Norms of weighted sums of log-concave random vectors

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Abstract

Let C and K be centrally symmetric convex bodies of volume 1 in $\mathbb{R}^n$. We provide upper bounds for the multi-integral expression

$$\|\mathbf{t}\|_{C^s,K} = \int_C \cdots \int_C \Big\| \sum_{j=1}^s t_j x_j \Big\|_K \, dx_1 \cdots dx_s$$

in the case where C is isotropic. Our approach provides an alternative proof of the sharp lower bound, due to Gluskin and V. Milman, for this quantity. We also present some applications to "randomized" vector balancing problems.

1 Introduction

Let K be a centrally symmetric convex body in $\mathbb{R}^n$. For any s-tuple $\mathcal{C} = (C_1, \dots, C_s)$ of centrally symmetric convex bodies C_j in $\mathbb{R}^n$ we consider the norm on $\mathbb{R}^s$, defined by

$$\|\mathbf{t}\|_{\mathcal{C},K} = \frac{1}{\prod_{j=1}^s \mathrm{vol}_n(C_j)} \int_{C_1} \cdots \int_{C_s} \Big\| \sum_{j=1}^s t_j x_j \Big\|_K \, dx_1 \cdots dx_s,$$

where $\mathbf{t} = (t_1, \dots, t_s)$. If $\mathcal{C} = (C, \dots, C)$ then we write $\|\mathbf{t}\|_{C^s,K}$ instead of $\|\mathbf{t}\|_{\mathcal{C},K}$. A question posed by V. Milman is to determine if, in the case $C = K$, one has that $\|\cdot\|_{K^s,K}$ is equivalent to the standard Euclidean norm up to a term which is logarithmic in the dimension, and in particular, if under some cotype condition on the norm induced by K to $\mathbb{R}^n$ one has equivalence between $\|\cdot\|_{K^s,K}$ and the Euclidean norm.

This question was studied by Bourgain, Meyer, V. Milman and Pajor in [9]; they obtained the lower bound

$$\|\mathbf{t}\|_{\mathcal{C},K} \geqslant c\sqrt{s} \Big(\prod_{j=1}^s |t_j| \Big)^{1/s} \Big(\prod_{j=1}^s \mathrm{vol}_n(C_j) \Big)^{\frac{1}{sn}} / \mathrm{vol}_n(K)^{1/n},$$

where $c > 0$ is an absolute constant. Gluskin and V. Milman studied the same question in [17] and obtained a better lower bound in a more general context.

Theorem 1.1 (Gluskin-Milman)**.** *Let $A_1, \dots, A_s$ be measurable sets in $\mathbb{R}^n$ and K be a star body in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$. Then, for all $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$,*

$$\|\mathbf{t}\|_{\mathcal{A},K} := \frac{1}{\prod_{j=1}^s \mathrm{vol}_n(A_j)} \int_{A_1} \cdots \int_{A_s} \Big\| \sum_{j=1}^s t_j x_j \Big\|_K \, dx_1 \cdots dx_s \geqslant c \Big(\sum_{j=1}^s t_j^2 \Big(\frac{\mathrm{vol}_n(A_j)}{\mathrm{vol}_n(K)} \Big)^{2/n} \Big)^{1/2},$$

where $c > 0$ is an absolute constant. Equivalently, if $\mathrm{vol}_n(A_j) = \mathrm{vol}_n(K)$ for all $1 \leqslant j \leqslant s$ then

$$\|\mathbf{t}\|_{\mathcal{A},K} \geqslant c\, \|\mathbf{t}\|_2 \tag{1.1}$$

for all $\mathbf{t} \in \mathbb{R}^s$.

In the statement above, when K is a star body with respect to 0 we use the notation $\|x\|_K$ for the gauge function of K, defined by $\inf\{r>0 : x/r \in K\}$. The proof of Theorem 1.1 actually shows that one can have $c \geqslant c(n)/\sqrt{2}$, where $c(n) \to 1$ as $n \to \infty$. Gluskin and V. Milman use a symmetrization type result which is a consequence of the Brascamp-Lieb-Luttinger inequality: under the assumptions of Theorem 1.1 and the additional assumption that $\mathrm{vol}_n(A_j) = \mathrm{vol}_n(K) = \mathrm{vol}_n(B_2^n)$ for all $1 \leqslant j \leqslant s$, one has

$$\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{1\leqslant j\leqslant s} : x_j \in A_j \ \text{ for all } j \ \text{ and } \Big\|\sum_{j=1}^{s} t_j x_j\Big\|_K < \alpha\Big\}\Big)$$
$$\leqslant \mathrm{vol}_{ns}\Big(\Big\{(x_j)_{1\leqslant j\leqslant s} : x_j \in B_2^n \ \text{ for all } j \ \text{ and } \Big\|\sum_{j=1}^{s} t_j x_j\Big\|_2 < \alpha\Big\}\Big)$$

for any $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$ and any $\alpha > 0$.

Our starting point is a simple but useful identity; one has

$$\|\mathbf{t}\|_{\mathcal{C},K} = \|\mathbf{t}\|_2 \int_{\mathbb{R}^n} \|x\|_K \, d\nu_{\mathbf{t}}(x), \tag{1.2}$$

where $\nu_{\mathbf{t}}$ is the distribution of the random vector $\frac{1}{\|\mathbf{t}\|_2}(t_1X_1 + \cdots + t_sX_s)$ and X_j are independent random vectors uniformly distributed on C_j. Starting with (1.2) we can actually give an alternative short proof of Theorem 1.1 in the case that we study.

Theorem 1.2. *Let $\mathcal{C} = (C_1, \ldots, C_s)$ be an s-tuple of centrally symmetric convex bodies and K be a centrally symmetric convex body in $\mathbb{R}^n$ with $\mathrm{vol}_n(C_j) = \mathrm{vol}_n(K) = 1$. Then, for any $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$,*

$$\|\mathbf{t}\|_{\mathcal{C},K} \geqslant \frac{n}{e(n+1)} \|\mathbf{t}\|_2.$$

We are mainly interested in upper bounds for the quantity $\|\mathbf{t}\|_{C^s,K}$. Since $\|\mathbf{t}\|_{C^s,K} = \|\mathbf{t}\|_{(TC)^s,TK}$ for any $T \in SL(n)$, we may restrict our attention to the case where C is isotropic (see Section 2 for the definition and background information). In this case

$$\|\mathbf{t}\|_{C^s,K} = \|\mathbf{t}\|_2 L_C \, I_1(\mu_{\mathbf{t}}, K), \tag{1.3}$$

where $\mu_{\mathbf{t}}$ is an isotropic, compactly supported log-concave probability measure depending on $\mathbf{t}$ and, for any centered log-concave probability measure μ on $\mathbb{R}^n$,

$$I_1(\mu, K) = \int_{\mathbb{R}^n} \|x\|_K d\mu(x).$$

In order to get a feeling of what one would expect, let us note that if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ and K is a centrally symmetric convex body of volume 1 in $\mathbb{R}^n$ then

$$\int_{O(n)} I_1(\mu, U(K)) \, d\nu(U) = \int_{\mathbb{R}^n} \int_{O(n)} \|x\|_{U(K)} d\nu(U) \, d\mu(x) = M(K) \int_{\mathbb{R}^n} \|x\|_2 d\mu(x) \approx \sqrt{n} M(K),$$

where

$$M(K) := \int_{S^{n-1}} \|\xi\|_K d\sigma(\xi)$$

and ν, σ denote the Haar probability measures on $O(n)$ and S^{n-1} respectively. It follows that

$$\int_{O(n)} \|\mathbf{t}\|_{U(C)^s,K} \approx (L_C \sqrt{n} M(K)) \, \|\mathbf{t}\|_2. \tag{1.4}$$

Therefore, our goal is to obtain a constant of the order of $L_C \sqrt{n} M(K)$ in our upper estimate for $\|\mathbf{t}\|_{C^s,K}$. Let us note here that the question to estimate the parameter $M(K)$ for an isotropic centrally symmetric convex body K in $\mathbb{R}^n$, which will appear frequently in our upper bounds, remains open; one may hope that $L_K\sqrt{n}M(K)\leqslant c(\log n)^b$ for some absolute constant $b>0$. However, the currently best known estimate is

$$M(K)\leqslant \frac{c\log^{2/5}(e+n)}{\sqrt[10]{n}L_K}.$$

This is proved in [15] (see also [16] for previous work on this question) and it is also shown that in the case where K is a ψ_2-body with constant ϱ one has

$$M(K)\leqslant \frac{c\sqrt[3]{\varrho}\log^{1/3}(e+n)}{\sqrt[6]{n}L_K}.$$

We pass now to our bounds for $\|\mathbf{t}\|_{C^s,K}$. Some straightforward upper and lower estimates are given in the next theorem.

Theorem 1.3. *Let C be an isotropic convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then, for any $s\geqslant 1$ and $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$,*

$$c_1L_CR(K^\circ)\,\|\mathbf{t}\|_2\leqslant \|\mathbf{t}\|_{C^s,K}\leqslant \sqrt{n}L_CR(K^\circ)\,\|\mathbf{t}\|_2,$$

where $c_1>0$ is an absolute constant and $R(K^\circ)$ is the radius of K°.

A class of centrally symmetric convex bodies for which the upper bound of Theorem 1.3 can be applied is the class of 2-convex bodies. More precisely, in Section 4.1 we see that if K is an isotropic convex body in $\mathbb{R}^n$, which is also 2-convex with constant α, then

$$\|\mathbf{t}\|_{C^s,K}\leqslant (c_2L_C/\sqrt{\alpha})\,\|\mathbf{t}\|_2$$

for any isotropic centrally symmetric convex body C and any $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$, where $c_2>0$ is an absolute constant. In particular, for any centrally symmetric convex body K in $\mathbb{R}^n$ which is 2-convex with constant α we have

$$\|\mathbf{t}\|_{K^s,K}\leqslant (c_3/\alpha)\,\|\mathbf{t}\|_2$$

for all $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$, where $c_3>0$ is an absolute constant.

