections of C° are 4-Euclidean. The next theorem of V. Milman and Schechtman from [109] shows that the dimension $k_*(C)$ is determined by the parameters $w(C)$ and $R(C)$ up to an absolute constant.

Theorem 5.11 (V. Milman-Schechtman)**.** *There exist absolute constants $c_1, c_2 > 0$ such that*

$$c_1 n \frac{w(C)^2}{R(C)^2} \leqslant k_*(C) \leqslant c_2 n \frac{w(C)^2}{R(C)^2},$$

for every symmetric convex body C in $\mathbb{R}^n$.

For every $p \neq 0$ we define

$$w_p(C) = \left(\int_{S^{n-1}} h_C(\theta)^p d\sigma(\theta)\right)^{1/p}.$$

Note that $w_1(C) = w(C)$. The parameters w_p, $p \geqslant 1$ were studied by Litvak, V. Milman and Schechtman in [93].

Theorem 5.12 (Litvak-V. Milman-Schechtman)**.** *Let C be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$\max\left\{w(C), c_1\frac{R(C)\sqrt{p}}{\sqrt{n}}\right\} \leqslant w_p(C) \leqslant \max\left\{2w(C), c_2\frac{R(C)\sqrt{p}}{\sqrt{n}}\right\}$$

for all $1 \leqslant p \leqslant n$, where $c_1, c_2 > 0$ are absolute constants.

Note that the behavior of $w_p(C)$ changes when $p \approx n(w(C)/R(C))^2$. This value of p is roughly equal to the "dual Dvoretzky dimension" $k_*(C)$ of C. One also has $w_n(C) \approx R(C)$, and since $w_p(C) \leqslant R(C)$ for every $p \geqslant 1$ we conclude that $w_p(C) \approx R(C)$ for all $p \geqslant n$.

Concerning $w_p(C)$ for negative values of p, Klartag and Vershynin [81] established the next important result.

Theorem 5.13 (Klartag-Vershynin)**.** *Let C be a symmetric convex body in $\mathbb{R}^n$. Then, $w_p(C) \approx w_{-p}(C)$ for all $1 \leqslant p \leqslant ck_*(C)$, where $c > 0$ is an absolute constant.*

We start with the next lemma, which relates the p-moment of the Euclidean norm with respect to μ with the parameters w_p and the L_p-centroid bodies of μ.

Lemma 5.14. *Let μ be a log-concave probability measure in $\mathbb{R}^n$. For every $p \geqslant 1$ we have*

$$w_p(Z_p(\mu)) = a_{n,p}\sqrt{\frac{p}{p+n}}I_p(\mu)$$

where $a_{n,p} \approx 1$.

Proof. Direct computation shows that for every $x \in \mathbb{R}^n$ we have

$$\left(\int_{S^{n-1}} |\langle x, \xi\rangle|^p d\sigma(\xi)\right)^{1/p} = a_{n,p}\frac{\sqrt{p}}{\sqrt{p+n}}|x|,$$

where $a_{n,p} \approx 1$. Since

$$w_p(Z_p(\mu)) = \left(\int_{S^{n-1}}\int_{\mathbb{R}^n} |\langle x, \xi\rangle|^p d\mu(x)\sigma(d\xi)\right)^{1/p},$$

the lemma follows. □

Proof of Theorem 5.10. We start with the formula $I_p(\mu) = c_{n,p} w_p(Z_p(\mu))$ from Lemma 5.14, where $c_{n,p} \approx \max\{1, \sqrt{n/p}\}$. Therefore, we need to show that

$$w_p(Z_p(\mu)) \leqslant C \min\{1, \sqrt{p/n}\} \left(I_2(\mu) + R(Z_p(\mu))\right).$$

Since $w_p(Z_p(\mu)) \leqslant R(Z_p(\mu))$, we clearly have the result when $p \geqslant n$, and hence in the sequel we may assume that p is an integer and $1 \leqslant p \leqslant n$.

From Theorem 5.12 we have that

$$w_p(Z_p(\mu)) \leqslant c_1 \max\{w(Z_p(\mu)), \sqrt{p/n} R(Z_p(\mu))\}. \tag{5.17}$$

Therefore, the theorem will follow if we show that, for all $1 \leqslant p \leqslant n$,

$$w(Z_p(\mu)) \leqslant C\sqrt{p/n}(I_2(\mu) + R(Z_p(\mu))). \tag{5.18}$$

If $p \geqslant k_*(Z_p(\mu))$ then we have

$$w(Z_p(\mu)) \leqslant c_2\sqrt{p/n} R(Z_p(\mu)) \tag{5.19}$$

by Theorem 5.11. If $p \leqslant k_*(Z_p(\mu))$ then by the definition of $k_*(Z_p(\mu))$ we can find some $F \in G_{n,p}$ that satisfies both

$$\int |P_F(x)|^2 \, d\mu(x) \leqslant c_3(p/n) I_2^2(\mu)$$

(this is justified by averaging over all $F \in G_{n,k}$ and then applying Markov's inequality) and

$$w(Z_p(\mu)) B_F \subseteq c_4 P_F(Z_p(\mu)). \tag{5.20}$$

Since $P_F(Z_p(\mu)) = Z_p(\pi_F(\mu))$ and $\pi_F(\mu)$ is a p-dimensional centered log-concave probability measure, we get

$$\text{vrad}(Z_p(\pi_F(\mu))) \approx \frac{\sqrt{p}}{\|\pi_F(\mu)\|_\infty^{1/p}} = \sqrt{p}\frac{\left(\det \text{Cov}(\pi_F(\mu))\right)^{\frac{1}{2p}}}{L_{\pi_F(\mu)}}. \tag{5.21}$$

Using the fact that $L_{\pi_F(\mu)} \geqslant c$ for an absolute constant $c > 0$, we see that

$$\text{vrad}(Z_p(\pi_F(\mu))) \leqslant c_5 \frac{\left(\int |x|^2 d\pi_F\mu(x)\right)^{1/2}}{L_{\pi_F(\mu)}} \leqslant c_6 \left(\int |P_F(x)|^2 \, d\mu(x)\right)^{1/2} \leqslant c_7\sqrt{p/n} I_2(\mu). \tag{5.22}$$

Combining (5.20) and (5.22) we see that

$$w(Z_p(\mu)) \leqslant c_8\sqrt{p/n} I_2(\mu). \tag{5.23}$$

This completes the proof. □

In the case where $\mu = \mu_K$ for an isotropic convex body K in $\mathbb{R}^n$, the inequality (5.13) takes the form

$$\text{vol}_n(\{x \in K : |x| \geqslant ct\sqrt{n} L_K\}) \leqslant \exp\left(-t\sqrt{n}\right) \tag{5.24}$$

for all $t \geqslant 1$. Taking also into account the fact that $L_K \approx 1$, we obtain a much stronger statement than that of the ellipsoid intersection theorem of § 5.1.

6 A reduction of the slicing problem

In this section we do not assume the affirmative answer to the isotropic constant conjecture. We discuss small ball probability estimates for isotropic log-concave probability measures, starting with the work of Paouris in [113]. In particular, we describe a reduction of the isotropic constant conjecture to such small ball estimates. An optimal result of this type turns out to be the key fact in the affirmative answer to the problem given by Bizeul in [15].

First we introduce two parameters that are essential for the study of these questions.

§ 6.1. The parameter $q_*(\mu)$. A parameter which was originally central in the work of Paouris is $q_*(\mu)$, which is defined for every centered log-concave probability measure μ in $\mathbb{R}^n$, as follows:

$$q_*(\mu) = \max\{p \in [2, n] : k_*(Z_p(\mu)) \geqslant p\}.$$

The next proposition provides a lower bound for $q_*(\mu)$.

Proposition 6.1. *There exists an absolute constant $c > 0$ with the following property: if μ is a centered log-concave probability measure on $\mathbb{R}^n$ then*

$$q_*(\mu) \geqslant c\sqrt{k_*(Z_2(\mu))}.$$

Proof. We set $q_* := q_*(\mu)$. From Theorem 5.12, Lemma 5.14, Hölder's inequality, and the simple observation that $I_2(\mu) = \sqrt{n}w_2(Z_2(\mu))$, we get

$$w(Z_{q_*}(\mu)) \geqslant c_1 w_{q_*}(Z_{q_*}(\mu)) \geqslant c_2\sqrt{q_*/n}I_{q_*}(\mu) \geqslant c_2\sqrt{q_*/n}I_2(\mu) = c_2\sqrt{q_*/n}\,\sqrt{n}w_2(Z_2(\mu)).$$

In other words,

$$w(Z_{q_*}(\mu)) \geqslant c_2\sqrt{q_*}w(Z_2(\mu)). \tag{6.1}$$

Since $R(Z_{q_*}(\mu)) \leqslant c_3 q_*\, R(Z_2(\mu))$, using the definition of q_* and Theorem 5.11 we write

$$2q_* \geqslant k_*(Z_{q_*}(\mu)) \geqslant c_4 n\left(\frac{w(Z_{q_*}(\mu))}{R(Z_{q_*}(\mu))}\right)^2 \geqslant c_4 n\frac{c_2^2 q_*}{c_3^2 q_*^2}\frac{w^2(Z_2(\mu))}{R^2(Z_2(\mu))} \geqslant c_5\frac{k_*(Z_2(\mu))}{q_*}. \tag{6.2}$$

This shows that $q_*(\mu) \geqslant c\,\sqrt{k_*(Z_2(\mu))}$ for some absolute constant $c > 0$. □

Note that if μ is isotropic then $k_*(Z_2(\mu)) = n$. Therefore, in the isotropic case we have:

Corollary 6.2. *There exists an absolute constant $c > 0$ with the following property: for every isotropic log-concave probability measure μ on $\mathbb{R}^n$,*

$$q_*(\mu) \geqslant c\sqrt{n}.$$

We close this subsection with the following observation.

Theorem 6.3. *Let μ be an isotropic log-concave probability measure in $\mathbb{R}^n$. If $1 \leqslant p \leqslant c\sqrt{n}$, then*

$$w(Z_p(\mu)) \approx \sqrt{p}. \tag{6.3}$$

For the proof we write $w(Z_p(\mu)) \approx w_p(Z_p(\mu)) \approx \sqrt{p/n}I_p(\mu) \approx \sqrt{p}$, where the first equality holds because $c\sqrt{n} \leqslant q_*(\mu)$ by Corollary 6.2, the second comes from Lemma 5.14 and the third follows from Theorem 5.9.

§ 6.2. Small ball probability estimates. Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. We extend the definition of $I_p(\mu)$, allowing negative values of p, in the obvious way: for every $p \in (-n, \infty)$, $p \neq 0$, we define

$$I_p(\mu) := \left(\int_{\mathbb{R}^n} |x|^p d\mu(x)\right)^{1/p}.$$

Also, given any $\zeta \geqslant 1$, we define the parameter

$$q_{-c}(\mu,\zeta) := \max\{p \geqslant 1 : I_2(\mu) \leqslant \zeta I_{-p}(\mu)\}. \tag{6.4}$$

The first main result of this section is the next theorem.

Theorem 6.4 (Paouris)**.** *Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For every integer $1 \leqslant k \leqslant q_*(\mu)$ we have*

$$I_{-k}(\mu) \approx I_k(\mu).$$

In particular, Theorem 6.4 shows that for every $1 \leqslant k \leqslant q_*(\mu)$ we have $I_k(\mu) \leqslant cI_2(\mu)$, where $c > 0$ is an absolute constant. Assuming that μ is isotropic and taking into account Corollary 6.2, we immediately obtain the assertion of Theorem 5.9.

The proof of Theorem 6.4 is based on two identities.

Claim 6.5. *If μ is a centered log-concave probability measure on $\mathbb{R}^n$ and $1 \leqslant k \leqslant n-1$ is a positive integer, then*

$$I_{-k}(\mu) = c_{n,k}\left(\int_{G_{n,k}} \pi_F(f_\mu)(0) d\nu_{n,k}(F)\right)^{-1/k}, \tag{6.5}$$

where

$$c_{n,k} = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k} \approx \sqrt{n}.$$

Proof. Let $1 \leqslant k \leqslant n-1$. Then, we have

$$\begin{aligned}
\int_{G_{n,k}} \pi_F(f_\mu)(0)\, d\nu_{n,k}(F) &= \int_{G_{n,n-k}} \pi_{E^\perp}(f_\mu)(0)\, d\nu_{n,n-k}(E) = \int_{G_{n,n-k}} \int_E f_\mu(y)\, dy\, d\nu_{n,n-k}(E)\\
&= \int_{G_{n,n-k}} (n-k)\omega_{n-k} \int_{S_E} \int_0^\infty r^{n-k-1} f_\mu(r\xi)\, dr\, d\sigma_E(\xi)\, d\nu_{n,n-k}(E)\\
&= \frac{(n-k)\omega_{n-k}}{n\omega_n} n\omega_n \int_{S^{n-1}} \int_0^\infty r^{n-k-1} f_\mu(r\xi)\, dr\, d\sigma(\xi)\\
&= \frac{(n-k)\omega_{n-k}}{n\omega_n} \int_{\mathbb{R}^n} |x|^{-k} f_\mu(x)\, dx = \frac{(n-k)\omega_{n-k}}{n\omega_n} I_{-k}^{-k}(\mu).
\end{aligned}$$

It follows that

$$I_{-k}(\mu) = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k} \left(\int_{G_{n,k}} \pi_F(f_\mu)(0)\, d\nu_{n,k}(F)\right)^{-1/k}.$$

Finally, we check that $c_{n,k} = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k} \approx \sqrt{n}$. $\square$

Claim 6.6. *If C is a symmetric convex body in $\mathbb{R}^n$ and $1 \leqslant k \leqslant n-1$ is a positive integer, then*

$$w_{-k}(C) \approx \sqrt{k}\left(\int_{G_{n,k}} \mathrm{vol}_k(P_F(C))^{-1} d\nu_{n,k}(F)\right)^{-\frac{1}{k}}. \tag{6.6}$$

Proof. Using the Blaschke-Santaló and the Bourgain-Milman inequality, we write

$$\begin{aligned}
w_{-k}^{-1}(C) &= \left(\int_{S^{n-1}} \frac{1}{h_C^k(\xi)} d\sigma(\xi)\right)^{1/k} = \left(\int_{G_{n,k}} \int_{S_F} \frac{1}{\|\xi\|_{(P_F C)^\circ}^k} d\sigma(\xi) d\nu_{n,k}(F)\right)^{1/k}\\
&= \left(\int_{G_{n,k}} \frac{\mathrm{vol}_k(P_F(C))^\circ}{\mathrm{vol}_k(B_2^k)} d\nu_{n,k}(F)\right)^{1/k} \approx \left(\int_{G_{n,k}} \frac{\mathrm{vol}_k(B_2^k)}{\mathrm{vol}_k(P_F(C))} d\nu_{n,k}(F)\right)^{1/k},
\end{aligned}$$

and the result follows. $\square$

Now, consider a centered log-concave probability measure μ on $\mathbb{R}^n$ and an integer $1 \leqslant k \leqslant n-1$. Recall that from Theorem 4.4 we also have

$$\frac{1}{\mathrm{vol}_k(P_F(Z_k(\mu)))^{1/k}} \approx \pi_F(f_\mu)(0)^{1/k}.$$

for every subspace $F \in G_{n,k}$. Combining Claim 6.5 and Claim 6.6 we get

$$\sqrt{k/n}\, I_{-k}(\mu) \approx w_{-k}(Z_k(\mu)). \tag{6.7}$$

Proof of Theorem 6.4. Let $1 \leqslant k \leqslant n-1$. We know that $w_k(Z_k(\mu)) \approx \sqrt{k/n}\, I_k(\mu)$ and (6.7) shows that we also have $w_{-k}(Z_k(\mu)) \approx \sqrt{k/n}\, I_{-k}(\mu)$

We set $k_0 = \lfloor q_* \rfloor$, where $q_* = q_*(\mu)$. Then,

$$k_*(Z_{k_0}(\mu)) \approx k_*(Z_{q_*}(\mu)) \geqslant c_1 q_* \geqslant c_1 k_0. \tag{6.8}$$

From Theorem 5.13 we have

$$w_{-k}(Z_{k_0}(\mu)) \approx w_k(Z_{k_0}(\mu)) \tag{6.9}$$

for every $1 \leqslant k \leqslant c_2 k_*(Z_{k_0}(\mu))$, and (6.8) shows that (6.9) holds for every $k \leqslant c_3 q_*(\mu)$. Setting $k_1 = \lfloor c_3 q_*(\mu) \rfloor \approx k_0$, and using the fact that $Z_{k_0}(\mu) \approx Z_{k_1}(\mu)$, we get

$$w_{-k_1}(Z_{k_1}(\mu)) \approx w_{k_1}(Z_{k_1}(\mu)). \tag{6.10}$$

It is now clear that $I_{-k_1}(\mu) \approx I_{k_1}(\mu)$ and since $k_1 \approx q_*(\mu)$ we see that $q \mapsto I_q(\mu)$ is "constant" in the range $1 \leqslant |q| \leqslant c q_*(\mu)$. □

Suppose that μ is isotropic. Then, $q_*(\mu) \geqslant c_1\sqrt{n}$, and since $I_2(\mu) \leqslant I_p(\mu) \leqslant c_2 I_{-p}(\mu)$ for all $1 \leqslant p \leqslant c q_*(\mu)$, we obtain the following.

Corollary 6.7. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then, $q_{-c}(\mu, \zeta_0) \geqslant c\sqrt{n}$, where $c, \zeta_0 > 0$ are absolute constants.*

Another consequence of Theorem 6.4 is the next small ball probability estimate:

Theorem 6.8. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then, for every $0 < \varepsilon < \varepsilon_0$ we have*

$$\mu(\{x \in \mathbb{R}^n : |x| \leqslant \varepsilon\sqrt{n}\}) \leqslant \varepsilon^{c\sqrt{n}}, \tag{6.11}$$

where $\varepsilon_0, c > 0$ are absolute constants.

Proof. We know that $I_2(\mu) \leqslant c_1 I_{-p}(\mu)$ for all $1 \leqslant p \leqslant c_2\sqrt{n}$. It follows that

$$\begin{aligned} \mu(\{x \in \mathbb{R}^n : |x| \leqslant \varepsilon I_2(\mu)\}) &\leqslant \mu(\{x : |x| \leqslant c_1 \varepsilon I_{-p}(\mu)\}) \\ &\leqslant (c_1\varepsilon)^p \leqslant \varepsilon^{p/2}, \end{aligned}$$

for every $0 < \varepsilon < c_1^{-2}$ and $p \leqslant c_2\sqrt{n}$. This gives the result with $\varepsilon_0 = c_1^{-2}$ and $c = c_2/2$. □

The next theorem shows that if the hyperplane conjecture is correct then there are absolute constants $\tau, \zeta_0 > 0$ such that, for every isotropic convex body K in $\mathbb{R}^n$, one has $q_{-c}(\mu_K, \zeta_0) \geqslant \tau n$.

Theorem 6.9 (Dafnis-Paouris)**.** *There exists an absolute constant $c > 0$ such that, for every n and every isotropic convex body K in $\mathbb{R}^n$,*

$$q_{-c}(\mu_K, cL_n) \geqslant n-1.$$

Proof. Using Theorem 5.2, for every $1 \leqslant s \leqslant n-1$ we write

$$\begin{aligned} I_{-s}(K) &\approx \sqrt{n}\left(\int_{G_{n,s}} \mathrm{vol}_{n-s}(K\cap F^{\perp})\, d\nu_{n,s}(F)\right)^{-1/s} \\ &\approx \sqrt{n}\left(\int_{G_{n,s}} \left(\frac{L_{\overline{K_{s+1}(\pi_F(\mu_K))}}}{L_K}\right)^{s} d\nu_{n,s}(F)\right)^{-1/s} \geqslant \frac{\sqrt{n}L_K}{L_s}. \end{aligned}$$

This shows that

$$I_{-s}(K) \geqslant \frac{c_1\sqrt{n}L_K}{L_s} \geqslant \frac{c_2\sqrt{n}L_K}{L_n}$$

because it is known that $L_s \leqslant c_3 L_n$ for all integers $1 \leqslant s \leqslant n-1$ (see [28]). Since $I_2(K) = \sqrt{n}L_K$, we get

$$q_{-c}(K, c_2^{-1}L_n) := \max\{p \geqslant 1 : I_2(K) \leqslant c_2^{-1}L_n I_{-p}(K)\} \geqslant n-1.$$

This is the claim of the theorem. $\square$

§ 6.3. A reduction of the isotropic constant problem. In the previous subsection we introduced the parameter

$$q_{-c}(K,\zeta) := \max\{p \geqslant 1 : I_2(K) \leqslant \zeta I_{-p}(K)\}.$$

We shall show that the hyperplane conjecture is equivalent to the following statement:

> *There exist absolute constants $c, \zeta_0 > 0$ such that $q_{-c}(K,\zeta_0) \geqslant cn$ for every isotropic convex body K in $\mathbb{R}^n$.*

We already know that there exists a parameter $q_* := q_*(K)$, related to the L_p-centroid bodies of K, with the following properties:

(i) $q_*(K) \geqslant c\sqrt{n}$,

(ii) $q_{-c}(K,\zeta_0) \geqslant q_*(K)$ for some absolute constant $\zeta_0 \geqslant 1$, and hence, $I_2(K) \leqslant \zeta_0 I_{-q_*}(K)$.

What is not clear is the behavior of $I_{-p}(K)$ when p lies in the interval $[q_*, n]$.

The main idea of Dafnis and Paouris in [38] is to start with an "extremal" isotropic convex body K in $\mathbb{R}^n$ with maximal isotropic constant $L_K \approx L_n$ which is at the same time in α-regular M-position. Their starting point, which has a rather technical proof, is the following precise statement.

Theorem 6.10 (Dafnis-Paouris)**.** *There exist absolute constants $\kappa, \tau > 1$ and $\delta > 0$ such that, for every $\alpha \in [1,2)$, we can find an isotropic convex body K_α in $\mathbb{R}^n$ with the following properties:*

(i) $L_{K_\alpha} \geqslant \delta L_n$,

(ii) *for every* $t \geqslant \tau(2-\alpha)^{-3/2}$

$$\ln N(K_\alpha, t\sqrt{n}B_2^n) \leqslant \frac{\kappa n}{(2-\alpha)^{2\alpha}\, t^{\alpha}}. \tag{6.12}$$

Sketch of the proof. We shall give a very rough sketch of the proof of Theorem 6.10 for the case $\alpha = 1$. We should note here that the idea of considering convex bodies with "maximal" isotropic constant and their M-ellipsoids had been previously used by Bourgain, Klartag and V. Milman in [28].

