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Gauss curvature flow: the fate of the rolling stones

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Abstract. We prove Firey's 1974 conjecture that convex surfaces moving by their Gauss curvature become spherical as they contract to points.

1. Introduction

W.J. Firey introduced the motion of convex surfaces by their Gauss curvature as a model for the changing shape of a tumbling stone subjected to collisions from all directions with uniform frequency [F]. He showed (assuming some existence and regularity of solutions) that surfaces which are symmetric about the origin contract to points, becoming spherical in shape in the process. Firey conjectured that the result should hold without any symmetry assumption. The existence and regularity aspects were resolved by K.S. Chou [T], but the conjecture has remained open despite approaches by many authors ([Ch1–2], [Ha2–3], [A2], [L], [LO], [O1–2]).

In this paper we prove Firey's conjecture:

Theorem 1. *Let $M_0 = x_0(M)$ be a compact, smooth, strictly convex surface in $\mathbb{R}^3$, given by an embedding x_0. Then there exists a unique, smooth solution $\{M_t = x(M,t)\}$ of the Gauss curvature flow*

$$\begin{aligned}\frac{\partial}{\partial t}x(p,t) &= -K(p,t)\nu(p,t)\\ x(p,0) &= x_0(p)\end{aligned}$$

for $t \in [0,T)$ where $T = V(M_0)/4\pi$ and $V(M_0)$ is the volume of the region enclosed by M_0. The surfaces M_t are strictly convex, and converge to $q \in \mathbb{R}^3$ as t approaches T. Rescaling about q gives smooth convergence to a sphere:

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$$\tilde{x}(p,t) = \frac{x(p,t) - q}{(3(T-t))^{1/3}} \to \tilde{x}_T(p)$$

in C^∞*, where* $\tilde{x}_T$ *is a smooth embedding with* $\tilde{x}_T(M) = S^2(1) \subset \mathbb{R}^3$.

The existence and uniqueness, smoothness and convexity of the solutions, and the uniform convergence to a point, were all proved in [T]. Our contribution is the last part of the above theorem. The key step in the proof is an estimate on the difference of the principal curvatures of the evolving surface, which we prove by a maximum principle argument.

The estimates we obtain also allow us to deal with non-smooth initial surfaces:

Theorem 2. *Let* M_0 *be the boundary of an open bounded convex region in* $\mathbb{R}^3$*. Then there exists a unique viscosity solution* $\{M_t\}$ *for* $0 < t < T = V(M_0)/4\pi$ *of the Gauss curvature flow which converges in Hausdorff distance to* M_0 *as t approaches zero.* M_t *is a* $C^{1,1}$ *surface for* $t \in (0,T)$*, and there exists* $t_0 \in (0,T)$ *depending only on* $V(M_0)$ *and* $\operatorname{diam}(M_0)$ *such that* M_t *is* C^∞ *and strictly convex for* $t \geq t_0$*, and the subsequent behaviour is the same as in Theorem 1.*

The regularity here is probably optimal, as non-strictly convex surfaces moving by Gauss curvature can contain stationary planar regions [Ha2] which persist for some time before disappearing, and approximate solutions indicate that the surface is no more regular than $C^{1,1}$ at the boundary of such a region.

2. Preliminary results

We denote by ν the outward-pointing unit normal to M_t, and by g and h the metric and second fundamental form, defined by

$$g_{ij} = \left\langle \frac{\partial x}{\partial p^i}, \frac{\partial x}{\partial p^j} \right\rangle$$

and

$$h_{ij} = -\left\langle \frac{\partial^2 x}{\partial p^i \partial p^j}, \nu \right\rangle$$

with respect to some local coordinates $\{p^1, p^2\}$ for a region of M. We denote by g^{ij} the inverse of the metric. The Weingarten map is then given by

$$h_i^j = g^{jk} h_{ki},$$

where we sum over repeated indices. The principal curvatures λ_1 and λ_2 are the eigenvalues of the Weingarten map. The surface is strictly convex when these are positive everywhere on M. The Gauss curvature K and the mean curvature H are then given by $K = \det h = h_1^1 h_2^2 - (h_1^2)^2 = \lambda_1\lambda_2$, and $H = h_i^i = \lambda_1 + \lambda_2$.