Starting again with (1.3) and using an argument which goes back to Bourgain (also, employing Paouris' inequality and Talagrand's comparison theorem) in Section 4.2 we obtain a general upper bound of different type.

Theorem 1.4. *Let C be an isotropic convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then,*

$$\|\mathbf{t}\|_{C^s,K}\leqslant c\left(L_C\max\left\{\sqrt[4]{n},\sqrt{\log(1+s)}\right\}\right)\sqrt{n}M(K)\|\mathbf{t}\|_2$$

for every $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$, where $c>0$ is an absolute constant.

In the case where C is a ψ_2-body with constant ϱ, a direct application of Talagrand's theorem leads to a stronger estimate: If C is an isotropic ψ_2-body with constant ϱ and K is a centrally symmetric convex body in $\mathbb{R}^n$ then

$$\|\mathbf{t}\|_{C^s,K}\leqslant c\varrho^2\sqrt{n}M(K)\,\|\mathbf{t}\|_2$$

for every $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$, where $c>0$ is an absolute constant.

Next, combining (1.3) with results of E. Milman from [24], we obtain some rather strong estimates in the case where K has bounded cotype-2 constant (see Section 5). In the case $C=K$ we get:

Theorem 1.5. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$. For any $s \geqslant 1$ and any $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$ we have that*

$$\frac{c_3}{C_2(X_K)} \|\mathbf{t}\|_2 \leqslant \|\mathbf{t}\|_{K^s,K} \leqslant \big(c_4 L_K C_2(X_K)\sqrt{n} M(K_{\rm iso})\big) \|\mathbf{t}\|_2,$$

where $C_2(X_K)$ is the cotype-2 constant of the normed space X_K with unit ball K, and $K_{\rm iso}$ is an isotropic image of K.

In Section 6 we consider the unconditional case; using an argument from [14] which is based on well-known results of Bobkov and Nazarov one has the following estimates.

Theorem 1.6. *If K and $C_1, \dots, C_s$ are isotropic unconditional convex bodies in $\mathbb{R}^n$ then,*

$$\|\mathbf{t}\|_{\mathcal{C},K} \leqslant c\sqrt{\log n} \cdot \max\{\|\mathbf{t}\|_2, \sqrt{\log n}\|\mathbf{t}\|_\infty\}$$

for every $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$, where $c > 0$ is an absolute constant.

As an application of Theorem 1.5 and of the "ψ_2-version" of Theorem 1.4 we can check that in the special case of the unit ball B_p^n of ℓ_p^n, $1 \leqslant p \leqslant \infty$, one has the upper bound

$$\|\mathbf{t}\|_{\overline{B_p^n}^s, \overline{B_p^n}} \leqslant c \min\{\sqrt{p}, \sqrt{\log n}\} \|\mathbf{t}\|_2$$

for every $s \geqslant 1$ and $\mathbf{t} \in \mathbb{R}^s$, where $c > 0$ is an absolute constant (and, generally, $\overline{K} = {\rm vol}_n(K)^{-1/n} K$).

In Section 7 we discuss applications of the previous results to some randomized versions of vector balancing problems. Given two centrally symmetric convex bodies C, K in $\mathbb{R}^n$, the parameter $\beta_s(C,K)$ is defined as follows:

$$\beta_s(C,K) := \min\Big\{r > 0 : \text{ for any } x_1, \dots, x_s \in C, \min_{\epsilon \in E_2^s} \Big\|\sum_{j=1}^s \epsilon_j x_j\Big\|_K \leqslant r\Big\},$$

where $E_2^s := \{-1,1\}^s$ is the discrete cube in $\mathbb{R}^s$. Given $x_1, \dots, x_n \in K$, by the triangle inequality it is clear that $\|\sum_{j=1}^n \epsilon_j x_j\|_K \leqslant n$ holds for every $\epsilon \in E_2^n$, thus $\beta_n(K,K) \leqslant n$. This bound is actually sharp: taking $K = B_1^n$ and $x_j = e_j$, the standard basis of $\mathbb{R}^n$, we get $\|\sum_{j=1}^n \epsilon_j e_j\|_1 = n$ for every choice of signs. However, the upper bound for $\beta_n(K,K)$ can be significantly better for certain convex bodies, as suggested for example by a theorem of Spencer [28]: one has $\beta_n(B_\infty^n, B_\infty^n) \leqslant 6\sqrt{n}$.

We further define $\tilde{\beta}(C,K) = \sup_{k \geqslant n} \beta_k(C,K)$. Clearly, $\beta_n(C,K) \leqslant \tilde{\beta}(C,K)$. By a theorem of Bárány and Grinberg [3], one has $\tilde{\beta}(K,K) \leqslant 2n$. This result can also be derived by the trivial bound on $\beta_n(K,K)$ mentioned earlier and the general observation that

$$\tilde{\beta}(C,K) \leqslant 2 \max_{k \leqslant n} \beta_k(C,K).$$

A related result is the Dvoretzky-Hanani lemma (see for example [18, Lemma 2.2.1]) which asserts that for every centrally symmetric convex body K in $\mathbb{R}^n$, for any $s \geqslant 1$ and any $x_1, \dots, x_s \in K$, there exist $\epsilon_1, \dots, \epsilon_s \in \{-1,1\}$ such that $\max_{k \leqslant s} \|\sum_{j=1}^k \epsilon_j x_j\|_K \leqslant 2n$.

The question that we discuss is whether one can achieve something better than the $O(n)$ bound for a random s-tuple $(x_1, \dots, x_s)$ from C. In order to make this question precise, for any $\delta \in (0,1)$ we introduce the parameter

$$\beta_{\delta,s}^{(R)}(C,K) := \min\Big\{r > 0 : {\rm vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j \in C \text{ for all } j \text{ and } \min_{\epsilon \in E_2^s} \Big\|\sum_{j=1}^s \epsilon_j x_j\Big\|_K \leqslant r\Big\}\Big) \geqslant 1 - \delta\Big\}.$$

The results of Section 4 and Section 5 allow us to obtain significantly better bounds for $\beta_{\delta,s}^{(R)}(C,K)$. In the statement below we restrict ourselves to the case $C = K$ and $s = n$; the reader may deduce analogous bounds for an arbitrary choice of C or s.

Theorem 1.7. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Then, for any $\delta \in (0,1)$,*

$$\beta^{(R)}_{\delta,n}(K,K) \leqslant (c\log(2/\delta)L_K n^{3/4})\,\sqrt{n}M(K_{\rm iso})$$

where $c > 0$ is an absolute constant and $K_{\rm iso}$ is an isotropic image of K. If K is a ψ_2-body with constant ϱ then

$$\beta^{(R)}_{\delta,n}(K,K) \leqslant (c\log(2/\delta)\varrho^2\sqrt{n})\,\sqrt{n}M(K_{\rm iso}).$$

Analogous results hold for 2-convex bodies with constant α, in which case we have

$$\beta^{(R)}_{\delta,n}(K,K) \leqslant (c\log(2/\delta)\sqrt{n}/\alpha),$$

or bodies with bounded cotype-2 constant; in this case we have

$$\beta^{(R)}_{\delta,n}(K,K) \leqslant (c\log(2/\delta)L_K C_2(X_K)\sqrt{n})\,\sqrt{n}M(K_{\rm iso}).$$

In fact, the proof of Theorem 1.7 shows that the same upper bounds hold for the parameter $\kappa^{(R)}_{\delta,s}(C,K)$ which is defined as the smallest $r > 0$ with the property that

$$\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j \in C \text{ for all } j \text{ and } \mathbb{P}\Big(\Big\{\epsilon \in E_2^s : \Big\|\sum_{j=1}^s \epsilon_j x_j\Big\|_K \leqslant r\Big\}\Big) \geqslant 1-\delta\Big\}\Big) \geqslant 1-\delta.$$

Note that, by definition, $\kappa^{(R)}_{\delta,s}(C,K) \geqslant \beta^{(R)}_{\delta,s}(C,K)$.

Finally, combining our approach with some classical results from asymptotic convex geometry we obtain variants of the main results of [12] as well as their dual estimates. We close this introductory section with the statements in the particular case $C = \overline{B_2^n}$.

Theorem 1.8. *Let $\mathbf{t} \in \mathbb{R}^s$. For any centrally symmetric convex body K in $\mathbb{R}^n$ and any $S \subseteq E_2^s$ with $|S| \leqslant e^{cd(K)}$ we have*

$$\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j \in \overline{B_2^n} \text{ for all } j \text{ and } \Big\|\sum_{j=1}^s \epsilon_j t_j x_j\Big\|_K \leqslant c_1 L_C\sqrt{n}M(K)\,\|\mathbf{t}\|_2 \text{ for some } \epsilon \in S\Big\}\Big) \leqslant e^{-c_2 d(K)}$$

and

$$\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j \in \overline{B_2^n} \text{ for all } j \text{ and } \Big\|\sum_{j=1}^s \epsilon_j t_j x_j\Big\|_K \geqslant c_3 L_C\sqrt{n}M(K)\,\|\mathbf{t}\|_2 \text{ for some } \epsilon \in S\Big\}\Big) \leqslant e^{-c_4 k(K)},$$

where $c_i > 0$ are absolute constants.

The quantities $k(K)$ and $d(K)$ are well-known parameters of a centrally symmetric convex body K which are introduced in Section 7; $k(K) = n(M(K)/b(K))^2$ is the Dvoretzky dimension of K and

$$d(K) = \min\Big\{n, -\log\gamma_n\Big(\frac{m(K)}{2}K\Big)\Big\},$$

where $m(K) \approx \sqrt{n}M(K)$ is the median (the Lévy mean) of $\|\cdot\|_K$ with respect to the standard Gaussian measure γ_n on $\mathbb{R}^n$.