We may assume that $n = 2m$ is even, and we start with an isotropic convex body K_0 that has isotropic constant $L_{K_0} \geqslant \delta_0 L_{2m}$, where $\delta_0 \in (0,1)$. Then, for every k-codimensional subspace F of $\mathbb{R}^{2m}$ we have

$$\mathrm{vol}_{2m-k}(K\cap F^{\perp})^{1/k} \approx \frac{L_{\overline{K_{k+1}(\pi_F(\mu_K))}}}{L_K} \leqslant c_1(\delta_0),$$

where $c_1(\delta_0)$ is a constant depending only on δ_0. If $\mathcal{E}_1$ is a 1-regular M-ellipsoid for K_0 then we see that for every $F\in G_{2m,m}$ we have

$$\mathrm{vol}_m(P_F(K_0))^{1/m}\leqslant c_1\,\mathrm{vol}_m(P_F(\mathcal{E}_1))^{1/m}.$$

From Theorem 5.3 we know that

$$\big(\mathrm{vol}_m(K_0\cap F)\mathrm{vol}_m(P_{F^\perp}(K_0))\big)^{1/m}\approx\big(\mathrm{vol}_m(\mathcal{E}_1\cap F)\mathrm{vol}_m(P_{F^\perp}(\mathcal{E}_1))\big)^{1/m}\approx 1.$$

Combining the above with the fact that, for any $1\leqslant k\leqslant n-1$, the maximal/minimal volume k-dimensional section and projection of an ellipsoid coincide, we conclude that

$$c_2(\delta_0)\leqslant \mathrm{vol}_m(K\cap F^\perp)^{1/m}\leqslant c_1(\delta_0)$$

for every $F\in G_{2m,m}$. Using this fact we can check that if $\lambda_1\geqslant\cdots\geqslant\lambda_{2m}>0$ are the semi-axes of $\mathcal{E}_1$ then $\lambda_m\approx_\delta\sqrt{n}$. Now, we use the standard fact that there exists $F_0\in G_{2m,m}$ such that $P_{F_0}(\mathcal{E}_1)=\lambda_m B_{F_0}$, and hence

$$c_3(\delta_0)\overline{B}_{F_0}\subseteq P_{F_0}(\mathcal{E}_1)\subseteq c_4(\delta_0)\overline{B}_{F_0}.$$

Then, we define $W=\overline{K_{m+1}}(\pi_{F_0}(\mu_{K_0})))$ and $K_1=W\times U(W)$, where $U\in O(2m)$ satisfies $U(F_0)=F_0^\perp$. The convex body K_1 satisfies the assertion of the theorem for $\alpha=1$. □

Then, Dafnis and Paouris try taking advantage of the fact that small ball estimates are closely related to estimates for covering numbers. The key lemma is the following.

Lemma 6.11. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Assume that, for some $s>0$,*

$$r_s:=\ln N(K,sB_2^n)<n. \tag{6.13}$$

Then,

$$I_{-r_s}(K)\leqslant 3es.$$

Proof. Let $z_0\in\mathbb{R}^n$ be such that $\mathrm{vol}_n(K\cap(-z_0+sB_2^n))\geqslant\mathrm{vol}_n(K\cap(z+sB_2^n))$ for every $z\in\mathbb{R}^n$. It follows that

$$\mathrm{vol}_n((K+z_0)\cap sB_2^n)\,N(K,sB_2^n)\geqslant\mathrm{vol}_n(K)=1. \tag{6.14}$$

Let $q=r_s$. Then, using Markov's inequality, the definition of $I_{-q}(K+z_0)$ and (6.13), we get

$$\mathrm{vol}_n((K+z_0)\cap 3^{-1}I_{-q}(K+z_0)B_2^n)\leqslant 3^{-q}<e^{-q}=e^{-r_s}\leqslant\frac{1}{N(K,sB_2^n)}.$$

and hence

$$\mathrm{vol}_n((K+z_0)\cap 3^{-1}I_{-q}(K+z_0)B_2^n)<\mathrm{vol}_n((K+z_0)\cap sB_2^n),$$

which implies that

$$3^{-1}I_{-q}(K+z_0)\leqslant s.$$

Finally, we can check that $I_{-k}(K+z)\geqslant e^{-1}I_{-k}(K)$ for any $1\leqslant k\leqslant n-1$ and $z\in\mathbb{R}^n$. Indeed, using Claim 6.5 and the fact that K is centered (more precisely, Fradelizi's inequality (4.1) for the centered log-concave function $\pi_{F^\perp}(\mathbb{1}_K)$ where $F\in G_{n,k}$) we write

$$\begin{aligned}I_{-k}(K+z)&=c_{n,k}\left(\int_{G_{n,k}}\mathrm{vol}_{n-k}(K+z)\cap F^\perp)\,d\nu_{n,k}(F)\right)^{-1/k}\\&\geqslant\frac{c_{n,k}}{e}\left(\int_{G_{n,k}}\mathrm{vol}_{n-k}(K\cap F^\perp)\,d\nu_{n,k}(F)\right)^{-1/k}=\frac{1}{e}I_{-k}(K).\end{aligned}$$

This proves the lemma. □

We can now prove the main theorem.

Theorem 6.12 (Dafnis-Paouris)**.** *Assume that $q_{-c}(K,\zeta)\geqslant \beta n$ for some $\zeta\geqslant 1$, some $\beta\in(0,1)$ and every isotropic convex body K in $\mathbb{R}^n$. Then,*

$$L_n\leqslant \frac{C\zeta}{\sqrt{\beta}}\ln^2(e/\beta), \tag{6.15}$$

where $C>0$ is an absolute constant.

Proof. We set $\alpha := 2-\ln(e/\beta)^{-1}$ and, for this value of α, we apply Theorem 6.10 to find an isotropic convex body K_α which satisfies its conclusion: for some absolute constants $\kappa,\tau\geqslant 1$ and $\delta>0$ it holds that $L_{K_\alpha}\geqslant \delta L_n$ and

$$\ln N(K_\alpha, t\sqrt{n}B_2^n)\leqslant \frac{\kappa n}{(2-\alpha)^{2\alpha}t^\alpha}\quad \text{for all } t\geqslant \tau\ln^{3/2}(e/\beta).$$

We may clearly assume that $\tau^2\leqslant e\kappa$ as well. We choose

$$t_1=(e\kappa)^{1/\alpha}\frac{1}{\sqrt{\beta}}\ln^2(e/\beta).$$

Note that $\tau\leqslant\sqrt{e\kappa}\leqslant(e\kappa)^{1/\alpha}$, and hence $t_1\geqslant\tau\frac{1}{\sqrt{\beta}}\ln^2(e/\beta)\geqslant\tau\ln^{3/2}(e/\beta)$. Therefore, (6.12) is valid for $t=t_1$, and this shows that

$$r_1:=\ln N(K_\alpha,t_1\sqrt{n}B_2^n)\leqslant\frac{\kappa n}{(2-\alpha)^{2\alpha}t_1^\alpha}\leqslant\frac{1}{e}(\sqrt{\beta})^\alpha n\leqslant\beta n.$$

Then, from Lemma 6.11 we get

$$I_{-r_1}(K_\alpha)\leqslant 3et_1\sqrt{n}.$$

On the other hand, since $q_{-c}(K_\alpha,\zeta)\geqslant\beta n$ and $r_1\leqslant\beta n$, we have that

$$\sqrt{n}L_{K_\alpha}=I_2(K_\alpha)\leqslant\zeta I_{-r_1}(K_\alpha).$$

It follows that

$$L_{K_\alpha}\leqslant 3e\zeta t_1=3e\zeta(e\kappa)^{1/\alpha}\frac{1}{\sqrt{\beta}}\ln^2(e/\beta)\leqslant\frac{3e^2\zeta\kappa}{\sqrt{\beta}}\ln^2(e/\beta).$$

Since $L_{K_\alpha}\geqslant\delta L_n$, the result follows. □

Remark 6.13. Since $q_{-c}(K,\zeta_0)\geqslant q_*(K)\geqslant c\sqrt{n}$ for some absolute constants $\zeta_0\geqslant 1$ and $c>0$, we may apply Theorem 6.12 with $\zeta=\zeta_0$ and $\beta=c/\sqrt{n}$ to get

$$L_n\leqslant\frac{C\zeta}{\sqrt{c}}\sqrt[4]{n}\ln^2\Big(\frac{e\sqrt{n}}{c}\Big)\leqslant c_1\sqrt[4]{n}(\ln n)^2, \tag{6.16}$$

where $c_1>0$ is an absolute constant. This is a non-trivial estimate for L_n, slightly weaker but close to the estimate $L_n=O(\sqrt[4]{n}\ln n)$ of Bourgain [26] and the estimate $L_n=O(\sqrt[4]{n})$ of Klartag [69].

We close this section with a much more direct reduction of the problem but with a much stronger assumption. We shall use the following corollary of Theorem 6.10.

Proposition 6.14. *For every $n\geqslant 1$ there exists an isotropic convex body K in $\mathbb{R}^n$ such that $L_K\geqslant c_1L_n$ and*

$$\mathrm{vol}_n(K\cap\sqrt{c_2n}B_2^n)>c_0^n$$

where $c_i>0$ are absolute constants.

Proof. We apply Theorem 6.10 with $\alpha = 1$ to find an isotropic convex body K in $\mathbb{R}^n$ such that $L_K \geqslant c_1 L_n$ and

$$\ln N(K, t\sqrt{n}B_2^n) \leqslant \frac{c_3 n}{t}$$

for all $t \geqslant c_3$. Then, it is clear that

$$1 = \mathrm{vol}_n(K) \leqslant e^n N(K, \sqrt{c_2 n}B_2^n)\,\mathrm{vol}_n(K \cap \sqrt{c_2 n}B_2^n) \leqslant e^{2n}\mathrm{vol}_n(K \cap \sqrt{c_2 n}B_2^n)$$

if $c_2 = c_3^2$, and the result follows. □

From Proposition 6.14 we deduce the following direct reduction of the isotropic constant conjecture to small ball estimates.

Theorem 6.15. *Assume that there exist $\varepsilon_0, c_0 > 0$ such that for every isotropic convex body K in $\mathbb{R}^n$ and any $0 < \varepsilon \leqslant \varepsilon_0$ we have that*

$$\mathrm{vol}_n(\{x \in K : |x| \leqslant \varepsilon\sqrt{n}L_K\}) \leqslant \varepsilon^{c_0 n}.$$

Then, $L_n \leqslant C$ for all $n \geqslant 1$, where $C = C(\varepsilon_0, c_0)$ depends only on ε_0 and c_0.

Proof. Proposition 6.14 shows that there exists an isotropic convex body K in $\mathbb{R}^n$ with $L_n \leqslant C_1 L_K$ and $\mathrm{vol}_n(K \cap \sqrt{C_2 n}B_2^n) > c_2^n$ for some absolute constants $C_1, C_2, c_2 > 0$.

On the other hand, applying the hypothesis we see that for any $0 < \varepsilon \leqslant \varepsilon_0$,

$$\mathrm{vol}_n(K \cap \varepsilon\sqrt{n}L_K\, B_2^n) \leqslant \varepsilon^{c_0 n}.$$

Choosing $\varepsilon_1 = \min\{\varepsilon_0, c_2^{\frac{1}{c_0}}\}$, and comparing the above inequalities, we get

$$L_n^2 \leqslant C_1^2 L_K^2 \leqslant \frac{C_1^2 C_2}{\varepsilon_1}.$$

□

7 Eldan's stochastic localization

Eldan's stochastic localization has been the key for the recent developments in Bourgain's slicing problem and other well-known isoperimetric problems about high-dimensional log-concave probability measures and convex bodies, such as the Kannan-Lovász-Simonovits conjecture that we shall discuss in the next section. In this section we briefly introduce the stochastic localization scheme that we shall use.

First we recall with some preliminaries from stochastic calculus. We refer to [111], [120] and [41] for definitions and background on semimartingales and stochastic integration.

Let x_t and y_t be real-valued stochastic processes. The quadratic variations $[x]_t$ and $[x,y]_t$ are real-valued stochastic processes defined by

$$[x]_t = \lim_{|P|\to 0}\sum_{i=1}^{\infty}(x_{\tau_n} - x_{\tau_{n-1}})^2 \quad \text{and} \quad [x,y]_t = \lim_{|P|\to 0}\sum_{n=1}^{\infty}(x_{\tau_n} - x_{\tau_{n-1}})(y_{\tau_n} - y_{\tau_{n-1}}),$$

where $P = \{0 = \tau_0 \leqslant \tau_1 \leqslant \tau_2 \cdots \uparrow t\}$ is a stochastic partition of the non-negative real numbers, $|P| = \max_n(\tau_n - \tau_{n-1})$ is the mesh of P, and the limit is defined using convergence in probability. Note that $[x]_t$ is non-decreasing in t and $[x,y]_t$ satisfies the equation

$$[x,y]_t = \frac{1}{4}([x+y]_t - [x-y]_t).$$

Assume that the processes x_t and y_t satisfy the differential equations $dx_t = a(x_t)dt + \sigma(x_t)dW_t$ and $dy_t = b(y_t)dt + \eta(y_t)dW_t$ where (W_t) is a standard Brownian motion. Then we have

$$[x]_t = \int_0^t \sigma^2(x_s)ds \quad \text{and} \quad [x,y]_t = \int_0^t \sigma(x_s)\eta(y_s)ds.$$

Also, $d[x,y]_t = \sigma(x_t)\eta(y_t)dt$.

Analogously, for vector valued differential equations $dx_t = a(x_t)dt + \Sigma(x_t)dW_t$ and $dy_t = b(y_t)dt + H(y_t)dW_t$ we have that

$$[x^i, x^j]_t = \int_0^t (\Sigma(x_s)\Sigma^T(x_s))_{ij}ds \quad \text{and} \quad d[x^i, y^j]_t = (\Sigma(x_t)H^T(y_t))_{ij}dt.$$

We shall use Itô's formula: If x is a semimartingale and f is a twice continuously differentiable function then

$$df(x_t) = \sum_i \frac{df(x_t)}{dx^i}\, dx^i + \frac{1}{2}\sum_{i,j} \frac{d^2 f(x_t)}{dx^i dx^j}\, d[x^i, x^j]_t. \tag{7.1}$$

§ 7.1. Eldan's stochastic localization. A variant of the stochastic localization scheme that we shall use was introduced by Eldan in [42]. Then, his idea was used in a number of subsequent works [90], [35], [76].

Given a probability measure $\mu = \mu_0$ on $\mathbb{R}^n$ with density $\varrho = \varrho_0$, the starting idea of localization is to restrict the distribution to a random half-space and then repeat this process. The discrete version of this procedure is to define

$$\varrho_{t+1}(x) = \varrho_t(x)\big(1 + \sqrt{h}\langle x - a_t, w\rangle\big)$$

where a_t is the barycenter of the measure μ_t, the parameter $h > 0$ is sufficiently small and w is a Gaussian random vector. So, each step is a renormalization of the measure with a linear function in a random direction. Using the approximation $1 + y \sim e^{y - \frac{1}{2}y^2}$ as $y \to 0$ we see that this process introduces a Gaussian factor in the exponent, which is more and more concentrated as $t \to \infty$. Eventually, μ_t becomes a Dirac measure, and then any set has measure 0 or 1. The idea is to stop at a large enough time t so that the density would include a strong Gaussian factor but at the same time we would still be able to compare the original measure μ with μ_t.

In stochastic localization, the discrete steps described above are replaced by infinitesimal steps. We are given a probability measure μ on $\mathbb{R}^n$, and a standard Brownian motion (W_t) on $\mathbb{R}^n$. We consider an infinite system of stochastic differential equations whose unknown is the family (ϱ_t) of functions from $\mathbb{R}^n$ to $\mathbb{R}_+$, with $\varrho_0(x) = 1$ and

$$d\varrho_t(x) = \varrho_t(x)\,\langle x - a_t, dW_t\rangle,$$

where

$$a_t = \frac{\int_{\mathbb{R}^n} \langle x, \varrho_t(x)\rangle\, d\mu(x)}{\int_{\mathbb{R}^n} \varrho_t(x)\, d\mu(x)}$$

is the barycenter of $\varrho_t(x)\, d\mu(x)$. Note that we have only one Brownian motion (W_t) which is used for every x. This simplified version of Eldan's process was introduced by Lee and Vempala in [90].

We may assume that $\varrho_t(x) > 0$ for all t, almost surely. We have

$$d\left(\int_{\mathbb{R}^n} \varrho_t(x)\, d\mu(x)\right) = \int_{\mathbb{R}^n} d\varrho_t(x)\, d\mu(x) = \Big\langle \int_{\mathbb{R}^n} (x - a_t)\varrho_t(x)\, d\mu(x), dW_t \Big\rangle = 0$$

by the definition of a_t, which shows that the total mass of $\varrho_t d\mu$ remains constant. So, $\mu_t := \varrho_t\, d\mu$ is a random probability measure for every $t > 0$. We can also check that $\varrho_t(x)$ is a martingale for all x. In particular,

$$\mathbb{E}(\varrho_t(x)) = \varrho_0(x) = 1$$

for all x. This shows that the random measure μ_t is equal to μ on average:

$$\mathbb{E}(\mu_t) = \mu.$$

Finally, we can solve the equation

$$d\varrho_t(x) = \varrho_t(x)\langle x - a_t, dW_t\rangle$$

explicitly: Applying Itô's formula to $\ln \varrho_t(x)$ we get

$$d(\ln \varrho_t(x)) = \frac{d\varrho_t(x)}{\varrho_t(x)} - \frac{1}{2}\frac{d[\varrho_t(x)]_t}{\varrho_t^2(x)} = \langle x - a_t, dW_t\rangle - \frac{1}{2}|x - a_t|^2\, dt,$$

which gives

$$\varrho_t(x) = \exp\left(\int_0^t \langle x - a_s, dW_s\rangle - \frac{1}{2}\int_0^t |x - a_s|^2\, ds\right) = \exp\left(c_t + \langle x, \xi_t\rangle - \frac{t}{2}|x|^2\right),$$

where (c_t) and (ξ_t) are random processes that do not depend on x. This shows that the density ϱ_t of μ_t with respect to μ is just a Gaussian factor. The linear term and the normalizing constant are random but the quadratic term is deterministic, equal to $\frac{t}{2}|x|^2$.

This means that if the original measure μ is log-concave then the measure μ_t is t-uniformly log-concave (i.e. the function $\varrho_t(x)e^{t|x|^2/2}$ is log-concave) almost surely, and the process becomes more and more peaked as t grows. For every $t > 0$ we have expressed the log-concave measure μ as a mixture of t-uniformly log-concave measures. Moreover, this mixture is constructed as a solution of a stochastic differential equation, and using Itô's formula we can try to control its behavior over time.

§ 7.3. Construction of the process. A rigorous construction of this stochastic localization process is provided in [77] (see also [80]). Consider a standard n-dimensional Brownian motion (ξ_t) defined on some probability space $(\Omega, \mathcal{A}, P)$ equipped with a filtration $(\mathcal{A}_t)$. For every $x \in \mathbb{R}^n$ the process (E_t) defined by

$$E_t = \exp\left(\langle x, \xi_t\rangle - \frac{t}{2}|x|^2\right)$$

is a martingale. Then, for any test function g, the process

$$N_t = \int_{\mathbb{R}^n} g(x) \exp\left(\langle x, \xi_t\rangle - \frac{t}{2}|x|^2\right)\, d\mu(x)$$

is also a martingale, and in particular we have that $\mathbb{E}(N_t) = N_0 = \mathbb{E}_\mu(g)$.

We define the random probability measure μ_t on $\mathbb{R}^n$ with

$$d\mu_t(x) = \frac{1}{Z_t}\exp\left(\langle x, \xi_t\rangle - \frac{t}{2}|x|^2\right)\, d\mu(x) \tag{7.2}$$

where

$$Z_t = \int_{\mathbb{R}^n} \exp\left(\langle x, \xi_t\rangle - \frac{t}{2}|x|^2\right)\, d\mu(x)$$

is a suitable normalization constant. Then,

$$N_t = Z_t\int_{\mathbb{R}^n} g(x)\, d\mu_t(x).$$

We fix $T > 0$. Since $(Z_t)_{t\leqslant T}$ is a positive martingale with expectation equal to 1, if Q is the probability measure on $(\Omega, \mathcal{A})$ with density Z_T with respect to P we have that the process (M_t) defined by

$$M_t = \int_{\mathbb{R}^n} g(x)\, d\mu_t(x)$$

is a martingale with respect to Q. Using Girsanov's change of measure formula we check that the Itô derivative of M_t is given by the formula

$$dM_t = \Big\langle \int_{\mathbb{R}^n} g(x)(x - a_t)\, d\mu_t(x), dW_t \Big\rangle$$

where $a_t = \int_{\mathbb{R}^n} x\, d\mu_t(x)$ is the barycenter of μ_t.

One can also obtain a concrete description of the law of (ξ_t). It is proved in [80] that (ξ_t) has the same law as the process $(tX + W_t)$ where (W_t) is a standard Brownian motion and X is a random vector which is independent from (W_t) and is distributed according to μ.

§ 7.4. Time reversal. Let ϱ denote the density of μ with respect to the Lebesgue measure. Then, we can reformulate the definition (7.2) of μ_t as follows:

$$\int_{\mathbb{R}^n} g\, d\mu_t = \frac{\int_{\mathbb{R}^n} g(x)\varrho(x) \exp\left(\langle \xi_t, x\rangle - \frac{t}{2}|x|^2\right)\, dx}{\int_{\mathbb{R}^n} \varrho(x) \exp\left(\langle \xi_t, x\rangle - \frac{t}{2}|x|^2\right)\, dx} \tag{7.3}$$

for any test function g. We introduce the heat semi-group

$$P_t f(x) = \mathbb{E}(f(x + W_t)) = f * \gamma_t$$

where $\gamma_t(x) = (2\pi t)^{-n/2} e^{-|x|^2/2t}$ is the density of the Gaussian measure with mean 0 and covariance matrix tI_n. Then (7.3) takes the form

$$\int_{\mathbb{R}^n} g\, d\mu_t = \frac{P_{1/t}(g\varrho)}{P_{1/t}\varrho} \left(\frac{\xi_t}{t}\right).$$

Now set $s = 1/t$. Since the process (ξ_t) has the same law as the process $(tX + W_t)$ we have

$$\frac{\xi_t}{t} = \frac{tX + W_t}{t} = X + sW_{1/s}$$

in law. Since $\tilde{W}_s := sW_{1/s}$ is again a standard Brownian motion we conclude that up to the time reversal $t = 1/s$, the process $(\int g\, d\mu_t)_{t\geqslant 0}$ has the same distribution as $(Q_s g(X + W_s))_{s\geqslant 0}$, where Q_s is the operator defined by

$$Q_s g = \frac{P_s(g\varrho)}{P_s\varrho}.$$

Since the heat semigroup is self-adjoint in $L^2(dx)$ we may also check that

$$Q_s g(X + W_s) = \mathbb{E}\big(g(X) \mid X + W_s\big).$$

Combining the above we see that the stochastic localization process (μ_t) initiated from μ has the same law as the measure-valued process obtained by looking at the conditional law of X given $X + W_s$ and then reversing time by setting $t = 1/s$. In particular, for every test function g the variable $\int_{\mathbb{R}^n} g\, d\mu_t$ has the same law as $\mathbb{E}(g(X) \mid X + \sqrt{s}G)$ where G is a standard Gaussian random vector independent of X.

8 Kannan-Lovász-Simonovits conjecture

Let μ be a Borel probability measure on $\mathbb{R}^n$. For every Borel set $A \subseteq \mathbb{R}^n$ we define the Minkowski content $\mu^+(A)$ as follows:

$$\mu^+(A) = \liminf_{t\to 0^+} \frac{\mu(A_t) - \mu(A)}{t},$$

where $A_t = \{x \in \mathbb{R}^n : \mathrm{dist}(x, A) < t\}$ is the t-extension of A. The isoperimetric ratio of A is defined by $\chi_\mu(A) := \mu^+(A)/\min\{\mu(A), 1-\mu(A)\}$. Then, the (reciprocal) Cheeger constant ψ_μ of μ is the quantity

$$\psi_\mu := \sup_A \frac{1}{\chi_\mu(A)} = \sup_A \frac{\min\{\mu(A), 1-\mu(A)\}}{\mu^+(A)},$$

where the supremum is over all open sets $A \subset \mathbb{R}^n$ with smooth boundary and $0 < \mu(A) < 1$. It follows from results of Rothaus, Cheeger, Maz'ya (see [19]) that if α_μ is the smallest constant with the property that for every integrable, locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ one has

$$\int_{\mathbb{R}^n} |f(x) - \mathbb{E}_\mu(f)|\, d\mu(x) \leqslant \alpha_\mu \int_{\mathbb{R}^n} |\nabla f(x)|\, d\mu(x),$$

then $\alpha_\mu/2 \leqslant \psi_\mu \leqslant 2\alpha_\mu$.

