

A Deterministic Proof of the Slicing Theorem

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Abstract

We give a deterministic proof of Bourgain’s slicing theorem. The proof uses neither stochastic localization nor stochastic calculus, does not rely on Guan’s covariance bound, and avoids M -ellipsoids, Pisier’s regularization machinery, and complex interpolation. Instead, the argument is based on a deterministic one-parameter quadratic perturbation of the measure, a covariance-survival principle, and a geometric analysis of L_p -centroid bodies.

1 Introduction

A convex body $K \subseteq \mathbb{R}^n$ is said to be in isotropic position if

$$|K| = 1, \quad \int_K x \, dx = 0,$$

and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 \, dx = L_K^2$$

for every $\theta \in S^{n-1}$. The number L_K is called the isotropic constant of K . More generally, for an arbitrary centered convex body one may define the affine-invariant quantity

$$L_K = |K|^{-1/n} \det \operatorname{Cov}(\lambda_K)^{1/(2n)},$$

where λ_K denotes the uniform probability measure on K . We write

$$L_n = \sup\{L_K : K \subseteq \mathbb{R}^n \text{ is a convex body}\}.$$

Bourgain’s slicing problem, also known as the hyperplane conjecture, asks whether

$$L_n \leq C$$

for a universal constant $C > 0$. Equivalently, every convex body $K \subseteq \mathbb{R}^n$ of volume one should admit a hyperplane section of $(n - 1)$ -dimensional volume bounded from below by a universal positive constant. The problem has played a central role in asymptotic convex geometry since Bourgain’s work in the 1980s [3, 4]. We refer to [6] for a systematic account of the theory of isotropic convex bodies, and to the recent surveys [16, 10] for the developments leading to the resolution of the problem and for its connections with concentration, isoperimetry, the thin-shell problem, and the Kannan–Lovász–Simonovits conjecture.

For many years the best general estimates for L_n were polynomial in the dimension. A sequence of major advances based on stochastic localization led eventually to polylogarithmic bounds. In particular, Klartag proved

$$L_n \leq C\sqrt{\log n}$$

in [14]. A key ingredient in his argument is the improved log-concave Lichnerowicz inequality: if ν is t -uniformly log-concave, then

$$C_P(\nu) \leq \sqrt{\frac{\|\operatorname{Cov}(\nu)\|_{\operatorname{op}}}{t}}.$$

Shortly afterwards, Guan in [12] obtained the striking estimate

$$L_n \leq C \log \log n.$$

Building on Guan's covariance estimate, Klartag and Lehec completed the affirmative resolution of Bourgain's slicing problem [15]. Their proof combines Milman's theory of M -ellipsoids, Eldan's stochastic localization, Guan's estimate, and stability estimates for the Shannon–Stam inequality.

Bizeul subsequently gave a different proof of the slicing conjecture based on exponential small-ball estimates for isotropic log-concave measures [2]. His argument also uses stochastic localization together with Guan's bound, but the final passage from small-ball probabilities to the slicing conjecture follows the reduction of Dafnis and Paouris through Milman's theory of M -ellipsoids. This provides a conceptually different interpretation of the information supplied by the stochastic localization process: a sufficiently strong lower-tail estimate for the Euclidean norm already forces the isotropic constant to be universally bounded.

The purpose of the present note is to give a genuinely deterministic route to the slicing theorem. The argument does not use stochastic localization, martingales, Itô calculus, or any other form of stochastic calculus, and it is independent of Guan's covariance estimate. It also avoids the theory of M -ellipsoids and regular positions altogether; in particular, no use is made of Pisier's machinery, complex interpolation, or the entropy and covering arguments underlying that theory. The only deformation considered in the proof is a deterministic one-parameter quadratic perturbation of the measure

$$d\mu_t(x) = \frac{e^{-t|x|^2/2}}{Z_t} d\mu(x), \quad Z_t = \int_{\mathbb{R}^n} e^{-t|x|^2/2} d\mu(x), \quad A_t = \text{Cov}(\mu_t)$$

and the proof proceeds from a direct analysis of the covariance along this one-parameter family, followed by a geometric argument based on L_p -centroid bodies. The idea of using deterministic perturbations in connection with the slicing problem has an important precedent in Klartag's work [13], where convex perturbations were used in the proof of the bound $L_n \lesssim n^{1/4}$, although the perturbation used here is of a different nature.

We prove that there are universal constants $c_1, c_2, \beta, t_0 > 0$ and a subspace $F \subseteq \mathbb{R}^n$ with

$$\dim F \geq \beta n$$

such that

$$c_1 \text{Id}_F \preceq A_{t_0} P_F \preceq c_2 \text{Id}_F.$$

The quantity used to detect this phenomenon is

$$\Phi(t) = \text{Tr}(\sqrt{\text{Id} + A_t} - \text{Id}) = \sum_{i=1}^n \left(\sqrt{1 + \lambda_i(A_t)} - 1 \right).$$

Since $A_0 = \text{Id}$,

$$\Phi(0) = (\sqrt{2} - 1)n.$$

The square-root potential is adapted to the available Poincaré estimate. Differentiation introduces the matrix

$$(\text{Id} + A_t)^{-1/2}.$$

On a spectral block where A_t has size s , this contributes a factor of order $s^{-1/2}$, while the improved Lichnerowicz inequality contributes the complementary factor $\sqrt{s/t}$. After a dyadic decomposition of the covariance spectrum, this matching yields

$$|\Phi'(t)| \leq Cn t^{-3/4}.$$

The singularity at the origin is integrable, and hence, for a sufficiently small universal t_0 ,

$$\Phi(t_0) \geq cn.$$

Together with the Brascamp–Lieb bound $A_{t_0} \preceq t_0^{-1} \text{Id}$, this gives the required result for the subspaces.

From this point the argument has two geometric steps. First, the above implies a dimension-proportional exponential small-ball estimate for every isotropic log-concave measure. Such a small-ball estimate is already sufficient to conclude the slicing conjecture via the reduction of Dafnis and Paouris [9]. We instead give a direct completion in order to keep the argument essentially self-contained. Using the small-ball estimate, we obtain lower Hessian bounds for L_p -centroid bodies on subspaces of codimension $O(p)$. A weighted dyadic combination of these bodies produces a norm whose squared support function has Hessian eigenvalues satisfying

$$\lambda_i \gtrsim i^2.$$

The resulting determinant estimate is then converted, through Legendre duality into a volume estimate. The reverse Santaló and Rogers–Shephard inequalities complete the proof. Note that for the reverse Santaló there is a purely convex geometric proof in [11], which is based on Klartag’s work in [13].