2 Backgound information and preliminary observations

In this section we introduce notation and terminology that we use throughout this work, and provide background information on isotropic convex bodies. We write $\langle\cdot,\cdot\rangle$ for the standard inner product in $\mathbb{R}^n$ and denote the Euclidean norm by $\|\cdot\|_2$. In what follows, B_2^n is the Euclidean unit ball, S^{n-1} is the unit sphere, and σ is the rotationally invariant probability measure on S^{n-1}. Lebesgue measure in $\mathbb{R}^n$ is denoted by vol_n. The letters c, c', c_j, c_j' etc. denote absolute positive constants whose value may change from line to line. Sometimes we might even relax our notation: $a \lesssim b$ will then mean "$a \leqslant cb$ for some (suitable) absolute constant $c > 0$", and $a \approx b$ will stand for "$a \lesssim b \wedge a \gtrsim b$". If A, B are sets, $A \approx B$ will similarly state that $c_1 A \subseteq B \subseteq c_2 A$ for some absolute constants $c_1, c_2 > 0$.

A convex body in $\mathbb{R}^n$ is a compact convex set $C \subset \mathbb{R}^n$ with non-empty interior. We say that C is **centrally symmetric** if $-C = C$. We say that C is unconditional with respect to the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ if $x = (x_1, \ldots, x_n) \in C$ implies that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in C$ for any choice of signs $\epsilon_j \in \{-1, 1\}$. The volume radius of C is the quantity $\mathrm{vrad}(C) = (\mathrm{vol}_n(C)/\mathrm{vol}_n(B_2^n))^{1/n}$. Integration in polar coordinates shows that if the origin is an interior point of C then the volume radius of C can be expressed as

$$\mathrm{vrad}(C) = \Big(\int_{S^{n-1}} \|\xi\|_C^{-n}\, d\sigma(\xi)\Big)^{1/n},$$

where $\|\xi\|_C = \inf\{t > 0 : \xi \in tC\}$. We also consider the parameter

$$M(C) = \int_{S^{n-1}} \|\xi\|_C d\sigma(\xi).$$

The support function of C is defined by $h_C(y) := \max\big\{\langle x, y\rangle : x \in C\big\}$, and the mean width of C is the average

$$w(C) := \int_{S^{n-1}} h_C(\xi)\, d\sigma(\xi)$$

of h_C on S^{n-1}. The radius $R(C)$ of a centrally symmetric convex body C is the smallest $R > 0$ such that $C \subseteq RB_2^n$. We shall use the fact that $R(C) \leqslant c\sqrt{n}w(C)$; equivalently, $b(C) \leqslant c\sqrt{n}M(C)$, where $b(C)$ is the smallest $b > 0$ with the property that $\|x\|_C \leqslant b\|x\|_2$ for all $x \in \mathbb{R}^n$. For notational convenience we write $\overline{C}$ for the homothetic image of volume 1 of a convex body $C \subseteq \mathbb{R}^n$, i.e. $\overline{C} := \mathrm{vol}_n(C)^{-1/n} C$.

The polar body C° of a centrally symmetric convex body C in $\mathbb{R}^n$ is defined by

$$C^\circ := \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\big\}.$$

The Blaschke-Santaló inequality states that $\mathrm{vol}_n(C)\mathrm{vol}_n(C^\circ) \leqslant \mathrm{vol}_n(B_2^n)^2$, with equality if and only if C is an ellipsoid. The reverse Santaló inequality of Bourgain and V. Milman [10] asserts that there exists an absolute constant $c > 0$ such that, conversely,

$$\Big(\mathrm{vol}_n(C)\mathrm{vol}_n(C^\circ)\Big)^{1/n} \geqslant c\,\mathrm{vol}_n(B_2^n)^{2/n} \approx 1/n.$$

A convex body C in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_C > 0$ such that

$$\|\langle\cdot,\xi\rangle\|_{L_2(C)}^2 := \int_C \langle x, \xi\rangle^2 dx = L_C^2 \tag{2.1}$$

for all $\xi \in S^{n-1}$. We shall use the fact that if C is isotropic then $R(C) \leqslant cnL_C$ for some absolute constant $c > 0$. The hyperplane conjecture asks if there exists an absolute constant $A > 0$ such that

$$L_n := \max\{L_C : C \text{ is isotropic in } \mathbb{R}^n\} \leqslant A \tag{2.2}$$

for all $n \geqslant 1$. Bourgain proved in [8] that $L_n \leqslant c\sqrt[4]{n}\log n$; later, Klartag [19] improved this bound to $L_n \leqslant c\sqrt[4]{n}$.

A Borel measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A+(1-\lambda)B)\geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda\in(0,1)$. A function $f:\mathbb{R}^n\to[0,\infty)$ is called log-concave if its support $\{f>0\}$ is a convex set and the restriction of $\log f$ to it is concave. It is known that if a probability measure μ is log-concave and $\mu(H)<1$ for every hyperplane H, then μ has a log-concave density f_μ. Note that if C is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_C$ is the density of a log-concave measure.

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x\in\mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}, \tag{2.3}$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\,dx}{\int_{\mathbb{R}^n} f_\mu(x)\,dx}. \tag{2.4}$$

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered, i.e. if

$$\int_{\mathbb{R}^n}\langle x,\xi\rangle d\mu(x) = \int_{\mathbb{R}^n}\langle x,\xi\rangle f_\mu(x)dx = 0 \tag{2.5}$$

for all $\xi\in S^{n-1}$, and $\mathrm{Cov}(\mu)$ is the identity matrix.

If C is a centered convex body of volume 1 in $\mathbb{R}^n$ then we say that a direction $\xi\in S^{n-1}$ is a ψ_α-direction (where $1\leqslant\alpha\leqslant 2$) for C with constant $\varrho>0$ if

$$\|\langle\cdot,\xi\rangle\|_{L_{\psi_\alpha}(C)} \leqslant \varrho \|\langle\cdot,\xi\rangle\|_{L_2(C)},$$

where

$$\|\langle\cdot,\xi\rangle\|_{L_{\psi_\alpha}(C)} := \inf\Big\{t>0 : \int_C \exp\left((|\langle x,\xi\rangle|/t)^\alpha\right)dx\leqslant 2\Big\}.$$

From Markov's inequality it is clear that if C satisfies a ψ_α-estimate with constant ϱ in the direction of ξ then for all $t\geqslant 1$ we have $\mathrm{vol}_n(\{x\in C : |\langle x,\xi\rangle|\geqslant t\|\langle\cdot,\xi\rangle\|_{L_2(C)}\})\leqslant 2e^{-t^\alpha/\varrho^\alpha}$. Conversely, it is a standard fact that tail estimates of this form imply that ξ is a ψ_α-direction for C. Similar definitions may be given in the context of a centered log-concave probability measure μ on $\mathbb{R}^n$. From log-concavity it follows that every $\xi\in S^{n-1}$ is a ψ_1-direction for any C or μ with an absolute constant ϱ: there exists $\varrho>0$ such that

$$\|\langle\cdot,\xi\rangle\|_{L_{\psi_1}(\mu)} \leqslant \varrho \|\langle\cdot,\xi\rangle\|_{L_2(\mu)}$$

for all $n\geqslant 1$, all centered log-concave probability measures μ on $\mathbb{R}^n$ and all $\xi\in S^{n-1}$. We refer the reader to the book [11] for an updated exposition of isotropic log-concave measures and more information on the hyperplane conjecture.

We close this introductory section with a lemma that may be viewed as a form of generalization of Khinchine's inequality, where the randomness is no longer that of Bernoulli $\{-1,1\}$ random variables but here is given by random vectors in the bodies $C_1,\dots,C_s$.

Lemma 2.1. *Let $C_1,\dots,C_s$ be convex bodies of volume 1 and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then,*

$$\Big(\mathbb{E}_{\mathcal{C}}\Big\|\sum_{j=1}^s t_j x_j\Big\|_K^q\Big)^{1/q} \leqslant cq\,\|\mathbf{t}\|_{\mathcal{C},K},$$

where $c>0$ is an absolute constant.

The lemma follows immediately from the fact (see [11, Theorem 2.4.6]) that if μ is a log-concave probability measure on $\mathbb{R}^k$ and $f:\mathbb{R}^k\to\mathbb{R}$ is a seminorm then, for any $q\geqslant 1$,

$$\|f\|_{L_q(\mu)}\leqslant cq\,\|f\|_{L_1(\mu)},$$

where $c>0$ is an absolute constant. We apply this fact on $\mathbb{R}^{ns}$ for the semi-norm

$$(x_1,\dots,x_s)\mapsto\Big\|\sum_{j=1}^s t_jx_j\Big\|_K$$

and the uniform measure on $C_1\times\cdots\times C_s$.

3 A basic identity and a proof of the lower bound

In this section we assume that $C_1,\dots,C_s$ are centrally symmetric convex bodies of volume 1 in $\mathbb{R}^n$ and study the quantity

$$\|\mathbf{t}\|_{\mathcal{C},K}=\int_{C_1}\cdots\int_{C_s}\Big\|\sum_{j=1}^s t_jx_j\Big\|_K\,dx_1\cdots dx_s$$

where $\mathbf{t}=(t_1,\dots,t_s)$ and K is a centrally symmetric convex body in $\mathbb{R}^n$. By the symmetry of the C_j's we have that

$$\int_{C_1}\cdots\int_{C_s}\Big\|\sum_{j=1}^s t_jx_j\Big\|_K\,dx_1\cdots dx_s=\int_{C_1}\cdots\int_{C_s}\Big\|\sum_{j=1}^s \epsilon_jt_jx_j\Big\|_K\,dx_1\cdots dx_s$$

for all $\epsilon=(\epsilon_1,\dots,\epsilon_s)\in E_2^s$, therefore we may always assume that $t_1,\dots,t_s\geqslant 0$. Our starting point is the next observation.