The Codazzi identity gives an important symmetry of the covariant derivatives of the second fundamental form:

$$\nabla_i h_{jk} = \nabla_j h_{ik}$$

so ∇h is totally symmetric. Here ∇ is the Levi-Civita connection of the metric g.

These quantities evolve by the following formulae, proved in [Ch1] and [A1]:

$$\begin{aligned}
&\frac{\partial}{\partial t} g_{ij} = -2Kh_{ij};\\
&\frac{\partial}{\partial t} K = \dot{K}^{kl}\nabla_k\nabla_l K + K^2 H;\\
&\frac{\partial}{\partial t} H = \dot{K}^{kl}\nabla_k\nabla_l H + g^{ij}\ddot{K}^{klmn}\nabla_i h_{kl}\nabla_j h_{mn} + KH^2 - K|A|^2.
\end{aligned}$$

Here $|A|^2 = h_i^j h_j^i = \lambda_1^2 + \lambda_2^2$, $\dot{K}^{ij} = \frac{\partial K}{\partial h_{ij}} = K\left(h^{-1}\right)^{ij}$, and $\ddot{K}^{klmn} = \frac{\partial^2 K}{\partial h_{kl}\partial h_{mn}}$. We observe that the last two terms in the evolution equation for H can be rewritten using the identity $H^2 - |A|^2 = (\lambda_1 + \lambda_2)^2 - \lambda_1^2 - \lambda_2^2 = 2\lambda_1\lambda_2 = 2K$, to give:

$$\frac{\partial}{\partial t} H = \dot{K}^{kl}\nabla_k\nabla_l H + g^{ij}\ddot{K}(\nabla_i h, \nabla_j h) + 2K^2.$$

3. The curvature estimate

Proposition 3. *Let $\{M_t = x(M,t)\}_{0\le t<T}$ be a smooth, strictly convex solution of the Gauss curvature flow. Then*

$$\sup_M |\lambda_1(p,t) - \lambda_2(p,t)| \le \sup_M |\lambda_1(p,0) - \lambda_2(p,0)|.$$

Proof. We will apply the maximum principle to the quantity

$$Q = H^2 - 4K = (\lambda_1 - \lambda_2)^2.$$

We first compute an evolution equation for Q, observing that in the evolution equation for

$$\begin{aligned}
\frac{\partial}{\partial t} Q &= 2H\frac{\partial}{\partial t}H - 4\frac{\partial}{\partial t}K\\
&= 2H\left(\dot{K}^{kl}\nabla_k\nabla_l H + g^{ij}\ddot{K}(\nabla_i h, \nabla_j h) + 2K^2\right)\\
&\quad - 4\left(\dot{K}^{kl}\nabla_k\nabla_l K + K^2 H\right)\\
&= \dot{K}^{kl}\nabla_k\nabla_l Q - 2\dot{K}^{kl}\nabla_k H\nabla_l H + 2Hg^{ij}\ddot{K}(\nabla_i h, \nabla_j h).
\end{aligned}$$

Suppose p is a point in M where a maximum of Q is attained at time $t \in [0, T)$. Choose local coordinates for M near Q such that $g_{ij}(p, t) = \delta_{ij}$ and h is diagonal. At this point the leading term on the right hand side of the above evolution equation is non-positive. We now estimate the remaining terms, using the fact that $\nabla Q = 0$ at p:

$$\begin{aligned} 0 = \nabla_1 Q &= 2H\nabla_1 H - 4\nabla_1 K \\ &= 2(\lambda_1 + \lambda_2)\,(\nabla_1 h_{11} + \nabla_2 h_{22}) - 4\lambda_2 \nabla_1 h_{11} - 4\lambda_1 \nabla_1 h_{22} \\ &= 2\,(\lambda_1 - \lambda_2)\,(\nabla_1 h_{11} - \nabla_1 h_{22})\,. \end{aligned}$$