We finally mention a recent development of a different nature. Chen and Klartag in [8] have obtained the sharp thin-shell inequality by a deterministic argument based on log-concave moment measures, a weighted Riemannian structure, and the Monge–Ampère equation.

2 Preliminaries

We collect here the few analytic and geometric facts that will be used throughout the proof.

Uniform log-concavity and Poincaré inequalities. A probability measure ν on $\mathbb{R}^d$ is called t -uniformly log-concave if it has a density of the form

$$d\nu(x) = e^{-t|x|^2/2} g(x) dx,$$

up to normalization, where g is log-concave. Equivalently, if $d\nu = e^{-V} dx$, then

$$x \mapsto V(x) - \frac{t}{2}|x|^2$$

is convex. In the smooth case this is equivalent to

$$\nabla^2 V \succeq t \text{Id}.$$

We denote by $C_P(\nu)$ the optimal constant in the Poincaré inequality

$$\text{Var}_\nu(f) \leq C_P(\nu) \int |\nabla f|^2 d\nu.$$

We shall repeatedly use the Brascamp–Lieb variance inequality [5]. If

$$d\nu(x) = Z^{-1} e^{-V(x)} dx$$

with V smooth and strictly convex, then

$$\text{Var}_\nu(f) \leq \int \langle (\nabla^2 V)^{-1} \nabla f, \nabla f \rangle d\nu. \tag{1}$$

The general log-concave case follows by approximation. In particular, if ν is t -uniformly log-concave, then

$$C_P(\nu) \leq \frac{1}{t}. \tag{2}$$

Applying (1) to linear functions also gives

$$\text{Cov}(\nu) \preceq \frac{1}{t} \text{Id}. \tag{3}$$

The covariance-sensitive improvement of (2) that we need is due to Klartag [14, Theorem 1.3]:

$$C_P(\nu) \leq \sqrt{\frac{\|\text{Cov}(\nu)\|_{\text{op}}}{t}}. \quad (4)$$

Notice that (3) recovers from this the classical bound $C_P(\nu) \leq t^{-1}$. In the proof below the genuinely improved estimate (4) is used only when the covariance-sensitive factor $\|\text{Cov}(\nu)\|_{\text{op}}^{1/2}$ is essential; the estimate $C_P(\nu) \leq t^{-1}$ itself follows directly from Brascamp–Lieb.

Gaussian concentration. The Bakry–Émery criterion [1] implies that a t -uniformly log-concave probability measure satisfies the logarithmic Sobolev inequality

$$\text{Ent}_\nu(h^2) \leq \frac{2}{t} \int |\nabla h|^2 d\nu.$$

The standard Herbst argument then yields the following Gaussian concentration estimate: for every 1-Lipschitz function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ and every $s > 0$,

$$\nu\{F \leq \mathbb{E}_\nu F - s\} \leq \exp\left(-\frac{ts^2}{2}\right), \quad (5)$$

and similarly for the upper tail. We refer also to [17] for this standard implication.

A one-dimensional moment comparison. We shall need the following elementary consequence of log-concavity. If Z is a centered log-concave real random variable and $p \geq q \geq 1$, then

$$\|Z\|_p \leq \frac{p}{q} \|Z\|_q. \quad (6)$$

See [19, Theorem 1.2].

Differentiation of spectral functions and max-min theorem We shall use the standard differentiation rule for spectral functions of symmetric matrices, see [18, Theorem 1.1]. Let φ be continuously differentiable on an interval containing the spectrum of a symmetric matrix A , and set

$$\mathcal{F}(A) = \text{Tr } \varphi(A).$$

Then, for every symmetric matrix H ,

$$D\mathcal{F}(A)[H] = \text{Tr}(\varphi'(A)H). \quad (7)$$

Moreover, we will use the Courant–Fischer formula

$$\lambda_{d+1}(H) = \max_{\substack{E \subseteq \mathbb{R}^n \\ \dim E = n-d}} \min_{\substack{v \in E \\ |v|=1}} \langle Hv, v \rangle$$

Finally, we use the standard duality facts for a norm and its dual: if R is an origin-symmetric convex body and

$$G(u) = \frac{1}{2} h_R(u)^2,$$

then

$$G^*(x) = \frac{1}{2} p_R(x)^2.$$

At points of twice differentiability,

$$D^2 G^*(x) = [D^2 G(\nabla G^*(x))]^{-1}.$$

These are standard facts of Legendre duality; see, for example, [20].

3 The uniformly log-concave deformation

Let μ be an even isotropic log-concave probability measure on $\mathbb{R}^n$. For $t \geq 0$, define

$$d\mu_t(x) = \frac{e^{-t|x|^2/2}}{Z_t} d\mu(x), \quad Z_t = \int_{\mathbb{R}^n} e^{-t|x|^2/2} d\mu(x),$$

and write

$$A_t = \text{Cov}(\mu_t).$$

Since μ_t is even, its barycenter is the origin.

The main quantity that we will use is

$$\Phi(t) = \text{Tr}(\sqrt{\text{Id} + A_t} - \text{Id}).$$

We first prove the following differential estimate.

Lemma 3.1. *For every $0 < t \leq 1$,*

$$|\Phi'(t)| \leq Cn t^{-3/4}. \quad (8)$$

Proof. We first observe that

$$\frac{d}{dt} \int_{\mathbb{R}^n} |x|^2 d\mu_t(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \left(|x|^2 - \int_{\mathbb{R}^n} |y|^2 d\mu_t(y) \right)^2 d\mu_t(x) \leq 0. \quad (9)$$

Since $\mu = \mu_0$ is isotropic,

$$\int_{\mathbb{R}^n} |x|^2 d\mu_t(x) \leq \int_{\mathbb{R}^n} |x|^2 d\mu_0(x) = n. \quad (10)$$

On the other hand, μ_t is t -uniformly log-concave. Indeed, if $d\mu = e^{-V(x)} dx$ with V convex, then the potential of μ_t is

$$V_t(x) = V(x) + \frac{t}{2}|x|^2,$$

and hence

$$\nabla^2 V_t \succeq t \text{Id}.$$

Therefore the Brascamp–Lieb inequality, applied to the linear function $x \mapsto \langle x, u \rangle$, gives

$$\langle A_t u, u \rangle = \text{Var}_{\mu_t} \langle X, u \rangle \leq \frac{|u|^2}{t}.$$

