Asymptotic shape of a random polytope in a convex body

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Abstract

Let $K$ be an isotropic convex body in $\mathbb{R}^n$ and let $Z_q(K)$ be the $L_q$-centroid body of $K$. For every $N > n$ consider the random polytope $K_N := \text{conv}\{x_1, \ldots, x_N\}$ where $x_1, \ldots, x_N$ are independent random points, uniformly distributed in $K$. We prove that a random $K_N$ is “asymptotically equivalent” to $Z_{\ln(N/n)}(K)$ in the following sense: there exist absolute constants $\rho_1, \rho_2 > 0$ such that, for all $\beta \in (0, \frac{1}{2}]$ and all $N \geq N(n, \beta)$, one has:

(i) $K_N \supseteq c(\beta) Z_q(K)$ for every $q \leq \rho_1 \ln(N/n)$, with probability greater than $1 - c_1 \exp(-c_2 N^{-\beta n^\beta})$.

(ii) For every $q \geq \rho_2 \ln(N/n)$, the expected mean width $E[w(K_N)]$ of $K_N$ is bounded by $c_3 w(Z_q(K))$.

As an application we show that the volume radius $|K_N|^{1/n}$ of a random $K_N$ satisfies the bounds $c_4 \sqrt{\frac{\ln(2N/n)}{\sqrt{n}}} \leq |K_N|^{1/n} \leq c_5 L_K \sqrt{\frac{\ln(2N/n)}{\sqrt{n}}}$ for all $N \leq \exp(n)$.

1 Introduction

Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$. For every $q \geq 1$ we define the $L_q$-centroid body $Z_q(K)$ of $K$ by its support function:

$$h_{Z_q(K)}(x) = \|\langle \cdot, x \rangle\|_q := \left( \int_K |\langle y, x \rangle|^q dy \right)^{1/q}.$$  

The aim of this article is to provide some precise quantitative information on the “asymptotic shape” of a random polytope $K_N = \text{conv}\{x_1, \ldots, x_N\}$ spanned by $N$ independent random points $x_1, \ldots, x_N$ uniformly distributed in $K$. Our approach is to compare $K_N$ with the $L_q$-centroid body $Z_q(K)$ of $K$ for $q \simeq \ln(N/n)$.

The origin of our work is in the study of the behavior of symmetric random $\pm 1$-polytopes, the absolute convex hulls of random subsets of the discrete cube $E_n^2 = \{-1, 1\}^n$. The natural way to produce these random polytopes is to fix $N > n$ and to consider the convex hull $K_{n,N} = \text{conv}\{ \pm \tilde{X}_1, \ldots, \pm \tilde{X}_N \}$ of $N$ independent random points $\tilde{X}_1, \ldots, \tilde{X}_N$, uniformly distributed over $E_n^2$. It turns out...
(see [9]) that a random polytope $K_{n,N}$ has the largest possible volume among all $\pm 1$-polytopes with $N$ vertices, at every scale of $n$ and $N$. This is a consequence of the following fact: If $n \geq n_0$ and if $N \geq n(\ln n)^2$, then

$$K_{n,N} \supseteq c\left(\sqrt{\ln(N/n)}B_2^n \cap B_2^n\right)$$

with probability greater than $1 - e^{-n}$, where $c > 0$ is an absolute constant, $B_2^n$ is the Euclidean unit ball in $\mathbb{R}^n$ and $B_\infty^n = [-1,1]^n$.

In [16], Litvak, Pajor, Rudelson, and Tomczak–Jaegermann worked in a more general setting which contains the previous Bernoulli model and the Gaussian model; let $K_{n,N}$ be the absolute convex hull of the rows of the random matrix $\Gamma_{n,N} = (\xi_{ij})_{1 \leq i \leq N, 1 \leq j \leq n}$, where $\xi_{ij}$ are independent symmetric random variables satisfying certain conditions ($\|\xi_{ij}\|_{L^2} \geq 1$ and $\|\xi_{ij}\|_{L^\psi} \leq \rho$ for some $\rho \geq 1$, where $\| \cdot \|_{L^\psi}$ is the Orlicz norm corresponding to the function $\psi(t) = e^{t^2} - 1$). For this larger class of random polytopes, the estimates from [9] were generalized and improved in two ways: the paper [16] provides estimates for all $N \geq (1+\delta)n$, where $\delta > 0$ can be as small as $1/\ln n$, and establishes the following inclusion: for every $0 < \beta < 1$,

$$K_{n,N} \supseteq c(\rho)\left(\sqrt{\beta \ln(N/n)}B_2^n \cap B_\infty^n\right)$$

with probability greater than $1 - \exp(-c_1 n^\beta N^{1-\beta}) - \exp(-c_2 N)$. The proof in [16] is based on a lower bound of the order of $\sqrt{N}$ for the smallest singular value of the random matrix $\Gamma_{n,N}$ with probability greater than $1 - \exp(-cN)$.

In a sense, both works correspond to the study of the size of a random polytope $K_N = \text{conv}\{x_1, \ldots, x_N\}$ spanned by $N$ independent random points $x_1, \ldots, x_N$ uniformly distributed in the unit cube $Q_n := [-1/2,1/2]^n$. The connection of the estimates (1.2) and (1.3) with $L_q$-centroid bodies comes from the following observation.

**Remark.** For $x \in \mathbb{R}^n$ and $t > 0$, define

$$K_{1,2}(x,t) := \inf \{\|u\|_1 + t\|x - u\|_2 : u \in \mathbb{R}^n\}.$$  

If we write $(x^*_j)_{j \leq n}$ for the decreasing rearrangement of $(|x_j|)_{j \leq n}$ we have Holmstrom’s approximation formula

$$K_{1,2}(x,t) \leq \sum_{j=1}^{[t^2]} x^*_j + t \left(\sum_{j=\lfloor t^2 \rfloor + 1}^{n} (x^*_j)^2\right)^{1/2} \leq K_{1,2}(x,t)$$

where $c > 0$ is an absolute constant (see [14]). Now, for any $\alpha \geq 1$ define $C(\alpha) = \alpha B_2^n \cap B_\infty^n$. Then,

$$h_{C(\alpha)}(\theta) = K_{1,2}(\theta, \alpha)$$
for every $\theta \in S^{n-1}$. On the other hand,

\begin{equation}
\|\langle \cdot, \theta \rangle\|_{L^q(Q_n)} \simeq \sum_{j \leq q} \theta_j^* + \sqrt{q} \left( \sum_{q < j \leq n} (\theta_j^*)^2 \right)^{1/2}
\end{equation}

for every $q \geq 1$ (see, for example, [6]). In other words,

\begin{equation}
C(\sqrt{q}) \simeq Z_q(Q_n)
\end{equation}

where $Z_q(K)$ is the $L_q$-centroid body of $K$. This shows that (1.3) or (1.2) can be written in the form

\begin{equation}
K_{n,N} \supseteq c(\rho) Z_{\beta \ln(N/n)}(Q_n).
\end{equation}

This observation leads us to consider a random polytope $K_N = \text{conv}\{x_1, \ldots, x_N\}$ spanned by $N$ independent random points $x_1, \ldots, x_N$ uniformly distributed in an isotropic convex body $K$ and try to compare $K_N$ with $Z_q(K)$ for a suitable value $q = q(N, n) \simeq \ln(N/n)$. Our first main result states that an analogue of (1.9) holds true in full generality.