§ 8.1. Isoperimetric profile. The isoperimetric profile $I_\mu : [0,1] \to [0,+\infty]$ of μ is the function

$$I_\mu(t) = \inf\{\mu^+(A) : A \text{ Borel},\ \mu(A) = t\}.$$

Note that

$$\psi_\mu = \sup_{0<t\leqslant 1/2} \frac{t}{\min\{I_\mu(t), I_\mu(1-t)\}}.$$

The next important result of E. Milman, whose proof employs deep tools from geometric measure theory, is very useful.

Theorem 8.1 (E. Milman)**.** *Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then, the isoperimetric profile I_μ of μ is concave on $(0,1)$, and for every $t \in (0,1)$ we have $I_\mu(t) = I_\mu(1-t)$. As a consequence,*

$$\psi_\mu = \sup_{0<t\leqslant 1/2} \frac{t}{I_\mu(t)} = \frac{1}{2I_\mu(1/2)}.$$

The concavity of the isoperimetric profile on a convex domain was first obtained by Sternberg and Zumbrun in [126]. They showed that if $n \geqslant 2$ and K is a convex body in $\mathbb{R}^n$, then I_{λ_K} is concave on $[0,1]$, where λ_K is the uniform probability measure on K. Kuwert later noted in [84] that $I_{\lambda_K}^{n/(n-1)}$ is also concave on $[0,1]$. This is precisely the correct power to use. Kuwert's result was extended to convex domains in Riemannian manifolds with non-negative Ricci curvature by Bayle and Rosales [14]. E. Milman showed in [101] that the concavity of I_μ remains valid for log-concave measures on $\mathbb{R}^n$.

Theorem 8.1 asserts that we can calculate the Cheeger constant of a log-concave probability measure μ by looking only at Borel sets A with $\mu(A) = 1/2$. In fact, E. Milman proved in [101] that one can determines the order of the Poincaré constant of a log-concave probability measure μ on $\mathbb{R}^n$ just by testing 1-Lipschitz functions of the form $x \mapsto d(x, A)$.

Theorem 8.2 (E. Milman)**.** *Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then,*

$$\psi_\mu \approx \sup\left\{ \int d(x,A)\,d\mu(x) : \mu(A) \geqslant \frac{1}{2} \right\}. \tag{8.1}$$

§ 8.2. Kannan-Lovász-Simonovits conjecture. Kannan, Lovász and Simonovits conjectured in [67] that the isoperimetric ratio of any Borel set A with respect to the uniform measure λ_K on a convex body K in $\mathbb{R}^n$ should be, up to an absolute constant, at least as large as the minimal isoperimetric ratio over all half-spaces. Recall that when K is an isotropic convex body in $\mathbb{R}^n$ then $1/L_K$ is approximately equal to the $(n-1)$-dimensional volume of the section of K with any hyperplane passing through the origin: we know that $\mathrm{vol}_{n-1}(K \cap \xi^\perp) \approx 1/L_K$ for every $\xi \in S^{n-1}$. On the other hand, $\mathrm{vol}_{n-1}(K \cap \xi^\perp)$ is the Minkowski content of the intersection of K with the half-space $H_\xi^+ := \{x : \langle x, \xi\rangle \geqslant 0\}$ or $H_\xi^- := \{x \in K : \langle x, \xi\rangle \leqslant 0\}$. Therefore, the KLS conjecture amounts to asking whether $\psi_{\lambda_K} \leqslant cL_K$. Taking into account the isotropic constant conjecture and the different normalization that we have chosen in the definition of isotropic log-concave measures, we can restate the KLS conjecture in the more general setting of isotropic log-concave probability measures as follows:

Conjecture 8.3 (KLS conjecture)**.** *If μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ then the reciprocal Cheeger constant of μ satisfies $\psi_\mu \leqslant C$ for some absolute constant $C > 0$. In other words,*

$$\psi_n := \sup\{\psi_\mu : \mu \textit{ is isotropic log-concave measure on } \mathbb{R}^n\} \leqslant C.$$

Kannan, Lovász and Simonovits [67] gave an upper bound for ψ_{λ_K} in terms of the quantity

$$I_1(K) := \frac{1}{\mathrm{vol}_n(K)} \int_K |x - \mathrm{bar}(K)|\, dx,$$

where $\mathrm{bar}(K)$ is the barycenter of K.

Theorem 8.4 (Kannan-Lovász-Simonovits)**.** *For every convex body K in $\mathbb{R}^n$ one has*

$$\psi_{\lambda_K} \leqslant cI_1(K). \tag{8.2}$$

The proof of Theorem 8.4 exploited the localization lemma of Lovász and Simonovits from [94], a very useful tool that allows one to reduce inequalities for log-concave functions in $\mathbb{R}^n$ to one-dimensional inequalities. Note that, if K is an isotropic convex body in $\mathbb{R}^n$ then

$$I_1(K) \leqslant \Big(\int_K |x|^2 dx\Big)^{1/2} = \sqrt{n} L_K,$$

and hence (8.2) implies that

$$\psi_{\lambda_K} \leqslant c\sqrt{n} L_K, \tag{8.3}$$

where $c > 0$ is an absolute constant.

The KLS-conjecture is stronger than the isotropic constant conjecture. Ball and Nguyen [10] showed that $L_n \leqslant \exp(c\psi_n^2)$ for an absolute constant $c > 0$. In fact, from their work one can conclude that for each individual isotropic log-concave probability measure μ, a bound for the KLS-constant implies a bound for the isotropic constant: one has that $L_\mu \leqslant \exp(c\psi_\mu^2)$. An important result of Eldan and Klartag [43] establishes a linear dependence of L_n on ψ_n.

Theorem 8.5 (Eldan-Klartag)**.** *There exists an absolute constant $C > 0$ such that, for every $n \geqslant 1$,*

$$L_n \leqslant C\psi_n.$$

Theorem 8.5 shows that any upper bound for ψ_n implies an equivalent, up to an absolute constant, upper bound for L_n. In fact, as we will see in this section, Klartag has established in [74] the upper bound $\psi_n \leqslant c\sqrt{\ln n}$.

§ 8.3. Poincaré constant and KLS constant. The Cheeger constant is closely related to another isoperimetric constant associated to μ, the Poincaré constant, defined to be the best, i.e. the smallest, constant ϑ_μ with the property that

$$\mathrm{Var}_\mu(f) = \int_{\mathbb{R}^n} f^2\, d\mu - \Big(\int_{\mathbb{R}^n} f\, d\mu\Big)^2 \leqslant \vartheta_\mu^2 \int_{\mathbb{R}^n} |\nabla f|^2\, d\mu$$

for all locally Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$ with $\int |\nabla f|^2 \, d\mu < \infty$. We shall use the fact that the Poincaré constant is subadditive: if μ, ν are probability measures then $\vartheta_{\mu*\nu}^2 \leqslant \vartheta_\mu^2 + \vartheta_\nu^2$, where $\mu * \nu$ is the convolution measure of μ and ν; this follows by a classical variance decomposition and the convexity of $t \mapsto t^2$ (see e.g. [22]).

A theorem of Maz'ya [99], [100] and Cheeger [37] shows that the Poincaré constant ϑ_μ of μ is bounded by the reciprocal Cheeger constant ψ_μ.

Theorem 8.6 (Maz'ya, Cheeger)**.** *Let μ be a Borel probability measure with reciprocal Cheeger constant ψ_μ. Then its Poincaré constant ϑ_μ satisfies*

$$\vartheta_\mu \leqslant 2\psi_\mu. \tag{8.4}$$

On the other hand, the assumption that μ is log-concave implies a reverse inequality with a constant that does not depend on the dimension. The next theorem is due to Buser [31] (see also Ledoux [86]).

Theorem 8.7 (Buser, Ledoux)**.** *Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then, $\psi_\mu \leqslant c\,\vartheta_\mu$, where $c > 0$ is an absolute constant.*

§ 8.4. Improved log-concave Lichnerowicz inequality. One of the crucial ingredients in Klartag's estimate for ψ_n is an improved log-concave Lichnerowicz inequality which we now describe.

In what follows we assume that μ is a log-concave probability measure with a density $\varrho = e^{-\varphi}$, where φ is a convex function on $\mathbb{R}^n$. There are some issues that we shall ignore in our discussion. What we will do requires that μ is regular. We assume that the density ϱ of μ is smooth and positive on $\mathbb{R}^n$, and there exists $\delta > 0$ such that $\delta I_n \leqslant \nabla^2\varphi \leqslant \delta^{-1} I_n$, where $\nabla^2\varphi$ denotes the Hessian of φ. Also, we assume that φ and all its partial derivatives grow at most polynomially at infinity. One can obtain a reduction to this case, because of the following lemma: For every μ and $\varepsilon > 0$ we may find a regular ν that satisfies $\vartheta_\nu^2 \geqslant \vartheta_\mu^2 - \varepsilon$ and $\|\mathrm{Cov}(\nu) - \mathrm{Cov}(\mu)\|_{\mathrm{op}} \leqslant \varepsilon$. A proof of this fact is given in [74, Lemma 2.1].

We consider the class $\mathcal{F}_\mu$ of all functions u on $\mathbb{R}^n$ which are smooth and have subexponential decay relative to ϱ, i.e. there exist $C, a > 0$ such that

$$|u(x)| \leqslant \frac{C}{\sqrt{\varrho(x)}} e^{-a|x|},$$

where ϱ is the density of μ. We also require that the same is true for the partial derivatives of u. The Laplace-Beltrami operator is defined for $u \in \mathcal{F}_\mu$ by

$$L_\mu u = \Delta u - \langle \nabla\varphi, \nabla u\rangle.$$

The Laplace-Beltrami operator satisfies the basic identity

$$\int_{\mathbb{R}^n} (L_\mu u)\, v \, d\mu = -\int_{\mathbb{R}^n} \langle \nabla u, \nabla v\rangle \, d\mu \tag{8.5}$$

for all $u, v \in \mathcal{F}_\mu$. We will also use Bochner's formula

$$\int_{\mathbb{R}^n} (L_\mu u)^2 \, d\mu = \int_{\mathbb{R}^n} \|\nabla^2 u\|_{\mathrm{HS}}^2 \, d\mu + \int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla u, \nabla u\rangle \, d\mu \tag{8.6}$$

which holds true for every $u \in \mathcal{F}_\mu$. Note that $-L_\mu$ is essentially self-adjoint and positive semi-definite in $\mathcal{F}_\mu \subset L^2(\mu)$, and has a discrete spectrum $\lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$. The minimal eigenvalue λ_0 of L_μ is 0, a simple eigenvalue with constant eigenfunctions. Let $\lambda_\mu = \lambda_1$ denote the smallest positive eigenvalue of L_μ. If $-L_\mu g = \lambda_\mu g$ with $\int g \, d\mu = 0$, then

$$\lambda_\mu \int_{\mathbb{R}^n} g^2 \, d\mu = \int_{\mathbb{R}^n} (-L_\mu g)\, g \, d\mu = \int_{\mathbb{R}^n} |\nabla g|^2 \, d\mu.$$

This shows that $1/\lambda_\mu \leqslant \vartheta_\mu^2$. In fact, we have equality by the next lemma.

Lemma 8.8. *For every log-concave probability measure μ on $\mathbb{R}^n$ we have that*

$$1/\lambda_\mu = \vartheta_\mu^2.$$

Proof. Define the energy functional

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^n} \langle \nabla f, \nabla g\rangle \, d\mu$$

for $f, g \in \mathcal{F}_\mu$. Note that $\mathcal{E}(\cdot,\cdot)$ is linear in each of its arguments and $\mathcal{E}(f,f) \geqslant 0$ for all $f \in \mathcal{F}_\mu$. We can write any $f \in \mathcal{F}_\mu$ as

$$f = \langle f, w_0\rangle w_0 + \sum_{j=1}^{\infty} \langle f, w_j\rangle w_j,$$

where $-L_\mu w_j = \lambda_j w_j$ and $\|w_j\|_{L^2(\mu)} = 1$ for all $j \geqslant 0$. Note that $w_0 \equiv 1$, and hence $\langle f, w_0\rangle = \int f \, d\mu$. For every $s \geqslant 1$ we have that

$$\begin{aligned} 0 &\leqslant \mathcal{E}\left(f - \sum_{j=1}^{s} \langle f, w_j\rangle w_j, f - \sum_{k=1}^{s} \langle f, w_k\rangle w_k \right) \\ &= \mathcal{E}(f,f) - 2\sum_{j=1}^{s} \langle f, w_j\rangle \mathcal{E}(w_j, f) + \sum_{j,k=1}^{s} \langle f, w_j\rangle\langle f, w_k\rangle \mathcal{E}(w_j, w_k). \end{aligned}$$

On observing that

$$\mathcal{E}(w_j, w_k) = \lambda_j \int_{\mathbb{R}^n} w_j w_k \, d\mu = \lambda_j \delta_{j,k}$$

and

$$\mathcal{E}(w_j, f) = \int_{\mathbb{R}^n} \langle \nabla w_j, \nabla f\rangle \, d\mu = \int_{\mathbb{R}^n} (-L_\mu w_j) \, f \, d\mu = \lambda_j \int_{\mathbb{R}^n} w_j f \, d\mu = \lambda_j \langle f, w_j\rangle$$

for all $1 \leqslant j, k \leqslant s$, we conclude that

$$0 \leqslant \mathcal{E}(f,f) - \sum_{j=1}^{s} \lambda_j \langle f, w_j\rangle^2,$$

and this shows that

$$\lambda_1 \mathrm{Var}_\mu(f) = \lambda_1 \left(\int_{\mathbb{R}^n} f^2 \, d\mu - \left(\int_{\mathbb{R}^n} f \, d\mu \right)^2 \right) \leqslant \sum_{j=1}^{\infty} \lambda_j \langle f, w_j\rangle^2 \leqslant \mathcal{E}(f,f) = \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu.$$

It follows that $\vartheta_\mu^2 \leqslant 1/\lambda_1$, and this completes the proof. $\square$

Definition 8.9. We say that a Borel probability measure μ on $\mathbb{R}^n$ with density $\varrho = e^{-\varphi}$ is t-uniformly log-concave if the function $\varrho(x) e^{t|x|^2/2}$ is log-concave. Then, $\varphi(x) - \frac{t|x|^2}{2}$ is convex, that is $\nabla^2 \varphi(x) \geqslant t I_n$ for all $x \in \mathbb{R}^n$.

A theorem of Bakry and Ledoux [4] asserts that if μ is t-uniformly log-concave then

$$\vartheta_\mu^2 = \frac{1}{\lambda_\mu} \leqslant \frac{1}{t}. \tag{8.7}$$

This inequality is known as the log-concave Lichnerowicz inequality, because it is analogous to some investigations of Lichnerowicz [92] in Riemannian geometry (see [5] and [87]). The following stronger "improved log-concave Lichnerowicz inequality" was proved by Klartag in [74].

Theorem 8.10 (Klartag)**.** *Let $t > 0$ and let μ be a t-uniformly log-concave probability measure on $\mathbb{R}^n$. Then,*

$$\vartheta_\mu^2 = \frac{1}{\lambda_\mu} \leqslant \sqrt{\frac{\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}}{t}}. \tag{8.8}$$

Note that (8.8) is stronger than (8.7), since $\|\mathrm{Cov}(\mu)\|_{\mathrm{op}} \leqslant 1/\lambda_\mu$, and hence $\sqrt{\frac{\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}}{t}} \leqslant \frac{1}{\sqrt{\lambda_\mu t}}$. So, from (8.8) we get $\frac{1}{\lambda_\mu} \leqslant \frac{1}{\sqrt{\lambda_\mu t}}$, which in turn gives (8.7). On the other hand, the Kannan-Lóvasz-Simonovits conjecture asks for the estimate

$$\frac{1}{\lambda_\mu} \leqslant c\,\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}$$

for an absolute constant $c > 0$. Therefore, in the context of t-uniformly log-concave measures, (8.8) stands between the Bakry-Ledoux theorem and the Kannan-Lóvasz-Simonovits conjecture.

Equality in Theorem 8.10 is attained when μ is a Gaussian measure, with any covariance matrix. In this case we have that ϑ_μ^2 and $\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}$ coincide, and they also coincide with the inverse lower bound on the Hessian of the potential. If $\gamma_n^{(s)}$ is the distribution of a Gaussian random vector of mean zero and covariance matrix sI_n in $\mathbb{R}^n$ then $\gamma_n^{(s)}$ satisfies the assumptions of Theorem 8.10 for $t = 1/s$ while $\vartheta^2_{\gamma_n^{(s)}} = \|\mathrm{Cov}(\gamma_n^{(s)})\|_{\mathrm{op}} = s$.

The proof of Theorem 8.10 is based on the next lemma.

Lemma 8.11. *Let μ be a regular log-concave probability measure on $\mathbb{R}^n$ with density $\varrho = e^{-\varphi}$, where φ is a convex function. If w_1 is an eigenfunction corresponding to $\lambda_1 = \lambda_\mu$, with $\|w_1\|_{L^2(\mu)} = 1$, then*

$$\int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla w_1, \nabla w_1\rangle \, d\mu \leqslant \lambda_1^3 \|\mathrm{Cov}(\mu)\|_{\mathrm{op}}.$$

Proof. Applying Bochner's formula (8.6) for w_1 we get

$$\lambda_1^2 = \int_{\mathbb{R}^n} (L_\mu w_1)^2 \, d\mu = \int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla w_1, \nabla w_1\rangle \, d\mu + \int_{\mathbb{R}^n} \|\nabla^2 w_1\|_{\mathrm{HS}}^2 \, d\mu.$$

Note that

$$\begin{aligned}
\int_{\mathbb{R}^n} \|\nabla^2 w_1\|_{\mathrm{HS}}^2 \, d\mu &= \sum_{i=1}^n \int_{\mathbb{R}^n} |\nabla \partial^i w_1|^2 \, d\mu \\
&\geqslant \lambda_1 \left(\sum_{i=1}^n \int_{\mathbb{R}^n} |\partial^i w_1|^2 \, d\mu - \sum_{i=1}^n \left| \int_{\mathbb{R}^n} \partial^i w_1 \, d\mu \right|^2 \right) \\
&= \lambda_1 \left(\int_{\mathbb{R}^n} |\nabla w_1|^2 \, d\mu - \left| \int_{\mathbb{R}^n} \nabla w_1 \, d\mu \right|^2 \right) \\
&\geqslant \lambda_1^2 \int_{\mathbb{R}^n} w_1^2 \, d\mu - \lambda_1 \left| \int_{\mathbb{R}^n} \nabla w_1 \, d\mu \right|^2 \\
&= \lambda_1^2 - \lambda_1 \left| \int_{\mathbb{R}^n} \nabla w_1 \, d\mu \right|^2.
\end{aligned}$$

Therefore,

$$\lambda_1^2 \geqslant \int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla w_1, \nabla w_1\rangle \, d\mu + \lambda_1^2 - \lambda_1 \left| \int_{\mathbb{R}^n} \nabla w_1 \, d\mu \right|^2,$$

which implies that

$$\int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla w_1, \nabla w_1\rangle \, d\mu \leqslant \lambda_1 \left| \int_{\mathbb{R}^n} \nabla w_1 \, d\mu \right|^2 . \tag{8.9}$$

Now, let $\xi \in S^{n-1}$ and consider the function $g_\xi(x) = \langle x, \xi\rangle$. Set $E_\xi = \int g_\xi \, d\mu$. We may write

$$\begin{aligned}\left| \left\langle \int_{\mathbb{R}^n} \nabla w_1 \, d\mu, \xi \right\rangle \right|^2 &= \left| \int_{\mathbb{R}^n} \langle \nabla w_1, \nabla g_\xi\rangle \, d\mu \right|^2 = \left| \int_{\mathbb{R}^n} (-L_\mu w_1) g_\xi \, d\mu \right|^2 \\ &= \lambda_1^2 \left| \int_{\mathbb{R}^n} w_1 g_\xi \, d\mu \right|^2 = \lambda_1^2 \left| \int_{\mathbb{R}^n} w_1 (g_\xi - E_\xi) \, d\mu \right|^2 \\ &\leqslant \lambda_1^2 \int_{\mathbb{R}^n} w_1^2 \, d\mu \cdot \int_{\mathbb{R}^n} (g_\xi - E_\xi)^2 \, d\mu = \lambda_1^2 \langle \mathrm{Cov}(\mu)\xi, \xi\rangle .\end{aligned}$$

It follows that

$$\begin{aligned}\left| \int_{\mathbb{R}^n} \nabla w_1 \, d\mu \right|^2 &= \sup_{\xi \in S^{n-1}} \left| \left\langle \int_{\mathbb{R}^n} \nabla w_1 \, d\mu, \xi \right\rangle \right|^2 \leqslant \lambda_1^2 \sup_{\xi \in S^{n-1}} \langle \mathrm{Cov}(\mu)\xi, \xi\rangle \\ &= \lambda_1^2 \|\mathrm{Cov}(\mu)\|_{\mathrm{op}}.\end{aligned} \tag{8.10}$$

Combining (8.9) and (8.10) we obtain the assertion of the lemma. □

Note. For every $\xi \in S^{n-1}$ we have $\nabla g_\xi = \xi$, and hence

$$\langle \mathrm{Cov}(\mu)\xi, \xi\rangle = \mathrm{Var}_\mu(g_\xi) \leqslant \frac{1}{\lambda_1} \int_{\mathbb{R}^n} |\nabla g_\xi|^2 \, d\mu = \frac{1}{\lambda_1}.$$

This shows that

$$\|\mathrm{Cov}(\mu)\|_{\mathrm{op}} \leqslant \frac{1}{\lambda_1}. \tag{8.11}$$

Proof of Theorem 8.10. By Lemma 8.11, using also the assumption that $\nabla^2\varphi \geqslant tI_n$, we have that

$$t\lambda_\mu = t \int_{\mathbb{R}^n} |\nabla w_1|^2 \, d\mu \leqslant \int_{\mathbb{R}^n} \langle (\nabla^2\varphi)\nabla w_1, \nabla w_1\rangle \, d\mu \leqslant \lambda_\mu^3 \|\mathrm{Cov}(\mu)\|_{\mathrm{op}},$$

which implies that

$$\frac{1}{\lambda_\mu^2} \leqslant \frac{\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}}{t}. \tag{8.12}$$

This is the claim of the theorem. □

§ 8.5. Stochastic localization. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ with a regular log-concave density $\varrho = e^{-\varphi}$. Set $\varrho_0 = \varrho$, $\mu_0 = \mu$, $a_0 = \mathrm{bar}(\mu)$, $A_0 = \mathrm{Cov}(\mu) = I_n$ and $\lambda_0 = \lambda_\mu$. Our goal is to show that

$$\lambda_0 \geqslant \frac{c}{\|A_0\|_{\mathrm{op}}} = c,$$

for some absolute constant $c > 0$. For $t \geqslant 0$ and $\xi \in \mathbb{R}^n$ consider the measure $\mu_{t,\xi}$ with density

$$\varrho_{t,\xi}(x) = \frac{1}{Z_{t,\xi}} e^{\langle x, \xi\rangle - t|x|^2/2} \varrho_0(x),$$

where $Z_{t,\xi}$ is a suitable normalizing constant. Note that $\mu_{t,\xi}$ is t-uniformly log-concave and if we set $\lambda_{t,\xi} = \lambda_{\mu_{t,\xi}}$, $A_{t,\xi} = \mathrm{Cov}(\mu_{t,\xi})$ etc., then

$$\lambda_{t,\xi} \geqslant \sqrt{\frac{t}{\|A_{t,\xi}\|_{\mathrm{op}}}} \geqslant t$$

by Theorem 8.10. As in Section 7, we choose ξ_t in a random way, by the stochastic differential equation

$$\xi_0 = 0, \qquad d\xi_t = dW_t + a_{t,\xi_t}dt,$$

where $(W_t)_{t\geqslant 0}$ is a standard Brownian motion in $\mathbb{R}^n$ with $W_0 = 0$. One can check that $(\xi_t)_{t\geqslant 0}$ coincides in law with the process $(tX + W_t)_{t\geqslant 0}$, where $X \sim \mu$ is independent from W_t. We set

$$\mu_t = \mu_{t,\xi_t}, \varrho_t = \varrho_{t,\xi_t}, a_t = \mathrm{bar}(\mu_t), A_t = \mathrm{Cov}(\mu_t), \lambda_t = \lambda_{\mu_t}.$$

From Itô's formula we see that

$$d\varrho_t(x) = \varrho_t(x)\langle x - a_t, dW_t\rangle, \qquad x \in \mathbb{R}^n.$$

The process $(\varrho_t(x))_{t\geqslant 0}$ is a martingale with respect to the filtration induced by the Brownian motion, and in particular

$$\mathbb{E}\big(\varrho_t(x)\big) = \varrho_0(x) = \varrho(x) \tag{8.13}$$

for all $t \geqslant 0$ and $x \in \mathbb{R}^n$. Recall that $A_0 = I_n$ because μ is isotropic, therefore $\|A_0\|_{\mathrm{op}} = 1$. One of the crucial properties of the covariance process $(A_t)_{t\geqslant 0}$ is the following fact.