Since this holds for every $u \in \mathbb{R}^n$,

$$A_t \preceq t^{-1} \text{Id}. \quad (11)$$

We next differentiate Φ . Consider

$$F(A) := \text{Tr}((I + A)^{1/2} - I).$$

If $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$, then

$$F(A) = g(\lambda(A)),$$

where

$$g(x_1, \dots, x_n) := \sum_{i=1}^n (\sqrt{1 + x_i} - 1).$$

The function g is symmetric and continuously differentiable on $(-1, \infty)^n$. Since A_t is positive semidefinite, all its eigenvalues belong to this interval. Write

$$A_t = U \text{diag}(\lambda_1, \dots, \lambda_n) U^T.$$

By [18, Theorem 1.1],

$$\begin{aligned}\nabla F(A_t) &= U \operatorname{diag} \left(\frac{1}{2\sqrt{1+\lambda_1}}, \dots, \frac{1}{2\sqrt{1+\lambda_n}} \right) U^T \\ &= \frac{1}{2}(I + A_t)^{-1/2}.\end{aligned}$$

Here the gradient is with respect to the Hilbert–Schmidt inner product. Consequently, by the chain rule,

$$\begin{aligned}\Phi'(t) &= \langle \nabla F(A_t), A'_t \rangle_{\text{HS}} \\ &= \frac{1}{2} \operatorname{Tr} \left((I + A_t)^{-1/2} A'_t \right),\end{aligned}\tag{12}$$

where we used $\langle C, D \rangle_{\text{HS}} = \operatorname{Tr}(CD)$ for symmetric matrices. Set

$$B_t = (\operatorname{Id} + A_t)^{-1/2}$$

and, for $s \geq 0$, define

$$H_t(s) = \operatorname{Tr}(B_t A_s).$$

Since B_t is fixed with respect to s ,

$$\frac{d}{ds} H_t(s) = \operatorname{Tr}(B_t A'_s),$$

and therefore

$$\Phi'(t) = \frac{1}{2} \frac{d}{ds} H_t(s) \Big|_{s=t}.$$

Since μ_s is centered,

$$A_s = \operatorname{Cov}(\mu_s) = \int_{\mathbb{R}^n} x \otimes x \, d\mu_s(x).$$

Therefore,

$$H_t(s) = \operatorname{Tr}(B_t A_s) = \int_{\mathbb{R}^n} \operatorname{Tr}(B_t(x \otimes x)) \, d\mu_s(x).$$

For every $x \in \mathbb{R}^n$,

$$\operatorname{Tr}(B_t(x \otimes x)) = \langle B_t x, x \rangle.$$

Hence

$$H_t(s) = \int_{\mathbb{R}^n} \langle B_t x, x \rangle \, d\mu_s(x).$$

Using the definition of μ_s , we may write

$$H_t(s) = \frac{\int_{\mathbb{R}^n} \langle B_t x, x \rangle e^{-s|x|^2/2} \, d\mu(x)}{\int_{\mathbb{R}^n} e^{-s|x|^2/2} \, d\mu(x)}.$$

Consequently,

$$\begin{aligned}\Phi'(t) &= \frac{1}{2} H'_t(t) \\ &= \frac{1}{2} \frac{d}{ds} \frac{\int_{\mathbb{R}^n} \langle B_t x, x \rangle e^{-s|x|^2/2} \, d\mu(x)}{\int_{\mathbb{R}^n} e^{-s|x|^2/2} \, d\mu(x)} \Big|_{s=t}.\end{aligned}\tag{13}$$

For notational convenience below we set $B := B_t$. Differentiating the quotient gives

$$\begin{aligned} \Phi'(t) = & -\frac{1}{4} \left[\int_{\mathbb{R}^n} \langle Bx, x \rangle |x|^2 d\mu_t(x) \right. \\ & \left. - \left(\int_{\mathbb{R}^n} \langle Bx, x \rangle d\mu_t(x) \right) \left(\int_{\mathbb{R}^n} |x|^2 d\mu_t(x) \right) \right]. \end{aligned} \quad (14)$$

Equivalently,

$$\begin{aligned} \Phi'(t) = & -\frac{1}{4} \int_{\mathbb{R}^n} \left(\langle Bx, x \rangle - \int_{\mathbb{R}^n} \langle By, y \rangle d\mu_t(y) \right) \\ & \times \left(|x|^2 - \int_{\mathbb{R}^n} |y|^2 d\mu_t(y) \right) d\mu_t(x). \end{aligned} \quad (15)$$

Hence, by Cauchy–Schwarz,

$$|\Phi'(t)| \leq \frac{1}{4} UV, \quad (16)$$

where

$$U^2 = \int_{\mathbb{R}^n} \left(\langle Bx, x \rangle - \int_{\mathbb{R}^n} \langle By, y \rangle d\mu_t(y) \right)^2 d\mu_t(x) \quad (17)$$

and

$$V^2 = \int_{\mathbb{R}^n} \left(|x|^2 - \int_{\mathbb{R}^n} |y|^2 d\mu_t(y) \right)^2 d\mu_t(x). \quad (18)$$

We first estimate U . Recall that the covariance ellipsoid $Z_2(\mu_t)$ has support function

$$h_{Z_2(\mu_t)}(u)^2 = \langle A_t u, u \rangle.$$

We group its principal axes according to their lengths. Let E_0 be the span of the principal axes of $Z_2(\mu_t)$ of length at most 1, and, for $j \geq 1$, let E_j be the span of those whose lengths belong to

$$\left(2^{(j-1)/2}, 2^{j/2} \right].$$

Let P_j denote the orthogonal projection onto E_j , and put

$$r_j = \dim E_j.$$

Thus

$$A|_{E_j} \preceq 2^j \text{Id}_{E_j}, \quad j \geq 0. \quad (19)$$

For $j \geq 1$, every principal axis belonging to E_j has squared length larger than 2^{j-1} . Since

$$\sum_{i=1}^n \lambda_i(A) = \int_{\mathbb{R}^n} |x|^2 d\mu_t(x) \leq n$$

by (10), we obtain that for all $j \geq 1$

$$r_j \leq 2^{1-j} n. \quad (20)$$

The marginal $(P_j)_* \mu_t$ is again t -uniformly log-concave. Indeed, writing the density of μ_t as

$$e^{-t|x|^2/2} g(x),$$

with g log-concave, its marginal on E_j has density proportional to

$$e^{-t|z|^2/2} \int_{E_j^\perp} e^{-t|w|^2/2} g(z+w) dw,$$

and the second factor is log-concave by Prékopa's theorem. Together with (19), Klartag's improved Lichnerowicz inequality gives

$$C_P((P_j)_*\mu_t) \leq \sqrt{\frac{2^j}{t}}. \quad (21)$$

Since $B = (\text{Id} + A)^{-1/2}$ is a function of A , each E_j is also B -invariant. Let B_j denote the restriction of B to E_j . Then

$$\langle Bx, x \rangle = \sum_{j \geq 0} \langle B_j P_j x, P_j x \rangle.$$

Consequently, by Minkowski's inequality,

$$U \leq \sum_{j \geq 0} \left[\int_{\mathbb{R}^n} \left(\langle B_j P_j x, P_j x \rangle - \int_{\mathbb{R}^n} \langle B_j P_j y, P_j y \rangle d\mu_t(y) \right)^2 d\mu_t(x) \right]^{1/2}. \quad (22)$$