**Theorem 1.1** Let $\beta \in (0, 1/2]$ and $\gamma > 1$. If

\begin{equation}
N \geq N(\gamma, n) = c_1 \gamma n,
\end{equation}

where $c > 0$ is an absolute constant, for every isotropic convex body $K$ in $\mathbb{R}^n$ we have

\begin{equation}
K_N \supseteq c_2 Z_q(K) \text{ for all } q \leq c_2 \beta \ln(N/n),
\end{equation}

with probability greater than

\begin{equation}
1 - \exp\left(-c_3 N^{1-\beta} n^3\right) - \mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^n\| \geq \gamma L_K \sqrt{N}),
\end{equation}

where $\Gamma : \ell_2^n \to \ell_2^n$ is the random operator $\Gamma(y) = (\langle x_1, y \rangle, \ldots, \langle x_N, y \rangle)$ defined by the vertices $x_1, \ldots, x_N$ of $K_N$.

The proof of Theorem 1.1 is given in Section 2, where we also collect what is known about the probability $\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^n\| \geq \gamma L_K \sqrt{N})$ which appears in (1.12).

It should be emphasized that a reverse inclusion of the form $K_N \subseteq c_4 Z_q(K)$ cannot be expected with probability close to 1, unless $q$ is of the order of $n$. This follows by a simple volume argument which makes use of the upper estimate of Paouris (see [20]) for the volume of $Z_q(K)$ and is presented in Section 3. However, one can easily see that $K_N$ is “weakly sandwiched” between $Z_{q_i}(K)$ ($i = 1, 2$), where $q_i \simeq \ln(N/n)$, in the following sense:

**Proposition 1.2** For every $\alpha > 1$ one has

\begin{equation}
\mathbb{E} \left[ \sigma(\theta : (h_{K_N}(\theta) \geq \alpha h_{Z_{q_i}(K)}(\theta))) \right] \leq N \sigma^{-q_i}.
\end{equation}
This shows that if \( q \geq c_5 \ln(N/n) \) then, for most \( \theta \in S^{n-1} \), one has \( h_{K_N}(\theta) \leq c_6 h_{Z_q(K)}(\theta) \). It follows that several geometric parameters of \( K_N \), e.g. the mean width, are controlled by the corresponding parameter of \( Z_{\ln(N/n)}(K) \).

As an application, we discuss the volume radius of \( K_N \): Let \( K \) be a convex body of volume 1 in \( \mathbb{R}^n \). The question to estimate the expected volume radius

\[
\mathbb{E}(K, N) = \int_K \cdots \int_K |\text{conv}(x_1, \ldots, x_N)|^{1/n} dx_N \cdots dx_1
\]

of \( K_N \) was studied in [12] where it was proved that for every isotropic convex body \( K \) in \( \mathbb{R}^n \) and every \( N \geq n + 1 \),

\[
\mathbb{E}(B(n), N) \leq \mathbb{E}(K, N) \leq cL_K \frac{\ln(2N/n)}{\sqrt{n}}
\]

where \( B(n) \) is a ball of volume 1. This estimate is rather weak for large values of \( N \): a strong conjecture is that

\[
\mathbb{E}(K, N) \simeq \min \left\{ \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}}, 1 \right\} L_K
\]

for every \( N \geq n + 1 \). This was verified in [10] in the unconditional case, where it was also shown that the general problem is related to the \( \psi_2 \)-behavior of linear functionals on isotropic convex bodies. Using a recent result of G. Paouris [21] on the negative moments of the support function of \( h_{Z_q(K)} \) we can settle the question for the full range of values of \( N \).

**Theorem 1.3** For every \( N \leq \exp(n) \), one has

\[
c_4 \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}} \leq |K_N|^{1/n} \leq c_5 L_K \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}}
\]

with probability greater than \( 1 - \frac{1}{N} \), where \( c_4, c_5 > 0 \) are absolute constants.

**Notation and terminology.** We work in \( \mathbb{R}^n \), which is equipped with a Euclidean structure \( \langle \cdot, \cdot \rangle \). We denote by \( \| \cdot \|_2 \) the corresponding Euclidean norm, and write \( B_2^n \) for the Euclidean unit ball, and \( S^{n-1} \) for the unit sphere. Volume is denoted by \( | \cdot | \). We write \( \omega_n \) for the volume of \( B_2^n \) and \( \sigma \) for the rotationally invariant probability measure on \( S^{n-1} \). We also write \( A \) for the homothetic image of volume 1 of a convex body \( A \subseteq \mathbb{R}^n \), i.e. \( A := \frac{A}{|A|^{1/n}} \).

A convex body is a compact convex subset \( C \) of \( \mathbb{R}^n \) with non-empty interior. We say that \( C \) is symmetric if \( -x \in C \) whenever \( x \in C \). We say that \( C \) has center of mass at the origin if \( \int_C \langle x, \theta \rangle dx = 0 \) for every \( \theta \in S^{n-1} \). The support function \( h_C : \mathbb{R}^n \to \mathbb{R} \) of \( C \) is defined by \( h_C(x) = \max \{ \langle x, y \rangle : y \in C \} \). The mean width of \( C \) is defined by

\[
w(C) = \int_{S^{n-1}} h_C(\theta) \sigma(d\theta).
\]
The radius of $C$ is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$, and the polar body $C^\circ$ of $C$ is

\begin{equation}
C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in C\}.
\end{equation}

Whenever we write $a \simeq b$, we mean that there exist universal constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. The letters $c, c', c_1, c_2 > 0$ etc., denote universal positive constants which may change from line to line.

A convex body $K$ in $\mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, center of mass at the origin, and there is a constant $L_K > 0$ such that

\begin{equation}
\int_K \langle x, \theta \rangle^2 dx = L_K^2.
\end{equation}

for every $\theta$ in the Euclidean unit sphere $S^{n-1}_2$. For every convex body $K$ in $\mathbb{R}^n$ there exists an affine transformation $T$ of $\mathbb{R}^n$ such that $T(K)$ is isotropic. Moreover, if we ignore orthogonal transformations, this isotropic image is unique, and hence, the isotropic constant $L_K$ is an invariant of the affine class of $K$. We refer to [18] and [8] for more information on isotropic convex bodies.