If $\lambda_1 = \lambda_2$ then $Q = 0$ and we have nothing to prove. So we can assume that $\nabla_1 h_{11} = \nabla_1 h_{22}$ at the point p. Similarly we have $\nabla_2 h_{11} = \nabla_2 h_{22}$. Now we compute:

$$\begin{aligned} \ddot{K}(\nabla_1 h, \nabla_1 h) &= 2\nabla_1 h_{11} \nabla_1 h_{22} - 2\,(\nabla_1 h_{12})^2 \\ &= 2\nabla_1 h_{11}^2 - 2\nabla_2 h_{11}^2 \\ &= 2\nabla_1 h_{11}^2 - 2\nabla_2 h_{22}^2 \end{aligned}$$

by using the $\nabla_1 Q = 0$ condition and the Codazzi identity. Similarly,

$$\ddot{K}(\nabla_2 h, \nabla_2 h) = 2\nabla_2 h_{22}^2 - 2\nabla_1 h_{11}^2.$$

Therefore at the point p we have

$$g^{ij}\ddot{K}(\nabla_i h, \nabla_j h) = \ddot{K}(\nabla_1 h, \nabla_1 h) + \ddot{K}(\nabla_2 h, \nabla_2 h) = 0.$$

Thus the last term on the right-hand side of the evolution equation for Q vanishes, and the second term is manifestly non-positive. The maximum principle (see for example [Ha1], Lemma 3.5) applies to show that the supremum of Q over M is a non-increasing function of time. □

4. Convergence

In this section we apply the curvature estimate to prove Theorem 1. We begin with an isoperimetric estimate:

Proposition 4. *If $\{M_t\}$ evolves by the Gauss curvature flow, then there exists $\tilde{q}(t) \in \mathbb{R}^3$ such that for any direction $p \in M$,*

$$\left| \langle x(p, t) - \tilde{q}, \nu(p, t)\rangle - \frac{1}{8\pi}\int_{M_t} H\, d\mu \right| \le \frac{C}{4\pi} A(M),$$

where $A(M_t)$ is the surface area of M_t, and

$$C = \sup_{p \in M} |\lambda_1(p, 0) - \lambda_2(p, 0)|.$$

Proof. For any direction $z \in S^2$, the width $w(z,t)$ of M_t in direction z is

$$w(z,t) = \sup_{x,y\in M_t} \langle x - y, z\rangle.$$

In [A1], Theorem 5.1 the author proved the following identity:

$$w(z,t) = \frac{1}{2\pi}\int_{M_t} \dot{K}(e_z(p), e_z(p))d\mu(p),$$

where $e_z(p)$ is the unit vector tangent to the image in M_t of a great circle through $\nu(p)$ and z under the inverse of the Gauss map. If we write $e_z^\perp$ for the orthogonal unit vector at each point, then

$$H = \dot{K}(e_z, e_z) + \dot{K}\left(e_z^\perp, e_z^\perp\right),$$

and Proposition 3 gives the estimate

$$\left|\dot{K}(e_z, e_z) - \dot{K}\left(e_z^\perp, e_z^\perp\right)\right| \le C,$$

since $\dot{K}(e,e)$ lies between λ_1 and λ_2 for any unit vector e. This gives:

$$\left| w(z,t) - \frac{1}{4\pi}\int_M H\, d\mu\right| \le \frac{C}{4\pi}A(M).$$

Now we need another identity:

Lemma 5. *Let M be a compact strictly convex smooth surface, and define $\tilde{q} = \frac{1}{4\pi}\int_M Kx d\mu(x)$. Let $x \in M$, and write $z = \nu(x)$. Then*

$$\langle x - \tilde{q}, z\rangle = \frac{1}{2}w(z) + \frac{1}{4\pi}\int_M \dot{K}(e_z, e_z)\langle \nu, z\rangle d\mu.$$

Proof. We work on S^2, choosing standard angle coordinates $\theta \in [0,\pi]$ and $\pi \in [0, 2\pi)$ such that z has coordinates $\theta = 0$. Let $s : S^2 \to \mathbb{R}$ be the support function of M, defined by

$$s(z') = \sup_{y\in M}\langle y, z'\rangle$$

for any $z' \in S^2$. Then (see [A1] or [T]) we have the identities

$$\langle y, \nu(y)\rangle = s(\nu(y))$$

for each $y \in M$,

$$w(z') = s(z') + s(-z')$$

for each $z \in S^2$, and

$$h^{-1}(e_z, e_z) = \frac{\partial^2 s}{\partial\theta^2} + s.$$

In these terms we have

$$\tilde{q} = \frac{3}{4\pi}\int_{S^2} s(z')z'd\mu(z').$$

Computing this integral in the θ and ϕ coordinates, we find:

$$\begin{aligned}\langle \tilde{q}, z\rangle &= \frac{3}{4\pi}\int_{S^2} s(z')\langle z, z'\rangle d\mu(z')\\ &= \frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi} s(\theta,\phi)\cos\theta\sin\theta d\theta d\phi\\ &= \frac{3}{8\pi}\int_0^{2\pi}\int_0^{\pi} s(\theta,\phi)\sin 2\theta d\theta d\phi.\end{aligned}$$