Applying the Poincaré inequality (21) to the function

$$z \mapsto \langle B_j z, z \rangle$$

gives

$$\int_{\mathbb{R}^n} \left(\langle B_j P_j x, P_j x \rangle - \int_{\mathbb{R}^n} \langle B_j P_j y, P_j y \rangle d\mu_t(y) \right)^2 d\mu_t(x) \leq 4 \sqrt{\frac{2^j}{t}} \int_{\mathbb{R}^n} |B_j P_j x|^2 d\mu_t(x). \quad (23)$$

If λ is an eigenvalue of A on E_j , the corresponding eigenvalue of B is $(1 + \lambda)^{-1/2}$. Let $(e_i)_{i=1}^n$ be an orthonormal eigenbasis of A_t , so that

$$A_t e_i = \lambda_i e_i.$$

Since $B = (\text{Id} + A_t)^{-1/2}$, we also have

$$B e_i = (1 + \lambda_i)^{-1/2} e_i.$$

Therefore

$$|B P_j x|^2 = \sum_{i: e_i \in E_j} \frac{\langle x, e_i \rangle^2}{1 + \lambda_i}.$$

Using that μ_t is centered and $A_t = \text{Cov}(\mu_t)$,

$$\int_{\mathbb{R}^n} \langle x, e_i \rangle^2 d\mu_t(x) = \lambda_i,$$

and hence

$$\int_{\mathbb{R}^n} |B P_j x|^2 d\mu_t(x) = \sum_{i: e_i \in E_j} \frac{\lambda_i}{1 + \lambda_i} \leq r_j.$$

Hence

$$\int_{\mathbb{R}^n} |B_j P_j x|^2 d\mu_t(x) = \sum_{\lambda_i \in E_j} \frac{\lambda_i}{1 + \lambda_i} \leq r_j.$$

It follows from (23) that

$$\left[\int_{\mathbb{R}^n} \left(\langle B_j P_j x, P_j x \rangle - \int_{\mathbb{R}^n} \langle B_j P_j y, P_j y \rangle d\mu_t(y) \right)^2 d\mu_t(x) \right]^{1/2} \leq 2t^{-1/4} 2^{j/4} \sqrt{r_j}. \quad (24)$$

For $j = 0$, this is bounded by

$$2t^{-1/4} \sqrt{n}.$$

For $j \geq 1$, using (20),

$$2^{j/4} \sqrt{r_j} \leq \sqrt{2n} 2^{-j/4}.$$

Therefore (22) yields

$$\begin{aligned} U &\leq Ct^{-1/4} \sqrt{n} \left(1 + \sum_{j \geq 1} 2^{-j/4} \right) \\ &\leq Ct^{-1/4} \sqrt{n}. \end{aligned} \tag{25}$$

Thus

$$U \leq Ct^{-1/4} \sqrt{n}. \tag{26}$$

We estimate V directly, without decomposing the covariance ellipsoid. By (11),

$$\|A_t\|_{\text{op}} \leq t^{-1}.$$

Klartag's improved Lichnerowicz inequality applied to μ_t itself therefore gives

$$C_P(\mu_t) \leq \sqrt{\frac{\|A_t\|_{\text{op}}}{t}} \leq \frac{1}{t}.$$

Applying Poincaré to the function $x \mapsto |x|^2$, and using (10), we obtain

$$\begin{aligned} V^2 &\leq \frac{1}{t} \int_{\mathbb{R}^n} |\nabla |x|^2|^2 d\mu_t(x) \\ &= \frac{4}{t} \int_{\mathbb{R}^n} |x|^2 d\mu_t(x) \leq \frac{4n}{t}. \end{aligned} \tag{27}$$

Hence

$$V \leq 2t^{-1/2} \sqrt{n}. \tag{28}$$

Combining (16), (26), and (28), we conclude that

$$|\Phi'(t)| \leq Cn t^{-3/4},$$

which proves (8). □

Theorem 3.2. *There exist universal constants $t_0, c_1, c_2, \beta > 0$ such that the following holds.*

For every even isotropic log-concave probability measure μ on $\mathbb{R}^n$, there exists a subspace $F \subseteq \mathbb{R}^n$ satisfying

$$\dim F \geq \beta n$$

and

$$c_1 \text{Id}_F \preceq A_{t_0} P_F \preceq c_2 \text{Id}_F. \tag{29}$$

Proof. Since $A_0 = \text{Id}$,

$$\Phi(0) = (\sqrt{2} - 1)n. \tag{30}$$

By Lemma 3.1,

$$\begin{aligned} \Phi(t) &\geq \Phi(0) - Cn \int_0^t s^{-3/4} ds \\ &= (\sqrt{2} - 1)n - C'n t^{1/4}. \end{aligned} \tag{31}$$

Choose a sufficiently small universal $t_0 > 0$ so that

$$C't_0^{1/4} \leq \frac{\sqrt{2} - 1}{2}.$$

Then

$$\Phi(t_0) \geq c_0 n, \quad c_0 = \frac{\sqrt{2} - 1}{2}. \quad (32)$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A_{t_0} . Choose a sufficiently small universal $\delta > 0$ such that

$$\sqrt{1 + \delta} - 1 \leq \frac{c_0}{2}.$$

Set

$$I = \{i : \lambda_i \geq \delta\}.$$

The eigenvalues outside I contribute at most $c_0 n/2$ to $\Phi(t_0)$. Hence (32) gives

$$\sum_{i \in I} \left(\sqrt{1 + \lambda_i} - 1 \right) \geq \frac{c_0 n}{2}. \quad (33)$$

On the other hand, (11) gives

$$\lambda_i \leq t_0^{-1},$$

and therefore

$$\sqrt{1 + \lambda_i} - 1 \leq \sqrt{1 + t_0^{-1}} - 1 =: C_0.$$

Together with (33), this implies

$$|I| \geq \frac{c_0}{2C_0} n =: \beta n.$$

Let F be the span of the corresponding principal directions of $Z_2(\mu_{t_0})$. Then

$$\dim F \geq \beta n,$$

and

$$\delta \text{Id}_F \preceq A_{t_0} P_F \preceq t_0^{-1} \text{Id}_F.$$

Thus (29) holds with

$$c_1 = \delta, \quad c_2 = t_0^{-1}.$$

□

4 A universal small-ball estimate

We next extract from Theorem 3.2 the only small-ball information needed in the remainder of the proof.