Lemma 3.1. *Let $X_1,\dots,X_s$ be independent random vectors, uniformly distributed on $C_1,\dots,C_s$ respectively. Given $\mathbf{t}=(t_1\dots,t_s)\in\mathbb{R}^s$, we write $\nu_{\mathbf{t}}$ for the distribution of the random vector $t_1X_1+\cdots+t_sX_s$. Then,*

$$\|\mathbf{t}\|_{\mathcal{C},K}=\int_{\mathbb{R}^n}\|x\|_K\,d\nu_{\mathbf{t}}(x).$$

Since $\|\mathbf{t}\|_{\mathcal{C},K}$ is a norm, we may always assume that $\|\mathbf{t}\|_2=1$. Note that $\nu_{\mathbf{t}}$ is an even log-concave probability measure on $\mathbb{R}^n$ (this is a consequence of the Prékopa-Leindler inequality; see [1]). We write $g_{\mathbf{t}}$ for the density of $\nu_{\mathbf{t}}$. The next lemma provides an upper bound for $\|g_{\mathbf{t}}\|_\infty=g_{\mathbf{t}}(0)$.

Lemma 3.2. *If $\|\mathbf{t}\|_2=1$ then $\|g_{\mathbf{t}}\|_\infty\leqslant e^n$.*

Proof. The proof employs a result of Bobkov and Madiman from [5] and the Shannon-Stam inequality (see [29]). Recall that the entropy functional of a random vector X in $\mathbb{R}^n$ with density $g(x)$ is defined by

$$h(X)=-\int_{\mathbb{R}^n}g(x)\log g(x)\,dx$$

provided the integral exists. Bobkov and Madiman have shown that if g is log-concave then

$$\log(\|g\|_\infty^{-1})\leqslant h(X)\leqslant n+\log(\|g\|_\infty^{-1})$$

(the assumption that g is log-concave is needed only for the right hand side inequality). Let $\mathbf{t}\in\mathbb{R}^s$ with $\|\mathbf{t}\|_2=1$ and $t_1,\dots,t_s\geqslant 0$. Then, if $X_1,\dots,X_s$ are independent random vectors with densities $g_1,\dots,g_s$ we have that

$$h(t_1X_1+\cdots+t_sX_s)\geqslant\sum_{j=1}^s t_j^2h(X_j).$$

This is an equivalent form of the Shannon-Stam inequality (see [22] and [13]). Since the density $g_{\mathbf{t}}$ of $t_1X_1 + \cdots + t_sX_s$ is also log-concave, we may write

$$\sum_{j=1}^{s} t_j^2 \log(\|g_j\|_\infty^{-1}) \leqslant \sum_{j=1}^{s} t_j^2 h(X_j) \leqslant h(t_1X_1 + \cdots + t_sX_s) \leqslant n + \log(\|g_{\mathbf{t}}\|_\infty^{-1}),$$

which implies that

$$\|g_{\mathbf{t}}\|_\infty \leqslant e^n \prod_{j=1}^{s} \|g_j\|_\infty^{t_j^2}.$$

In our case, $g_j = \mathbf{1}_{C_j}$, therefore $\|g_j\|_\infty = 1$ and the lemma follows. $\square$

The next lemma is an immediate consequence of [9, Lemma 2.3] (see also [25, Lemma 2.1]).

Lemma 3.3. *Let f be a bounded positive density of a probability measure μ on $\mathbb{R}^n$. For any centrally symmetric convex body K in $\mathbb{R}^n$ and any $p > 0$ one has*

$$\left(\frac{n}{n+p}\right)^{1/p} \leqslant \left(\int_{\mathbb{R}^n} \|x\|_K^p f(x)\, dx\right)^{1/p} \|f\|_\infty^{1/n} \mathrm{vol}_n(K)^{1/n}.$$

We apply Lemma 3.3 for the log-concave probability measure $\nu_{\mathbf{t}}$. For any $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$ we have $\|g_{\mathbf{t}}\|_\infty = g_{\mathbf{t}}(0) \leqslant e^n$, therefore

$$\frac{n}{n+1} \leqslant e\, \mathrm{vol}_n(K)^{1/n} \int_{\mathbb{R}^n} \|x\|_K \, d\nu_{\mathbf{t}}(x).$$

Combining this inequality with Lemma 3.1 we see that if $\mathcal{C} = (C_1, \ldots, C_s)$ is an s-tuple of centrally symmetric convex bodies of volume 1 and K is a centrally symmetric convex body in $\mathbb{R}^n$ then, for any $s \geqslant 1$ and any $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$

$$\|\mathbf{t}\|_{\mathcal{C},K} \geqslant \frac{n}{e(n+1)} \mathrm{vol}_n(K)^{-1/n} \, \|\mathbf{t}\|_2.$$

This proves Theorem 1.2.

4 Upper bounds

In this section we assume that C is an isotropic convex body in $\mathbb{R}^n$. We shall further exploit the identity of Lemma 3.1 to give upper estimates for $\|\mathbf{t}\|_{C^s,K}$, where K is a centrally symmetric convex body in $\mathbb{R}^n$.

As in the previous section, let $X_1, \ldots, X_s$ be independent random vectors, uniformly distributed on C. Given $\mathbf{t} = (t_1 \ldots, t_s) \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$, we write $\nu_{\mathbf{t}}$ for the distribution of the random vector $t_1X_1 + \cdots + t_sX_s$. It is then easily verified that the covariance matrix $\mathrm{Cov}(\nu_{\mathbf{t}})$ of $\nu_{\mathbf{t}}$ is a multiple of the identity: more precisely,

$$\mathrm{Cov}(\nu_{\mathbf{t}}) = L_C^2\, I_n.$$

It follows that if $g_{\mathbf{t}}$ is the density of $\nu_{\mathbf{t}}$ then $f_{\mathbf{t}}(x) = L_C^n g_{\mathbf{t}}(L_C x)$ is the density of an isotropic log-concave probability measure on $\mathbb{R}^n$. Indeed, we have

$$\int_{\mathbb{R}^n} f_{\mathbf{t}}(x) x_i x_j \, dx = L_C^n \int_{\mathbb{R}^n} g_{\mathbf{t}}(L_C x) x_i x_j \, dx = L_C^{-2} \int_{\mathbb{R}^n} g_{\mathbf{t}}(y) y_i y_j \, dy = \delta_{ij}$$

for all $1 \leqslant i, j \leqslant n$. From Lemma 3.2 we see that

$$L_{\mu_{\mathbf{t}}} = \|f_{\mathbf{t}}\|_\infty^{\frac{1}{n}} = L_C \|g_{\mathbf{t}}\|_\infty^{\frac{1}{n}} \leqslant e L_C$$

for all $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$. We also have

$$\|\mathbf{t}\|_{C^s,K} = \int_{\mathbb{R}^n} \|x\|_K \, d\nu_{\mathbf{t}}(x) = L_C^{-n} \int_{\mathbb{R}^n} \|x\|_K f_{\mathbf{t}}(x/L_C) \, dx = L_C \int_{\mathbb{R}^n} \|y\|_K d\mu_{\mathbf{t}}(y).$$

Definition 4.1. Let μ be a centered log-concave probability measure on $\mathbb{R}^n$. For any star body K in $\mathbb{R}^n$ we define

$$I_1(\mu, K) = \int_{\mathbb{R}^n} \|x\|_K d\mu(x).$$

With this definition, we can write

$$\|\mathbf{t}\|_{C^s,K} = L_C\, I_1(\mu_{\mathbf{t}}, K) \tag{4.1}$$

for all $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$. Then, our aim is to establish an upper bound for $I_1(\mu_{\mathbf{t}}, K)$.

4.1 Simple upper and lower bounds

A first upper bound for $I_1(\mu_{\mathbf{t}}, K)$ can be obtained if we use the simple inequality $\|y\|_K \leqslant b\|y\|_2$, where $b = b(K) = R(K^\circ)$. We observe that

$$I_1(\mu_{\mathbf{t}}, K) \leqslant b \int_{\mathbb{R}^n} \|y\|_2 d\mu_{\mathbf{t}}(y) \leqslant b\sqrt{n}.$$

because the last integral is bounded by $\sqrt{n}$: this follows immediately from the Cauchy-Schwarz inequality and the isotropicity of $\mu_{\mathbf{t}}$. On the other hand,

$$\begin{aligned} I_1(\mu_{\mathbf{t}}, K) &= \int_{\mathbb{R}^n} \max_{x\in K^\circ} |\langle x, y\rangle|\, d\mu_{\mathbf{t}}(y) \geqslant \max_{x\in K^\circ} \int_{\mathbb{R}^n} |\langle x, y\rangle|\, d\mu_{\mathbf{t}}(y) \geqslant \max_{x\in K^\circ} c_1 \Big(\int_{\mathbb{R}^n} |\langle x, y\rangle|^2\, d\mu_{\mathbf{t}}(y)\Big)^{1/2} \\ &= c_1 \max_{x\in K^\circ} \|x\|_2 = c_1 R(K^\circ) = c_1 b, \end{aligned}$$

where in the second inequality we are using [11, Theorem 2.4.6]. Inserting these two bounds into (4.1) we have the next theorem.

Theorem 4.2. *Let C be an isotropic convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then, for any $s \geqslant 1$ and $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$,*

$$c_1 L_C R(K^\circ)\, \|\mathbf{t}\|_2 \leqslant \|\mathbf{t}\|_{C^s,K} \leqslant \sqrt{n} L_C R(K^\circ)\, \|\mathbf{t}\|_2,$$

where $c_1 > 0$ is an absolute constant.

There are some classes of centrally symmetric convex bodies that behave well with respect to the upper bound of Theorem 4.2. We discuss one of them in the next subsection.