Theorem 8.12. *For every $0 < t \leqslant c/(\psi_n^2 \ln n)$ we have*

$$\mathbb{E}\,\|A_t\|_{\mathrm{op}} \leqslant C, \tag{8.14}$$

where $C > 0$ is an absolute constant.

A sketch of the proof of Theorem 8.12 will be presented in the Appendix (Section 11). We also need the next lemma.

Lemma 8.13. *Let $f \in L^2(\mu)$. For every $t > 0$ we have that*

$$\mathbb{E}\left(\mathrm{Var}_{\varrho_t}(f)\right) \leqslant \mathrm{Var}_{\varrho_0}(f) \leqslant \left(2 + \frac{t}{\lambda_0}\right)\mathbb{E}\left(\mathrm{Var}_{\varrho_t}(f)\right).$$

Proof. For $f \in L^2(\mu)$ we set

$$M_{t,\xi}(f) = \int_{\mathbb{R}^n} f(x)\varrho_{t,\xi}(x)\,dx.$$

This is a smooth function of ξ, and differentiation under the integral sign shows that

$$\nabla_\xi M_{t,\xi}(f) = \int_{\mathbb{R}^n}(x - a_{t,\xi})f(x)\varrho_{t,\xi}(x)\,dx = \int_{\mathbb{R}^n}(x - a_{t,\xi})(f(x) - M_{t,\xi}(f))\varrho_{t,\xi}(x)\,dx. \tag{8.15}$$

From (8.13) and Fubini's theorem we see that

$$\begin{aligned}\mathrm{Var}_{\varrho_0}(f) &= \int_{\mathbb{R}^n}(f - M_0)^2\varrho_0 = \mathbb{E}\left(\int_{\mathbb{R}^n}(f - M_0)^2\varrho_t\right)\\ &= \mathbb{E}\left(\int_{\mathbb{R}^n}(f - M_t)^2\varrho_t\right) + \mathbb{E}(M_t - M_0)^2.\end{aligned} \tag{8.16}$$

Note that

$$\mathbb{E}\left(\int_{\mathbb{R}^n}(f-M_t)^2\varrho_t\right)=\mathbb{E}\left(\mathrm{Var}_{\varrho_t}(f)\right)\leqslant \mathrm{Var}_{\varrho_0}(f).$$

Next, we observe that

$$\mathbb{E}(M_t-M_0)^2=\mathrm{Var}(M_t)=\mathrm{Var}(M_{t,\xi_t}(f)),$$

where $\xi_t\sim tX+W_t$, where X has distribution μ and W_t is a Gaussian random vector in $\mathbb{R}^n$, independent from X, with mean zero and covariance matrix tI_n. From the subbaditivity property of the Poincaré constant we get

$$\vartheta_{\xi_t}^2=\vartheta_{tX+W_t}^2\leqslant \vartheta_{tX}^2+\vartheta_{W_t}^2=\frac{t^2}{\lambda_0}+t. \tag{8.17}$$

From (8.15) and the Cauchy-Schwartz inequality we see that

$$\begin{aligned}|\nabla_\xi M_{t,\xi}(f)|&=\sup_{u\in S^{n-1}}\int_{\mathbb{R}^n}\langle x-a_{t,\xi},u\rangle(f(x)-M_{t,\xi}(f))\varrho_{t,\xi}(x)\,dx\\&\leqslant\sqrt{\|A_{t,\xi}\|_{\mathrm{op}}}\sqrt{\mathrm{Var}_{\varrho_{t,\xi}}(f)}\end{aligned}$$

for any $\xi\in\mathbb{R}^n$. Using the Poincaré inequality for the random vector ξ_t and (8.17), we get

$$\begin{aligned}\mathrm{Var}(M_{t,\xi_t}(f))&\leqslant\left(1+\frac{t}{\lambda_0}\right)t\mathbb{E}|\nabla_{\xi_t}M_{t,\xi_t}(f)|^2\\&\leqslant\left(1+\frac{t}{\lambda_0}\right)t\mathbb{E}\big(\|A_t\|_{\mathrm{op}}\mathrm{Var}_{\varrho_t}(f)\big)\\&\leqslant\left(1+\frac{t}{\lambda_0}\right)\mathbb{E}\left(\mathrm{Var}_{\varrho_t}(f)\right),\end{aligned} \tag{8.18}$$

where we have also used the fact that $\|A_t\|_{\mathrm{op}}\leqslant 1/t$. Combining (8.16) and (8.18) we see that

$$\mathrm{Var}_{\varrho_0}(f)\leqslant\mathbb{E}\left(\mathrm{Var}_{\varrho_t}(f)\right)+\left(1+\frac{t}{\lambda_0}\right)\mathbb{E}\left(\mathrm{Var}_{\varrho_t}(f)\right)=\left(2+\frac{t}{\lambda_0}\right)\mathbb{E}\left(\mathrm{Var}_{\varrho_t}(f)\right),$$

as claimed. □

A consequence of Lemma 8.13 is the next proposition.

Proposition 8.14. *For any $\alpha>0$ and $0<t\leqslant\alpha\lambda_0$,*

$$\frac{1}{\lambda_0}\leqslant c_1(\alpha+2)\mathbb{E}(1/\lambda_t)\leqslant c_1(\alpha+2)\frac{\mathbb{E}(\sqrt{\|A_t\|_{\mathrm{op}}})}{\sqrt{t}}, \tag{8.19}$$

where $c_1>0$ is an absolute constant.

Proof. Recall E. Milman's Theorem 8.2 which states that

$$c\lambda_0^{-1}\leqslant\sup_{\varphi}\mathrm{Var}_\mu(f), \tag{8.20}$$

where the supremum is over all 1-Lipschitz functions $f:\mathbb{R}^n\to\mathbb{R}$ and $c>0$ is an absolute constant. Therefore, we may choose a 1-Lipschitz function $f:\mathbb{R}^n\to\mathbb{R}$ such that

$$\mathrm{Var}_\mu(f)\geqslant c\lambda_0^{-1}/2. \tag{8.21}$$

Lemma 8.13 and the Poincaré inequality imply that

$$\text{Var}_\mu(f) = \text{Var}_{\varrho_0}(f) \leqslant (2+\alpha)\mathbb{E}\left(\text{Var}_{\varrho_t}(f)\right) \leqslant (2+\alpha)\mathbb{E}\left(\frac{1}{\lambda_t}\int_{\mathbb{R}^n}|\nabla f|^2\varrho_t\right) \leqslant (2+\alpha)\mathbb{E}(\lambda_t^{-1}). \tag{8.22}$$

Now, the left-hand side inequality in (8.19) is a consequence of (8.21) and (8.22), while the right-hand side inequality follows from (8.8). □

§ 8.6. Upper bound for the KLS-constant. We are now ready to prove the main theorem.

Theorem 8.15. *For any $n \geqslant 2$ we have that*

$$\psi_n \leqslant C\sqrt{\ln n},$$

where $C > 0$ is an absolute constant.

Proof. We may choose an isotropic log-concave probability measure μ on $\mathbb{R}^n$ so that

$$\psi_\mu \geqslant \frac{\psi_n}{2}. \tag{8.23}$$

From (8.14) we know that, for $t = c\psi_n^{-2}/\ln n$,

$$\mathbb{E}\left(\sqrt{\|A_t\|_{\text{op}}}\right) \leqslant \sqrt{\mathbb{E}\|A_t\|_{\text{op}}} \leqslant C_1, \tag{8.24}$$

where $C_1 > 0$ is an absolute constant. By the definition of t and the fact that $\psi_n \approx \psi_\mu \approx \vartheta_\mu$ and $\lambda_0 = \vartheta_\mu^{-2}$ we get

$$t \leqslant c_1\psi_n^{-2} \leqslant c_2\psi_\mu^{-2} \leqslant c_3\lambda_0. \tag{8.25}$$

Then, we can apply Proposition 8.14 with $\alpha = c_3$. Combining (8.19) with (8.24) and the choice of t gives

$$\lambda_0^{-1} \leqslant C_2\frac{\mathbb{E}(\sqrt{\|A_t\|_{\text{op}}})}{\sqrt{t}} \leqslant \frac{C_3}{\sqrt{t}} \leqslant C_4\psi_n\sqrt{\ln n}. \tag{8.26}$$

Then, from (8.23), (8.26) and the equivalence $\lambda_0^{-1} = \vartheta_\mu^2 \approx \psi_\mu^2$ we get

$$\psi_n^2 \leqslant 4\psi_\mu^2 \leqslant C_5\lambda_0^{-1} \leqslant C_6\psi_n\sqrt{\ln n},$$

which shows that $\psi_n \leqslant C\sqrt{\ln n}$ for an absolute constant $C > 0$. □

9 The slicing theorem

Our aim in this section is to describe the affirmative answer to the slicing problem. More precisely, we shall show that the equivalent isotropic constant conjecture is true.

Theorem 9.1. *There exists an absolute constant $C > 0$ such that $L_n \leqslant C$ for all $n \geqslant 1$.*

We shall present the proof of P. Bizeul [15] which makes use of the reduction to small ball estimates that we presented in Section 6. In what follows, we say that a random vector X in $\mathbb{R}^n$ is b-subgaussian if for every $p \geqslant 1$ and any $\xi \in S^{n-1}$

$$\left(\mathbb{E}\,|\langle X - \mathbb{E}(X), \xi\rangle|^p\right)^{1/p} \leqslant b\sqrt{p}.$$

The small ball estimate of Theorem 6.8 can be also stated for a not necessarily isotropic log-concave random vector. Then, it takes the following form (see Paouris [113]).

Theorem 9.2 (Paouris). *Let X be a b-subgaussian log-concave random vector in $\mathbb{R}^n$ with covariance matrix A. For any $0 < \varepsilon \leqslant c_0$ and any $y \in \mathbb{R}^n$,*

$$\mathbb{P}\left(|X-y|^2 \leqslant \varepsilon\, \mathrm{tr}(A)\right) \leqslant \varepsilon^{\frac{c_0 \mathrm{tr}(A)}{b^2 \|A\|_{\mathrm{op}} \|A^{-1}\|_{\mathrm{op}}}}$$

where $c_0 > 0$ is an absolute constant.

One can check that the theorem is tight, up to absolute constants, by considering the case where X is a standard Gaussian random vector. However, for general isotropic log-concave random vectors, it provides a suboptimal exponent of order $\sqrt{n}$. Lee and Vempala [90] obtained an improved estimate using stochastic localization.

Theorem 9.3 (Lee-Vempala). *Let X be an isotropic log-concave random vector in $\mathbb{R}^n$. For any $y \in \mathbb{R}^n$ and any $0 < \varepsilon \leqslant c_0$,*

$$\mathbb{P}\left(|X-y|^2 \leqslant \varepsilon n\right) \leqslant \varepsilon^{\frac{c_0 n}{\psi_n^2 \ln(n)}}$$

where $c_0 > 0$ is an absolute constant.

In the previous section we discussed the best known bound for the KLS constant, due to Klartag [74]: we saw that

$$\psi_n^2 \leqslant C \ln n$$

for some absolute constant $C > 0$. Inserting this estimate into Theorem 9.3 we get an exponent which is of the order of n up to a factor $(\ln n)^2$. However, even if we knew that the KLS conjecture has an affirmative answer, the estimate of Lee and Vempala would still include a suboptimal factor $\ln n$.

§ 9.1. The slicing theorem. Following Bizeul [15] we shall see how one can remove the extra logarithmic factor that appears in Theorem 9.3 and establish an optimal small ball estimate.

Theorem 9.4. *Let X be an isotropic log-concave random vector in $\mathbb{R}^n$. For any $0 < \varepsilon \leqslant c_0$ and any $y \in \mathbb{R}^n$,*

$$\mathbb{P}\left(|X-y|^2 \leqslant \varepsilon n\right) \leqslant \varepsilon^{c_0 n}$$

where $c_0 > 0$ is an absolute constant.

Then, we will show that Theorem 9.4 implies the isotropic constant conjecture. In Section 6 we described in detail the work of Dafnis and Paouris and the reduction of the isotropic constant conjecture to the optimal small ball estimate. Theorem 9.4 is actually equivalent to Theorem 9.1. For the inverse direction note that if L_n is bounded above by some constant $C_1 > 0$ then we have seen that the same holds true for $\tilde{L}_n$. This means that for any isotropic log-concave random vector X with density f and any $\varepsilon > 0$ and $y \in \mathbb{R}^n$, we have that

$$\mathbb{P}\left(|X-y|^2 \leqslant \varepsilon n\right) = \int_{B(y,\sqrt{\varepsilon n})} f(x)\, dx \leqslant C_1^n \mathrm{vol}_n(\sqrt{\varepsilon n} B_2^n) \leqslant (C_2 \varepsilon)^{n/2},$$

which implies Theorem 9.4.

We start with some preliminary facts that will be used in the proof. The first one is a variant of Theorem 9.2.

Lemma 9.5. *Let X be a b-subgaussian log-concave random vector with covariance matrix A. Then, for any $y \in \mathbb{R}^n$ and any $0 < \varepsilon \leqslant c_0$ we have that*

$$\mathbb{P}\left(|X-y|^2 \leqslant \varepsilon\, \mathrm{tr}(A)\right) \leqslant \left(\frac{4n\|A\|_{\mathrm{op}}}{\mathrm{tr}(A)} \varepsilon\right)^{\frac{c_0 \mathrm{tr}(A)^3}{8n^2 b^2 \|A\|_{\mathrm{op}}^2}}$$

where $c_0 > 0$ is the constant from Theorem 9.2.

Proof. Let $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ be the eigenvalues of the matrix A. For any $2 \leqslant k \leqslant n$ we have that

$$n\lambda_k \geqslant \sum_{i=k}^{n} \lambda_i = \mathrm{tr}(A) - \sum_{i=1}^{k-1} \lambda_i \geqslant \mathrm{tr}(A) - k\lambda_1. \tag{9.1}$$

We choose

$$k = \max\left\{ \left\lfloor \frac{\mathrm{tr}(A)}{2\lambda_1} \right\rfloor, 1 \right\}.$$

If $k \geqslant 2$ then (9.1) implies that

$$\lambda_k \geqslant \frac{\mathrm{tr}(A)}{2n},$$

which remains true when $k = 1$. Let E be the k-dimensional subspace associated to the eigenvalues $\lambda_1, \ldots, \lambda_k$, and let $X_E = P_E X$ be the projection of X onto E, and $A_E = P_E A P_E$ the covariance matrix of X_E. Then, X_E is b-subgaussian and its covariance matrix satisfies :

$$\|A_E\|_{\mathrm{op}} \leqslant \|A\|_{\mathrm{op}}, \qquad \|A_E^{-1}\|_{\mathrm{op}} \leqslant \frac{2n}{\mathrm{tr}(A)}, \qquad \mathrm{tr}(A_E) \geqslant \frac{k}{n}\mathrm{tr}(A) \geqslant \frac{\mathrm{tr}(A)^2}{4n\|A\|_{\mathrm{op}}}.$$

Let $y \in \mathbb{R}^n$ and define $y_E = P_E y$. If $0 < \varepsilon \leqslant c_0$ then

$$\begin{aligned}
\mathbb{P}\left(|X - y|^2 \leqslant \varepsilon\, \mathrm{tr}(A)\right) &\leqslant \mathbb{P}\left(|X_E - y_E|^2 \leqslant \varepsilon \mathrm{tr}(A)\right) \\
&\leqslant \left(\frac{\varepsilon\, \mathrm{tr}(A)}{\mathrm{tr}(A_E)}\right)^{\frac{c_0 \mathrm{tr}(A_E)}{b^2 \|A_E\|_{\mathrm{op}} \|A_E^{-1}\|_{\mathrm{op}}}} \\
&\leqslant \left(\frac{4n\varepsilon \|A\|_{\mathrm{op}}}{\mathrm{tr}(A)}\right)^{\frac{c_0 \mathrm{tr}(A)^3}{8b^2 n^2 \|A\|_{\mathrm{op}}^2}}
\end{aligned}$$

where the second inequality follows by applying Theorem 9.2 to X_E. □

We shall also use some classical facts about log-concave vectors. The next lemma is a special case of (4.3).

Lemma 9.6. *There exists an absolute constant $\kappa_0 > 1$ such that for any log-concave real random variable Y and any $p \geqslant 2$*

$$\left(\mathbb{E}|Y|^p\right)^{1/p} \leqslant \kappa_0 p\, \mathbb{E}\left(|Y|^2\right)^{1/2}.$$

Recall also that a random vector, or its density, is called t-uniformly log-concave for some $t > 0$ if it is log-concave with respect to a Gaussian of variance $\frac{1}{t}$. The next lemma follows from a log-concave/convex correlation inequality of Hargé [65].

Lemma 9.7. *Let X be a t-uniformly log-concave random vector. Then it is $\frac{1}{\sqrt{t}}$-subgaussian.*

Proof. Without loss of generality we may assume that $t = 1$ so that X is log-concave with respect to the standard Gaussian measure denoted by γ_n. We may also assume that $\mathbb{E}(X) = 0$. Hargé's result states that if f is log-concave and g is convex then

$$\mathrm{Cov}_{\gamma_n}(f, g) \leqslant 0. \tag{9.2}$$

Let f be the relative density of X with respect to γ_n. Applying (9.2) with $g(x) = |\langle x, \xi \rangle|^p$ for all $p \geqslant 1$ and all $\xi \in S^{n-1}$ we obtain the result. □

§ 9.2. Reduction to small diameter. A first observation, which has been also used in previous approaches to the problem, is that in order to prove Theorem 9.4 we may assume that X is supported in a ball of radius of the order of $\sqrt{n}$ and that $y = 0$. In this subsection we explain this reduction.

Lemma 9.8. *Let X be an isotropic log-concave random vector in $\mathbb{R}^n$. There exists an isotropic log-concave random vector Y which has support inside the ball $B(0,\sqrt{c_1 n})$, where $c_1>0$ is an absolute constant, and satisfies*

$$\mathbb{P}\left(|X-y|\leqslant \varepsilon\sqrt{n}\right)^2 \leqslant \mathbb{P}\left(|Y|\leqslant 2\varepsilon\sqrt{n}\right)$$

for every $0<\varepsilon<1$ and any $y\in\mathbb{R}^n$.

Proof. We consider the symmetrized version $X_1=\frac{X-\tilde{X}}{\sqrt{2}}$ of X, where $\tilde{X}$ is an independent copy of X. Then, X_1 is an isotropic symmetric log-concave random vector with the property that, for any $r>0$,

$$\begin{aligned}\mathbb{P}\left(|X-y|\leqslant r\right) &= \left[\mathbb{P}\left(|X-y|\leqslant r,\ |\tilde{X}-y|\leqslant r\right)\right]^{1/2}\\ &\leqslant \left[\mathbb{P}\left(\left|\tfrac{1}{\sqrt{2}}(X-\tilde{X})\right|\leqslant \sqrt{2}r\right)\right]^{1/2}\\ &= \left[\mathbb{P}\left(|X_1|\leqslant\sqrt{2}r\right)\right]^{1/2}.\end{aligned} \tag{9.3}$$

We set $K=B(0,8\kappa_0^2\sqrt{n})=\{x\in\mathbb{R}^n : |x|\leqslant 8\kappa_0^2\sqrt{n}\}$, where $\kappa_0\geqslant 1$ is the constant from Lemma 9.6, and define the random vector

$$X_2 = X_1\cdot \mathbb{1}_{X_1\in K}.$$

If f_1 is the density of X_1, then X_2 has density

$$f_2=\frac{f_1\mathbb{1}_K}{\int_K f_1}\leqslant 2f_1\mathbb{1}_K, \tag{9.4}$$

where the inequality above holds because $\int_K f_1\geqslant \frac{1}{2}$. To see this, we first use the fact that X_1 is isotropic and Markov's inequality to write

$$\mathbb{P}(X_1\in K^c)=\mathbb{P}(|X_1|^2>64\kappa_0^4 n)\leqslant \frac{1}{64\kappa_0^4 n}\mathbb{E}|X_1^2| = \frac{1}{64\kappa_0^4}<\frac{1}{2}, \tag{9.5}$$

and this implies that

$$\int_K f_1 = 1-\mathbb{P}(X_1\in K^c)\geqslant 1-\frac{1}{2}=\frac{1}{2}.$$

We know that X_2 is symmetric, and hence, from (9.4) we get

$$\mathrm{Cov}(X_2)\leqslant 2I_n.$$

In fact, we claim that

$$\frac{1}{2}I_n\leqslant \mathrm{Cov}(X_2)\leqslant 2I_n. \tag{9.6}$$

To see the left-hand side inequality, for any $\xi\in S^{n-1}$ we write

$$\begin{aligned}\int_{\mathbb{R}^n}\langle x,\xi\rangle^2 f_2(x)\,dx &\geqslant \int_K \langle x,\xi\rangle^2 f_1(x)\,dx\\ &= 1-\int_{K^c}\langle x,\xi\rangle^2 f_1(x)\,dx\\ &\geqslant 1-\mathbb{P}(X_1\in K^c)^{1/2}\left(\int_{\mathbb{R}^n}\langle x,\xi\rangle^4 f_1(x)\,dx\right)^{1/2}\\ &\geqslant 1-\frac{1}{8\kappa_0^2}\,(2\kappa_0)^2\int_{\mathbb{R}^n}\langle x,\xi\rangle^2 f_1(x)\,dx=\frac{1}{2},\end{aligned}$$

where we have used the Cauchy-Schwarz inequality, Lemma 9.6 and (9.5). Finally, we set

$$X_3 = \mathrm{Cov}(X_2)^{-1/2} X_2.$$

Note that X_3 is an isotropic log-concave random vector by construction. Furthermore, using (9.6), we see that it has support inside the ball $B(0, 8\sqrt{2}\kappa_0^2\sqrt{n})$. For any $0 \leqslant \varepsilon < 1$, setting $r = \varepsilon\sqrt{n}$ we get

$$\begin{aligned} \mathbb{P}\left(X \in B(y,r)\right)^2 &\leqslant \mathbb{P}\left(X_1 \in B(0, \sqrt{2}r\right) \\ &\leqslant \mathbb{P}\left(X_2 \in B(0, \sqrt{2}r\right) \\ &= \mathbb{P}\left(X_3 \in \mathrm{Cov}(X_2)^{-1/2} B(0, \sqrt{2}r)\right) \\ &\leqslant \mathbb{P}\left(X_3 \in B(0, 2r)\right) \end{aligned} \tag{9.7}$$

where we used successively (9.3), the fact that $B(0, \sqrt{2}r) \subset K = B(0, 8\kappa_0^2\sqrt{n})$ and in the last line the inequality (9.6). □

It is now clear from (9.7) that in order to prove Theorem 9.4 we may assume X has support inside a ball of diameter $\sqrt{c_1 n}$, where $c_1 > 0$ is an absolute constant.