2 The main inclusion

In this Section we prove Theorem 1.1. Let $K$ be an isotropic convex body in $\mathbb{R}^n$. For every $q \geq 1$ consider the $L_q$–centroid body $Z_q(K)$ of $K$; recall that

\begin{equation}
h_{Z_q(K)}(x) = \|\langle \cdot, x \rangle\|_q := \left(\int_K |\langle y, x \rangle|^q dy\right)^{1/q}.
\end{equation}

Since $|K| = 1$, we readily see that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for every $1 \leq p \leq q \leq \infty$, where $Z_\infty(K) = \text{conv}\{K, -K\}$. On the other hand, one has the reverse inclusions

\begin{equation}
Z_q(K) \subseteq \frac{cq}{p} Z_p(K)
\end{equation}

for every $1 \leq p < q < \infty$, as a consequence of the $\psi_1$–behavior of $y \mapsto \langle y, x \rangle$. Observe that $Z_q(K)$ is always symmetric, and $Z_q(TK) = T(Z_q(K))$ for every $T \in SL(n)$ and $q \in [1, \infty]$. Also, if $K$ has its center of mass at the origin, then $Z_q(K) \supseteq cZ_\infty(K)$ for all $q \geq n$, where $c > 0$ is an absolute constant. We refer to [8] for proofs of these assertions and further information.

Lemma 2.1 Let $0 < t < 1$. For every $\theta \in S^{n-1}$ one has

\begin{equation}
P \left(\{x \in K : |\langle x, \theta \rangle| \geq t \|\langle \cdot, \theta \rangle\|_q\}\right) \geq \frac{(1 - t^q)^2}{Cq}.
\end{equation}
Proof. We apply the Paley-Zygmund inequality

\[ P(g(x) \geq t^q \mathbb{E}(g)) \geq (1 - t^q)^2 \frac{\mathbb{E}(g)^2}{\mathbb{E}(g^2)} \]

for the function \( g(x) = |\langle x, \theta \rangle|^q \). Since, by (2.2),

\[ \mathbb{E}(g^2) = \mathbb{E}|\langle x, \theta \rangle|^2q \leq C^q (\mathbb{E}|\langle x, \theta \rangle|^q)^2 = C^q [\mathbb{E}(g)]^2 \]

for some absolute constant \( C > 0 \), the lemma is proved. \( \square \)

**Lemma 2.2** For every \( \sigma \subseteq \{1, \ldots, N\} \) and any \( \theta \in S^{n-1} \) one has

\[ P(\{ \tilde{X} = (x_1, \ldots, x_N) \in K_N : \max_{j \in \sigma} |\langle x_j, \theta \rangle| \leq \frac{1}{2} \|\langle \cdot, \theta \rangle\|_q \}) \leq \exp \left( -|\sigma|/(4C^q) \right), \]

where \( C > 0 \) is an absolute constant.

**Proof.** Applying Lemma 2.1 with \( t = 1/2 \) we see that

\[ P \left( \max_{j \in \sigma} |\langle x_j, \theta \rangle| \leq \frac{1}{2} \|\langle \cdot, \theta \rangle\|_q \right) = \prod_{j \in \sigma} P \left( |\langle x_j, \theta \rangle| \leq \frac{1}{2} \|\langle \cdot, \theta \rangle\|_q \right) \]

\[ \leq \left( 1 - \frac{1}{4C^q} \right)^{|\sigma|} \leq \exp \left( -|\sigma|/(4C^q) \right), \]

since \( 1 - v < e^{-v} \) for every \( v > 0 \). \( \square \)

**Proof of Theorem 1.1.** Let \( \Gamma : \ell_2^n \rightarrow \ell_2^N \) be the random operator defined by

\[ \Gamma(y) = (\langle x_1, y \rangle, \ldots, \langle x_N, y \rangle). \]

We modify an idea from [16]. Define \( m = \lceil 8(N/n)^{2/3} \rceil \) and \( k = \lceil N/m \rceil \). Fix a partition \( \sigma_1, \ldots, \sigma_k \) of \( \{1, \ldots, N\} \) with \( m \leq |\sigma_i| \) for all \( i = 1, \ldots, k \) and define the norm

\[ \|u\|_0 = \frac{1}{k} \sum_{i=1}^k \|P_{\sigma_i}(u)\|_\infty. \]

Since

\[ h_{K_N}(z) = \max_{1 \leq j \leq N} |\langle x_j, z \rangle| \geq \|P_{\sigma_i}(\Gamma(z))\|_\infty \]

for all \( z \in \mathbb{R}^n \) and \( i = 1, \ldots, k \), we observe that

\[ h_{K_N}(z) \geq \|\Gamma(z)\|_0. \]
If \( z \in \mathbb{R}^n \) and \( \| \Gamma(z) \|_0 < \frac{1}{4} \| \langle \cdot, z \rangle \|_q \), then, Markov’s inequality implies that there exists \( I \subset \{1, \ldots, k\} \) with \( |I| > k/2 \) such that \( \| P_{\sigma_i} \Gamma(z) \|_\infty < \frac{1}{2} \| \langle \cdot, z \rangle \|_q \), for all \( i \in I \). It follows that, for fixed \( z \in S^{n-1} \) and \( \alpha \geq 1 \),

\[
P \left( \| \Gamma(z) \|_0 < \frac{1}{4} \| \langle \cdot, z \rangle \|_q \right) \leq \sum_{|I|=[(k+1)/2]} \prod_{i \in I} P \left( \| P_{\sigma_i} \Gamma(z) \|_\infty < \frac{1}{2} \| \langle \cdot, z \rangle \|_q \right)
\]

\[
\leq \sum_{|I|=[(k+1)/2]} \prod_{i \in I} \exp \left( -|\sigma_i|/(4C^q) \right)
\]

\[
\leq \left( \frac{k}{[(k+1)/2]} \right) \exp \left( -c_1 km/C^q \right)
\]

\[
\leq \exp \left( k \ln 2 - c_1 km/C^q \right).
\]

Choosing

(2.11) \[ q \simeq \beta \ln(N/n) \]

we see that

(2.12) \[ P \left( \| \Gamma(z) \|_0 < \frac{1}{4} \| \langle \cdot, z \rangle \|_q \right) \leq \exp \left( -c_2 N^{1-\beta} n^\beta \right). \]

Let \( S = \{ z : \| \langle \cdot, z \rangle \|_q/2 = 1 \} \) and consider a \( \delta \)-net \( U \) of \( S \) with cardinality \( |U| \leq (3/\delta)^n \). For every \( u \in U \) we have