Corollary 4.1. *There exist universal constants $\varepsilon_0, c > 0$ such that every centered isotropic log-concave random vector X in $\mathbb{R}^m$ satisfies*

$$\mathbb{P} \{ |X| \leq \varepsilon_0 \sqrt{m} \} \leq e^{-cm}. \quad (34)$$

Proof. We first assume that the law ρ of X is even.

Apply Theorem 3.2 to ρ in dimension m . There exists a subspace $F \subseteq \mathbb{R}^m$, with

$$k := \dim F \geq \beta m,$$

such that for the marginal ν of ρ_{t_0} onto F we have

$$c_1 \text{Id}_F \preceq \text{Cov}(\nu) \preceq c_2 \text{Id}_F. \quad (35)$$

Since ρ is even, so are ρ_{t_0} and ν . In particular, ν is centered. Let Y have distribution ν . Then

$$\mathbb{E}|Y|^2 = \text{Tr Cov}(\nu) \geq c_1 k.$$

By reverse Hölder's inequality for seminorms of log-concave random vectors,

$$(\mathbb{E}|Y|^2)^{1/2} \leq C \mathbb{E}|Y|.$$

Consequently, there exists a universal constant $a > 0$ such that

$$\mathbb{E}|Y| \geq a\sqrt{k}. \quad (36)$$

Moreover, ν is t_0 -uniformly log-concave. Indeed, ρ_{t_0} is t_0 -uniformly log-concave, and this property is preserved under taking marginals by Prékopa's theorem. Hence the Gaussian concentration inequality for uniformly log-concave measures gives, for every $r > 0$,

$$\nu\{|Y| \leq \mathbb{E}|Y| - r\} \leq \exp(-ct_0 r^2).$$

Taking

$$r = \frac{a}{2}\sqrt{k}$$

and using (36), we obtain

$$\nu\left(\frac{a}{2}\sqrt{k} B_2^F\right) \leq e^{-c_0 k}, \quad (37)$$

where $c_0 > 0$ is universal.

Choose now a universal $\varepsilon_0 > 0$ sufficiently small so that

$$\varepsilon_0 \leq \frac{a\sqrt{\beta}}{2} \quad \text{and} \quad \frac{t_0 \varepsilon_0^2}{2} \leq \frac{c_0 \beta}{2}. \quad (38)$$

Since $k \geq \beta m$, the first condition in (38) implies

$$\varepsilon_0 \sqrt{m} \leq \frac{a}{2}\sqrt{k}.$$

Therefore, using that orthogonal projection does not increase the Euclidean norm,

$$\begin{aligned} \rho_{t_0}(\varepsilon_0 \sqrt{m} B_2^m) &\leq \nu(\varepsilon_0 \sqrt{m} B_2^F) \\ &\leq \nu\left(\frac{a}{2}\sqrt{k} B_2^F\right) \\ &\leq e^{-c_0 k} \leq e^{-c_0 \beta m}. \end{aligned} \quad (39)$$

We now return from the measure ρ_{t_0} to ρ . By definition,

$$d\rho_{t_0}(x) = \frac{e^{-t_0|x|^2/2}}{Z_{t_0}} d\rho(x),$$

and hence

$$d\rho(x) = Z_{t_0} e^{t_0|x|^2/2} d\rho_{t_0}(x).$$

Since $Z_{t_0} \leq 1$, on the ball $\varepsilon_0 \sqrt{m} B_2^m$ we have

$$d\rho \leq e^{t_0 \varepsilon_0^2 m/2} d\rho_{t_0}.$$

Together with (39) and the second condition in (38), this gives

$$\begin{aligned} \rho(\varepsilon_0 \sqrt{m} B_2^m) &\leq \exp\left(\frac{t_0 \varepsilon_0^2}{2} m - c_0 \beta m\right) \\ &\leq e^{-c_0 \beta m/2}. \end{aligned} \quad (40)$$

Thus (34) holds when ρ is even.

We finally remove the symmetry assumption. Let X' be an independent copy of X and set

$$W = \frac{X - X'}{\sqrt{2}}.$$

The random vector W is even and isotropic. Its law is also log-concave, since log-concavity is preserved under reflection, convolution, and linear images. Hence the even case gives

$$\mathbb{P} \{ |W| \leq \varepsilon_0 \sqrt{m} \} \leq e^{-cm}.$$

On the other hand,

$$\left\{ |X| \leq \frac{\varepsilon_0}{\sqrt{2}} \sqrt{m}, |X'| \leq \frac{\varepsilon_0}{\sqrt{2}} \sqrt{m} \right\} \subseteq \{ |W| \leq \varepsilon_0 \sqrt{m} \}.$$

By independence,

$$\mathbb{P} \left\{ |X| \leq \frac{\varepsilon_0}{\sqrt{2}} \sqrt{m} \right\}^2 \leq e^{-cm}.$$

Taking square roots and renaming the universal constants yields (34). $\square$

For completeness, let us point out that the proof could already be concluded at this stage by appealing to the reduction of Dafnis and Paouris, which shows that an exponential small-ball estimate of the form obtained above implies a universal bound for the isotropic constant. We prefer not to use this reduction here, since its proof relies on the theory of M -ellipsoids and Pisier's α -regular positions. Instead, in order to keep the remainder of the argument essentially self-contained and geometric, we derive the slicing bound directly from the small-ball estimate by means of a weighted combination of L_p -centroid bodies, Legendre duality, and the reverse Santaló inequality.

We now complete the proof of the slicing theorem. By affine invariance, it is enough to consider a convex body $K \subseteq \mathbb{R}^n$ whose uniform probability measure μ_K is centered and satisfies

$$\text{Cov}(\mu_K) = \text{Id}. \quad (41)$$

Let $X \sim \mu_K$.

For $p \geq 2$, recall that the L_p -centroid body $Z_p(K)$ is the origin-symmetric convex body whose support function is

$$h_{Z_p(K)}(u) = \left(\int_K |\langle x, u \rangle|^p d\mu_K(x) \right)^{1/p}.$$

Fix a sufficiently small universal constant $a > 0$, and let p_* be the largest dyadic integer satisfying

$$p_* \leq an.$$

We may clearly assume that n is larger than a universal constant, so that $p_* \geq 4$.