§ 9.3. Bounds for the shrinkage of sets. The main idea for the proof of Theorem 9.4 is again to use stochastic localization as in the previous section. We recall some basic facts from Section 7. If μ is a log-concave probability measure with density ϱ with respect to the Lebesgue measure, for $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ we define a density

$$\varrho_{t,\xi}(x) = \frac{1}{Z_{t,\xi}} e^{\langle x,\xi\rangle - \frac{t}{2}|x|^2} \varrho(x), \qquad x \in \mathbb{R}^n \tag{9.8}$$

where $Z_{t,\xi}$ is a normalizing factor. Note that $\varrho_{t,\xi}$ is t-uniformly log-concave. We also define

$$a_{t,\xi} = \int_{\mathbb{R}^n} x \varrho_{t,\xi}(x)\, dx$$

and

$$A_{t,\xi} = \int_{\mathbb{R}^n} (x - a_{t,\xi})^{\otimes 2} \varrho_{t,\xi}(x)\, dx,$$

the barycenter and covariance matrix of $\varrho_{t,\xi}$ respectively. The tilt process $(\xi_t)_{t\geqslant 0}$ is defined as the solution of the stochastic differential equation:

$$d\xi_t = a_{t,\xi_t} dt + dW_t \tag{9.9}$$

where $(W_t)_{t\geqslant 0}$ is a standard Brownian motion. The process ϱ_{t,ξ_t} is the stochastic localization process starting at ϱ. We abbreviate $\varrho_t = \varrho_{t,\xi_t}$ and similarly for the barycenter and covariance, a_t and A_t. We also define μ_t to be the random measure with density ϱ_t. For any $x \in \mathbb{R}^n$, ϱ_t is an Itô process satisfying the stochastic differential equation

$$d\varrho_t(x) = \langle x - a_t, dW_t\rangle \varrho_t(x). \tag{9.10}$$

Integrating (9.10) we see that for any integrable function φ, the process $(\int_{\mathbb{R}^n} \varphi d\mu_t)_{t\geqslant 0}$ is a martingale. In particular, for any $t \geqslant 0$

$$\mathbb{E}_\mu(\varphi) = \mathbb{E}\big(\mathbb{E}_{\mu_t}(\varphi)\big). \tag{9.11}$$

Therefore, for any $t \geqslant 0$ we have decomposed the original log-concave probability measure μ into a mixture of measures μ_t which are t-uniformly log-concave.

By the log-concave Lichnerowicz inequality (8.7) we have that

$$A_t \leqslant \frac{1}{t} I_n \tag{9.12}$$

almost surely, in the sense of symmetric matrices. Using Klartag's improved log-concave Lichnerowicz inequality of Theorem 8.10, Guan [62] obtained a lower-bound on the trace of A_t up to a time of order 1:

Theorem 9.9. *Assume that the starting measure μ is an isotropic log-concave probability measure. Then,*

$$\mathbb{E}\big(\mathrm{tr}(A_{\delta_1})\big) \geqslant \delta_1 n,$$

where $\delta_1 > 0$ is an absolute constant.

A sketch of the proof of Theorem 9.9 will be presented in the Appendix (Section 11).

Our goal in this subsection is to obtain estimates for the evolution of the measure of sets along the stochastic localization process under the additional assumption that μ has bounded support of diameter D.

Proposition 9.10. *Let μ be an isotropic log-concave probability measure with bounded support of dimameter D. For any measurable set S and any $\lambda > 1$ we have that*

$$\mu(S) \leqslant e^{\frac{D^2 t}{2}} \mu_t(S)^{\frac{1}{\lambda}}$$

with probability at least $1 - \frac{1}{\lambda}$, where $(\mu_t)_{t\geqslant 0}$ is the stochastic localization process starting at μ.

Proof. Let $S \subset \mathbb{R}^n$ be a set of positive measure and let $g_t = \mu_t(S)$. Using (9.10), we can check that

$$dg_t = \Big\langle \int_S (x - a_t) d\mu_t(x), dW_t \Big\rangle.$$

Then,

$$d[g]_t = \left| \int_S (x - a_t) d\mu_t(x) \right|^2 dt.$$

Note that μ_t is supported in the same set as μ, which has diameter D, and hence

$$\left| \int_S (x - a_t) d\mu_t(x) \right|^2 \leqslant \sup_{x \in \mathrm{supp}(\mu_t)} |x - a_t|^2 \left(\int_S d\mu_t(x) \right)^2 \leqslant D^2 \mu_t(S)^2 = D^2 g_t^2.$$

It follows that

$$d[g]_t \leqslant D^2 g_t^2 dt. \tag{9.13}$$

Using Itô's formula (7.1) we compute $d\big(\ln(g_t^{-1})\big) = -\frac{dg_t}{g_t} + \frac{1}{2}\frac{d[g]_t}{g_t^2}$, which implies that $d\mathbb{E}\big(\ln(g_t^{-1})\big) \leqslant \frac{D^2 t}{2}$, and hence

$$\mathbb{E}\big(\ln(g_t^{-1})\big) \leqslant \ln\left(g_0^{-1}\right) + \frac{D^2 t}{2}. \tag{9.14}$$

Now, let $\lambda > 1$. From (9.14), using Markov's inequality, we see that with probability at least $1 - \frac{1}{\lambda}$

$$\ln(g_t^{-1}) \leqslant \lambda \left(\ln(g_0^{-1}) + \frac{D^2 t}{2} \right).$$

Equivalently,

$$\ln g_0 \leqslant \frac{1}{\lambda} \ln g_t + \frac{D^2 t}{2}$$

and the result follows. □

§ 9.4. Proof of the slicing theorem. We start with the proof of the optimal small ball estimate of Theorem 9.4.

Proof of Theorem 9.4. Lemma 9.8 shows that it is enough to consider an isotropic log-concave random vector X distributed according to an isotropic log-concave probability measure μ with support of diameter $\sqrt{c_1 n}$, where $c_1 > 0$ is an absolute constant. Let $(\mu_t)_{t\geqslant 0}$ be the stochastic localization process starting at μ and let X_t be the random vector with law μ_t. Let δ_1 be the constant in Guan's Theorem 9.9. We consider the event

$$E_0 = \{\mathrm{tr}(A_{\delta_1}) \geqslant \delta_1 n/2\}. \tag{9.15}$$

Lemma 9.11. *We have* $\mathbb{P}(E_0) \geqslant \delta_1^2/2$.

Proof. We set $p_0 = \mathbb{P}(E_0)$. From (9.12) we know that $A_{\delta_1} \leqslant \frac{1}{\delta_1} I_n$ almost surely, and hence

$$\delta_1 n \leqslant \mathbb{E}\left(\mathrm{tr}(A_{\delta_1})\right) = \mathbb{E}\left(\mathrm{tr}(A_{\delta_1})\mathbb{1}_{E_0}\right) + \mathbb{E}\left(\mathrm{tr}(A_{\delta_1})\mathbb{1}_{E_0^c}\right) \leqslant \frac{np_0}{\delta_1} + \frac{\delta_1 n}{2},$$

which implis that $p_0 \geqslant \delta_1^2/2$. □

Let $\bar{c}_0 > 0$ be the constant from Lemma 9.5. For every $0 < \varepsilon < \bar{c}_0$ we set $S_\varepsilon = B(0, \sqrt{\varepsilon n})$. We also fix $\lambda = \frac{4}{\delta_1^2}$. Lemma 9.10 shows that the probability p_1 of the event

$$E_1 = \left\{\mu(S_\varepsilon) \leqslant e^{\frac{c_1^2\delta_1 n}{2}} \mu_{\delta_1}(S_\varepsilon)^{\frac{1}{\lambda}}\right\}$$

satisfies $p_1 \geqslant 1 - \frac{\delta_1^2}{4}$, therefore

$$\mathbb{P}(E_0 \cap E_1) \geqslant p_0 + p_1 - 1 \geqslant \frac{\delta_1^2}{4} > 0.$$

On the event $E_0 \cap E_1$ we have $\mathrm{tr}(A_{\delta_1}) \geqslant \delta_1 n/2$, and also $\|A_{\delta_1}\|_{\mathrm{op}} \leqslant 1/\sqrt{\delta_1}$. Moreover, X_{δ_1} is $1/\sqrt{\delta_1}$-subgaussian by Lemma 9.7. Inserting all these estimates into Lemma 9.5 we get that

$$\mu(S_\varepsilon) \leqslant e^{\frac{c_1^2\delta_1 n}{2}} \mu_{\delta_1}(S_\varepsilon)^{\frac{4}{\delta_1^2}} \leqslant e^{\frac{c_1^2\delta_1 n}{2}} \left(\frac{8\varepsilon}{\delta_1^2}\right)^{\frac{\bar{c}_0\delta_1^6 n}{64}} \leqslant \varepsilon^{\frac{\bar{c}_0\delta_1^6 n}{128}}$$

if we assume that

$$\varepsilon < \varepsilon_0 := \min\left\{\bar{c}_0, \frac{\delta_1^4}{64}\exp\left(-\frac{64c_1^2}{\bar{c}_0\delta_1^5}\right)\right\}.$$

This establishes Theorem 9.4 with $c_0 = \min\left\{\bar{c}_0, \frac{\delta_1^4}{64}\exp\left(-\frac{64c_1^2}{\bar{c}_0\delta_1^5}\right), \frac{\bar{c}_0\delta_1^6}{128}\right\}$. □

Proof of Theorem 9.1. Basically, we repeat the proof of Theorem 6.15. From Proposition 6.14 we know that there exists an isotropic convex body K in $\mathbb{R}^n$ with $L_n \leqslant C_1 L_K$ and $\mathrm{vol}_n(K \cap \sqrt{C_2 n}B_2^n) > c_2^n$ for some absolute constants $C_1, C_2, c_2 > 0$.

Theorem 9.4 shows that there exists an absolute constant $c_0 > 0$ such that for any $0 < \varepsilon \leqslant c_0$,

$$\mathrm{vol}_n(K \cap \varepsilon\sqrt{n}L_K\, B_2^n) \leqslant \varepsilon^{c_0 n}.$$

Choosing $\varepsilon_1 = \min\left\{c_0, c_2^{\frac{1}{c_0}}\right\}$, and comparing the above inequalities, we get $L_K \leqslant \sqrt{C_2}/\varepsilon_1$, and hence

$$L_n^2 \leqslant C_1^2 L_K^2 \leqslant \frac{C_1^2 C_2}{\varepsilon_1^2}.$$

This shows that L_n is bounded by an absolute constant. □

§ 9.5. The thin-shell conjecture. A third well-known conjecture about isotropic log-concave probability measures is the thin-shell conjecture which has its origin in the central limit problem, the question to identify those high-dimensional distributions that have approximately Gaussian marginals. A typical example is given by the random vector $X = (X_1, \dots, X_n)$ which is distributed uniformly in the cube $Q(n) = [-\sqrt{3}, \sqrt{3}]^n$ (the normalization is so that $\mathrm{Var}(X_j^2) = 1$ for all $1 \leqslant j \leqslant n$). It is well-known that, if the ξ_j's satisfy e.g. Lindeberg's condition, then the distribution of

$$\langle X, \xi\rangle = \sum_{j=1}^{n} \xi_j X_j$$

is approximately Gaussian. A second example is given by the ball $D(n) = \sqrt{n+2}B_2^n$. Let X be a random vector which is uniformly distributed in $D(n)$. From Maxwell's observation that, if n is large enough, then

$$\sigma\left(\{\xi \in S^{n-1} : \xi_j \leqslant t\}\right) \sim \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{t} \exp(-s^2 n/2)\, ds$$

for all $t \in [-1, 1]$, as well as the symmetry of $D(n)$, one can check that the distribution of $\langle X, \xi\rangle$ is close to the standard normal distribution for any $\xi \in S^{n-1}$.

Assume now that μ is an isotropic Borel probability measure on $\mathbb{R}^n$, i.e. normalized so that

$$\mathbb{E}_\mu(x_j) = 0 \quad \text{and} \quad \mathbb{E}_\mu(x_i x_j) = \delta_{ij}, \qquad i, j = 1, \dots, n.$$

It has been observed that if μ satisfies a "thin shell bound" then the central limit problem has an affirmative answer for μ. More precisely, if

$$\mu\left(\left\{x \in \mathbb{R}^n : \big|\, |x| - \sqrt{n}\, \big| \geqslant \varepsilon\sqrt{n}\right\}\right) \leqslant \varepsilon$$

for some $\varepsilon \in (0, 1/2)$, then, for all directions ξ in a subset A of S^{n-1} of measure $\sigma(A) \geqslant 1 - \exp(-c_1\sqrt{n})$, we have

$$|\mathbb{P}\left(\langle X, \xi\rangle \leqslant t\right) - \Phi(t)| \leqslant c_2(\varepsilon + n^{-\alpha}) \qquad \text{for all } t \in \mathbb{R},$$

where $\Phi(t)$ is the standard Gaussian distribution function and $c_1, c_2, \alpha > 0$ are absolute constants.

Note that in the statement above we have assumed that the dimension is large enough but we have not assumed independence of the coordinate functions $x \mapsto x_j$ and we have not made any symmetry assumptions about μ. Sudakov's work [127] is probably the first place where it is observed that a thin-shell condition implies that most marginals of a high-dimensional distribution are approximately Gaussian. Related early works are the ones by Diaconis and Freedman [40], and by von Weizsäker [130]. The case where $\mu = \mu_K$ is the measure with density $L_K^n \mathbb{1}_{\frac{K}{L_K}}$ for an isotropic symmetric convex body K in $\mathbb{R}^n$ was studied by Anttila, Ball and Perissinaki in [1] who showed that a thin-shell estimate implies an affirmative answer to the central limit problem for μ_K. This work made the central limit problem widely known among people working in convex geometry. A clear exposition of both the general and the log-concave case can be found in Bobkov's article [18].

As Klartag notes in [73], it is not hard to construct simple examples of isotropic distributions for which a thin-shell estimate is not possible. If we write σ_t for the uniform probability measure on the sphere tS^{n-1} then, for $t_1 = \sqrt{n}/2$ and $t_2 = \sqrt{7n}/2$, the isotropic probability measure $\mu = \frac{1}{2}(\sigma_{t_1} + \sigma_{t_2})$ does not have Gaussian marginals, and hence it does not satisfy a thin shell bound. However, it was conjectured that the assumption of log-concavity guarantees a thin-shell bound. Bobkov and Koldobsky [20] defined the parameter

$$\sigma_\mu^2 = \frac{1}{n}\mathrm{Var}_\mu(|x|^2)$$

and asked if

$$\sigma_n = \sup\{\sigma_\mu : \mu \text{ is an isotropic log-concave probability measure on } \mathbb{R}^n\} \leqslant C$$

for some absolute constant $C>0$. It is easily checked that this is equivalent to the question if there exists an absolute constant $C>0$ such that, for any $n\geqslant 1$ and any isotropic log-concave probability measure μ on $\mathbb{R}^n$, one has

$$\mathbb{E}_\mu(|x|-\sqrt{n})^2\leqslant C^2.$$

Moreover, applying the Poincaré inequality for the function $f(x)=|x|^2$ we see that

$$\mathrm{Var}_\mu(|x|^2)\leqslant 4\vartheta_\mu^2\int_{\mathbb{R}^n}|x|^2d\mu(x)=4\vartheta_\mu^2 n,$$

which shows that $\sigma_\mu\leqslant 2\vartheta_\mu$, and hence

$$\sigma_n\leqslant C\psi_n$$

for all n, where $C>0$ is an absolute constant. In other words the KLS conjecture implies the thin-shell conjecture. An important fact is that the thin-shell conjecture is also related to the isotropic constant conjecture. Eldan and Klartag [43] proved that the there exists an absolute constant $C>0$ such that

$$L_n\leqslant C\sigma_n$$

for every $n\geqslant 1$.

The first non-trivial upper bound for σ_n was given by Klartag in [70] in his proof of the central limit theorem for convex bodies. He proved that

$$\sigma_n\leqslant C\sqrt{n}/\ln n.$$

This estimate was then improved to $\sigma_n\leqslant Cn^{2/5+o(1)}$ by Klartag [71], to $\sigma_n\leqslant Cn^{3/8}$ by Fleury [46], to $\sigma_n\leqslant Cn^{1/3}$ by Guédon and E. Milman [63], and to $\sigma_n\leqslant Cn^{1/4}$ by Lee and Vempala [89]. A consequence of the more recent developments on the KLS conjecture is that $\sigma_n\leqslant C\sqrt{\ln n}$ by the corresponding bound $\psi_n=O(\sqrt{\ln n})$ of Klartag in [74]. An even more recent breakthrough of Guan [62] showed that

$$\sigma_n\leqslant C\ln\ln n.$$

Finally, Klartag and Lehec [79] announced an affirmative answer to the thin-shell conjecture.

Theorem 9.12 (Klartag-Lehec)**.** *There exists an absolute constant $C>0$ such that*

$$\mathrm{Var}_\mu(|x|^2)\leqslant Cn$$

for every $n\geqslant 1$ and every isotropic log-concave probability measure μ on $\mathbb{R}^n$. Equivalently, $\sup_n\sigma_n\leqslant C$.

Using reverse Hölder inequalities for polynomials of a random vector distributed uniformly in a convex body (see [26], [110]) one can deduce from Theorem 9.12 that, for any isotropic log-concave probability measure μ on $\mathbb{R}^n$ and for any $t>0$,

$$\mu(\{x\in\mathbb{R}^n:\big|\,|x|-\sqrt{n}\,\big|\geqslant t\})\leqslant c_1\exp(-c_2\sqrt{t}).$$

This is a thin-shell estimate because it shows that if $1\ll t\ll\sqrt{n}$ then most of the mass of μ is concentrated on a thin spherical cell $\{x\in\mathbb{R}^n:\sqrt{n}-t\leqslant|x|\leqslant\sqrt{n}+t\}$ whose width t is much smaller than the central radius $\sqrt{n}$.

The proof of Theorem 9.12 starts from ideas that Klartag used in [72], where he established an optimal thin-shell bound for unconditional log-concave measures. This approach was extended by Barthe and Cordero-Erausquin in [12] who adapted Klartag's techniques to provide spectral gap estimates for log-concave measures with many symmetries, and by Barthe and Klartag in [13]. Given a log-concave probability measure μ on $\mathbb{R}^n$, let $H^1(\mu)$ be the space of all functions $f\in L^2(\mu)$ with weak partial derivatives in $L^2(\mu)$, equipped with the norm

$$\|f\|_{H^1(\mu)}^2=\left(\int_{\mathbb{R}^n}|f|^2d\mu+\int_{\mathbb{R}^n}|\nabla f|^2d\mu\right)^{1/2}.$$

Barthe and Klartag showed in [13] that $C_c^\infty(\mathbb{R}^n)$ is dense in $H^1(\mu)$. Now, for every $f \in L^2(\mu)$ with $\int f\,d\mu = 0$, define

$$\begin{aligned}\|f\|_{H^{-1}(\mu)} &= \sup\left\{\int_{\mathbb{R}^n} fg\,d\mu : g \in H^1(\mu),\ \int_{\mathbb{R}^n} |\nabla g|^2 d\mu \leqslant 1\right\}\\ &= \sup\left\{\int_{\mathbb{R}^n} fg\,d\mu : g \in C_c^\infty(\mathbb{R}^n),\ \int_{\mathbb{R}^n} |\nabla g|^2 d\mu \leqslant 1\right\}.\end{aligned}$$

In [13] and [72] it is shown, by using Bochner's formula, that

$$\|f\|_{L^2(\mu)}^2 \leqslant \|\nabla f\|_{H^{-1}(\mu)}^2 := \sum_{i=1}^n \|\partial_i f\|_{H^{-1}(\mu)}^2 \tag{9.16}$$

for every smooth function $f \in H^1(\mu)$ that satisfies $\int f\,d\mu = 0$ and $\int \nabla f\,d\mu = 0$. Applying (9.16) for an isotropic log-concave probability measure μ and the function $f(x) = |x|^2 - n$, we see that

$$\mathrm{Var}_\mu(|x|^2) \leqslant 4\sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2.$$

Klartag and Lehec prove the next inequality, which immediately implies Theorem 9.12.

Theorem 9.13 (Klartag-Lehec)**.** *There exists an absolute constant $C > 0$ such that*

$$\sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2 \leqslant Cn$$

for every $n \geqslant 1$ and every isotropic log-concave probability measure μ on $\mathbb{R}^n$.

The proof is based again on stochastic localization. Klartag and Lehec consider the family of exponential tilts of log-affine perturbations of the original measure μ and construct suitable couplings between these tilts. Then, they use these couplings to bound the $H^{-1}(\mu)$-norm by the growth of the covariance process $(A_t)_{t\geqslant 0}$ using a variant of Guan's technique from [62].

10 The MM^*-estimate for isotropic convex bodies

Let K be a convex body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. Recall that p_K is the Minkowski functional of K, defined by $p_K(x) = \inf\{t > 0 : x \in tK\}$, and h_K is the support function $h_K(x) = \max\{\langle x, y\rangle : y \in K\}$ of K. The parameters

$$M(K) = \int_{S^{n-1}} p_K(\xi)\,d\sigma(\xi) \quad \text{and} \quad M^*(K) = \int_{S^{n-1}} h_K(\xi)\,d\sigma(\xi)$$

play a central role in the asymptotic theory of finite dimensional normed spaces.

Recall that $\mathrm{vrad}(K) = (\mathrm{vol}_n(K)/\mathrm{vol}_n(B_2^n))^{1/n}$ denotes the volume radius of K. It is known that

$$M(K)^{-1} \leqslant \mathrm{vrad}(K) \leqslant M^*(K) = M(K^\circ),$$

where $K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\}$ is the polar body of K. The right-hand side inequality is a classical inequality of Urysohn, while the left-hand side inequality follows by integration in spherical coordinates and an application of Hölder's inequality. Therefore,

$$M(K)M^*(K) \geqslant 1$$

for every convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. A fundamental fact in the other direction, following from results of Figiel–Tomczak-Jaegermann [45], Lewis [91] and Pisier's estimate [116] on the norm of the Rademacher projection, states that for any symmetric convex body K in $\mathbb{R}^n$ there exists $T \in GL_n$ such that

$$M(T(K))\, M^*(T(K)) \leqslant c \ln n,$$

where $c > 0$ is an absolute constant.

In the general case, without the symmetry assumption for K, it is natural to consider the parameter

$$E(K) = \inf M(T(K))\, M^*(T(K))$$

where the infimum is taken over all invertible affine transformations of $\mathbb{R}^n$ for which $0 \in \mathrm{int}(T(K))$. The question to obtain a sharp upper bound for $\max E(K)$ remains open in the nonsymmetric case. Banaszczyk, Litvak, Pajor and Szarek showed in [11] that if K is a convex body in $\mathbb{R}^n$ which is in John's position (i.e. the ellipsoid of maximal volume inscribed in K is the Euclidean unit ball) then

$$M^*(K) \leqslant c\sqrt{n \ln n}.$$

When K is in John's position, from the inclusion $K \supseteq B_2^n$ we also obtain the trivial upper bound $M(K) \leqslant M(B_2^n) = 1$, and this implies that $E(K) \leqslant c\sqrt{n \ln n}$. This estimate was improved by Rudelson in [121]: he showed that if K is a convex body in $\mathbb{R}^n$ then

$$E(K) \leqslant c\sqrt[3]{n}(\ln n)^b$$

where $b > 0$ is an absolute constant.