(2.13) \[ P \left( \| \Gamma(u) \|_0 < \frac{1}{2} \right) \leq \exp \left( -c_2 N^{1-\beta} n^\beta \right), \]

and hence,

(2.14) \[ P \left( \bigcup_{u \in U} \left\{ \| \Gamma(u) \|_0 < \frac{1}{2} \right\} \right) \leq \exp \left( n \ln(3/\delta) - c_2 N^{1-\beta} n^\beta \right). \]

Fix \( \gamma > 1 \) and set

(2.15) \[ \Omega_\gamma = \{ \Gamma : \| \Gamma : \ell_2^N \to \ell_2^N \| \leq \gamma L_K \sqrt{N} \}. \]

Since \( Z_q(K) \supseteq c_L K B_2^q \), we have

(2.16) \[ \| \Gamma(z) \|_0 \leq \frac{1}{\sqrt{k}} \| \Gamma(z) \|_2 \leq c_L K \sqrt{N/k} \| z \|_2 \leq c_L \sqrt{N/k} \| \langle \cdot, z \rangle \|_q \]

for all \( z \in \mathbb{R}^n \) and all \( \Gamma \) in \( \Omega_\gamma \).
Let \( z \in S \). There exists \( u \in U \) such that \( \frac{1}{2}\|\langle \cdot, z - u \rangle\|_q < \delta \), which implies that

\[(2.17) \quad \|\Gamma(u)\|_0 \leq \|\Gamma(z)\|_0 + c\gamma\delta\sqrt{N/k}\]
on \( \Omega_\gamma \). Now, choose \( \delta = \sqrt{k/N/(4c\gamma)} \). Then,

\[
\mathbb{P}\left( \{ \Gamma \in \Omega_\gamma : \exists z \in \mathbb{R}^n : \|\Gamma(z)\|_0 \leq \|\langle \cdot, z \rangle\|_q/8 \} \right)
\]
\[
= \mathbb{P}\left( \{ \Gamma \in \Omega_\gamma : \exists z \in \mathbb{R}^n : \|\Gamma(z)\|_0 \leq 1/4 \} \right)
\]
\[
\leq \mathbb{P}\left( \{ \Gamma \in \Omega_\gamma : \exists u \in U : \|\Gamma(u)\|_0 \leq 1/2 \} \right)
\]
\[
\leq \exp \left( n \ln(12c\gamma\sqrt{N/k}) - c_2N^{1-\beta}n^\beta \right)
\]
\[
\leq \exp \left( -c_3N^{1-\beta}n^\beta \right)
\]

provided that \( N \) is large enough. Since \( h_{K_N}(z) \geq \|\Gamma(z)\|_0 \) for every \( z \in \mathbb{R}^n \), we get that \( K_N \geq cZ_q(K) \) with probability greater than \( 1 - \exp \left( -c_4N^{1-\beta}n^\beta \right) - \mathbb{P}(\|\Gamma : \ell_2^n \rightarrow \ell_2^N \| \geq \gamma L_K\sqrt{N}) \).

We now analyze the restriction for \( N \); we need \( n \ln(12c\gamma\sqrt{N/k}) \leq CN^{1-\beta}n^\beta \) for some suitable constant \( C > 0 \). Assuming

\[(2.18) \quad N \geq 12c\gamma n,\]

and since \( \beta \in (0, \frac{1}{2}] \), using the definitions of \( k \) and \( m \) we see that it is enough to guarantee

\[\ln(N/n) \leq C\sqrt{N/n},\]

which is valid if \( N/n \geq c_5 \) for a suitable absolute constant \( c_5 > 0 \). We get the result taking (2.18) into account.

**Remark 2.3** The statement of Theorem 1.1 raises the question to estimate the probability

\[(2.19) \quad \mathbb{P}(\Omega_\gamma) = \mathbb{P}(\|\Gamma : \ell_2^n \rightarrow \ell_2^N \| \geq \gamma L_K\sqrt{N}).\]

In [16] it was proved that if \( I_{n,N} = (\xi_{ij})_{1 \leq i \leq n, 1 \leq j \leq N} \) is a random matrix, where \( \xi_{ij} \) are independent symmetric random variables satisfying \( \|\xi_{ij}\|_{L_2} \geq 1 \) and \( \|\xi_{ij}\|_{L_\infty} \leq \rho \) for some \( \rho \geq 1 \), then \( \mathbb{P}(\Omega_\gamma) \leq \exp(-c(\rho, \gamma)N) \). In our case, \( \Gamma \) is a random \( N \times n \) matrix whose rows are \( N \) uniform random points from an isotropic convex body \( K \) in \( \mathbb{R}^n \). Then, the question is to estimate the probability that, \( N \) random points \( x_1, \ldots, x_N \) from \( K \) satisfy

\[(2.20) \quad \frac{1}{N} \sum_{j=1}^{N} (x_j, \theta)^2 \leq \gamma^2L_K^2\]

for all \( \theta \in S^{n-1} \). This is related to the following well-studied question of Kannan, Lovász and Simonovits [15] which has its origin in the problem of finding a fast
algorithm for the computation of the volume of a given convex body: given \( \delta, \varepsilon \in (0, 1) \), find the smallest positive integer \( N_0(n, \delta, \varepsilon) \) so that if \( N \geq N_0 \) then with probability greater than \( 1 - \delta \) one has

\[
(1 - \varepsilon)L_K^2 \leq \frac{1}{N} \sum_{j=1}^{N} (x_j, \theta)^2 \leq (1 + \varepsilon)L_K^2
\]

for all \( \theta \in S^{n-1} \). In [15] it was proved that one can have \( N_0 \simeq c(\delta, \varepsilon)n^2 \), which was later improved to \( N_0 \simeq c(\delta, \varepsilon)n(\ln n)^3 \) by Bourgain [2] and to \( N_0 \simeq c(\delta, \varepsilon)n(\ln n)^2 \) by Rudelson [24]. One can actually check (see [11]) that this last estimate can be obtained by Bourgain’s argument if we also use Alesker’s concentration inequality. For subsequent developments, see see, for instance, [20], [13], [17] and [1].

Here, we are only interested in the upper bound of (2.21): actually, we need an isomorphic version of this upper estimate, since we are allowed to choose an absolute constant \( \gamma \gg 1 \) in (2.20). An application of the main result of [17] to the isotropic case gives such an estimate: If \( N \geq c_1 n \ln^2 n \), then

\[
P(\| \Gamma : \ell_2^n \rightarrow \ell_2^N \| \geq \gamma L_K \sqrt{N}) \leq \exp \left( -c_2 \gamma \left( \frac{N}{(\ln N)(n \ln n)} \right)^{1/4} \right)
\]

A slightly better estimate can be extracted from the work of Guédon and Rudelson in [13]. It should be emphasized that this term does not allow us to fully exploit the second term \( \exp \left( -c_3 N^{1-\beta} n^{\delta} \right) \) in the probability estimate of Theorem 1.1. However, it is not clear if it is optimal.