Define the origin-symmetric convex body R by

$$h_R(u)^2 = |u|^2 + \sum_{\substack{p=2^j \\ 4 \leq p \leq p_*}} p h_{Z_p(K)}(u)^2. \quad (42)$$

Set

$$K_s = \text{conv}(K \cup (-K)).$$

Then,

$$h_{Z_p(K)}(u) \leq h_{K_s}(u)$$

for every $p \geq 2$. Moreover, by isotropy,

$$h_{Z_2(K)}(u)^2 = \int_K |\langle x, u \rangle|^2 d\mu_K(x) = |u|^2,$$

and hence

$$|u| \leq h_{K_s}(u).$$

Since the sum in (42) is dyadic,

$$\sum_{\substack{p=2^j \\ 4 \leq p \leq p_*}} p \leq 2p_* \leq 2an.$$

It follows that

$$h_R(u)^2 \leq Cn h_{K_s}(u)^2,$$

and therefore

$$R \subseteq C\sqrt{n} K_s. \quad (43)$$

Since the barycenter of K is the origin, we have $0 \in K$. Thus

$$K_s = \text{conv}(K \cup (-K)) \subseteq K - K.$$

Hence, by the Rogers–Shephard inequality,

$$|K_s| \leq |K - K| \leq \binom{2n}{n} |K| \leq 4^n |K|.$$

Consequently, (43) gives

$$|R|^{1/n} \leq C\sqrt{n} |K_s|^{1/n} \leq C\sqrt{n} |K|^{1/n}.$$

It is therefore enough to prove

$$|R|^{1/n} \geq c\sqrt{n}. \quad (44)$$

Indeed, (44) would imply $|K|^{1/n} \geq c$, and since

$$L_K = |K|^{-1/n}$$

under the normalization (41), this would yield $L_K \leq C$.

Proposition 4.2. *For the body R defined in (42), one has*

$$|R|^{1/n} \geq c\sqrt{n},$$

where $c > 0$ is a universal constant.

Proof. For every dyadic p with $4 \leq p \leq p_*$, set

$$\Phi_p(u) = \frac{1}{2} h_{Z_p(K)}(u)^2 = \frac{1}{2} \left(\int_K |\langle x, u \rangle|^p d\mu_K(x) \right)^{2/p}.$$

Also define

$$\Phi(u) = \frac{1}{2} h_R(u)^2 = \frac{1}{2} |u|^2 + \sum_{\substack{p=2^j \\ 4 \leq p \leq p_*}} p \Phi_p(u).$$

Fix $p \geq 4$ and $u \neq 0$, and write

$$m = \left(\int_K |\langle x, u \rangle|^p d\mu_K(x) \right)^{1/p}.$$

For $v \in \mathbb{R}^n$, define

$$Q_{p,u}(v) = m^{2-p} \int_K |\langle x, u \rangle|^{p-2} \langle x, v \rangle^2 d\mu_K(x)$$

and

$$\ell_{p,u}(v) = m^{1-p} \int_K |\langle x, u \rangle|^{p-2} \langle x, u \rangle \langle x, v \rangle d\mu_K(x).$$

A direct differentiation gives

$$D^2\Phi_p(u)[v, v] = (p-1)Q_{p,u}(v) - (p-2)\ell_{p,u}(v)^2. \quad (45)$$

By Cauchy–Schwarz,

$$\ell_{p,u}(v)^2 \leq Q_{p,u}(v).$$

In particular,

$$D^2\Phi_p(u) \succeq 0. \quad (46)$$

We shall use the following lemma, which was also proved in [7]. For completeness, we give a self-contained proof.

Lemma 4.3. *There exists a subspace $E_{p,u} \subseteq \mathbb{R}^n$ such that*

$$\text{codim } E_{p,u} \leq C_0 p \quad (47)$$

and

$$D^2\Phi_p(u)[v, v] \geq c_0 p |v|^2, \quad v \in E_{p,u}, \quad (48)$$

where $c_0, C_0 > 0$ are universal.

Proof. Set

$$W = \int_K |\langle x, u \rangle|^{p-2} d\mu_K(x).$$

The one-dimensional reverse moment inequality for centered log-concave random variables gives

$$\|\langle X, u \rangle\|_p \leq \frac{p}{p-2} \|\langle X, u \rangle\|_{p-2}.$$

Therefore

$$W = \|\langle X, u \rangle\|_{p-2}^{p-2} \geq \left(\frac{p-2}{p} \right)^{p-2} m^{p-2} \geq e^{-2} m^{p-2}.$$

Define a probability measure ν on K by

$$d\nu(x) = \frac{|\langle x, u \rangle|^{p-2}}{W} d\mu_K(x).$$

Then, by the moment comparison

$$Q_{p,u}(v) = \frac{W}{m^{p-2}} \int_K \langle x, v \rangle^2 d\nu(x) \geq e^{-2} \int_K \langle x, v \rangle^2 d\nu(x). \quad (49)$$

Let F be the spectral subspace of the quadratic form $Q_{p,u}$ corresponding to eigenvalues smaller than a sufficiently small universal constant $b > 0$, and put

$$r = \dim F.$$

We shall show that

$$r \leq Cp.$$

This is immediate if $r = 0$, so assume for the moment that $r \geq 1$. Let $e_1, \dots, e_r$ be an orthonormal basis of F . From (49),

$$\begin{aligned} \int_K |P_F x|^2 d\nu(x) &= \sum_{j=1}^r \int_K \langle x, e_j \rangle^2 d\nu(x) \\ &\leq e^2 \sum_{j=1}^r Q_{p,u}(e_j) \leq e^2 b r. \end{aligned} \quad (50)$$

Let $\varepsilon_0 > 0$ be the universal constant from Corollary 4.1, and fix

$$0 < \delta < \varepsilon_0.$$

Choosing $b > 0$ sufficiently small, depending only on δ , Markov's inequality and (50) give

$$\nu \{ |P_F x| > \delta \sqrt{r} \} \leq \frac{e^2 b}{\delta^2} \leq \frac{1}{4}.$$

Hence if $A = \{x \in K : |P_F x| \leq \delta \sqrt{r}\}$, we have

$$\nu(A) \geq \frac{3}{4}. \quad (51)$$

By the definition of ν and the moment comparison,

$$\begin{aligned} \int_A |\langle x, u \rangle|^{p-2} d\mu_K(x) &= W \nu(A) \\ &\geq \frac{3}{4} W \geq \frac{3}{4e^2} m^{p-2}. \end{aligned} \quad (52)$$