In this section we review the known upper bounds for the parameters $M^*(K)$ and $M(K)$, when K is in the isotropic position. E. Milman proved in [103] that if K is a symmetric isotropic convex body in $\mathbb{R}^n$ then

$$M^*(K) \leqslant c_1\sqrt{n}(\ln n)^2 L_K \leqslant c_2\sqrt{n}(\ln n)^2$$

where the second inequality takes into account the fact that now we know that $L_n \leqslant C$. Urysohn's inequality shows that the dependence on n is optimal up to the logarithmic term. The dual problem, to estimate $M(K)$ in the isotropic position, is not well-understood. Some first non-trivial results were obtained in [57]. The currently best known estimate appears in [55]:

$$M(K) \leqslant \frac{C(n \ln n)^{1/3}}{\sqrt{n}}$$

(see also [123]).

§ 10.1. The general approach. A starting point for the question to bound $M^*(K)$ is Dudley's entropy estimate (see e.g. [118, Theorem 5.5]):

$$\sqrt{n} M^*(K) \leqslant C_1 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} e_k(K, B_2^n) \tag{10.1}$$

where $e_k(A, B)$ is the k-th entropy number of A and B, defined for $k \geqslant 1$ as

$$e_k(A, B) = \inf\{t > 0 : N(A, tB) \leqslant 2^{k-1}\}.$$

We set $e_k(K) := e_k(K, B_2^n)$. Although (10.1) is usually stated for symmetric convex bodies, it should be noted that the same estimate holds true for non-symmetric convex bodies too. This follows from the observation that $M^*(K) = \frac{1}{2} M^*(K - K)$ and

$$e_{k-1}(K - K, B_2^n) \leqslant 2 e_{k/2}(K, B_2^n) \tag{10.2}$$

for every even integer k, as a consequence of the simple estimate

$$N(K-K,2t_0B_2^n)\leqslant N(K,t_0B_2^n)N(-K,t_0B_2^n)\leqslant 2^{k-2}$$

if $t_0=e_{k/2}(K,B_2^n)$.

Our MM^*-estimates will depend on the following volumetric parameters that can be defined for any convex body K in $\mathbb{R}^n$ with $0\in \mathrm{int}(K)$:

$$w_k(K):=\sup\left\{\mathrm{vrad}(K\cap E):E\in G_{n,k}\right\},\quad v_k(K):=\sup\left\{\mathrm{vrad}(P_E(K)):E\in G_{n,k}\right\},$$

and

$$w_k^-(K):=\inf\left\{\mathrm{vrad}(K\cap E):E\in G_{n,k}\right\},\quad v_k^-(K):=\inf\left\{\mathrm{vrad}(P_E(K)):E\in G_{n,k}\right\}.$$

If K is symmetric then we have that

$$0<c\leqslant w_k^-(K)v_k(K^\circ)\leqslant 1\quad\text{and}\quad 0<c\leqslant v_k^-(K)w_k(K^\circ)\leqslant 1$$

where $c>0$ is an absolute constant, by the Blaschke-Santaló inequality and the Bourgain-Milman inequality. Note that, for every $F\in G_{n,k}$,

$$\begin{aligned}\mathrm{vol}_k(P_F(K))&\leqslant N(P_F(K),e_k(K)P_F(B_2^n))\mathrm{vol}_k(e_k(K)B_F)\leqslant N(K,e_k(K)B_2^n)e_k(K)^k\mathrm{vol}_k(B_F)\\&\leqslant (2e_k(K))^k\mathrm{vol}_k(B_F),\end{aligned}\tag{10.3}$$

and hence,

$$v_k(K)\leqslant 2e_k(K).\tag{10.4}$$

In view of (10.4), the next theorem of V. Milman and Pisier [108] is an alternative, and sometimes stronger, version of Dudley's bound.

Theorem 10.1 (V. Milman-Pisier). *For every symmetric convex body K in $\mathbb{R}^n$ one has*

$$\sqrt{n}M^*(K)\leqslant c_2\sum_{k=1}^n\frac{1}{\sqrt{k}}\mathrm{Rad}_k(K)v_k(K),\tag{10.5}$$

where $\mathrm{Rad}_k(K):=\sup\{\mathrm{Rad}(X_{P_F(K)}):F\in G_{n,k}\}$, *and* $\mathrm{Rad}(Y)\leqslant c_3\ln(d(Y,\ell_2^{\dim(Y)})+1)$ *is the Rademacher constant of* Y*.*

We refer to [2, Chapter 6] for more information on the Rademacher constant; what we really use here is that $\mathrm{Rad}_k(K)=O(\ln n)$ for all $1\leqslant k\leqslant n$. When K is assumed isotropic, Theorem 10.1 is the basis for the M^*-estimate of E. Milman [103], which is essentially optimal and, as we will see, can be easily transferred to nonsymmetric convex bodies.

For the M-estimate we need to introduce the so-called Gelfand numbers. Given any pair of convex bodies K,L in $\mathbb{R}^n$ with $0\in\mathrm{int}(K)\cap\mathrm{int}(L)$, the Gelfand numbers $c_k(K,L)$, $0\leqslant k\leqslant n-1$ are defined as follows:

$$c_k(K,L)=\inf\{\mathrm{diam}_{L\cap F}(K\cap F):F\in G_{n,n-k}\},$$

where $\mathrm{diam}_A(B):=\inf\{R>0:B\subseteq RA\}$. We set $c_k(K):=c_k(K,B_2^n)$.

In the symmetric case, Carl's theorem [34] relates any reasonable Lorentz norm of the sequence of entropy numbers $\{e_m(K,L)\}$ with that of the Gelfand numbers $\{c_m(K,L)\}$. In particular, for any $\alpha>0$, there exist constants $c(\alpha),c'(\alpha)>0$ such that, for any $n\geqslant 1$

$$\sup_{k=1,\dots,n}k^\alpha e_k(K,L)\leqslant c(\alpha)\sup_{k=1,\dots,n}k^\alpha c_k(K,L),\tag{10.6}$$

and

$$\sum_{k=1}^{n} k^{-1+\alpha} e_k(K,L) \leqslant c'(\alpha) \sum_{k=1}^{n} k^{-1+\alpha} c_k(K,L). \tag{10.7}$$

In fact, Pisier deduces the covering estimates of Theorem 3.2 from an application of Carl's theorem, after establishing the following estimates:

$$\max\{c_k(K,\mathcal{E}), c_k(K^\circ,\mathcal{E}^\circ)\} \leqslant c_1(\alpha)\left(\frac{n}{k}\right)^{1/\alpha} \quad \text{for all } k \in \{1,\dots,n\}. \tag{10.8}$$

The next theorem from [55] can serve as a basis for the M-estimate.

Theorem 10.2. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then for any $k = 1,\dots,n/2$,*

$$c_{2k}(K) \leqslant c\frac{n}{k} \ln\left(e + \frac{n}{k}\right) w_k(K). \tag{10.9}$$

In other words, there exists $F \in G_{n,n-2k}$ such that

$$K \cap F \subseteq c\frac{n}{k} \ln\left(e + \frac{n}{k}\right) w_k(K) B_F, \tag{10.10}$$

and dually, there exists $F \in G_{n,n-2k}$ such that

$$P_F(K) \supseteq \frac{1}{c\frac{n}{k}\ln(e+\frac{n}{k})} v_k^-(K) B_F. \tag{10.11}$$

Proof. Given $k = 1,\dots,n/2$, let $\mathcal{E}$ be an α-regular M-ellipsoid for K, for some $\alpha \in [1,2)$ to be determined. Consider any $H \in G_{n,k}$. We know that $N_t = N(K,t\mathcal{E}) \leqslant \exp(c(\alpha)n/t^\alpha)$ for every $t \geqslant c(\alpha)^{1/\alpha}$. Let $x_1,\dots,x_{N_t}$ be points in $\mathbb{R}^n$ such that $K \subseteq \bigcup_{i=1}^{N_t}(x_i + t\mathcal{E})$. Then, $P_H(K) \subseteq \bigcup_{i=1}^{N_t}(P_H(x_i) + tP_H(\mathcal{E}))$, and hence

$$\mathrm{vol}_k(P_H(K)) \leqslant N_t\,\mathrm{vol}_k(tP_H(\mathcal{E})) \leqslant \exp(c(\alpha)n/t^\alpha) t^k \mathrm{vol}_k(P_H(\mathcal{E})).$$

Choosing $t = c(\alpha)^{\frac{1}{\alpha}}(n/k)^{\frac{1}{\alpha}}$ we see that

$$\mathrm{vrad}(P_H(\mathcal{E})) \geqslant \frac{c_1}{c(\alpha)^{\frac{1}{\alpha}}}\left(\frac{k}{n}\right)^{1/\alpha} \mathrm{vrad}(P_H(K)) \geqslant \frac{c_1}{c(\alpha)^{\frac{1}{\alpha}}}\left(\frac{k}{n}\right)^{1/\alpha} v_k^-(K). \tag{10.12}$$

By the second estimate in (10.8), we know that there exists $E \in G_{n,n-k}$ so that:

$$P_E(K) \supseteq \frac{1}{c(\alpha)^{\frac{1}{\alpha}}}\left(\frac{k}{n}\right)^{1/\alpha} P_E(\mathcal{E}). \tag{10.13}$$

Consider the ellipsoid $\mathcal{E}' = P_E(\mathcal{E})$, let H be the subspace spanned by the k shortest axes of $\mathcal{E}'$, and set F to be its orthogonal complement into E. Then, $F \in G_{n,n-2k}$ and

$$P_F(\mathcal{E}') \supseteq \mathrm{vrad}(P_H(\mathcal{E}'))\, B_F. \tag{10.14}$$

From (10.14) and (10.12) we get

$$P_F(\mathcal{E}') \supseteq \frac{c_1}{c(\alpha)^{\frac{1}{\alpha}}}\left(\frac{k}{n}\right)^{1/\alpha} v_k^-(K)\, B_F$$

and then (10.13) shows that

$$P_F(K) \supseteq \frac{c_1}{c(\alpha)^{\frac{2}{\alpha}}}\left(\frac{k}{n}\right)^{2/\alpha} v_k^-(K)\, B_F. \tag{10.15}$$

Choosing $\alpha = 2 - \frac{1}{\ln(e+n/k)}$ we obtain (10.11). The estimate (10.10) then follows by duality. □

Corollary 10.3. *Let K be a symmetric convex body in $\mathbb{R}^n$. For every $k=1,\ldots,n$ and $\alpha>0$,*

$$e_k(K,B_2^n)\leqslant c(\alpha)\sup_{m=1,\ldots,k}\left(\frac{m}{k}\right)^{\alpha}\frac{n}{m}\ln\left(e+\frac{n}{m}\right)w_m(K),$$

where $c(\alpha)>0$ is a constant depending only on α.

Proof. The claim follows from Theorem 10.2 and Carl's theorem (see (10.6)). Since $k\mapsto c_k(K,B_2^n)$ is non-increasing, there is no difference whether we take the supremum on the right-hand-side just on the even integers. □

Corollary 10.4. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that $K\subseteq RB_2^n$. Then,*

$$\sqrt{n}M^*(K)\leqslant c\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\min\left\{R,\frac{n}{k}\ln\left(e+\frac{n}{k}\right)w_k(K)\right\}.$$

Dually, for every symmetric convex body K in $\mathbb{R}^n$ with $K\supseteq rB_2^n$ we have that

$$\sqrt{n}M(K)\leqslant c\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\min\left\{\frac{1}{r},\frac{n}{k}\ln\left(e+\frac{n}{k}\right)\frac{1}{v_k^-(K)}\right\}.$$

Proof. We explain the first claim, and then the second follows by duality. For the proof it is enough to combine Dudley's entropy estimate (10.1) with Carl's theorem (see (10.7)): we have

$$\sqrt{n}M^*(K)\leqslant c_1\sum_{k=1}^{n}\frac{1}{\sqrt{k}}e_k(K)\leqslant c_2\sum_{k=1}^{n}\frac{1}{\sqrt{k}}c_k(K).$$

Since $c_k(K)\leqslant R$ for all k, the assertion follows from the estimate (10.9) of Theorem 10.2. □

§ 10.2. MM^*-estimate for isotropic symmetric convex bodies. Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. First we describe a simplified version of E. Milman's proof of the next almost sharp estimate for the mean width of K.

Theorem 10.5 (E. Milman)**.** *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. Then,*

$$M^*(K)\leqslant c\sqrt{n}(\ln n)^2 \tag{10.16}$$

where $c>0$ is an absolute constant.

Proof. A direct consequence of Theorem 10.1 is the inequality

$$\sqrt{n}M^*(K)\leqslant c_1(\ln n)\sum_{k=1}^{n}\frac{1}{\sqrt{k}}v_k(K). \tag{10.17}$$

We shall apply (10.17) for an isotropic convex body K in $\mathbb{R}^n$. Let $F\in G_{n,k}$. From Theorem 5.4 we know that $\mathrm{vol}_k(P_F(K))^{1/k}\leqslant c_1\frac{n}{k}$ where $c_1>0$ is an absolute constant, and hence

$$\mathrm{vrad}(P_F(K))\approx\sqrt{k}\mathrm{vol}_k(P_F(K))^{1/k}\leqslant c_2\frac{n}{\sqrt{k}}.$$

Therefore, $v_k(K)\leqslant c_2n/\sqrt{k}$ for all $1\leqslant k\leqslant n-1$. Going back to (10.17) we get

$$\sqrt{n}M^*(K)\leqslant c_3(\ln n)\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\frac{n}{\sqrt{k}}=c_3n\,(\ln n)\sum_{k=1}^{n}\frac{1}{k}\leqslant c_4n(\ln n)^2,$$

and the theorem follows. □

Next, we prove the upper bound for $M(K)$.

Theorem 10.6 (Giannopoulos-E. Milman)**.** *Let K be an isotropic symmetric convex body in $\mathbb{R}^n$. Then,*

$$M(K) \leqslant \frac{c(n\,\ln n)^{1/3}}{\sqrt{n}} \tag{10.18}$$

where $c > 0$ is an absolute constant.

Proof. From Corollary 10.4 we know that

$$\sqrt{n}M(K) \leqslant c_1 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \min\left\{\frac{1}{r}, \frac{n}{k}\ln\left(e+\frac{n}{k}\right)\frac{1}{v_k^-(K)}\right\} \tag{10.19}$$

where $v_k^-(K) := \inf\{\mathrm{vrad}(P_F(K)) : F \in G_{n,k}\}$ and r is the "inradius" of K. Since K is isotropic and symmetric, we know that

$$h_K(\xi) = \|\langle\cdot,\xi\rangle\|_{L_\infty(K)} \geqslant \|\langle\cdot,\xi\rangle\|_{L_2(K)} = L_K \approx 1,$$

and hence $K \supseteq c_1 B_2^n$. Therefore, we may use (10.19) with $r \approx 1$. From Theorem 5.4 we also know that $\mathrm{vol}_k(P_F(K))^{1/k} \geqslant c_2$ for every $F \in G_{n,k}$, and hence $\mathrm{vrad}(P_F(K)) \geqslant c_3\sqrt{k}$, which gives

$$v_k^-(K) \geqslant c_3\sqrt{k}.$$

Set $k_n = (n\,\ln n)^{2/3}$. Inserting the above estimates into (10.19) we get

$$\begin{aligned}\sqrt{n}M(K) &\leqslant c_4 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \min\left\{1, \frac{n\,(\ln n)}{k^{3/2}}\right\}\\ &\leqslant c_5\left(\sum_{k=1}^{k_n}\frac{1}{\sqrt{k}} + \sum_{k=k_n}^{n}\frac{n\,\ln n}{k^2}\right) \approx (n\,\ln n)^{1/3}.\end{aligned}$$

This proves the theorem. □

§ 10.3. The nonsymmetric case. In the nonsymmetric case the upper bound for $M^*(K)$ remains the same. In fact, the proof does not present any difficulties. Let K be an isotropic convex body in $\mathbb{R}^n$ and consider the difference body $K-K$ of K. Since $0 \in K$ we have that $K \subseteq K-K$ and hence $M^*(K) \leqslant M^*(K-K)$. In fact, $M^*(K) = \frac{1}{2}M^*(K-K)$. We apply Theorem 10.1 for $K-K$ to get

$$\sqrt{n}M^*(K-K) \leqslant c_1(\ln n)\sum_{k=1}^{n}\frac{1}{\sqrt{k}}v_k(K-K). \tag{10.20}$$

Given $1 \leqslant k \leqslant n-1$, for every $F \in G_{n,k}$ we have

$$\mathrm{vol}_k(P_F(K-K))^{1/k} = \mathrm{vol}_k(P_F(K)-P_F(K))^{1/k} \leqslant 4\mathrm{vol}_k(P_F(K))^{1/k},$$

by the Rogers-Shephard inequality $\mathrm{vol}_k(C-C) \leqslant \binom{2k}{k}\mathrm{vol}_k(C)$ where C is a k-dimensional convex body. Then, as in the proof of Theorem 10.5, we see that $v_k(K-K) \leqslant c_2 n/\sqrt{k}$, and inserting these bounds into (10.17) we get $\sqrt{n}M^*(K) \leqslant c_3 n(\ln n)^2$.

The best known upper bound for $M(K)$, when K is isotropic but not necessarily symmetric, is due to Vritsiou [131]. For every isotropic convex body K in $\mathbb{R}^n$ one has that

$$M(K) \leqslant \frac{cn^{10/22}(\ln n)^{5/22}}{\sqrt{n}} \tag{10.21}$$

where $c > 0$ is an absolute constant. The proof is based on the existence of β-regular M-ellipsoids for nonsymmetric convex bodies. It is proved in [131] that for every $\beta \in \left(0, \frac{2}{5}\right)$ there exists a constant $d(\beta) \geqslant 1$ such that the following holds: For every convex body K in $\mathbb{R}^n$ with either $\mathrm{bar}(K) = 0$ or $s(K) = 0$ there exists a linear image K_β of K such that

$$\max\{N(K_\beta, tB_2^n), N(K_\beta^\circ, tB_2^n), N(B_2^n, tK_\beta), N(B_2^n, tK_\beta^\circ)\} \leqslant \exp\big(d(\beta)\, n/t^\beta\big) \tag{10.22}$$

for every $t \geqslant d(\beta)^{1/\beta}$. Moreover, the constants $d(\beta)$ satisfy $d(\beta) \approx \big(c\big(\frac{4\beta}{2-3\beta}\big)\big)^{\frac{2-3\beta}{2}} = O\big((2-5\beta)^{-\beta}\big)$ as $\beta \to \frac{2}{5}^-$, where $c(\alpha)$ with $\alpha = \frac{4\beta}{2-3\beta}$ are the constants appearing in Theorem 3.2. The optimal form that this result may have is not clear.

Another ingredient in the proof of (10.21) is a Blaschke-Santaló inequality for projections of not-necessarily symmetric convex bodies, which is again established in [131]: If K is a convex body in $\mathbb{R}^n$ such that either $\mathrm{bar}(K) = 0$ or $s(K) = 0$ then for every $1 \leqslant k \leqslant n-1$ and any $F \in G_{n,k}$ we have that

$$\big(\mathrm{vol}_k(P_F(K))\,\mathrm{vol}_k(K^\circ \cap F)\big)^{1/k} \leqslant c\frac{n}{k}\,\omega_k^{2/k}$$

where $c > 0$ is an absolute constant.

Having developed these tools, one can obtain a weaker variant of (10.19), namely,

$$\sqrt{n}M(K) \leqslant c\sum_{k=1}^n \frac{1}{\sqrt{k}} \min\left\{\frac{1}{r(K)}, \left(\frac{n}{k}\right)^5 \ln^2\left(\frac{en}{k}\right) \frac{1}{v_k^-(K)}\right\}$$

and (10.21) follows.

§ 10.4. A multi-integral norm. Let K be a symmetric convex body in $\mathbb{R}^n$. For any s-tuple $\mathcal{C} = (C_1, \ldots, C_s)$ of symmetric convex bodies C_j of volume 1 in $\mathbb{R}^n$, consider the norm on $\mathbb{R}^s$, defined by

$$\|\mathbf{t}\|_{\mathcal{C},K} = \int_{C_1} \cdots \int_{C_s} \Big\|\sum_{j=1}^s t_j x_j\Big\|_K \, dx_s \cdots dx_1$$

where $\mathbf{t} = (t_1, \ldots, t_s)$. If $\mathcal{C} = (C, \ldots, C)$ then we write $\|\mathbf{t}\|_{C^s,K}$ instead of $\|\mathbf{t}\|_{\mathcal{C},K}$. A question posed by V. Milman is to determine if, in the case $C = K$, one has that $\|\cdot\|_{K^s,K}$ is equivalent to the standard Euclidean norm up to a term which is logarithmic in the dimension, and in particular, if under some cotype condition on the norm induced by K to $\mathbb{R}^n$ one has equivalence between $\|\cdot\|_{K^s,K}$ and the Euclidean norm.

This question was studied by Bourgain, Meyer, V. Milman and Pajor. For simplicity let us assume that $\mathrm{vol}_n(K) = 1$ (this is only a matter of normalization). It was proved in [29] that

$$\|\mathbf{t}\|_{\mathcal{C},K} \geqslant c\sqrt{s}\Big(\prod_{j=1}^s |t_j|\Big)^{1/s}$$

where $c > 0$ is an absolute constant. Later, Gluskin and V. Milman obtained a better lower bound in [58], in fact working in a more general context: Let $\mathcal{A} = (A_1, \ldots, A_s)$ be an s-tuple of measurable sets of volume 1 in $\mathbb{R}^n$ and let K be a star body of volume 1 in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. Then,

$$\|\mathbf{t}\|_{\mathcal{A},K} \geqslant c\,|\mathbf{t}| \tag{10.23}$$

for all $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$. Their argument was based on the Brascamp-Lieb-Luttinger inequality (see also [3, Chapter 4]).

The question to obtain upper bounds for the quantity $\|\mathbf{t}\|_{C^s,K}$ is open. Since $\|\mathbf{t}\|_{C^s,K} = \|\mathbf{t}\|_{(TC)^s,TK}$ for any $T \in SL_n$, we may restrict our attention to the case where C is isotropic. In fact, we are particularly interested in the case where C is isotropic and $K = C$, which corresponds to V. Milman's original question.

For any centered log-concave probability measure μ on $\mathbb{R}^n$ and any symmetric convex body K in $\mathbb{R}^n$, consider the parameter

$$I_1(\mu, K) := \int_{\mathbb{R}^n} \|x\|_K d\mu(x). \tag{10.24}$$

Let C be an isotropic symmetric convex body in $\mathbb{R}^n$ and let $X_1, \ldots, X_s$ be independent random vectors, uniformly distributed in C. For any $\mathbf{t} = (t_1 \ldots, t_s) \in \mathbb{R}^s$ we write $\nu_{\mathbf{t}}$ for the distribution of the random vector $t_1X_1 + \cdots + t_sX_s$. Since $\|\mathbf{t}\|_{C^s,K}$ is a norm, we may always assume that $|\mathbf{t}| = 1$. Note that $\nu_{\mathbf{t}}$ is an even log-concave probability measure on $\mathbb{R}^n$ (this is a consequence of the Prékopa-Leindler inequality; see [2]). We write $g_{\mathbf{t}}$ for the density of $\nu_{\mathbf{t}}$. Our starting point is the next observation from [36].