4.2 2-convex bodies

Recall that if K is a centrally symmetric convex body in $\mathbb{R}^n$ then the modulus of convexity of K is the function $\delta_K : (0, 2] \to \mathbb{R}$ defined by

$$\delta_K(\varepsilon) = \inf \Big\{ 1 - \Big\| \frac{x+y}{2} \Big\|_K \,:\, \|x\|_K, \|y\|_K \leqslant 1, \|x-y\|_K \geqslant \varepsilon \Big\}.$$

Then, K is called 2-convex with constant α if, for every $\varepsilon \in (0, 2]$,

$$\delta_K(\varepsilon) \geqslant \alpha\varepsilon^2.$$

Examples of 2-convex bodies are given by the unit balls of subspaces of L_p-spaces, $1 < p \leqslant 2$; one can check that the definition is satisfied with $\alpha \approx p - 1$. Klartag and E. Milman have proved in [20] that if K is a centrally symmetric convex body of volume 1 in $\mathbb{R}^n$, which is also 2-convex with constant α, then

$$L_K \leqslant c_1/\sqrt{\alpha},$$

where $c_1 > 0$ is an absolute constant. Moreover, if K is isotropic then

$$c_2\sqrt{\alpha}\sqrt{n}B_2^n \subseteq K,$$

for an absolute constant $c_2 > 0$ (see, again, [20]). From Theorem 4.2 we immediately get the next estimate.

Theorem 4.3. *Let C be an isotropic convex body in $\mathbb{R}^n$ and K be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ which is also 2-convex with constant α. Then for any $s \geqslant 1$ and $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$,*

$$\|\mathbf{t}\|_{C^s,K} \leqslant \frac{cL_C}{\sqrt{\alpha}} \|\mathbf{t}\|_2$$

where $c > 0$ is an absolute constant. In particular, for any centrally symmetric convex body K in $\mathbb{R}^n$ which is 2-convex with constant α, we have that

$$\|\mathbf{t}\|_{K^s,K} \leqslant \frac{c}{\alpha} \|\mathbf{t}\|_2.$$

Proof. The first claim follows from the fact that $R(K^\circ) \leqslant c_2^{-1}/(\sqrt{\alpha}\sqrt{n})$. For the second assertion we use the observation that $\mathbb{E}_{K^s} \left\| \sum_{j=1}^s t_j x_j \right\|_K = \mathbb{E}_{(TK)^s} \left\| \sum_{j=1}^s t_j x_j \right\|_{TK}$ for any $T \in SL(n)$, and hence we may assume that K is isotropic. Since $L_K \leqslant c_1/\sqrt{\alpha}$ we see that

$$\mathbb{E}_{K^s} \Big\| \sum_{j=1}^s t_j x_j \Big\|_K \leqslant \frac{c_2^{-1} L_K}{\sqrt{\alpha}} \|\mathbf{t}\|_2 \leqslant \frac{c_3}{\alpha} \|\mathbf{t}\|_2,$$

where $c_3 = c_2^{-1} c_1$. $\square$

4.3 A general upper bound

In this subsection we prove Theorem 1.4. By homogeneity it is enough to consider the case $\|\mathbf{t}\|_2 = 1$. Our starting point will be again (4.1). We have

$$\|\mathbf{t}\|_{C^s,K} = L_C \, I_1(\mu_{\mathbf{t}}, K),$$

and hence our aim is to establish an upper bound for $I_1(\mu_{\mathbf{t}}, K)$. We shall use a well-known inequality of Paouris from [26].

Theorem 4.4 (Paouris)**.** *If μ is an isotropic log-concave probability measure on $\mathbb{R}^n$, then*

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 \geqslant c_1 \, r\sqrt{n}\}) \leqslant e^{-r\sqrt{n}}$$

for every $r \geqslant 1$, where $c_1 > 0$ is an absolute constant.

Note also that, since $R(C) \leqslant c_2 n L_C$ and $\mathrm{supp}(\nu_{\mathbf{t}}) \subseteq sC$, we have that

$$\mathrm{supp}(\mu_{\mathbf{t}}) \subseteq \frac{s}{L_C} \, C \subseteq (c_2 ns) \, B_2^n$$

for any $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$. Therefore, if we fix $r \geqslant 1$ and set $C_{\mathbf{t}}(r) = \mathrm{supp}(\mu_{\mathbf{t}}) \cap c_1 r\sqrt{n} B_2^n$, we may write

$$\begin{aligned}
\int_{\mathbb{R}^n} \|x\|_K \, d\mu_{\mathbf{t}}(x) &= \int_{C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) + \int_{\mathrm{supp}(\mu_{\mathbf{t}}) \setminus C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) \\
&\leqslant \int_{C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) + b(K) \int_{\mathrm{supp}(\mu_{\mathbf{t}}) \setminus C_{\mathbf{t}}(r)} \|x\|_2 d\mu_{\mathbf{t}}(x) \\
&\leqslant \int_{C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) + b(K) \, (c_2 ns) \, e^{-r\sqrt{n}}.
\end{aligned}$$

Turning our attention to the first term, we consider the log-concave probability measure $\mu_{\mathbf{t},r}$ with density

$$\frac{1}{\mu_{\mathbf{t}}(C_{\mathbf{t}}(r))} \, \mathbf{1}_{C_{\mathbf{t}}(r)} f_{\mathbf{t}}$$

and the stochastic process $(w_y)_{y\in K^\circ}$ on $(\mathbb{R}^n,\mu_{\mathbf{t},r})$, where $w_y(x)=\langle x,y\rangle$. We also consider a standard Gaussian random vector G in $\mathbb{R}^n$, and for $y\in K^\circ$ set $h_y(G)=\langle G,y\rangle$. Note that (see e.g. [1, Lemma 9.1.3])

$$\mathbb{E}\Big(\max_{y\in K^\circ} h_y(G)\Big)=\mathbb{E}\,\|G\|_K\approx\sqrt{n}M(K). \tag{4.2}$$

To bound $\mathbb{E}(\max_{y\in K^\circ} w_y)$, we will use Talagrand's comparison theorem (see [30]).

Theorem 4.5 (Talagrand's comparison theorem)**.** *If $(Y_t)_{t\in T}$ is a Gaussian process and $(X_t)_{t\in T}$ is a stochastic process such that*

$$\|X_s-X_t\|_{\psi_2}\leqslant\alpha\,\|Y_s-Y_t\|_2$$

for some $\alpha>0$ and every $s,t\in T$, then

$$\mathbb{E}\Big(\max_{t\in T}X_t\Big)\leqslant c\alpha\,\mathbb{E}\Big(\max_{t\in T}Y_t\Big).$$

It is easily checked that $\|h_y-h_z\|_2=\|y-z\|_2$ for all $y,z\in K^\circ$. To bound the ψ_2 norm of w_y-w_z, we use the inequality $\|h\|_{\psi_2}\leqslant\sqrt{\|h\|_{\psi_1}\|h\|_\infty}$. Note that

$$\|w_y-w_z\|_{L^\infty(\mu_{\mathbf{t},r})}\leqslant R(C_{\mathbf{t}}(r))\|y-z\|_2\leqslant c_1r\sqrt{n}\|y-z\|_2$$

and we also have

$$\|w_y-w_z\|_{L^{\psi_1}(\mu_{\mathbf{t},r})}\leqslant c_3\|w_y-w_z\|_{L^2(\mu_{\mathbf{t},r})}\leqslant 2c_3\|y-z\|_2$$

for some absolute constant $c_3>0$ (here we also use the fact that $\mu(C_{\mathbf{t}}(r))\geqslant 1-e^{-r\sqrt{n}}\geqslant 1/2$). It follows that

$$\|w_y-w_z\|_{L^{\psi_2}(\mu_{\mathbf{t},r})}\leqslant c_4\sqrt{r}\,\sqrt[4]{n}\,\|h_y-h_z\|_2.$$

Theorem 4.5 then implies that

$$\begin{aligned}\int_{C_{\mathbf{t}}(r)}\|x\|_K\,d\mu_{\mathbf{t}}(x)&=\mu_{\mathbf{t}}(C_{\mathbf{t}}(r))\,\mathbb{E}_{\mu_{\mathbf{t},r}}\Big(\max_{y\in K^\circ}w_y\Big)\leqslant c_5\sqrt{r}\,\sqrt[4]{n}\,\mathbb{E}\Big(\max_{y\in K^\circ}h_y\Big)\\&\approx\sqrt{r}\,\sqrt[4]{n}\,\sqrt{n}M(K).\end{aligned}$$

Finally,

$$\int_{\mathbb{R}^n}\|x\|_K\,d\mu_{\mathbf{t}}(x)\leqslant c_1'\Big(\sqrt{r}\,\sqrt[4]{n}\,\sqrt{n}M(K)+b(K)\,ns\,e^{-r\sqrt{n}}\Big).$$

Since $b(K)\leqslant c_6\sqrt{n}M(K)$ we have that

$$b(K)\,ns\,e^{-r\sqrt{n}}\leqslant c_6nse^{-r\sqrt{n}}\,\sqrt{n}M(K)\leqslant\sqrt{r}\,\sqrt[4]{n}\,\sqrt{n}M(K)$$

if we choose

$$r\approx\max\Big\{1,\frac{\log(1+s)}{\sqrt{n}}\Big\}.$$

Therefore,

$$\|\mathbf{t}\|_{C^s,K}=L_C\,I_1(\mu_{\mathbf{t}},K)\leqslant\Big(c_2'L_C\max\Big\{1,\frac{\sqrt{\log(1+s)}}{\sqrt[4]{n}}\Big\}\,\sqrt[4]{n}\Big)\,\sqrt{n}M(K)$$

as claimed. $\square$

Adapting the proof of Theorem 1.4 we can show that if C is assumed a ψ_2-body with constant ϱ, which means that every direction ξ is a ψ_2-direction for C with constant ϱ, then a much better estimate is available.

Theorem 4.6. *Let C be an isotropic convex body in $\mathbb{R}^n$, which is a ψ_2-body with constant ϱ, and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then for any $s \geqslant 1$ and every $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$,*

$$\|\mathbf{t}\|_{C^s,K} \leqslant c\varrho^2 \sqrt{n} M(K)\, \|\mathbf{t}\|_2$$

where $c > 0$ is an absolute constant.