On the other hand, Hölder's inequality gives

$$\begin{aligned} \int_A |\langle x, u \rangle|^{p-2} d\mu_K(x) &\leq \left(\int_A |\langle x, u \rangle|^p d\mu_K(x) \right)^{(p-2)/p} \mu_K(A)^{2/p} \\ &\leq m^{p-2} \mu_K(A)^{2/p}. \end{aligned} \quad (53)$$

Comparing (52) and (53), we obtain

$$\mu_K(A) \geq e^{-Cp}. \quad (54)$$

Since μ_K is centered and isotropic, the random vector $P_F X$ is centered and isotropic in the r -dimensional space F . It is also log-concave, by Prékopa's theorem. Since $\delta < \varepsilon_0$, Corollary 4.1 therefore gives

$$\mu_K(A) = \mathbb{P} \{ |P_F X| \leq \delta \sqrt{r} \} \leq e^{-cr}. \quad (55)$$

Together with (54), this yields

$$r \leq Cp. \quad (56)$$

We have thus proved in all cases that

$$r = \dim F \leq Cp.$$

Now set

$$E_{p,u} = F^\perp \cap \ker \ell_{p,u}.$$

Notice that $\ell_{p,u}$ is nonzero, since

$$\ell_{p,u}(u) = m^{1-p} \int |\langle x, u \rangle|^p d\mu_K(x) = m > 0.$$

Hence $\ker \ell_{p,u}$ has codimension one, and therefore

$$\text{codim } E_{p,u} \leq r + 1 \leq Cp + 1 \leq C_0p.$$

By the definition of F , for all $v \in F^\perp$ we have

$$Q_{p,u}(v) \geq b|v|^2.$$

Thus, for $v \in E_{p,u}$, we have $\ell_{p,u}(v) = 0$, and consequently

$$D^2\Phi_p(u)[v, v] = (p-1)Q_{p,u}(v) \geq b(p-1)|v|^2 \geq c_0p|v|^2.$$

This proves (47) and (48). □

We now combine the information coming from all dyadic scales. Fix $u \neq 0$ and put

$$H(u) = D^2\Phi(u).$$

By (46),

$$H(u) = \text{Id} + \sum_{\substack{p=2^j \\ 4 \leq p \leq p_*}} p D^2\Phi_p(u) \succeq \text{Id}.$$

Moreover, for $v \in E_{p,u}$,

$$\begin{aligned} \langle H(u)v, v \rangle &\geq |v|^2 + p D^2\Phi_p(u)[v, v] \\ &\geq (1 + c_0p^2)|v|^2. \end{aligned} \tag{57}$$

Let

$$\lambda_1(H(u)) \leq \dots \leq \lambda_n(H(u))$$

be the eigenvalues of $H(u)$. We use the following elementary consequence of the min-max principle: if a symmetric matrix H satisfies

$$\langle Hv, v \rangle \geq L|v|^2$$

on a subspace of codimension at most d , then by the Courant-Fisher formula

$$\#\{i : \lambda_i(H) < L\} \leq d.$$

Applying this observation to (57) and (47), we obtain

$$\#\{i : \lambda_i(H(u)) < 1 + c_0p^2\} \leq C_0p. \tag{58}$$

We now fix the universal constant $a > 0$ in the definition of p_* . By decreasing it if necessary, we may assume that

$$C_0a \leq \frac{1}{4} \quad \text{and} \quad 64c_0C_0^2a^2 \leq 1. \tag{59}$$

Since p_* is the largest dyadic integer satisfying $p_* \leq an$, we have, for n larger than a universal constant,

$$\frac{a}{2}n < p_* \leq an. \tag{60}$$

In particular,

$$C_0p_* \leq C_0an \leq \frac{n}{4}.$$

We claim that, for every $1 \leq i \leq n$,

$$\lambda_i(H(u)) \geq \frac{c_0a^2}{4} i^2. \tag{61}$$

We distinguish three ranges.

First, suppose that

$$16C_0 \leq i \leq C_0 p_*.$$

Choose a dyadic integer p such that

$$\frac{i}{4C_0} \leq p < \frac{i}{2C_0}.$$

Such a p exists by dyadicity. Moreover, $p \geq 4$, while

$$p < \frac{i}{2C_0} \leq \frac{p_*}{2},$$

so $4 \leq p \leq p_*$. Also

$$C_0 p < \frac{i}{2} < i.$$

Hence (58) implies

$$\lambda_i(H(u)) \geq 1 + c_0 p^2 \geq c_0 p^2 \geq \frac{c_0}{16C_0^2} i^2.$$

Since $C_0 a \leq 1/4$, we have

$$\frac{1}{16C_0^2} \geq \frac{a^2}{4},$$

and therefore

$$\lambda_i(H(u)) \geq \frac{c_0 a^2}{4} i^2. \tag{62}$$

Next, suppose that

$$i > C_0 p_*.$$

Applying (58) with $p = p_*$ gives

$$\lambda_i(H(u)) \geq 1 + c_0 p_*^2 \geq c_0 p_*^2.$$

By (60),

$$p_* > \frac{a}{2} n,$$

and hence

$$\lambda_i(H(u)) \geq \frac{c_0 a^2}{4} n^2 \geq \frac{c_0 a^2}{4} i^2,$$

since $i \leq n$.

Finally, if

$$1 \leq i < 16C_0,$$

then $H(u) \succeq \text{Id}$, and consequently

$$\lambda_i(H(u)) \geq 1.$$

On the other hand, by (59),

$$\frac{c_0 a^2}{4} i^2 \leq \frac{c_0 a^2}{4} (16C_0)^2 = 64c_0 C_0^2 a^2 \leq 1.$$

Thus (61) holds in this range as well.

We have therefore proved, uniformly for every $u \neq 0$, that

$$\lambda_i(H(u)) \geq \frac{c_0 a^2}{4} i^2, \quad 1 \leq i \leq n.$$

Consequently,

$$\det H(u) = \prod_{i=1}^n \lambda_i(H(u)) \quad (63)$$

$$\geq \left(\frac{c_0 a^2}{4} \right)^n \prod_{i=1}^n i^2 \quad (64)$$

$$= \left(\frac{c_0 a^2}{4} \right)^n (n!)^2. \quad (65)$$

Using $n! \geq (n/e)^n$, we conclude that for $u \neq 0$

$$\det H(u) \geq \left(\frac{c_0 a^2}{4e^2} n^2 \right)^n. \quad (66)$$

We now pass to the Legendre transform

$$\Psi = \Phi^*.$$

Since

$$\Phi(u) = \frac{1}{2} h_R(u)^2$$

and $h_R = \|\cdot\|_{R^\circ}$, the dual norm of h_R is the Minkowski functional $\|\cdot\|_R$. Therefore