Lemma 10.7. *For any $\mathbf{t} = (t_1 \ldots, t_s) \in \mathbb{R}^s$, we write $\nu_{\mathbf{t}}$ for the distribution of the random vector $t_1X_1 + \cdots + t_sX_s$. Then,*

$$\|\mathbf{t}\|_{C^s,K} = \int_{\mathbb{R}^n} \|x\|_K d\nu_{\mathbf{t}}(x).$$

It is easily verified that the covariance matrix $\mathrm{Cov}(\nu_{\mathbf{t}})$ of $\nu_{\mathbf{t}}$ is a multiple of the identity: more precisely,

$$\mathrm{Cov}(\nu_{\mathbf{t}}) = L_C^2\, I_n.$$

It follows that the function $f_{\mathbf{t}}(x) = L_C^n g_{\mathbf{t}}(L_C x)$ is the density of an isotropic log-concave probability measure $\mu_{\mathbf{t}}$ on $\mathbb{R}^n$. Indeed, we have

$$\int_{\mathbb{R}^n} f_{\mathbf{t}}(x) x_i x_j\, dx = L_C^n \int_{\mathbb{R}^n} g_{\mathbf{t}}(L_C x) x_i x_j\, dx = L_C^{-2} \int_{\mathbb{R}^n} g_{\mathbf{t}}(y) y_i y_j\, dy = \delta_{i,j}$$

for all $1 \leqslant i, j \leqslant n$.

Note. It is proved in [36, Lemma 3.2] that if $|\mathbf{t}| = 1$ then $\|g_{\mathbf{t}}\|_\infty \leqslant e^n$. From this inequality we see that $L_{\mu_{\mathbf{t}}} = \|f_{\mathbf{t}}\|_\infty^{\frac{1}{n}} = L_C \|g_{\mathbf{t}}\|_\infty^{\frac{1}{n}} \leqslant e L_C$ for all $\mathbf{t} \in \mathbb{R}^s$ with $|\mathbf{t}| = 1$.

One may easily check that if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ and K is a symmetric convex body in $\mathbb{R}^n$ then

$$\begin{aligned}\int_{O(n)} I_1(\mu, U(K))\, d\nu(U) &= \int_{\mathbb{R}^n} \int_{O(n)} \|x\|_{U(K)} d\nu(U)\, d\mu(x) \\ &= M(K) \int_{\mathbb{R}^n} |x|\, d\mu(x) \approx \sqrt{n} M(K)\end{aligned}$$

where ν, σ denote the Haar probability measures on $O(n)$ and S^{n-1} respectively. It follows that

$$\int_{O(n)} \|\mathbf{t}\|_{C^s,U(K)}\, d\nu(U) \approx (L_C \sqrt{n} M(K))\, |\mathbf{t}|. \tag{10.25}$$

Therefore, one might hope to obtain a quantity of the order of $L_C \sqrt{n} M(K)\, |\mathbf{t}|$ as an upper estimate for $\|\mathbf{t}\|_{C^s,K}$. The next theorem of Skarmogiannis [123] provides a logarithmic in n, and independent from s, upper bound for $\|\mathbf{t}\|_{C^s,K}$.

Theorem 10.8. *Let C be an isotropic symmetric convex body in $\mathbb{R}^n$ and K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$\|\mathbf{t}\|_{C^s,K} \leqslant c_1 \sqrt{n} (\ln n)\, M(K) |\mathbf{t}|$$

for every $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$, where $c_1 > 0$ is an absolute constant.

In the case $C = K$, Theorem 10.8 and (10.23) show that, for any symmetric convex body K of volume 1 in $\mathbb{R}^n$,

$$c_1|\mathbf{t}| \leqslant \|\mathbf{t}\|_{K^s,K} \leqslant c_2\sqrt{n}(\ln n)\, M(K^*)L_{K^*}|\mathbf{t}|$$

for every $s \geqslant 1$ and every $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$, where $c_i > 0$ are absolute constants and K^* is an isotropic linear image of K (note that $L_{K^*} = L_K$). Thus, we have a reduction of V. Milman's question to the problem of estimating the parameter $M(K^*)$ for an isotropic symmetric convex body K^* in $\mathbb{R}^n$, the main question that we discuss in this section.

The estimate of Theorem 10.8 is based on the next result.

Theorem 10.9. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any symmetric convex body K in $\mathbb{R}^n$ we have that*

$$I_1(\mu, K) \leqslant c_2\sqrt{n}(\ln n)\, M(K)$$

where $c_2 > 0$ is an absolute constant.

Theorem 10.9 follows from an upper bound for the same quantity, due to Eldan and Lehec [44], which involves the constant

$$\tau_n^2 = \sup_{\mu} \sup_{\xi \in S^{n-1}} \sum_{i,j=1}^{n} \mathbb{E}_\mu (x_i x_j \langle x, \xi\rangle)^2$$

where the first supremum is over all isotropic log-concave probability measures μ on $\mathbb{R}^n$.

Theorem 10.10 (Eldan-Lehec)**.** *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For any symmetric convex body K in $\mathbb{R}^n$ we have that*

$$\int_{\mathbb{R}^n} \|x\|_K \, d\mu(x) \leqslant c_1\sqrt{\ln n}\,\tau_n \int_{\mathbb{R}^n} \|x\|_K \, d\gamma_n(x)$$

where γ_n is the standard Gaussian measure on $\mathbb{R}^n$ and $c_1 > 0$ is an absolute constant.

Proof of Theorem 10.9. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. A result of Eldan [42] relates the constant τ_n with the thin-shell constant

$$\sigma_n = \sup_{\mu} \sqrt{\mathrm{Var}_\mu(|x|)}$$

where the supremum is over all isotropic log-concave probability measures μ on $\mathbb{R}^n$. Eldan proved that

$$\tau_n^2 \leqslant c_2 \sum_{k=1}^{n} \frac{\sigma_k^2}{k}$$

where $c_2 > 0$ is an absolute constant. Klartag and Lehec [79] have recently confirmed the thin-shell conjecture: it is now known that $\sigma_n \leqslant C$. where $C > 0$ is an absolute constant. Combining these estimates, one gets

$$\tau_n^2 \leqslant c_2 \sum_{k=1}^{n} \frac{\sigma_k^2}{k} \leqslant c_3 \ln n.$$

Therefore, the estimate of Eldan and Lehec immediately implies that

$$I_1(\mu, K) := \int_{\mathbb{R}^n} \|x\|_K \, d\mu(x) \leqslant c_4 \ln n \int_{\mathbb{R}^n} \|x\|_K \, d\gamma_n(x)$$

where $c_4 > 0$ is an absolute constant. Finally, integration in spherical coordinates shows that

$$\int_{\mathbb{R}^n} \|x\|_K \, d\gamma_n(x) \approx \sqrt{n} \int_{S^{n-1}} \|\xi\|_K \, d\sigma(\xi) \approx \sqrt{n} M(K)$$

and hence the proof of Theorem 10.9 is complete. $\square$

We can now complete the proof of Theorem 10.8. It combines the approach of [36] with Theorem 10.9.

Proof of Theorem 10.8. We compute

$$\|\mathbf{t}\|_{C^s,K} = \int_{\mathbb{R}^n} \|x\|_K \, d\nu_{\mathbf{t}}(x) = L_C^{-n} \int_{\mathbb{R}^n} \|x\|_K f_{\mathbf{t}}(x/L_C)\, dx = L_C \int_{\mathbb{R}^n} \|y\|_K d\mu_{\mathbf{t}}(y)$$

and hence we get

$$\|\mathbf{t}\|_{C^s,K} = L_C I_1(\mu_{\mathbf{t}}, K) \tag{10.26}$$

for all $\mathbf{t} \in \mathbb{R}^s$ with $|\mathbf{t}| = 1$. Now, we use Theorem 10.9 to estimate $I_1(\mu_{\mathbf{t}}, K)$. As a result, we obtain the upper bound

$$\|\mathbf{t}\|_{C^s,K} \leqslant c_1 L_C \sqrt{n}(\ln n)\, M(K) \approx \sqrt{n}(\ln n)\, M(K),$$

which is the assertion of Theorem 10.8. □

11 Appendix: Covariance estimates

In this last section we sketch the proofs of the two main technical covariance estimates that were used for the bound $\psi_n = O(\sqrt{\ln n})$ and the bound $L_n = O(1)$. The first one is Theorem 8.12 which we state again below for the convenience of the reader.

Theorem 11.1. *For every* $0 < t \leqslant c/(\psi_n^2 \ln n)$ *we have*

$$\mathbb{E}\, \|A_t\|_{\mathrm{op}} \leqslant C, \tag{11.1}$$

where $C > 0$ *is an absolute constant.*

In fact, using the improved log-concave Lichnerowicz inequality (Theorem 8.10) we can show that the statement of Theorem 11.1 is true for all $0 < t \leqslant c/(\ln n)^2$.

As in Section 9, if μ is a log-concave probability measure with density ϱ with respect to the Lebesgue measure, we consider the stochastic localization process starting at ϱ. The tilt process $(\xi_t)_{t\geqslant 0}$ is defined as the solution of the stochastic differential equation $d\xi_t = a_{t,\xi_t} dt + dW_t$ where $(W_t)_{t\geqslant 0}$ is a standard Brownian motion. For $t \geqslant 0$ we denote by $\mu_t := \mu_{t,\xi_t}$ the measure with density $\varrho_{t,\xi_t}(x) = \frac{1}{Z_{t,\xi_t}} e^{\langle x,\xi_t\rangle - \frac{t}{2}|x|^2} \varrho(x)$, and we write $a_t := a_{t,\xi_t}$ and $A_t := A_{t,\xi_t} = \mathrm{Cov}(\mu_{t,\xi_t})$ for the barycenter and covariance matrix of μ_{t,ξ_t} respectively.

§ 11.1. Preliminary observations. In Section 7 we saw that, for any test function f, the martingale $M_t = \int_{\mathbb{R}^n} f\, d\mu_t$ satisfies

$$dM_t = \Big\langle \int_{\mathbb{R}^n} f(x)(x - a_t)\, d\mu_t, dW_t \Big\rangle,$$

where (W_t) is some standard Brownian motion. This extends to vector valued functions as follows. If $F : \mathbb{R}^n \to \mathbb{R}^k$ is a vector valued function that grows mildly at infinity, then the process (M_t) given by

$$M_t = \int_{\mathbb{R}^n} F\, d\mu_t$$

is a martingale, and

$$dM_t = \Big\langle \int_{\mathbb{R}^n} F(x) \otimes (x - a_t)\, d\mu_t, dW_t \Big\rangle.$$

To see this, writing x_i for the i-th coordinate of a vector $x \in \mathbb{R}^n$ we get

$$dM_t = \sum_{i=1}^n \left(\int_{\mathbb{R}^n} F(x)(x - a_t)_i\, d\mu_t \right) dW_{t,i}. \tag{11.2}$$

Applying (11.2) to the tensors $F(x) = x$ and $F(x) = x \otimes x$ and then rearranging the terms appropriately, we obtain the next lemma.

Lemma 11.2. *If a_t and A_t are the barycenter and the covariance matrix of μ_t, respectively, then $da_t = A_t dW_t$ and*

$$dA_t = \sum_{i=1}^{n} \left(\int_{\mathbb{R}^n} (x - a_t)^{\otimes 2} \, (x - a_t)_i \, d\mu_t \right) dW_{t,i} - A_t^2 \, dt.$$

As Lemma 11.2 shows, the derivative of the barycenter is expressed in terms of the covariance, and the derivative of the covariance depends on 3-tensors.

§ 11.2. Auxiliary results. We start with an inequality about the Hessian of the function $\phi(A) = \mathrm{tr}(e^A)$ where A is a symmetric matrix.

Lemma 11.3. *Let ϕ be the function defined on the space $S_n(\mathbb{R})$ of symmetric $n \times n$ matrices by $\phi(A) = \mathrm{tr}(e^A)$. For every pair of symmetric matrices A, H we have that*

$$\nabla^2 \phi(A)(H, H) \leqslant \langle \nabla \phi(A), H^2 \rangle = \mathrm{tr}(e^A H^2),$$

where $\nabla^2 \phi(A)$ is the Hessian matrix of ϕ at A, viewed as a bilinear form on $S_n(\mathbb{R})$.

Proof. Assume first that the matrix A is positive. We use the following fact (see [77] for a simple proof): If K and H are symmetric matrices, and K is positive semi-definite, then for any $s, m \in \mathbb{N}$ we have

$$\mathrm{tr}(K^s H K^m H) \leqslant \mathrm{tr}(K^{s+m} H^2).$$

Applying this inequality we get

$$\begin{aligned} \nabla^2 \phi(A)(H, H) &= \sum_{k \geqslant 1} \frac{1}{k!} \sum_{s=0}^{k-1} \mathrm{tr}(A^s H A^{k-s-1} H) \\ &\leqslant \sum_{k \geqslant 1} \frac{1}{k!} \, k \, \mathrm{tr}(A^{k-1} H^2) = \mathrm{tr}(e^A H^2). \end{aligned}$$

For the case where A has some negative eigenvalues, we observe that ϕ satisfies the identity

$$\phi(A + tI_n) = e^t \phi(A)$$

and differentiating this equality with respect to A we see that $\nabla \phi$ and $\nabla^2 \phi$ also satisfy the same equation, which means that adding a multiple of the identity to A leads to an equivalent inequality. This reduces the proof of the lemma to the case of positive A. □

Recall the equation for A_t

$$dA_t = \sum_{i=1}^{n} H_{i,t} dW_i - A_t^2 \, dt,$$

where

$$H_{i,t} = \int_{\mathbb{R}^n} (x - a_t)^{\otimes 2} (x - a_t)_i \, d\mu_t$$

and a_t is the barycenter of μ_t. Therefore, the matrix $H_{i,t}$ is of the form $\mathbb{E}(X_i X^{\otimes 2})$ for some random vector with mean 0. In what follows, for any $u \in S^{n-1}$ we define $H_u = \mathbb{E}(\langle X, u \rangle X^{\otimes 2})$.

Lemma 11.4. *Let X be a centered log-concave random vector. Then,*

$$\sup_{u \in S^{n-1}} \|H_u\|_{\mathrm{op}} \leqslant c_1 \|\mathrm{Cov}(X)\|_{\mathrm{op}}^{3/2}$$

where $c_1 > 0$ is an absolute constant.

Proof. Let $u, v \in S^{n-1}$. Using the Cauchy-Schwarz inequality we get

$$\langle H_u(v), v\rangle = \mathbb{E}\big(\langle X, u\rangle \, \langle X, v\rangle^2\big) \leqslant \big(\mathbb{E}\langle X, u\rangle^2\big)^{1/2}\big(\mathbb{E}\langle X, v\rangle^4\big)^{1/2}.$$

Since the random variable $\langle X, v\rangle$ is centered and log-concave, its fourth moment and the square of its second moment are of the same order. It follows that

$$\langle H_u(v), v\rangle \leqslant c_1\big(\mathbb{E}\langle X, u\rangle^2\big)^{1/2}\big(\mathbb{E}\langle X, v\rangle^2\big) \leqslant c_1\|\mathrm{Cov}(X)\|_{\mathrm{op}}^{3/2}$$

for some absolute constant $c_1 > 0$. Taking the supremum over all u and v in S^{n-1} we conclude the proof. $\square$

Lemma 11.5. *Let X be a centered random vector with finite Poincaré constant $\vartheta(X)$. Then,*

$$\left\|\sum_{i=1}^{n}\big(\mathbb{E}(X_i X^{\otimes 2})\big)^2\right\|_{\mathrm{op}} \leqslant 4\vartheta^2(X)\,\|\mathrm{Cov}(X)\|_{\mathrm{op}}^2.$$

Proof. For every coordinate vector e_i we set $H_i := H_{e_i}$. In this notation the lemma asserts that

$$\sum_{i=1}^{n}\langle H_i^2 u, u\rangle \leqslant 4\vartheta^2(X)\,\|\mathrm{Cov}(X)\|_{\mathrm{op}}^2$$

for every $u \in S^{n-1}$. We check that $\sum_{i=1}^{n}\langle H_i^2 u, u\rangle = \mathrm{tr}(H_u^2)$ and, using the assumption that X is centered, from the Cauchy-Schwarz inequality and the Poincaré inequality we get

$$\begin{aligned}\mathrm{tr}(H_u^2) &= \mathbb{E}\big(\langle X, u\rangle\,\langle H_u X, X\rangle\big)\\ &\leqslant \big(\mathbb{E}\langle X, u\rangle)^2\big)^{1/2}\big(\mathrm{Var}(\langle H_u X, X\rangle)\big)^{1/2}\\ &\leqslant \big(\mathbb{E}\langle X, u\rangle)^2\big)^{1/2}\big(4\vartheta^2(X)\mathbb{E}|H_u X|^2\big)^{1/2}\\ &= \langle \mathrm{Cov}(X)u, u\rangle^{1/2}\big(4\vartheta^2(X)\,\mathrm{tr}(H_u^2\mathrm{Cov}(X))\big)^{1/2}\\ &\leqslant \|\mathrm{Cov}(X)\|_{\mathrm{op}}\,\big(4\vartheta^2(X)\,\mathrm{tr}(H_u^2)\big)^{1/2}.\end{aligned}$$

This shows that $\mathrm{tr}(H_u^2) \leqslant 4\vartheta^2(X)\|\mathrm{Cov}(X)\|_{\mathrm{op}}^2$, which is exactly the assertion of the lemma. $\square$

The last result that we need is a deviation inequality for martingales (see Freedman [48]).

Lemma 11.6. *Let $(M_t)_{t\geqslant 0}$ be a continuous local martingale with $M_0 = 0$. For every $u > 0$ and $\sigma \neq 0$ we have*

$$\mathbb{P}(\exists t > 0 : M_t \geqslant u \text{ and } \langle M\rangle_t \leqslant \sigma^2) \leqslant e^{-u^2/(2\sigma^2)}.$$

For the proof, we first check that if (Z_t) is a square integrable martingale such that $\langle Z\rangle_t \leqslant \sigma^2$ for all $t > 0$ and almost surely, then $Z_\infty = \lim_{t\to+\infty} Z_t$ exists and satisfies

$$\mathbb{P}(Z_\infty \geqslant u) \leqslant e^{-u^2/(2\sigma^2)}$$

for all $u > 0$. Then, we consider the stopping time

$$\tau = \inf\{t > 0 : \langle M\rangle_t > \sigma^2\}$$

and apply the previous claim to the martingale (M_t) stopped at time τ.

§ 11.3. Proof of Theorem 11.1. The basic step is to prove the next theorem.

Theorem 11.7. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let (A_t) be the covariance process of the stochastic localization starting from μ. Then,*

$$\mathbb{P}\left(\exists s \leqslant t : \|A_s\|_{\mathrm{op}} \geqslant 2\right) \leqslant \exp\left(-\frac{1}{c_0 t}\right)$$

for all $0 \leqslant t \leqslant \frac{1}{c_0(\ln n)^2}$, where $c_0 > 0$ is an absolute constant.

Proof. We consider the function $h_\beta(M) := \frac{1}{\beta}\ln(\mathrm{tr}\,(e^{\beta M}))$ on $S_n(\mathbb{R})$. Then, h_β is a smooth function and

$$\lambda_{\max}(M) \leqslant \frac{1}{\beta}\ln(\mathrm{tr}\,(e^{\beta M})) \leqslant \lambda_{\max}(M) + \frac{\ln n}{\beta}.$$

Therefore, if $\beta \approx \ln n$ then we have that $h_\beta(M) \approx \lambda_{\max}(M)$ up to an additive absolute constant.

From Itô's formula we see thar

$$dh_\beta(A) = \Big\langle \nabla h_\beta(A), \sum_{i=1}^n H_i dB_i \Big\rangle - \Big\langle \nabla h_\beta(A), A^2\, dt \Big\rangle + \frac{1}{2}\sum_{i=1}^n \nabla^2 h_\beta(A)(H_i, H_i)\, dt.$$

The matrix

$$M = \nabla h_\beta(A) = \frac{e^{\beta A}}{\mathrm{tr}(e^{\beta A})}$$

is positive semi-definite and has trace 1. Using Lemma 11.3 we see that the second derivative of h_β satisfies

$$\nabla^2 h_\beta(A)(H_i, H_i) \leqslant \beta\, \mathrm{tr}(MH_i^2).$$

This implies that

$$dh_\beta(A) \leqslant \sum_{i=1}^n \mathrm{tr}(MH_i)\, dB_i + \frac{\beta}{2}\,\mathrm{tr}\left(M\sum_{i=1}^n H_i^2\right)\, dt.$$

We concentrate on the absolutely continuous part. Since M is positive and has trace 1, from Lemma 11.5 we see that

$$\mathrm{tr}\left(M\sum_{i=1}^n H_i^2\right) \leqslant \left\|\sum_{i=1}^n H_i^2\right\|_{\mathrm{op}} \leqslant 4\vartheta_{\mu_t}^2\, \|A_t\|_{\mathrm{op}}^2.$$

Since μ_t is t-uniformly log-concave almost surely, from the improved log-concave Lichnerowicz inequality (Theorem 8.10) we get

$$\vartheta_{\mu_t}^2 \leqslant \left(\frac{\|A_t\|_{\mathrm{op}}}{t}\right)^{1/2},$$

and hence

$$dh_\beta(A) \leqslant \sum_{i=1}^n \mathrm{tr}(MH_i)\, dB_i + \frac{2\beta}{\sqrt{t}}\, \|A_t\|_{\mathrm{op}}^{5/2} dt.$$

Next, we give an upper bound for the quadratic variation of the martingale part. For any $u \in S^{n-1}$ we set $H_u = \sum H_i u_i$. Then, Lemma 11.4 shows that

$$\sum_{i=1}^n \mathrm{tr}(MH_i)\, u_i = \mathrm{tr}(MH_u) \leqslant \|H_u\|_{\mathrm{op}} \leqslant c_1\|A_t\|_{\mathrm{op}}^{3/2}.$$

It follows that

$$\sum_{i=1}^n \mathrm{tr}(MH_i)^2 \leqslant c_1^2\|A_t\|_{\mathrm{op}}^3.$$

The above calculations show that

$$\|A_t\|_{\text{op}} \leqslant h_\beta(A_t) \leqslant h_\beta(A_0) + Z_t + 2\beta \int_0^t r^{-1/2} \|A_r\|_{\text{op}}^{5/2}\, dr \tag{11.3}$$
$$= 1 + \frac{\ln n}{\beta} + Z_t + 2\beta \int_0^t r^{-1/2}\|A_r\|_{\text{op}}^{5/2}\, dr$$

where (Z_t) is a continuous martingale starting from 0 with quadratic variation that satisfies

$$[Z]_t \leqslant c_2 \int_0^t \|A_r\|_{\text{op}}^3 \, dr. \tag{11.4}$$

We choose $\beta = 2\ln n$, and assume that there exists $s \leqslant t$ such that $\|A_s\|_{\text{op}} \geqslant 2$. If s is the smallest such time, then before time s the operator norm of A is less than 2, and then (11.3) shows that

$$2 = \|A_s\|_{\text{op}} \leqslant \frac{3}{2} + Z_s + c_3 s^{1/2} \ln n \leqslant \frac{3}{2} + Z_s + c_3 t^{1/2} \ln n,$$

where $c_3 > 0$ is an absolute constant. If t is a sufficiently small multiple of $(\ln n)^{-2}$ then this last inequality implies that $Z_s \geqslant \frac{1}{4}$. Moreover, (11.4) shows that $[Z]_s \leqslant c_4 s \leqslant c_4 t$. Therefore,

$$\mathbb{P}(\exists s \leqslant t : \|A_s\|_{\text{op}} \geqslant 2) \leqslant \mathbb{P}\left(\exists s > 0 : Z_s \geqslant \frac{1}{4} \text{ and } [Z]_s \leqslant c_4 t\right).$$

Applying Lemma 11.6 we conclude the proof. □

Theorem 11.7 implies the upper bound for the expectation of A_t.