Now note that $\frac{\partial^2}{\partial\theta^2}\sin 2\theta + \sin 2\theta = -3\sin 2\theta$, so that

$$\begin{aligned}\langle \tilde{q}, z\rangle &= -\frac{1}{8\pi}\int_0^{2\pi}\int_0^{\pi} s(\theta,\phi)\left(\frac{\partial^2}{\partial\theta^2}\sin 2\theta + \sin 2\theta\right)d\theta d\phi\\ &= -\frac{1}{8\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2 s}{\partial\theta^2}+s\right)\sin 2\theta d\theta d\phi - \frac{1}{2}s(-z) + \frac{1}{2}s(z)\\ &= -\frac{1}{4\pi}\int_{S^2} h^{-1}(e_z, e_z)\langle z, z'\rangle d\mu(z') + \frac{1}{2}s(z) - \frac{1}{2}s(-z).\end{aligned}$$

So

$$\begin{aligned}\langle x - \tilde{q}, z\rangle &= \frac{1}{4\pi}\int_{S^2} h^{-1}(e_z, e_z)\langle z, z'\rangle d\mu(z') + \frac{1}{2}s(z) + \frac{1}{2}s(-z)\\ &= \frac{1}{4\pi}\int_M Kh^{-1}(e_z, e_z)\langle z, \nu\rangle d\mu + \frac{1}{2}w(z)\\ &= \frac{1}{4\pi}\int_m \dot{K}(e_z, e_z)\langle z, \nu\rangle d\mu + \frac{1}{2}w(z).\end{aligned}$$

□

This gives, in combination with the previous identity,

$$\begin{aligned}&\left|\langle x - \tilde{q}, z\rangle - \frac{1}{8\pi}\int_M H\,d\mu\right|\\ &\quad\le \frac{1}{2}\left|w(z) - \frac{1}{4\pi}\int_M H\,d\mu\right| + \left|\langle x - \tilde{q}, z\rangle - \frac{1}{2}w(z)\right|\\ &\quad\le \frac{C}{8\pi}A(M) + \frac{1}{4\pi}\left|\int_M \dot{K}(e_z, e_z)\langle z, \nu\rangle d\mu\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{4\pi}A(M) + \frac{1}{8\pi}\left|\int_M H\langle z, \nu\rangle d\mu\right| \\
&\quad + \frac{1}{8\pi}\int_M \left|\dot{K}(e_z, e_z) - \dot{K}\left(e_z^\perp, e_z^\perp\right)\right| |\langle z, \nu\rangle|\, d\mu \\
&\leq \frac{C}{4\pi}A(M)
\end{aligned}
$$

where we applied the estimates $\left|\dot{K}(e_z, e_z) - \dot{K}\left(e_z^\perp, e_z^\perp\right)\right| \leq C$ and $|\langle z, \nu\rangle| \leq 1$ in the second integral in the second-last line, and $\int_M H\nu d\mu = 0$ in the first integral. □

Corollary 6. *For* $(T - t) \leq \frac{1}{3(8C)^3}$, M_t *is contained between concentric spheres centred at* $\tilde{q}(t)$ *with radii* $r_- \leq r_+$ *satisfying*

$$\frac{r_+}{r_-} \leq 1 + 8C(3(T-t))^{1/3}.$$

Proof. We can take

$$r_- = \frac{1}{8\pi}\int_{M_t} Hd\mu - \frac{C}{4\pi}A(M)$$

and

$$r_+ = \frac{1}{8\pi}\int_{M_t} Hd\mu + \frac{C}{4\pi}A(M).$$

Then, since $A(M) \leq 4\pi r_+^2$, we find $0 \leq 2Cr_+^2 - r_+ + r_-$, which implies

$$r_+ \leq \frac{2r_-}{1 + \sqrt{1 - 8Cr_-}}.$$

Since we also have $r_- \leq \left(\frac{3V}{4\pi}\right)^{1/3} = (3(T-t))^{1/3}$, the result follows from the inequality $2/(1 + \sqrt{1-x}) \leq 1 + x$ which holds for $x \in [0, 1]$. □