$$\Psi(x) = \frac{1}{2} \|x\|_R^2. \quad (67)$$

The Euclidean term in the definition of Φ gives that for $u \neq 0$

$$D^2\Phi(u) \succeq \text{Id}.$$

Thus Φ is strictly convex, and $\nabla\Phi$ is invertible away from the origin, with inverse $\nabla\Psi$. At corresponding points

$$x = \nabla\Phi(u), \quad u = \nabla\Psi(x),$$

differentiating these inverse relations yields

$$D^2\Psi(x) = [D^2\Phi(\nabla\Psi(x))]^{-1}.$$

Hence (66) gives that for $x \neq 0$

$$\det D^2\Psi(x) \leq \left(\frac{C}{n^2} \right)^n, \quad (68)$$

We next identify the image of R under $\nabla\Psi$. By (67),

$$R = \left\{ x : \Psi(x) \leq \frac{1}{2} \right\},$$

whereas

$$R^\circ = \left\{ u : \Phi(u) \leq \frac{1}{2} \right\}.$$

If

$$u = \nabla\Psi(x),$$

then Euler's identity for the 2-homogeneous function Ψ gives

$$\langle x, u \rangle = \langle x, \nabla\Psi(x) \rangle = 2\Psi(x).$$

On the other hand, equality in Legendre duality gives

$$\Phi(u) + \Psi(x) = \langle x, u \rangle.$$

Consequently,

$$\Phi(u) = \Psi(x). \quad (69)$$

Since $\nabla\Phi$ and $\nabla\Psi$ are inverse maps, (69) shows that

$$\nabla\Psi(R) = R^\circ. \quad (70)$$

The change-of-variables formula, together with (70), gives

$$\begin{aligned} |R^\circ| &= \int_R \det D^2\Psi(x) \, dx \\ &\leq \left(\frac{C}{n^2}\right)^n |R|. \end{aligned} \quad (71)$$

Finally, the Bourgain–Milman reverse Santaló inequality yields

$$|R| |R^\circ| \geq c^n |B_2^n|^2.$$

Combining this with (71), we obtain

$$c^n |B_2^n|^2 \leq \left(\frac{C}{n^2}\right)^n |R|^2.$$

Therefore

$$|R|^{1/n} \geq cn |B_2^n|^{1/n}.$$

Since

$$|B_2^n|^{1/n} \simeq \frac{1}{\sqrt{n}},$$

we conclude that

$$|R|^{1/n} \geq c\sqrt{n}.$$

This proves the proposition. □

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References

- [1] Dominique Bakry and Michel Émery. Diffusions hypercontractives. In Jacques Azéma and Marc Yor, editors, *Séminaire de Probabilités XIX, 1983/84*, volume 1123 of *Lecture Notes in Mathematics*, pages 177–206. Springer, Berlin, 1985. doi : 10.1007/BFb0075847.
- [2] Pierre Bizeul. The slicing conjecture via small ball estimates, 2025. Preprint; to appear in *Annals of Probability*. URL: <https://arxiv.org/abs/2501.06854>.
- [3] Jean Bourgain. On high-dimensional maximal functions associated to convex bodies. *American Journal of Mathematics*, 108(6):1467–1476, 1986. doi : 10.2307/2374532.
- [4] Jean Bourgain. On the distribution of polynomials on high-dimensional convex sets. In *Geometric Aspects of Functional Analysis (1989–90)*, volume 1469 of *Lecture Notes in Mathematics*, pages 127–137. Springer, Berlin, 1991. doi : 10.1007/BFb0089219.

- [5] Herm Jan Brascamp and Elliott H. Lieb. On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *Journal of Functional Analysis*, 22(4):366–389, 1976. doi:10.1016/0022-1236(76)90004-5.
- [6] Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou. *Geometry of Isotropic Convex Bodies*, volume 196 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014. doi:10.1090/surv/196.
- [7] Silouanos Brazitikos and Christos Pandis. Geometric bounds for the mean gauge and the mean width in isotropic position, 2026. Preprint. URL: <https://arxiv.org/abs/2608.07216>.
- [8] Yuansi Chen and Bo’az Klartag. Digesting the proof of the sharp thin-shell inequality, 2026. Preprint. URL: <https://arxiv.org/abs/2607.23307>.
- [9] Nikos Dafnis and Grigoris Paouris. Small ball probability estimates, ψ_2 -behavior and the hyperplane conjecture. *Journal of Functional Analysis*, 258(6):1933–1964, 2010. doi:10.1016/j.jfa.2009.06.038.
- [10] Apostolos Giannopoulos, Minas Pafis, and Natalia Tziotziou. The isotropic constant in the theory of high-dimensional convex bodies. *Bulletin of the Hellenic Mathematical Society*, 69:89–188, 2025.
- [11] Apostolos Giannopoulos, Grigoris Paouris, and Beatrice-Helen Vritsiou. The isotropic position and the reverse Santaló inequality. *Israel Journal of Mathematics*, 203(1):1–22, 2014. doi:10.1007/s11856-012-0173-2.
- [12] Qingyang Guan. A note on bourgain’s slicing problem, 2024. Preprint. URL: <https://arxiv.org/abs/2412.09075>.
- [13] Bo’az Klartag. On convex perturbations with a bounded isotropic constant. *Geometric and Functional Analysis*, 16(6):1274–1290, 2006. doi:10.1007/s00039-006-0588-1.
- [14] Bo’az Klartag. Logarithmic bounds for isoperimetry and slices of convex sets. *Ars Inveniendi Analytica*, pages 1–17, 2023. Paper No. 4.
- [15] Bo’az Klartag and Joseph Lehec. Affirmative resolution of bourgain’s slicing problem using guan’s bound. *Geometric and Functional Analysis*, 35(4):1147–1168, 2025. doi:10.1007/s00039-025-00718-w.
- [16] Bo’az Klartag and Joseph Lehec. Isoperimetric inequalities in high-dimensional convex sets. *Bulletin of the American Mathematical Society*, 62(4):575–642, 2025. doi:10.1090/bull/1869.
- [17] Michel Ledoux. *The Concentration of Measure Phenomenon*, volume 89 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.
- [18] Adrian S. Lewis. Derivatives of spectral functions. *Mathematics of Operations Research*, 21(3):576–588, 1996. doi:10.1287/moor.21.3.576.
- [19] Daniel Murawski. Comparing moments of real log-concave random variables. *Bernoulli*, 32(3), 2026. doi:10.3150/25-BEJ1958.
- [20] R. Tyrrell Rockafellar. *Convex Analysis*, volume 28 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1970.