**Remark 2.4** G. Paouris and E. Werner [22] have recently studied the relation between the family of \( L_q \)-centroid bodies and the family of floating bodies of a convex body \( K \). Given \( \delta \in (0, \frac{1}{2}] \), the floating body \( K_\delta \) of \( K \) is the intersection of all halfspaces whose defining hyperplanes cut off a set of volume \( \delta \) from \( K \). It was observed in [18] that \( K_\delta \) is isomorphic to an ellipsoid as long as \( \delta \) stays away from 0. In [22] it is proved that

\[
c_1 Z_{\ln(1/\delta)}(K) \subseteq K_\delta \subseteq c_2 Z_{\ln(1/\delta)}(K)
\]

where \( c_1, c_2 > 0 \) are absolute constants. From Theorem 1.1 it follows that if \( K \) is isotropic and if, for example, \( N \geq n^2 \) then

\[
K_N \supseteq c_3 K_{1/N}
\]

with probability greater than \( 1 - o_n(1) \), where \( c_3 > 0 \) is an absolute constant. This fact should be compared with the following well-known result from [3]: for any convex body \( K \) in \( \mathbb{R}^n \) one has \( c|K_{1/N}| \leq E|K_N| \leq c_n|K_{1/N}| \) (where the constant on the left is absolute and the right hand side inequality holds true with a constant \( c_n \) depending on the dimension, for \( N \) large enough; the critical value of \( N \) is exponential in \( n \)).
2.1 Unconditional case

In this subsection we consider separately the case of unconditional convex bodies: we assume that $K$ is centrally symmetric and that, after a linear transformation, the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ is a 1-unconditional basis for $\| \cdot \|_K$, i.e. for every choice of real numbers $t_1, \ldots, t_n$ and every choice of signs $\varepsilon_j = \pm 1$,

$$\|\varepsilon_1 t_1 e_1 + \cdots + \varepsilon_n t_n e_n\|_K = \|t_1 e_1 + \cdots + t_n e_n\|_K.$$  

Then, a diagonal operator brings $K$ to the isotropic position. It is also known that the isotropic constant of an unconditional convex body $K$ satisfies $L_K \simeq 1$.

Bobkov and Nazarov have proved that $K \supseteq c_2 Q_n$, where $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ (see [4]). The following argument of R. Latała (private communication) shows that the family of $L_q$-centroid bodies of the cube $Q_n$ is extremal in the sense that $Z_q(K) \supseteq cZ_q(Q_n)$ for all $q \geq 1$, where $c > 0$ is an absolute constant: Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be independent and identically distributed $\pm 1$ random variables, defined on some probability space $(\Omega, \mathcal{F}, P)$, with distribution $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$. For every $\theta \in S^{n-1}$, by the unconditionality of $K$, Jensen’s inequality and the contraction principle, one has

$$\|\langle \cdot, \theta \rangle\|_{L^q(K)} = \left( \int_K \left| \sum_{i=1}^n \theta_i x_i \right|^q dx \right)^{1/q} \geq \left( \int_{Q_n} \sum_{i=1}^n t_i \varepsilon_i \int_K |x_i| dx \right)^{1/q} = \left( \int_{Q_n} \sum_{i=1}^n t_i \varepsilon_i \right)^{1/q} \geq \left( \int_{Q_n} \sum_{i=1}^n t_i \theta_i y_i \right)^{1/q} = \|\langle \cdot, \theta \rangle\|_{L^q(Q_n)},$$

where $t_i = \int_K |x_i| dx$ and $\theta = (t_1 \theta_1, \ldots, t_n \theta_n)$. Since $t_i \simeq 1$ for all $i = 1, \ldots, n$, from (1.7) we readily see that

$$\|\langle \cdot, \theta \rangle\|_{L^q(K)} \geq \|\langle \cdot, (\theta) \rangle\|_{L^q(K)} \geq c \|\langle \cdot, \theta \rangle\|_{L^q(Q_n)}.$$}

In view of (1.8), this observation and Theorem 1.1 show that, if $K$ is unconditional, then a random $K_N$ contains $Z_{\ln(n/n)}(Q_n)$:

**Theorem 2.5** Let $\beta \in (0, 1/2]$ and $\gamma > 1$. There exists an absolute constant $c > 0$ so that if

$$N \geq N(\gamma, n) = c \gamma n,$$

and if $K_N = \text{conv}\{x_1, \ldots, x_N\}$ is a random polytope spanned by $N$ independent random points $x_1, \ldots, x_N$ uniformly distributed in an unconditional isotropic convex body $K$ in $\mathbb{R}^n$, then we have

$$K_N \supseteq c_1 C(\alpha) = c_1 \left( \alpha B^n_2 \cap B^n_\infty \right) \text{ for all } \alpha \leq c_2 \sqrt{\beta \ln(n/n)},$$

\[10]
with probability greater than
\begin{equation}
1 - \exp\left(-c_3N^{1-\beta}n^\beta\right) - \mathbb{P}(\|\Gamma: \ell_2^n \to \ell_2^N\| \geq \gamma \sqrt{N}),
\end{equation}
where $\Gamma: \ell_2^n \to \ell_2^N$ is the random operator $\Gamma(y) = (\langle x_1, y \rangle, \ldots, \langle x_N, y \rangle)$ defined by the vertices $x_1, \ldots, x_N$ of $K_N$.

Next, we outline a direct proof of Theorem 2.5 (in which $L_q$-centroid bodies are not involved): For $k \in \mathbb{N}$ and $y \in \mathbb{R}^n$, define
\begin{equation}
\|y\|_{P(k)} := \sup \left\{ \sum_{i=1}^k \left( \sum_{j \in B_i} y_j^2 \right)^{1/2} : \bigcup_{i=1}^k B_i = [n], B_i \cap B_j = \emptyset (i \neq j) \right\},
\end{equation}
where we write $[n]$ for the set $\{1, 2, \ldots, n\}$. Montgomery–Smith has shown (see [19]) that: For any $y \in \mathbb{R}^n$ and $k \in \mathbb{N}$, one has
\begin{equation}
\mathbb{P}\left( \sum_{i=1}^n \epsilon_i y_i \geq \lambda \|y\|_{P(k)} \right) \geq \left( \frac{1}{3} \right)^k (1 - 2\lambda^2)^{2k} \quad (0 \leq \lambda \leq 1/\sqrt{2}).
\end{equation}
Also, for $y \in \mathbb{R}^n$, one has
\begin{equation}
\|y\|_{P(t^2)} \leq K_{1,2}(y, t) \leq \sqrt{2} \|y\|_{P(t^2)}
\end{equation}
when $t^2 \in \mathbb{N}$, from where one concludes the following:

**Lemma 2.6** There exists a constant $c > 0$ such that, for all $y \in \mathbb{R}^n$ and any $t > 0$,
\begin{equation}
\mathbb{P}\left( \sum_{i=1}^n \epsilon_i y_i \geq \lambda K_{1,2}(y, t) \right) \geq e^{-\phi(\lambda)t^2},
\end{equation}
where $\phi(\lambda) = 4 \ln(3(1 - 2\lambda^2)^{-2})$ for $0 < \lambda < 1/\sqrt{2}$.