Proof. We consider the Gaussian process $h_y(G) = \langle G, y\rangle$, where G is a standard Gaussian random vector in $\mathbb{R}^n$, and recall that $\|h_y - h_z\|_2 = \|y - z\|_2$ and

$$\mathbb{E}\Big(\max_{y\in K^\circ} h_y(G)\Big) \approx \sqrt{n} M(K).$$

The main observation is that if $\|\mathbf{t}\|_2 = 1$ then $\mu_{\mathbf{t}}$ is a ψ_2-measure with constant ϱ. Indeed, for any $\xi \in S^{n-1}$ we have (see [31, Proposition 2.6.1]) that if $w_\xi(x) = \langle x, \xi\rangle$ then

$$\|\langle x, \xi\rangle\|_{L_{\psi_2}(\mu_{\mathbf{t}})}^2 = \Big\|\Big\langle \sum_{j=1}^s L_C^{-1} t_j X_j, \xi\Big\rangle\Big\|_{L_{\psi_2}(C^s)}^2 \leqslant \sum_{j=1}^s L_C^{-2} t_j^2 \|\langle X_j, \xi\rangle\|_{L_{\psi_2}(C)}^2 \leqslant \varrho^2,$$

and hence, for any $y, z \in K^\circ$, the ψ_2 norm of $w_y - w_z$ can be directly estimated as follows:

$$\|w_y - w_z\|_{\psi_2} \leqslant c_1\varrho\|y - z\|_2 = c_1\varrho\|h_y - h_z\|_2.$$

Then, Theorem 4.5 and the fact that $L_C \leqslant c_2\varrho$ (see [11, Chapter 7]) imply that

$$\|\mathbf{t}\|_{C^s,K} = L_C \int_{\mathbb{R}^n} \|x\|_K \, d\mu_{\mathbf{t}}(x) = L_C\, \mathbb{E}\Big(\max_{y\in K^\circ} w_y\Big) \leqslant c_3\varrho^2\, \mathbb{E}\Big(\max_{y\in K^\circ} h_y(G)\Big) \approx \varrho^2 \sqrt{n} M(K).$$

as claimed. $\square$

5 Bodies with bounded cotype-2 constant

Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Recall that if X_K is the normed space with unit ball K, we write $C_{2,k}(X_K)$ for the best constant $C > 0$ such that

$$\Big(\mathbb{E}_\epsilon \Big\|\sum_{i=1}^k \epsilon_i x_i\Big\|_K^2\Big)^{1/2} \geqslant \frac{1}{C}\Big(\sum_{i=1}^k \|x_i\|_K^2\Big)^{1/2}$$

for all $x_1, \dots, x_k \in X$. Then, the cotype-2 constant of X_K is defined as $C_2(X_K) := \sup_k C_{2,k}(X_K)$. Replacing the ϵ_j's by independent standard Gaussian random variables g_j in the definition above, one may define the Gaussian cotype-2 constant $\alpha_2(X_K)$ of X_K. One can check that $\alpha_2(X_K) \leqslant C_2(X_K)$. E. Milman has proved in [24] that if μ is a finite, compactly supported isotropic measure on $\mathbb{R}^n$ then, for any centrally symmetric convex body K in $\mathbb{R}^n$,

$$I_1(\mu, K) \leqslant c_1\alpha_2(X_K)\sqrt{n} M(K) \leqslant c_1 C_2(X_K)\sqrt{n} M(K). \tag{5.1}$$

Using (5.1) we can prove the following.

Theorem 5.1. *Let C be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then for any $s \geqslant 1$ and $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$,*

$$\frac{c_1}{C_2(X_K)} \mathrm{vol}_n(K)^{-1/n} \|\mathbf{t}\|_2 \leqslant \mathbb{E}_{C^s}\Big\|\sum_{j=1}^s t_j x_j\Big\|_K \leqslant \big(c_2 L_C C_2(X_K)\sqrt{n} M(K)\big)\, \|\mathbf{t}\|_2$$

where $c_1, c_2 > 0$ are absolute constants. In particular, for any centrally symmetric convex body K of volume 1 in $\mathbb{R}^n$ we have that

$$\frac{c_1}{C_2(X_K)}\|\mathbf{t}\|_2 \leqslant \mathbb{E}_{K^s}\Big\|\sum_{j=1}^{s} t_j x_j\Big\|_K \leqslant \big(c_2 L_K C_2(X_K)\sqrt{n}M(K_{\rm iso})\big)\,\|\mathbf{t}\|_2,$$

where $K_{\rm iso}$ is an isotropic image of K.

Proof. Combining (5.1) with (4.1) we get

$$\|\mathbf{t}\|_{C^s,K} \leqslant c_1 L_C C_2(X_K)\,\sqrt{n}M(K)$$

for all $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$. On the other hand, for any $\mathbf{t} \in \mathbb{R}^s$, by the symmetry of C we have that

$$\begin{aligned}\|\mathbf{t}\|_{C^s,K} &= \int_C \cdots \int_C \int_{E_2^s} \Big\|\sum_{j=1}^{s} \epsilon_j t_j x_j\Big\|_K d\mu_s(\epsilon)\,dx_1\cdots dx_s\\ &\geqslant \int_C \cdots \int_C \frac{1}{\sqrt{2}}\Big(\int_{E_2^s} \Big\|\sum_{j=1}^{s} \epsilon_j t_j x_j\Big\|_K^2 d\mu_s(\epsilon)\Big)^{1/2} dx_1\cdots dx_s\\ &\geqslant \frac{1}{\sqrt{2}C_2(X_K)}\int_C \cdots \int_C \Big(\sum_{j=1}^{s} t_j^2\|x_j\|_K^2\Big)^{1/2} dx_1\cdots dx_s\\ &\geqslant \frac{c}{C_2(X_K)}\Big(\sum_{j=1}^{s} t_j^2 \int_C \|x_j\|_K^2 dx_j\Big)^{1/2}\\ &\geqslant \frac{c}{C_2(X_K)}\|\mathbf{t}\|_2 \int_C \|x\|_K\,dx,\end{aligned}$$

where $c > 0$ is an absolute constant (in the first inequality we are using the Kahane-Khintchine inequality and in the third inequality we are using [11, Theorem 2.4.6] for the semi-norm $(x_1, \ldots, x_s) \mapsto \Big(\sum_{j=1}^{s} t_j^2\|x_j\|_K^2\Big)^{1/2}$ on C^s, while in the last step we are using the Cauchy-Schwarz inequality for $\|\cdot\|_K$ on C). From Lemma 3.3 with $f = \mathbf{1}_C$ we see that

$$\int_C \|x\|_K\,dx \geqslant \frac{n}{n+1}\mathrm{vol}_n(K)^{-1/n},$$

and the result follows.

In the case $C = K$, we may assume that K is isotropic and these bounds take the form

$$\frac{c_1}{C_2(X_K)}\|\mathbf{t}\|_2 \leqslant \mathbb{E}_{K^s}\Big\|\sum_{j=1}^{s} t_j x_j\Big\|_K \leqslant \big(c_2 L_K C_2(X_K)\sqrt{n}M(K)\big)\,\|\mathbf{t}\|_2.$$

This completes the proof. $\square$

Remark 5.2. Another interesting case is when K has bounded type-2 constant. Recall that if X_K is the normed space with unit ball K, we write $T_{2,k}(X_K)$ for the best constant $T > 0$ such that

$$\Big(\mathbb{E}_\epsilon\Big\|\sum_{i=1}^{k} \epsilon_i x_i\Big\|_K^2\Big)^{1/2} \leqslant T\Big(\sum_{i=1}^{k} \|x_i\|_K^2\Big)^{1/2}$$

for all $x_1, \ldots, x_k \in X$. Then, the type-2 constant of X_K is defined as $T_2(X_K) := \sup_k T_{2,k}(X_K)$. E. Milman has proved in [24] that if μ is a finite, compactly supported isotropic measure on $\mathbb{R}^n$ then, for any centrally symmetric convex body K in $\mathbb{R}^n$,

$$I_1(\mu, K) \geqslant c\sqrt{n}\frac{M(K)}{T_2(X_K)}. \tag{5.2}$$

Using this inequality and following a similar argument, as in the cotype-2 case, we get:

Theorem 5.3. *Let C be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then for any $s \geqslant 1$ and $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$,*

$$\frac{c_1 L_C \sqrt{n} M(K)}{T_2(X_K)} \, \|\mathbf{t}\|_2 \leqslant \mathbb{E}_{C^s} \Big\| \sum_{j=1}^s t_j x_j \Big\|_K \leqslant c_2 T_2(X_K) \Big(\int_C \|x\|_K dx \Big) \, \|\mathbf{t}\|_2$$

where $c_1, c_2 > 0$ are absolute constants. In particular, for any centrally symmetric convex body K of volume 1 in $\mathbb{R}^n$ we have that

$$\frac{c_1 L_K \sqrt{n} M(K)}{T_2(X_K)} \|\mathbf{t}\|_2 \leqslant \mathbb{E}_{K^s} \Big\| \sum_{j=1}^s t_j x_j \Big\|_K \leqslant c_2 T_2(X_K) \|\mathbf{t}\|_2.$$

Note that if $\mathrm{vol}_n(K) = 1$ then $\sqrt{n} M(K) \geqslant c > 0$, therefore the estimate is exact, up to the type-2 constant, and actually implies an upper bound for L_K.

6 The unconditional case

The case where $C_1, \dots, C_s$ and K are isotropic unconditional convex bodies in $\mathbb{R}^n$ has been essentially studied in [14, Theorem 4.1].