Proof of Theorem 11.1. Since μ_t is t-uniformly log-concave, we have $A_t \leqslant \frac{1}{t} I_n$, and in particular $\|A_t\|_{\text{op}} \leqslant 1/t$, almost surely. Therefore,

$$\mathbb{E}\|A_t\|_{\text{op}} \leqslant 2 + \frac{1}{t}\mathbb{P}(\|A_t\|_{\text{op}} > 2).$$

Since $x \exp\left(-\frac{1}{c_0}x\right)$ is a bounded function of x, Theorem 11.7 shows that

$$\mathbb{E}\|A_t\|_{\text{op}} \leqslant C$$

on the time interval $\left[0, \frac{1}{c_0(\ln n)^2}\right]$. □

Note. Instead of the improved log-concave Lichnerowicz inequality, we could have bounded ϑ_{μ_t} by the KLS constant ψ_n. It is not hard to see that for any log-concave random vector X

$$\vartheta^2(X) \leqslant \psi_n^2 \|\text{Cov}(X)\|_{\text{op}},$$

so we can use the inequality

$$\vartheta_{\mu_t}^2 \leqslant \psi_n^2 \|A_t\|_{\text{op}}.$$

Using this estimate instead of Theorem 8.10 we would get Theorem 11.1 in the (weaker) form that we have stated it.

The second main estimate, which played a key role in the proof of the isotropic constant conjecture, is Guan's theorem [62], which we state again below for the convenience of the reader.

Theorem 11.8. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then,*

$$\mathbb{E}\big(\text{tr}(A_{\delta_1})\big) \geqslant \delta_1 n,$$

where $\delta_1 > 0$ is an absolute constant.

§ 11.4. Auxiliary results. For $i,j,k=1,\dots,n$ we denote

$$R_{ij}(t,\xi_t)=\int_{\mathbb{R}^n}(x-a)(x-a)_i(x-a)_j\varrho_{t,\xi_t}(x)\,dx$$

and

$$R_{ijk}(t,\xi_t)=\int_{\mathbb{R}^n}(x-a)_i(x-a)_j(x-a)_k\varrho_{t,\xi_t}(x)\,dx$$

where x_i are the coordinates of x with respect to an orthonormal basis $\{u_i(t,\xi_t)\}_{i=1}^n$ of eigenvectors corresponding to the eigenvalues $0<\lambda_1(t,\xi_t)\leqslant\cdots\leqslant\lambda_n(t,\xi_t)\leqslant 1/t$ of A_{t,ξ_t}.

For any $r>0$ let

$$d(r)=d(r)(t,\xi_t)=\max\{i\geqslant 1:\lambda_i(t,\xi_t)\leqslant r\}.$$

Lemma 11.9. *Let $r>0$. For any $1\leqslant k\leqslant n$ we have*

$$\sum_{i,j=1}^{d(r)}R_{ijk}^2\leqslant 4t^{-1/2}r^{3/2}\lambda_k.$$

Proof. Let E be the subspace of $\mathbb{R}^n$ spanned by the first $d(r)$ eigenvectors of A_{t,ξ_t}. The measure $\nu_E=(P_E)_*\mu_{t,\xi_t}$ is t-uniformly log-concave and $\mathrm{Cov}(\nu_E)\leqslant rP_E$. We set

$$g_k(x)=\sum_{i,j=1}^{d(r)}R_{ijk}(x-a)_i(x-a)_j.$$

Using the Cauchy-Schwarz inequality and Theorem 8.10 we write

$$\begin{aligned}\sum_{i,j=1}^{d(r)}R_{ijk}^2&=\int_E g_k(x)(x-a)_k d\nu_E(x)\leqslant\sqrt{\mathrm{Var}_{\nu_E}(g_k)}\left(\int_E(x-a)_k^2d\mu_{t,\xi_t}(x)\right)^{1/2}\\&=\sqrt{\lambda_k}\,\sqrt{\mathrm{Var}_{\nu_E}(g_k)}\leqslant\sqrt{\lambda_k}\,\left(\frac{\|\mathrm{Cov}(\nu_E)\|_{\mathrm{op}}}{t}\right)^{1/4}\left(\int_E|\nabla g_k|^2d\nu_E\right)^{1/2}\\&\leqslant 2\sqrt{\lambda_k}\,\left(\frac{r}{t}\right)^{1/4}\left(\sum_{i,j=1}^{d(r)}\lambda_jR_{ijk}^2\right)^{1/2}\leqslant 2\sqrt{\lambda_k}\,t^{-1/4}r^{3/4}\left(\sum_{i,j=1}^{d(r)}R_{ijk}^2\right)^{1/2}\end{aligned}$$

and the lemma follows. □

The starting point of Guan's work is the next lemma.

Lemma 11.10. *For any smooth function $f:[0,\infty)\to\mathbb{R}$ with bounded second derivative we have*

$$\frac{d}{dt}\mathbb{E}(\mathrm{tr}(f(A_t)))=\frac{1}{2}\sum_{i,j=1}^n\mathbb{E}(|R_{i,j}|^2)\frac{f'(\lambda_i)-f'(\lambda_j)}{\lambda_i-\lambda_j}-\sum_{i=1}^n\lambda_i^2f'(\lambda_i),$$

where the quotient in the above formula is interpreted as $f''(\lambda_i)$ if $\lambda_i=\lambda_j$.

Lemma 11.10 is applied for a function $f(\lambda)$ which is quadratic when λ is relatively large and exponential when λ is small. In this way, the analysis of A_t is restricted on the large eigenvalues. The precise construction of such an f is given in [62, Lemma 2.2].

Lemma 11.11. *Let $D_0>4$ and $\frac{7}{3}\leqslant r_0\leqslant\frac{8}{3}$. There exists $b\in\left[\frac{1}{20},\frac{1}{5}\right]$ and an increasing twice differentiable positive function f on $[0,\infty)$ such that $f(r)=e^{D_0(r-r_0)}$ if $0\leqslant r\leqslant r_0-1/D_0$ and $f(r)=br^2$ if $r\geqslant r_0$, which satisfies $|f''(r)|\leqslant D_0^2f(r)$ for all $r>0$.*

A direct consequence of Lemma 11.9 and of the fact that $\lambda_i \leqslant 1/t$ is that if $i \geqslant d(r_0)+1$ then

$$b \sum_{j,k=1}^{i} R_{ijk}^2 \leqslant 4bt^{-1}\lambda_i^{3/2}\lambda_i \leqslant 4bt^{-1/2}\lambda_i^2 = 4t^{-1}f(\lambda_i). \tag{11.5}$$

Instead of estimating $\mathbb{E}(\mathrm{tr}(A_t))$ directly, Guan is using the above in order to estimate $\mathbb{E}(F_t)$, where

$$F_t = \sum_{i=1}^{n} f(\lambda_i) = \mathrm{tr}(f(A_t)).$$

To this end, we apply the formula of Lemma 11.10. The first step is the next lemma (see [75]) which exploits (11.5) and the fact that, for any x,

$$\sum_{\substack{i,j,k=1\\ \lambda_j > x}}^{n} R_{ijk}^2 \leqslant 3 \sum_{i=d(x)}^{n} \sum_{j,k=1}^{i} R_{ijk}^2.$$

This last inequality holds true because R_{ijk}^2 is symmetric in i,j,k, therefore we may sum over those triples that satisfy $i \geqslant \max\{j,k\}$ and then multiply by 3.

Lemma 11.12. *For any $t>0$,*

$$\frac{d}{dt}\mathbb{E}(F_t) \leqslant \left(\frac{300}{t} + \frac{400D_0^2}{\sqrt{t}}\right)\mathbb{E}(F_t).$$

Proof. Since the second summand in Lemma 11.10 is negative, we have

$$\frac{d}{dt}\mathbb{E}(\mathrm{tr}(f(A_t))) \leqslant \frac{1}{2}\sum_{i,j=1}^{n} \mathbb{E}(|R_{i,j}|^2)\frac{f'(\lambda_i)-f'(\lambda_j)}{\lambda_i-\lambda_j}. \tag{11.6}$$

Consider the right-hand side sum

$$\sum_{i,j=1}^{n} |R_{i,j}|^2 \frac{f'(\lambda_i)-f'(\lambda_j)}{\lambda_i-\lambda_j}. \tag{11.7}$$

We split this quantity into four sums, according to whether $i,j \leqslant d(r_0)$ or not. From (11.5) we see that

$$b\sum_{i,j=d(r_0)+1}^{n} |R_{i,j}|^2 = b\sum_{i,j=d(r_0)+1}^{n}\sum_{k=1}^{n} R_{ijk}^2 \leqslant 3b\sum_{i=d(r_0)+1}^{n}\sum_{j,k=1}^{i} R_{ijk}^2 \leqslant 12t^{-1}\sum_{i=d(r_0)+1}^{n} f(\lambda_i). \tag{11.8}$$

Recall that $r_0 \leqslant \frac{8}{3}$. Using again (11.5) and the fact that $x/(x-y) \leqslant 3\left(r_0+\frac{1}{3}\right)$ when $x \geqslant r_0+\frac{1}{3}$ and $y \leqslant r_0$, we see that the indices with $i \leqslant d(r_0)$ and $j > d(r_0+1/3)$ or $i > d(r_0+1/3)$ and $j \leqslant d(r_0)$ contribute to the sum (11.7) at most

$$\begin{aligned}\sum_{i=1}^{d(r_0)}\sum_{j=d(r_0+1/3)+1}^{n}\sum_{k=1}^{n} R_{ijk}^2\frac{2b\lambda_j}{\lambda_i-\lambda_j} &\leqslant 2b\cdot 3(r_0+1/3)\sum_{i=1}^{d(r_0)}\sum_{j=d(r_0+1/3)+1}^{n}\sum_{k=1}^{n} R_{ijk}^2 \\ &\leqslant 54b\sum_{i=d(r_0+1/3)+1}^{n}\sum_{j,k=1}^{n} R_{ijk}^2 \leqslant 216t^{-1}\sum_{i=d(r_0)+1}^{n} f(\lambda_i).\end{aligned} \tag{11.9}$$

Since $|f''(r)| \leqslant D_0^2 f(r)$ for all $r > 0$, the indices with $i, j \leqslant d(r_0 + 1/3)$ contribute to the sum (11.7) at most

$$\frac{1}{2}\sum_{i,j=1}^{d(r_0+1/3)}\sum_{k=1}^{n} R_{ijk}^2 \frac{|f'(\lambda_j) - f'(\lambda_i)|}{|\lambda_j = \lambda_i|} \leqslant \frac{D_0^2}{2}\sum_{i,j=1}^{d(r_0+1/3)}\sum_{k=1}^{n} R_{ijk}^2 f(\max\{\lambda_i, \lambda_j\}). \tag{11.10}$$

We split the last sum into two parts. From Lemma 11.9 we see that the terms with $k \leqslant d(r_0+1/3)$ contribute at most

$$\begin{aligned} \frac{3D_0^2}{2}\sum_{i=1}^{d(r_0+1/3)} f(\lambda_i)\sum_{j,k=1}^{d(\lambda_i)} (R_{ijk}^2 &\leqslant 6D_0^2 t^{-1/2}\sum_{i=1}^{d(r_0+1/3)} \lambda_i^{3/2}\lambda_i f(\lambda_i) \\ &\leqslant 6D_0^2 t^{-1/2}(r_0+1/3)^{5/2}\sum_{i=1}^{d(r_0+1/3)} f(\lambda_i) \leqslant 6\cdot 3^{5/2} D_0^2 t^{-1/2}\sum_{i=1}^{d(r_0+1/3)} f(\lambda_i). \end{aligned} \tag{11.11}$$

Using Lemma 11.9 again, and taking into account that $b \in \left[\frac{1}{20}, \frac{1}{5}\right]$, $f(3) \leqslant 2$ and $\frac{7}{60}x \leqslant bx^2 = f(x)$ for $x \geqslant \frac{8}{3}$, we see that the terms with $k > d(r_0 + 1/3)$ contribute at most

$$\begin{aligned} D_0^2 \sum_{k=d(r_0+1/3)+1}^{n}\sum_{i=1}^{d(r_0+1/3)} f(\lambda_i)\sum_{j=1}^{d(r_0+1/3)} \mathbb{E}(R_{ijk}^2) &\leqslant 4D_0^2 f(3) t^{-1/2}(r_0+1/3)^{3/2}\sum_{k=d(r_0+1/3)+1}^{n}\lambda_k \\ &\leqslant 8\cdot 3^{3/2}\cdot\frac{60}{7} D_0^2 t^{-1/2}\sum_{k=d(r_0+1/3)+1}^{n} f(\lambda_k). \end{aligned} \tag{11.12}$$

From (11.8), (11.9),(11.10) and (11.11) we finally get

$$\begin{aligned} \frac{d}{dt}\mathbb{E}(F_t) &\leqslant \left(\frac{12+216}{t} + \frac{D_0^2}{\sqrt{t}}\max\left\{6\cdot 3^{5/2}, 8\cdot 3^{3/2}\cdot\frac{60}{7}\right\}\right)\mathbb{E}\left(\sum_{i=1}^{n} f(\lambda_i)\right) \\ &\leqslant \left(\frac{300}{t} + \frac{400D_0^2}{\sqrt{t}}\right)\mathbb{E}(F_t), \end{aligned}$$

and the proof is complete. □

A consequence of Lemma 11.12 is that

$$\mathbb{E}(F_s) \leqslant \left(\frac{s}{t}\right)^{700}\mathbb{E}(F_t) \tag{11.13}$$

for all $0 < t \leqslant s \leqslant D_0^{-4}$. It is known (see [76, Lemma 5.2]) that this implies the bound

$$\mathbb{P}(\|A_t\|_{\mathrm{op}} \geqslant 2) \leqslant c_1\exp(-c_2/t) \tag{11.14}$$

for all $0 < t \leqslant c_3/(\ln n)^2$. In particular, if $0 \leqslant t \leqslant t_1 = \overline{c}/(\ln n)^2$, where $\overline{c} > 0$ is an absolute constant, then

$$\mathbb{P}(\|A_t\|_{\mathrm{op}} \geqslant 2) \leqslant \frac{1}{n}. \tag{11.15}$$

§ 11.5. Proof of Theorem 11.8. We set $s_1 = 7/3$ and consider the function $f_1 = f_{D_0,r_0}$ from Lemma 11.11 with $r_0 = s_1$ and $D_0 = (\ln t_1)^4$. If we assume that $n \geqslant n_0$, where n_0 is a large enough absolute constant, then we know that $D_0 \geqslant 3$. We define $F_{1,t} = \mathrm{tr}(f_1(A_t))$ and using (11.13) and the fact that $A_t \leqslant \frac{1}{t}I_n$ we see that

$$\mathbb{E}(F_{1,t_1}) \leqslant \left(1 - \frac{1}{n}\right) n\, f_1(2) + \frac{1}{n}\, n\, f_1(1/t_1) = (n-1)e^{-|\ln t_1|^4/3} + bt_1^{-2}. \tag{11.16}$$

For $k \geqslant 2$ we define

$$t_k = |\ln t_{k-1}|^{-16}, \quad s_k = \frac{7}{3} + \sum_{i=2}^{k} |\ln t_k|^{-1/2}, \quad f_k = f_{(\ln t_k)^4, s_k}, \quad F_{k,t} = \mathrm{tr}(f_k(A_t)).$$

We denote by b_k the constant in $\left[\frac{1}{20}, \frac{1}{5}\right]$ from the construction of the function $f_k = f_{(\ln t_k)^4, s_k}$. We set $k_0 = \max\{k \geqslant 1 : t_k \leqslant e^{-100}\}$. Since the summands in the sum of the definition of s_k decay very fast, if we assume that $n \geqslant n_0$ for a large enough absolute constant $n_0 > 0$ then we see that $s_k \in \left[\frac{7}{3}, \frac{8}{3}\right]$ and $0 < t_1 < t_2 \leqslant \cdots \leqslant t_{k_0} \leqslant e^{-100}$. Also, for every $2 \leqslant k \leqslant k_0$ we have

$$s_{k-1} = s_k - \frac{1}{\sqrt{|\ln t_k|}} \leqslant s_k - \frac{1}{(\ln t_k)^4}. \tag{11.17}$$

Lemma 11.13. *Let $n \geqslant n_0$, where n_0 is a large enough absolute constant. For all $1 \leqslant k \leqslant k_0$ and $t \in [t_k, t_{k+1}]$,*

$$\mathbb{E}(F_{k,t}) \leqslant n \exp(-(\ln t_k)^2). \tag{11.18}$$

Proof. We shall show that for every $k \geqslant 2$ and $r \geqslant s_{k-1}$,

$$f_k(r) \leqslant 5 f_{k-1}(r). \tag{11.19}$$

If $r \geqslant s_k$ then

$$\frac{f_k(r)}{f_{k-1}(r)} = \frac{b_k}{b_{k-1}} \leqslant 4.$$

If $s_{k-1} \leqslant r \leqslant s_k$ then

$$f_k(r) \leqslant f_k(s_k) = b_k s_k^2 \leqslant 4 b_{k-1} s_k^2 \leqslant 5 b_{k-1} r^2 = 5 f_{k-1}(r)$$

So, (11.19) is true. We shall prove (11.18) by induction on k. For the case $k = 1$, recall that $t_1 = \bar{c}/(\ln n)^2$. Using (11.13), (11.16) and the fact that $b \leqslant 1/5$ we see that

$$\mathbb{E}(F_{1,t}) \leqslant \left(\frac{t_2}{t_1}\right)^{700} \mathbb{E}(F_{1,t_1}) \leqslant t_1^{-700}\Big((n-1)e^{-|\ln t_1|^4/3} + t_1^{-2}/5\Big) \leqslant n\, e^{-(\ln t_1)^2}$$

for every $t \in [t_1, t_2]$, provided that $n \geqslant n_0$. Now, let $k \geqslant 2$ and assume that (11.18) holds true for $k-1$. From (11.17) and (11.19) we have

$$\begin{aligned}
\mathbb{E}(F_{t,k}) = \mathbb{E}\left(\sum_{i=1}^{n} f_k(\lambda_{i,t_k})\right) &= \mathbb{E}\left(\sum_{i=1}^{d(s_{k-1})} f_{(\ln t_k)^4, s_k}(\lambda_{i,t_k})\right) + \sum_{i=d(s_{k-1})+1}^{n} f_k(\lambda_{i,t_k}) \\
&\leqslant d(s_{k-1})\, e^{(\ln t_k)^4 (s_{k-1} - s_k)} + 5 \sum_{i=d(s_{k-1})+1}^{n} f_{k-1}(\lambda_{i,t_k}) \\
&\leqslant n\, e^{-|\ln t_k|^{7/2}} + 5\mathbb{E}(F_{k-1,t_k}).
\end{aligned}$$

From (11.13) and the induction hypothesis we see that if $t \in [t_k, t_{k+1}]$ then

$$\begin{aligned}
\mathbb{E}(F_{k,t}) &\leqslant \left(\frac{t_{k+1}}{t_k}\right)^{700} \mathbb{E}(F_{k,t_k}) \leqslant \left(\frac{t_{k+1}}{t_k}\right)^{700} \left[n\, e^{-|\ln t_k|^{7/2}} + 5n\, e^{-(\ln t_{k-1})^2}\right] \\
&\leqslant n\, \left[t_k^{-700}\, e^{-|\ln t_k|^{7/2}} + 5 t_k^{-700}\, e^{-t_k^{-1/8}}\right] \leqslant n\, \left[\frac{1}{2} e^{-|\ln t_k|^2} + \frac{1}{2} e^{-|\ln t_k|^2}\right],
\end{aligned} \tag{11.20}$$

using also the fact that $t_k \leqslant \exp(-100)$ and $t_k = |\ln t_{k-1}|^{-16}$. Now, (11.20) gives (11.18) and the proof is complete. $\square$

We can proceed now with Guan's main estimate in [62].

Theorem 11.14. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. If $(A_t)_{t\geqslant 0}$ is the covariance process associated with the stochastic localization starting from μ, then*

$$\mathbb{E}(\mathrm{tr}(A_t^2)) \leqslant cn$$

for all $t > 0$, where $c > 0$ is an absolute constant.

Proof. We first assume that $n \geqslant n_0$ where n_0 is a large enough absolute constant. Let $c_1 > 0$ be an absolute constant such that $c_1 < t_{k_0+1}$. If $t \in [t_1, c_1]$ then there exists $1 \leqslant k \leqslant k_0$ such that $t_k \leqslant t \leqslant t_{k+1}$. From Lemma 11.13 we know that

$$\mathbb{E}(\mathrm{tr}(A_t^2)) = \mathbb{E}\left(\sum_{i=1}^n \lambda_i^2\right) \leqslant \left(\frac{8}{3}\right)^2 n + \mathbb{E}\left(\sum_{\{i:\lambda_i\geqslant 8/3\}} \lambda_i^2\right) \leqslant \frac{64}{9}n + 20\mathbb{E}(F_{k,t}) \leqslant n\left(\frac{64}{9} + e^{-(\ln t_k)^2}\right) \leqslant \frac{n}{8}.$$

On the other hand, if $0 < t \leqslant t_1$ then (11.15) and the fact that $A_t \leqslant \frac{1}{t}I_n$ show that

$$\mathbb{E}(\mathrm{tr}(A_t^2)) \leqslant 4n\,\mathbb{P}(\|A_t\|_{\mathrm{op}} \leqslant 2) + \frac{1}{t^2}\mathbb{P}(\|A_t\|_{\mathrm{op}} \geqslant 2) \leqslant 4n + \frac{c_2 n}{t^2}\exp(-c_3/t) \leqslant 8n,$$

where we have taken into account that $t \leqslant \overline{c}/(\ln n)^2$ and $n \geqslant n_0$. Note also that if $t > c_1$ then

$$\mathbb{E}(\mathrm{tr}(A_t^2)) \leqslant \frac{1}{c_1}n$$

because $A_t \leqslant \frac{1}{c_1}I_n$. Finally, the case $n < n_0$ is covered e.g. by [76, Corollary 5.4] which provides a bound for $\mathbb{E}\|A_t\|_{\mathrm{op}}^2$ and hence for $\mathbb{E}(\mathrm{tr}(A_t^2))$. □

It is not hard now to prove Theorem 11.8.

Proof of Theorem 11.8. Using the formula

$$dA_t = \int_{\mathbb{R}^n} (x - a_t)^{\otimes 2}\langle x - a_t, dW_t\rangle\, d\mu_t(x) - A_t^2 dt$$

and taking into account that $A_0 = I_n$, from the estimate $\mathbb{E}(\mathrm{tr}(A_t^2)) \leqslant Cn$ for $0 \leqslant t \leqslant \delta_1$ we see that

$$\mathbb{E}(\mathrm{tr}(A_t)) \geqslant \delta_1 n$$

for all $0 \leqslant t \leqslant \delta_1$, where δ_1 is a suitably small positive absolute constant. □

Acknowledgements. The first named author acknowledges support by the Hellenic Foundation for Research and Innovation (H.F.R.I.) in the framework of the call "Basic research Financing (Horizontal support of all Sciences)" under the National Recovery and Resilience Plan "Greece 2.0" funded by the European Union –NextGenerationEU (H.F.R.I. Project Number: 15445). The third named author acknowledges support by a PhD scholarship from the National Technical University of Athens.

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