P. Pivovarov [23] has recently obtained the following result: There exists an absolute constant $C \geq 1$ such that for any unconditional isotropic convex body $K$ in $\mathbb{R}^n$, the spherical measure of the set of $\theta \in S^{n-1}$ such that
\begin{equation}
\mathbb{P}(\|x, \theta\| \geq t) \geq \exp(-Ct^2)
\end{equation}
whenever $C \leq t \leq \sqrt{\frac{2\ln n}{2n}}$, is at least $1 - 2^{-n}$. The proof of the next Lemma follows more or less the same lines.

**Lemma 2.7** Let $K$ be an isotropic unconditional convex body in $\mathbb{R}^n$. For every $\theta \in S^{n-1}$ and any $\alpha \geq 1$, we have
\begin{equation}
\mathbb{P}_x(\langle x, \theta \rangle \geq h_{C(\alpha)}(\theta)) \geq c_1 e^{-c_2 \alpha^2}.
\end{equation}
Proof. For $\theta = (\theta_i)_{i=1}^n \in S^{n-1}$, $x \in K$ and $0 < s < 1/\sqrt{2}$ define the set

\begin{equation}
K_s(\theta) = \{ x \in K : K_{1,2}(\theta, \alpha) \leq sK_{1,2}(x \theta, \alpha) \},
\end{equation}

where by “$x \theta$” we mean the vector with coordinates $x_i \theta_i$ and $s$ is to be chosen. We have:

\begin{equation}
\mathbb{P}_x \left( \sum_{i=1}^n x_i \theta_i \geq h_{C(\alpha)}(\theta) \right) = \mathbb{P}_x \left( \sum_{i=1}^n \varepsilon_i x_i \theta_i \geq h_{C(\alpha)}(\theta) \right)
\end{equation}

\begin{align*}
&= \int_K \mathbb{P}_x \left( \sum_{i=1}^n \varepsilon_i (x_i \theta_i) \geq h_{C(\alpha)}(\theta) \right) dx \\
&= \int_K \mathbb{P}_x \left( \sum_{i=1}^n \varepsilon_i (x_i \theta_i) \geq K_{1,2}(\theta, \alpha) \right) dx \\
&\geq \int_{K_{1,2}(\theta)} \mathbb{P}_x \left( \sum_{i=1}^n \varepsilon_i (x_i \theta_i) \geq sK_{1,2}(x \theta, \alpha) \right) dx \\
&\geq e^{-\phi(s)\alpha^2} |K_s(\theta)|,
\end{align*}

by Lemma 2.6.

Assume first that $m := \alpha^2$ is an integer and let $B_1, B_2, \ldots, B_m$ be a partition of the set $\{1, 2, \ldots, n\}$ so that

\begin{equation}
K_{1,2}(\theta, \alpha) = \sum_{i=1}^m \left( \sum_{j \in B_i} |\theta_j|^2 \right)^{1/2} =: A.
\end{equation}

Consider the seminorm

\begin{equation}
f(x) = \sum_{i=1}^m \left( \sum_{j \in B_i} |x_j \theta_j|^2 \right)^{1/2}.
\end{equation}

Then, using the reverse Hölder inequality $c_1 \|f\|_{L^2(K)} \leq \|f\|_{L^1(K)}$ and the fact that $L_K \simeq 1$, we get

\begin{align*}
\int_K K_{1,2}(x \theta, \alpha) dx &\geq \int_K \sum_{i=1}^m \left( \sum_{j \in B_i} |x_j \theta_j|^2 \right)^{1/2} \\
&\geq c_1 \sum_{i=1}^m \left( \sum_{j \in B_i} |\theta_j|^2 \int_K |x_j|^2 \right)^{1/2} \\
&\geq cA.
\end{align*}

We now apply the Paley-Zygmund inequality to get

\begin{equation}
|K_s(\theta)| = \mathbb{P}_x (f > sA) \geq \frac{(\mathbb{E}|f|^2 - (sA)^2)^2}{\mathbb{E}[f^4]}.
\end{equation}
Choosing \( s = \frac{1}{2\sqrt{2}} \min\{c, 1\} \) we get

\[
|K_s(\theta)| \geq cA^4 \frac{E[f^4]}{\|f\|^4},
\]

for a suitable new absolute constant \( c > 0 \). On the other hand, we can estimate \( E[f^4] \) from above, by the reverse Hölder inequality:

\[
\left( E[f^4] \right)^{1/4} \leq 4cL K m \sum_{i=1}^m \left( \sum_{j \in B_i} |\theta_j|^2 \right)^{1/2} \leq 4cA.
\]

As a result, \( |K_s(\theta)| \geq c \). Returning to the estimate (2.38)

\[
P_X \left( \sum_{i=1}^n x_i \theta_i \geq h_{C(\alpha)}(\theta) \right) \geq e^{-\phi(s)\alpha^2} |K_s(\theta)|,
\]

we get:

(2.39)

\[
P_X \left( \sum_{i=1}^n x_i \theta_i \geq h_{C(\alpha)}(\theta) \right) \geq ce^{-c\alpha^2}.
\]

This proves the Lemma for \( \alpha^2 \in \mathbb{N} \) and the result follows easily for every \( \alpha \). \( \square \)

**Proof of Theorem 2.5.** Now, using the procedure of the proof of Theorem 1.1 we complete the proof of Theorem 2.5. \( \square \)

**Remark 2.8** Regarding the probability \( \mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^n \| \geq \gamma \sqrt{N}) \), in the unconditional case Aubrun has proved in [1] that for every \( \rho > 1 \) and \( N \geq \rho n \), one has

(2.40)

\[
\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^n \| \geq c_1(\rho) \sqrt{N}) \leq \exp(-c_2(\rho)n^{1/5}).
\]

In particular, one can find \( c, C > 0 \) so that, if \( N \geq Cn \), then

(2.41)

\[
\mathbb{P}(\|\Gamma : \ell_2^n \to \ell_2^n \| \geq C \sqrt{N}) \leq \exp(-cn^{1/5}).
\]

This allows us to use Theorem 2.5 with a probability estimate \( 1 - \exp(-cn^c) \) for values of \( N \) which are proportional to \( n \).
3 Weakly sandwiching $K_N$