Theorem 6.1. *There exists an absolute constant $c > 0$ with the following property: if K and $C_1, \dots, C_s$ are isotropic unconditional convex bodies in $\mathbb{R}^n$ then, for every $q \geqslant 1$,*

$$\Big(\int_{C_1} \cdots \int_{C_s} \Big\| \sum_{j=1}^s t_j x_j \Big\|_K^q dx_1 \dots dx_s \Big)^{1/q} \leqslant c n^{1/q} \sqrt{q} \cdot \max\{\|\mathbf{t}\|_2, \sqrt{q} \|\mathbf{t}\|_\infty\} \leqslant c n^{1/q} q \, \|\mathbf{t}\|_2,$$

for every $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$. In particular,

$$\|\mathbf{t}\|_{\mathcal{C},K} \leqslant c \sqrt{\log n} \cdot \max\{\|\mathbf{t}\|_2, \sqrt{\log n} \|\mathbf{t}\|_\infty\} \leqslant c \log n \, \|\mathbf{t}\|_2.$$

Proof. We briefly sketch the argument, which is essentially the same as in [14]. We write μ_n for the uniform distribution on B_1^n, with density $\frac{d\mu_n(x)}{dx} = \frac{n!}{2^n} \mathbf{1}_{B_1^n}(x)$. If we set $\Delta_n = \{x \in \mathbb{R}^n_+ : x_1 + \cdots + x_n \leqslant 1\}$ then a simple computation shows that for every n-tuple of non-negative integers $p_1, \dots, p_n$, one has

$$\int_{\Delta_n} x_1^{p_1} \dots x_n^{p_n} dx = \frac{p_1! \cdots p_n!}{(n + p_1 + \cdots + p_n)!}.$$

In [6] it is proved that for every isotropic unconditional convex body K in $\mathbb{R}^n$ one has $c B_\infty^n \subseteq K \subseteq V_n$, where $V_n = \sqrt{3/2} n B_1^n$ and $c > 0$ is an absolute constant. Therefore, $\| \cdot \|_K \leqslant c_1 \| \cdot \|_\infty \leqslant c_1 \| \cdot \|_q$, where $c_1 > 0$ is an absolute constant. We proceed to give an upper bound for

$$F_{\mathcal{C},q}(\mathbf{t}) := \int_{C_1} \cdots \int_{C_s} \| \sum_{i=1}^s t_i x_i \|_{2q}^{2q} dx_1 \cdots dx_s,$$

where $q \geq 1$ is an integer. We write $x_i = (x_{i1}, \dots, x_{in})$ and define $y_j = (x_{1j}, \dots, x_{sj})$ for all $j = 1, \dots, n$. Then,

$$F_{\mathcal{C},q}(\mathbf{t}) = \int_{C_1} \cdots \int_{C_s} \sum_{j=1}^n \langle \mathbf{t}, y_j \rangle^{2q} \, dx_1 \cdots dx_s = \sum_{j=1}^n \sum_{q_1 + \cdots + q_s = q} \frac{(2q)!}{(2q_1)! \cdots (2q_s)!} \prod_{i=1}^s t_i^{2q_i} \int_{C_i} x_{ij}^{2q_i} \, dx_i.$$

Next, we apply a comparison theorem from [7]: for every function $F : \mathbb{R}^n \to \mathbb{R}$ which is centrally symmetric, coordinatewise increasing and absolutely continuous, we have that

$$\int F(x) d\mu_A(x) \leqslant \int F(x) d\mu_{V_n}(x),$$

where μ_A is the uniform measure on the isotropic unconditional convex body A. It follows that

$$\int_{C_i} x_{ij}^{2q_i}\, dx_i \leqslant \int_{V_n} x_1^{2q_i} d\mu_{V_n}(x) \leqslant (c_1 n)^{2q_i} n! \int_{\Delta_n} x_1^{2q_i} dx = (c_1 n)^{2q_i} \frac{n!(2q_i)!}{(n+2q_i)!},$$

where $c_1 = \sqrt{3/2}$. Combining the above we see that

$$F_{\mathcal{C},q}(\mathbf{t}) \leqslant n(n!)^s (c_1 n)^{2q} (2q)! \sum_{q_1+\cdots+q_s=q} \frac{t_1^{2q_1}\cdots t_s^{2q_s}}{(n+2q_1)!\cdots(n+2q_s)!}.$$

Using the estimate $(n+2r)! \geq n! n^{2r}$ which holds for every $r \geq 0$, we get

$$F_{\mathcal{C},q}(\mathbf{t}) \leqslant n c_1^{2q} (2q)! \sum_{q_1+\cdots+q_s=q} t_1^{2q_1}\cdots t_s^{2q_s}.$$

We now use another observation from [7]: if $q \geq 1$ is an integer and $P_q(y) = \sum_{q_1+\cdots+q_s=q} y_1^{q_1}\cdots y_s^{q_s}$, $y=(y_1,\ldots,y_s)\in\mathbb{R}_+^s$, then for any $y\in\mathbb{R}_+^s$ with $y_1+\cdots+y_s=1$ we have

$$P_q(y) \leqslant (2e\max\{1/q, \|y\|_\infty\})^q.$$

Applying this inequality to the s-tuple $y = \frac{1}{\|\mathbf{t}\|_2^2}\left(t_1^2,\ldots,t_s^2\right)$ we get

$$F_{\mathcal{C},q}^{\frac{1}{2q}}(\mathbf{t}) \leqslant c_1 n^{\frac{1}{2q}} \sqrt[2q]{(2q)!}\left(2e\max\{\|\mathbf{t}\|_2^2/q, \|\mathbf{t}\|_\infty^2\}\right)^{1/2} \leqslant c_2 n^{\frac{1}{2q}} \sqrt{q}\max\{\|\mathbf{t}\|_2, \sqrt{q}\|\mathbf{t}\|_\infty\}.$$

Then, we easily conclude the proof. $\square$

Remark 6.2. Using our approach we can obtain a similar upper bound directly. Consider $\mathbf{t}\in\mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$. As usual, we have

$$\|\mathbf{t}\|_{C^s,K} = L_C\, I_1(\mu_{\mathbf{t}}, K),$$

where $\mu_{\mathbf{t}}$ is an unconditional isotropic log-concave probability measure. Since K is also unconditional and isotropic, we have $c_1 B_\infty^n \subseteq K$ and hence $\|x\|_K \leqslant c_1^{-1}\|x\|_\infty$ for all $x\in\mathbb{R}^n$. Therefore,

$$I_1(\mu_{\mathbf{t}}, K) = \int_{\mathbb{R}^n} \|x\|_K d\mu_{\mathbf{t}}(x) \leqslant c_1^{-1}\int_{\mathbb{R}^n} \max_{1\leqslant i\leqslant n} |\langle x, e_i\rangle|\, d\mu_{\mathbf{t}}(x) \leqslant c_2\log n$$

because $\mu_{\mathbf{t}}$ is an isotropic ψ_1-measure with an absolute constant ϱ (see [1, Proposition 3.5.8]). Since C is unconditional, we also have $L_C \leqslant c_3$ for some absolute constant $c_3 > 0$; it follows that

$$\|\mathbf{t}\|_{C^s,K} \leqslant c_4 \log n\, \|\mathbf{t}\|_2$$

for every $\mathbf{t}\in\mathbb{R}^s$. Of course the estimate of Theorem 6.1 is more delicate, and can be better by a $\sqrt{\log n}$-term, as it depends on the coordinates of $\mathbf{t}$.

Remark 6.3. In [14] it is observed that the ℓ_∞-term in the estimate provided by Theorem 6.1 cannot be removed. If $C = \overline{B_1^n}$ and $K = \frac{1}{2}B_\infty^n$ then choosing the vector $\mathbf{e_1} = (1,0,0,\ldots,0)$ we have

$$\|\mathbf{e_1}\|_{C^s,K} = \int_{\overline{B_1^n}} 2\|x\|_\infty\, dx \geqslant c\log n\, \|\mathbf{e_1}\|_\infty$$

for some absolute constant $c > 0$.

The example of the cube shows that the term $\sqrt{\log n}\|\mathbf{t}\|_2$ is necessary. Gluskin and V. Milman show in [17] that if $C = K = \frac{1}{2}B_\infty^n$ then

$$\|\mathbf{t}\|_{K^n,K} \approx q_n(\mathbf{t}) = \sum_{i=1}^u t_i^* + \sqrt{u}\left(\sum_{i=u+1}^n (t_i^*)^2\right)^{1/2}$$

where $u \approx \log n$ and $(t_i^*)_{i\leqslant n}$ is the decreasing rearrangement of $(|t_j|)_{j=1}^n$. It is observed in [14, Remark 4.5] that this implies the lower bound

$$\int_{S^{n-1}} \|\mathbf{t}\|_{K^n,K}\, d\sigma(\mathbf{t}) \geqslant c\sqrt{\log n}.$$

Remark 6.4. It is interesting to test the results of Section 4 and Section 5 on the example of the ℓ_p^n-balls B_p^n. Let us first assume that $1 \leqslant p \leqslant 2$. Then, ℓ_p^n has cotype-2 constant bounded by an absolute (independent from p and n) constant. It is also known (see [1, Chapter 5]) that $M(B_p^n) \approx n^{\frac{1}{p}-\frac{1}{2}}$ and $\mathrm{vol}_n(B_p^n)^{1/n} \approx n^{-\frac{1}{p}}$. It follows that

$$M(\overline{B_p^n}) = \mathrm{vol}_n(B_p^n)^{1/n} M(B_p^n) \approx 1/\sqrt{n}.$$

Since $\overline{B_p^n}$ is isotropic and its isotropic constant is also bounded by an absolute constant, Theorem 5.1 shows that

$$\|\mathbf{t}\|_{\overline{B_p^n}^s,\overline{B_p^n}} \leqslant c_1 \|\mathbf{t}\|_2$$

for every $s \geqslant 1$ and $\mathbf{t} \in \mathbb{R}^s$, where $c_1 > 0$ is an absolute constant.