Next we show that $\tilde{M}_t = \tilde{x}(M, t)$ becomes uniformly smooth and convex for t close to T:

Proposition 7. *For* $T - t \leq \min\{C_0 C^{-3}, T/2\}$,

$$\left|K(p,t) - (3(T-t))^{-2/3}\right| \leq C_1 C^{1/2}(T-t)^{-1/2}$$

and for any unit vector $e \in TM_t$,

$$\left|h(e,e) - (3(T-t))^{-1/3}\right| \leq C_2 C^{1/2}(T-t)^{-1/6}$$

where C_0, C_1, *and* C_2 *are constants.*

Proof. We will prove this in several stages: First an upper bound on the Gauss curvature, then a lower bound, and then upper and lower bounds on the principal curvatures.

We will use the Harnack estimate from [Ch2], which implies that if $z \in S^2$ and $p(z,t) \in M$ is the point with $\nu(p(z,t),t) = z$ for each t, then

$$\frac{d}{dt}\left(t^{2/3}K(p(z,t),t)\right) \geq 0.$$

We also note that

$$K(p(z,t),t) = -\frac{d}{dt}s(z,t)$$

so we can bound an average of $K(p(z,t),t)$ over a time interval by controlling the distance moved by the tangent plane of M_t in direction z.

Fix a time t close to T, and translate the origin to $\tilde{q}(t)$ M_t lies between spheres of radii r_- and r_+, so by the comparison principle $M_{t+\tau}$ lies outside the sphere of radius $(r_-^3 - 3\tau)^{1/3}$ for $\tau > 0$. Therefore the distance moved by the tangent plane in a direction z is at most $r_+ - (r_-^3 - 3\tau)^{1/3} \leq \tau r_-^{-2} + \tau^2 r_-^{-5} + r_+ - r_-$. Therefore

$$\begin{aligned}\inf_{t \leq t' \leq t+\tau} &\leq \frac{1}{\tau}\int_t^{t+\tau} K(p(z,t'),t')dt' \\ &\leq r_-^{-2} + \tau r_-^{-5} + \frac{r_+ - r_-}{\tau}.\end{aligned}$$

Choosing $\tau = r_-^{5/2}(r_+ - r_-)^{1/2}$ and using the bounds on r_+ and r_- from Corollary 6, we find $\tau \propto C^{1/2}(T-t)^{1/6}$, and

$$\begin{aligned}K(p(z,t),t) &\leq \left(\frac{t+\tau}{t}\right)^{2/3} \inf_{t \leq t' \leq t+\tau} K(p(z,t'),t') \\ &\leq (3(T-t))^{-2/3} + C_1 C^{1/2}(T-t)^{-1/2}.\end{aligned}$$

where we used the Harnack estimate in the first line.

A lower bound on K follows similarly, by obtaining a lower bound on the distance travelled by the tangent plane in a direction z over a time interval $[t-\tau, t]$ with $\tau \propto C^{1/2}(T-t)^{1/6}$.

Finally, the bound on the principal curvatures follows immediately by combining the bound on the difference between the principal curvatures with these bounds above and below on the Gauss curvature, since

$$H^2 = 4K + (\lambda_1 - \lambda_2)^2$$

and

$$\begin{aligned}\lambda_2 &= \frac{1}{2}H + \frac{1}{2}(\lambda_2 - \lambda_1); \\ \lambda_1 &= \frac{1}{2}H - \frac{1}{2}(\lambda_2 - \lambda_1).\end{aligned}$$

□

C^∞ regularity of the rescaled solutions now follows as in [T], and the convergence of $\tilde{M}_t$ to the unit sphere as $t \to T$, as well as the convergence of the embeddings $\tilde{x}(., t)$ to a smooth limit $\tilde{x}_T$, follow as in [A1].

This completes the proof of Theorem 1.

5. Viscosity solutions

Theorem 2 is a consequence of the following curvature bound, which establishes a bound on the curvature of a smooth solution depending only on time and the volume and diameter of the initial surface:

Proposition 8. *Let $\{M_t\}$ be smooth, strictly convex hypersurfaces moving under the Gauss curvature flow. Then for each $t \in (0, T)$ there exists $C(t)$ depending only on $V(M_0)$ and* $\mathrm{diam}(M_0)$ *such that*

$$\sup_{M_t} |A|^2 \leq C(t).$$

Proof. First note that we have bounds on K for any $t > 0$ (see [A2], Theorem 6), so without loss of generality we can work on an inerval where K is bounded. Choose the origin such that M_0 is contained in a ball of radius R about the origin.