We proceed to the question whether the inclusion given by Theorem 1.1 is sharp. It was already mentioned in the Introduction that we cannot expect a reverse inclusion of the form $K_N \subseteq c_4 Z_q(K)$ with probability close to 1, unless $q$ is of the order of $n$. To see this, observe that, for any $\alpha > 0$,

$$\mathbb{P}(K_N \subseteq \alpha Z_q(K)) = \mathbb{P}(x_1, x_2, \ldots, x_N \in \alpha Z_q(K)) = \left(\mathbb{P}(x \in \alpha Z_q(K))\right)^N \leq |\alpha Z_q(K)|^N.$$  

It was proved in [20] that, for every $q \leq n$, the volume of $Z_q(K)$ is bounded by $(c \sqrt{q/n} L_K)^n$. This leads immediately to the estimate

$$(3.1) \quad \mathbb{P}(K_N \subseteq \alpha Z_q(K)) \leq (c\alpha \sqrt{q/n} L_K)^n N^N,$$

where $c > 0$ is an absolute constant. Assume that $K$ has bounded isotropic constant and we want to keep $\alpha \approx 1$. Then, (3.1) shows that, independently from the value of $N$, we have to choose $q$ of the order of $n$ so that it might be possible to show that $\mathbb{P}(K_N \subseteq \alpha Z_q(K))$ is really close to 1. Actually, if $q \sim n$ then this is always the case, because $Z_n(K) \supseteq cK$.

**Lemma 3.1** Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$ and let $N > n$. Fix $\alpha > 1$. Then, for every $\theta \in S^{n-1}$ one has

$$(3.2) \quad \mathbb{P}(h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta)) \leq N \alpha^{-q}.$$  

**Proof.** Markov’s inequality shows that

$$(3.3) \quad \mathbb{P}(\alpha, \theta) := \mathbb{P}(x \in K : |\langle x, \theta \rangle| \geq \alpha \|\langle \cdot, \theta \rangle\|_q) \leq \alpha^{-q}.$$  

Then,

$$\mathbb{P}(h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta)) = \mathbb{P}(\max_{j \leq N} |\langle x_j, \theta \rangle| \geq \alpha \|\langle \cdot, \theta \rangle\|_q) \leq N \mathbb{P}(\alpha, \theta)$$

and the result follows. \hfill $\square$

**Lemma 3.2** Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$ and let $N > n$. For every $\alpha > 1$ one has

$$(3.4) \quad \mathbb{E} \left[ \sigma(\theta : (h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta))) \right] \leq N \alpha^{-q}.$$  

**Proof.** Immediate: observe that

$$\mathbb{E} \left[ \sigma(\theta : (h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta))) \right] = \int_{S^{n-1}} \mathbb{P}(h_{K_N}(\theta) \geq \alpha h_{Z_q(K)}(\theta)) d\sigma(\theta)$$

14
by Fubini’s theorem. □

The estimate of Lemma 3.2 is already enough to show that if \( q \geq c \ln N \) then, on the average, \( h_{K_N}(\theta) \leq c h_{Z_q(K)}(\theta) \) with probability greater than \( 1 - N^{-c} \). In particular, the mean width of a random \( K_N \) is bounded by the mean width of \( Z_{\ln(N/n)}(K) \):

**Proposition 3.3** Let \( K \) be an isotropic convex body in \( \mathbb{R}^n \). If \( q \geq 2 \ln N \) then

\[
E[w(K_N)] \leq c w(Z_q(K)),
\]

where \( c > 0 \) is an absolute constant.

**Proof.** We write

\[
w(K_N) \leq \int_{A_N} h_{K_N}(\theta) \, d\sigma(\theta) + c\sigma(A_N) n L_K,
\]

where \( A_N = \{ \theta : h_{K_N}(\theta) \leq a h_{Z_q(K)}(\theta) \} \). Then,

\[
w(K_N) \leq \alpha \int_{A_N} h_{Z_q(K)}(\theta) \, d\sigma(\theta) + c\sigma(A_N) n L_K,
\]

and hence, by Lemma 3.2,

\[
E[w(K_N)] \leq \alpha w(Z_q(K)) + cNn\alpha^{-q}L_K.
\]

Since \( w(Z_q(K)) \geq w(Z_2(K)) = L_K \), we get

\[
E[w(K_N)] \leq (\alpha + cNn\alpha^{-q})w(Z_q(K)).
\]

The result follows if we choose \( \alpha = e \).

**3.1 Volume radius of \( K_N \)**

Next, we discuss the volume radius of \( K_N \). A lower bound follows by comparison with the Euclidean ball. It was proved in [12, Lemma 3.3] that if \( K \) is a convex body in \( \mathbb{R}^n \) with volume 1, then

\[
\mathbb{P}(|K_N| \geq t) \geq \mathbb{P}(||B^n_2|| \geq t)
\]

for every \( t > 0 \). Therefore, it is enough to consider the case of \( B^n_2 \). In [10] it is shown that there exist \( c_1 > 1 \) and \( c_2 > 0 \) such that if \( N \geq c_1 n \) and \( x_1, \ldots, x_N \) are independent random points uniformly distributed in \( B^n_2 \), then

\[
[B^n_2]] \geq c_2 \min \left\{ \frac{\ln(2N/n)}{\sqrt{n}}, 1 \right\} B^n_2
\]

15
with probability greater than $1 - \exp(-n)$. It follows that if $N \geq c_1 n$ then, with probability greater than $1 - \exp(-n)$ we have

$$|K_N|^{1/n} \geq c_2 \min \left\{ \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}}, 1 \right\},$$

where $c_1 > 1$ and $c_2 > 0$ are absolute constants.

The case $n < N < c_1 n$ was studied in [7] where it was proved that (3.11) continues to hold true with probability greater than $1 - \exp(-cn/\ln n)$, where $c > 0$ is an absolute constant. Combining this fact with (3.10), we see that (3.12) is valid for all $N > n$.

We now pass to the upper bound; Proposition 3.3, combined with Urysohn’s inequality, yields the following:

**Proposition 3.4** Let $K$ be an isotropic convex body in $\mathbb{R}^n$. If $N > n$ and $q \geq 2 \ln N$, then

$$\mathbb{E}(K, N) \leq c_1 \frac{\mathbb{E}[w(K_N)]}{\sqrt{n}} \leq c_2 \frac{w(Z_q(K))}{\sqrt{n}},$$

where $c_1, c_2 > 0$ are absolute constants.

Proposition 3.4 reduces, in a sense, the question to that of giving upper bounds for $w(Z_q(K))$. It is proved in [20] that, if $q = \ln N \leq \sqrt{n}$ then $w(Z_q(K)) \leq c\sqrt{q}L_K$. It follows that

$$\mathbb{E}(K, N) \leq c\frac{\sqrt{\ln(N/n)}L_K}{\sqrt{n}},$$

which is the conjectured estimate for $N \leq e^{\sqrt{n}}$. For $q = \ln N > \sqrt{n}$ we know that $w(Z_q(K)) \leq \frac{qL_K}{\sqrt{n}}$ since $Z_q(K) \subseteq (q/\sqrt{n})Z_{\sqrt{n}}(K)$. This is most probably a non-optimal bound.

However, we can further exploit the simple estimate of Lemma 3.1 to obtain a sharp estimate for larger values of $N$. We will make use of the following facts:

**Fact 1.** Let $A$ be a symmetric convex body in $\mathbb{R}^n$. For any $1 \leq q < n$, set

$$w_{-q}(A) = \left( \int_{S^{n-1}} \frac{1}{h_A^q(\theta)} d\sigma(\theta) \right)^{-1/q}.$$

An application of Hölder’s inequality shows that

$$\left( \frac{|A|^q}{|B_2|^q} \right)^{1/n} = \left( \int_{S^{n-1}} \frac{1}{h_A^q(\theta)} d\sigma(\theta) \right)^{1/n} \geq \left( \int_{S^{n-1}} \frac{1}{h_{B_2}^q(\theta)} d\sigma(\theta) \right)^{1/q} \geq \frac{1}{w_{-q}(A)}.$$

From the Blaschke–Santaló inequality, it follows that

$$|A|^{1/n} \leq |B_2|^{1/n} w_{-q}(A) \leq \frac{c_1 w_{-q}(A)}{\sqrt{n}}.$$
Fact 2. A recent result of G. Paouris (see [21, Proposition 5.4]) shows that if $A$ is an isotropic convex body in $\mathbb{R}^n$ then, for any $1 \leq q < n/2$,

\[(3.18) \quad w_q(Z_q(A)) \simeq \sqrt[2]{I_{-q}(A)}\]

where

\[(3.19) \quad I_p(A) = \left( \int_A \|x\|_p^p \, dx \right)^{1/p}, \quad p > -n.\]

Fact 3. Let $K$ be an isotropic convex body in $\mathbb{R}^n$, let $N > n^2$ and $q = 2 \ln(2N)$. We write

\[
[w_{-q/2}(Z_q(K))]^{-q} = \left( \int_{S^{n-1}} \frac{1}{h^{q/2}_{Z_q(K)}(\theta)} \, d\sigma(\theta) \right)^2 \leq \left( \int_{S^{n-1}} \frac{1}{h^{q}_{K_N}(\theta)} \, d\sigma(\theta) \right) \left( \int_{S^{n-1}} \frac{h^{q}_{K_N}(\theta)}{h^{q}_{Z_q(K)}(\theta)} \, d\sigma(\theta) \right).
\]

Observe that $K_N \subseteq K \subseteq (n + 1)L_K$ and $Z_q(K) \supseteq Z_2(K) \supseteq L_K B_{n/2}$, and hence, $h_{K_N}(\theta) \leq (n + 1)h_{Z_q(K)}(\theta)$ for all $\theta \in S^{n-1}$. Therefore,

\[(3.20) \quad \int_{S^{n-1}} \frac{h^{q}_{K_N}(\theta)}{h^{q}_{Z_q(K)}(\theta)} \, d\sigma(\theta) = \int_{0}^{n+1} q^{t-q-1} \left[ \sigma(\theta : h_{K_N}(\theta) \geq th_{Z_q(K)}(\theta)) \right] \, dt.
\]

Fact 4. Taking expectations in (3.20) and using Lemma 3.2, we see that, for every $a > 1$,

\[
\mathbb{E} \left[ \int_{S^{n-1}} \frac{h^{q}_{K_N}(\theta)}{h^{q}_{Z_q(K)}(\theta)} \, d\sigma(\theta) \right] \leq a^q + \int_{a}^{n+1} q^{t-q-1} Nt^{-q} \, dt = a^q + qN \ln \left( \frac{n+1}{a} \right).
\]

Choosing $a = 2e$ and using the fact that $e^q = (2N)^2$ by the choice of $q$, we see that

\[(3.21) \quad \mathbb{E} \left[ \int_{S^{n-1}} \frac{h^{q}_{K_N}(\theta)}{h^{q}_{Z_q(K)}(\theta)} \, d\sigma(\theta) \right] \leq c_2^q \]

where $c_2 > 0$ is an absolute constant. Then, Markov’s inequality implies that

\[(3.22) \quad \int_{S^{n-1}} \frac{h^{q}_{K_N}(\theta)}{h^{q}_{Z_q(K)}(\theta)} \, d\sigma(\theta) \leq (c_2e)^q\]
with probability greater than $1 - e^{-q}$. Going back to Fact 3, we conclude that
\[
[w_{-q/2}(Z_q(K))]^{-q} \leq c_3[w_{-q}(K_N)]^{-q}, \text{ i.e.}
\]
(3.23) \[w_{-q}(K_N) \leq c_4 w_{-q/2}(Z_q(K))\]
with probability greater than $1 - e^{-q}$.

**Proof of Theorem 1.3.** Assume that $K_N$ satisfies (3.23) and set $S_N = K_N - K_N$.
From Fact 1 we have
\[
|K_N|^{1/n} \leq |S_N|^{1/n} \leq \frac{c_1}{\sqrt{n}} w_{-q}(S_N) = 2 \frac{c_1}{\sqrt{n}} w_{-q}(K_N).
\]
(3.24)

Now, Fact 4 shows that
\[
|K_N|^{1/n} \leq \frac{c_5}{\sqrt{n}} w_{-q/2}(Z_q(K))
\]
(3.25)
with probability greater than $1 - e^{-q}$. Since $Z_q(K) \subseteq cZ_{q/2}(K)$, using Fact 2 we write
\[
w_{-q/2}(Z_q(K)) \leq c_6 w_{-q/2}(Z_{q/2}(K)) \leq \frac{c_7 \sqrt{q}}{\sqrt{n}} I_{-q/2}(K).
\]
(3.26)

Since $K$ is isotropic, we have $I_{-q/2}(K) \leq I_2(K) = \sqrt{n} L_K$, which implies
\[
w_{-q/2}(Z_q(K)) \leq c_7 \sqrt{q} L_K.
\]
(3.27)

Putting everything together, we have
\[
|K_N|^{1/n} \leq \frac{c_8 \sqrt{q}}{\sqrt{n}} L_K \simeq \frac{\sqrt{\ln(N/n)} L_K}{\sqrt{n}},
\]
(3.28)
with probability greater than $1 - e^{-q} \geq 1 - \frac{1}{N}$. This completes the proof. \qed

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19


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