We consider the evolution of the quantity $S = \frac{H}{2R-|x|^2}$: Since $|x|$ is decreasing, the denominator remains positive. We have the evolution equations

$$\frac{\partial}{\partial t}|x|^2 = -2K\langle x, \nu\rangle = \dot{K}^{kl}\nabla_k\nabla_l|x|^2 + 2K\langle x, \nu\rangle - 2H,$$

so that

$$\begin{aligned}\frac{\partial}{\partial t}S = \dot{K}^{kl}\nabla_k\nabla_l S &+ \frac{2}{2R-|x|^2}\dot{K}^{ij}\nabla_i(2R-|x|^2)\nabla_j S \\ &+ \frac{1}{(2R-|x|^2)}\left(\ddot{K}(\nabla_i h, \nabla_i h) + 2K^2\right) \\ &- \frac{H}{(2R-|x|^2)^2}\left(-2K\langle x, \nu\rangle + 2H\right).\end{aligned}$$

The last term gives us a very strong negative term. The first term is elliptic, so non-positive at a maximum of S, and the second term is a gradient term which vanishes at a maximum. Since K is bounded, the only dangerous term is the one involving $\ddot{K}(\nabla_i h, \nabla_i h)$, which we now proceed to estimate using the fact the ∇S vanishes at a maximum point: This says that

$$\nabla_1 h_{11} = -\nabla_1 h_{22} - \frac{H}{2R-|x|^2}\nabla_1|x|^2$$

and

$$\nabla_2 h_{22} = -\nabla_2 h_{11} - \frac{H}{2R-|x|^2}\nabla_2|x|^2.$$

So we have at the maximum point

$$
\begin{aligned}
\ddot{K}(\nabla_i h, \nabla_i h) &= 2\nabla_1 h_{11}\nabla_1 h_{22} + 2\nabla_2 h_{11}\nabla_2 h_{22} - 2\nabla_1 h_{22}^2 - 2\nabla_2 h_{11}^2 \\
&= -\frac{2H}{2R-|x|^2}\nabla_1|x|^2\nabla_1 h_{22} - \frac{2H}{2R-|x|^2}\nabla_2|x|^2\nabla_2 h_{11} \\
&\quad - 4\nabla_1 h_{22}^2 - 4\nabla_2 h_{11}^2 \\
&\le \frac{1}{4}\left(\frac{H}{2R-|x|^2}\right)^2 |\nabla_1|x|^2|^2 + \frac{1}{4}\left(\frac{H}{2R-|x|^2}\right)^2 |\nabla_2|x|^2|^2 \\
&= S^2\left(|x|^2 - \langle x, \nu\rangle^2\right).
\end{aligned}
$$

Thus at the maximum point we have

$$
\frac{\partial}{\partial t}S \le \left(\frac{|x|^2 - \langle x, \nu\rangle^2}{2R-|x|^2} - 2\right)S^2 + \frac{2K^2}{2R-|x|^2} + \frac{2KH\langle x, \nu\rangle^{\,2}}{(2R-|x|)^2}.
$$

Our choice of R means that $|x|^2 \le C - |x|^2$, so that the coefficient of S^2 is less than or equal to -1. So we have

$$
\frac{\partial}{\partial t}S^2 \le -\frac{1}{2}S^2 + C_3(\sup K, R)
$$

and hence $\sup S \le C_4(\sup K, r) + 2/t$ for t sufficiently small. □

Corollary 9. *Any viscosity solution $\{M_t\}$ of Gauss curvature flow is $C^{1,1}$ for $t > 0$, has $Q = H^2 - 4K$ bounded for $t > 0$, and has Q uniformly bounded on (t, T) for any $t > 0$.*

This follows immediately, since we have proved each of these statements for smooth solutions with bounds independent of the regularity of the initial solution. Hence the same bounds apply for the viscosity solution.

Theorem 2 follows, because the isoperimetric estimates and convergence results of Section 4 depend only on a bound on Q.

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