Next, let us assume that $2 \leqslant q \leqslant \infty$. It is then known (see [1, Chapter 5]) that $\mathrm{vol}_n(B_q^n)^{1/n} \approx n^{-\frac{1}{q}}$ and

$$M(B_q^n) \approx \min\{\sqrt{q}, \sqrt{\log n}\} n^{\frac{1}{q}-\frac{1}{2}}.$$

It follows that

$$M(\overline{B_q^n}) = \mathrm{vol}_n(B_q^n)^{1/n} M(B_q^n) \approx \min\{\sqrt{q}, \sqrt{\log n}\}/\sqrt{n}.$$

Since $\overline{B_q^n}$ is an isotropic ψ_2-convex body with constant $\varrho \approx 1$ (independent from q and n – see [4]) and its isotropic constant is also bounded by an absolute constant, Theorem 4.6 shows that

$$\|\mathbf{t}\|_{\overline{B_q^n}^s,\overline{B_q^n}} \leqslant c_2 \min\{\sqrt{q}, \sqrt{\log n}\}\, \|\mathbf{t}\|_2$$

for every $s \geqslant 1$ and $\mathbf{t} \in \mathbb{R}^s$, where $c_2 > 0$ is an absolute constant.

7 Applications to vector balancing problems

Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Our starting observation is that

$$\int_{O(n)} I_1(\mu, U(K))\, d\nu(U) = \int_{\mathbb{R}^n}\int_{O(n)} \|x\|_{U(K)} d\nu(U)\, d\mu(x) = M(K)\int_{\mathbb{R}^n} \|x\|_2 d\mu(x) \approx \sqrt{n}M(K).$$

Applying this fact for the measure $\mu_{\mathbf{t}}$, from (4.1) we immediately get the following.

Proposition 7.1. *Let C be an isotropic convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. For every $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$ there exists $U \in O(n)$ such that*

$$\|\mathbf{t}\|_{U(C)^s,K} \geqslant cL_C\sqrt{n}M(K)\, \|\mathbf{t}\|_2. \tag{7.1}$$

We know that if $\mathrm{vol}_n(K) = 1$ then the quantity $\sqrt{n}M(K)$ is always greater than c. Therefore, Proposition 7.1 provides many examples in which the lower bound of Gluskin and V. Milman can be improved (note also the presence of L_C in the right hand side of the inequality). For example, in the classical example of the cube $K = \dfrac{1}{2}B_\infty^n$ we have that $\sqrt{n}M(K) \approx \sqrt{\log n}$, which implies the following:

Corollary 7.2. *For every isotropic convex body C in $\mathbb{R}^n$ and any $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$ there exists $U \in O(n)$ such that*

$$\int_{U(C)} \cdots \int_{U(C)} \Big\| \sum_{j=1}^s t_j x_j \Big\|_\infty dx_1 \cdots dx_s \geqslant cL_C\sqrt{\log n}\, \|\mathbf{t}\|_2,$$

where $c > 0$ is an absolute constant.

In this section we explore further this idea. We shall use a number of important facts from asymptotic convex geometry (see [11] for proofs and additional references). For any centrally symmetric convex body K in $\mathbb{R}^n$ and any $q \neq 0$ we define

$$M_q(K) = \left(\int_{S^{n-1}} \|\xi\|_K^q d\sigma(\xi) \right)^{1/q}.$$

Litvak, V. Milman and Schechtman have proved in [23] that

$$M_q(K) \approx M(K) \tag{7.2}$$

for every $1 \leqslant q \leqslant c_1 k(K)$, where $c_1 > 0$ is an absolute constant and $k(K) = n(M(K)/b(K))^2$ is the Dvoretzky dimension of K. Moreover, Klartag and Vershynin have proved in [21] that

$$M_{-q}(K) \approx M(K) \tag{7.3}$$

for every $1 \leqslant q \leqslant c_2 d(K)$, where $d(K) \geqslant c_3 k(K)$ is a parameter of K defined by

$$d(K) = \min \Big\{ n, -\log \gamma_n \Big(\frac{m(K)}{2} K \Big) \Big\},$$

and $m(K) \approx \sqrt{n} M(K)$ is the median of $\| \cdot \|_K$ with respect to the standard Gaussian measure γ_n on $\mathbb{R}^n$.

For any isotropic log-concave probability measure μ on $\mathbb{R}^n$ and any $q \neq 0$, $q > -n$, let

$$I_q(\mu) := \Big(\int_{\mathbb{R}^n} \|x\|_2^q \, d\mu(x) \Big)^{1/q}.$$

Paouris has proved in [26] and [27] that

$$I_{-q}(\mu) \approx I_q(\mu) \approx \sqrt{n} \tag{7.4}$$

for every $1 \leqslant q \leqslant c_4 q_*(\mu)$, where $q_*(\mu) := \max\{q : k(Z_q^\circ(\mu) \geqslant q\}$. It is known that $q_*(\mu) \geqslant c_5 \sqrt{n}$. Moreover, if μ is a ψ_2-measure with constant ϱ then $q_*(\mu) \geqslant c_6 n/\varrho^2$.

Theorem 7.3. *Let C be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then, for every $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$ and $S \subseteq E_2^n$ with $|S| \leqslant e^{q(\mathbf{t})}$, a random $U \in O(n)$ satisfies*

$$\mathrm{vol}_{ns}\Big(\Big\{ (x_j)_{j=1}^s : x_j \in U(C) \ \text{for all } j \text{ and} \Big\| \sum_{j=1}^s \epsilon_j t_j x_j \Big\|_K \leqslant c L_C \sqrt{n} M(K) \, \|\mathbf{t}\|_2 \ \text{for some } \epsilon \in S \Big\} \Big) \leqslant e^{-q(\mathbf{t})}$$

with probability greater than $1 - e^{-2q(\mathbf{t})}$, where

$$q(\mathbf{t}) := \min\{q_*(\mu_{\mathbf{t}}), d(K)\}.$$

Proof. We may assume that $\|\mathbf{t}\|_2 = 1$. We start by writing

$$\int_C \cdots \int_C \Big\| \sum_{j=1}^s t_j x_j \Big\|_K^{-q(\mathbf{t})} dx_1 \cdots dx_s = \int_{\mathbb{R}^n} \|x\|_K^{-q(\mathbf{t})} d\nu_{\mathbf{t}}(x) = L_C^{-q(\mathbf{t})} \int_{\mathbb{R}^n} \|x\|_K^{-q(\mathbf{t})} d\mu_{\mathbf{t}}(x).$$

It follows that

$$\begin{aligned} \int_{O(n)} \int_C \cdots \int_C \Big\| \sum_{j=1}^s t_j x_j \Big\|_{U(K)}^{-q(\mathbf{t})} dx_1 \cdots dx_s \, d\nu(U) &= L_C^{-q(\mathbf{t})} \int_{\mathbb{R}^n} \int_{O(n)} \|x\|_{U(K)}^{-q(\mathbf{t})} d\nu(U) \, d\mu_{\mathbf{t}}(x) \\ &= L_C^{-q(\mathbf{t})} M_{-q(\mathbf{t})}^{-q(\mathbf{t})}(K) \int_{\mathbb{R}^n} \|x\|_2^{-q(\mathbf{t})} d\mu_{\mathbf{t}}(x) \\ &= L_C^{-q(\mathbf{t})} I_{-q(\mathbf{t})}^{-q(\mathbf{t})}(\mu_{\mathbf{t}}) M_{-q(\mathbf{t})}^{-q(\mathbf{t})}(K). \end{aligned}$$

From Markov's inequality, a random $U\in O(n)$ satisfies

$$\int_C\cdots\int_C\Big\|\sum_{j=1}^s t_jx_j\Big\|_{U(K)}^{-q(\mathbf{t})}dx_1\cdots dx_s\leqslant e^{2q(\mathbf{t})}L_C^{-q(\mathbf{t})}I_{-q(\mathbf{t})}^{-q(\mathbf{t})}(\mu_{\mathbf{t}})M_{-q(\mathbf{t})}^{-q(\mathbf{t})}(K)$$

with probability greater than $1-e^{-2q(\mathbf{t})}$. Since

$$\int_C\cdots\int_C\Big\|\sum_{j=1}^s t_jx_j\Big\|_{U(K)}^{-q(\mathbf{t})}dx_1\cdots dx_s=\int_C\cdots\int_C\Big\|\sum_{j=1}^s \epsilon_jt_jx_j\Big\|_{U(K)}^{-q(\mathbf{t})}dx_1\cdots dx_s$$

for every $\epsilon\in E_2^s$, we conclude that a random $U\in O(n)$ satisfies

$$\int_C\cdots\int_C\Big\|\sum_{j=1}^s \epsilon_jt_jx_j\Big\|_{U(K)}^{-q(\mathbf{t})}dx_1\cdots dx_s\leqslant e^{2q(\mathbf{t})}L_C^{-q(\mathbf{t})}I_{-q(\mathbf{t})}^{-q(\mathbf{t})}(\mu_{\mathbf{t}})M_{-q(\mathbf{t})}^{-q(\mathbf{t})}(K)$$

for all $\epsilon\in E_2^s$, with probability greater than $1-e^{-2q(\mathbf{t})}$.

Next, fix any such U and let $S\subseteq E_2^n$ with $|S|\leqslant e^{q(\mathbf{t})}$. Using (7.3), (7.4) and Markov's inequality, we see that a random s-tuple $(x_1,\dots,x_s)\in C^s$ satisfies

$$\Big\|\sum_{j=1}^s \epsilon_jt_jx_j\Big\|_{U(K)}\geqslant e^{-3}L_CI_{-q(\mathbf{t})}(\mu_{\mathbf{t}})M_{-q(\mathbf{t})}(K)\geqslant c_1L_C\sqrt{n}M(K)$$

for all $\epsilon\in S$, with probability greater than $1-e^{-q(\mathbf{t})}$. □

Recall that if C is a ψ_2-body with constant ϱ then $\mu_{\mathbf{t}}$ is a ψ_2 isotropic log-concave probability measure with constant ϱ. In this case $q_*(\mu_{\mathbf{t}})\geqslant cn/\varrho^2$, and hence, in Theorem 7.3 we have $q(\mathbf{t})\geqslant c\min\{n/\varrho^2,d(K)\}$. Moreover, if $C=\overline{B_2^n}$ we have that $U(C)=\overline{B_2^n}$ for all $U\in O(n)$ and $\varrho\approx 1$. Therefore, we have the following corollary.

Corollary 7.4. *Let C be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ which is ψ_2 with constant ϱ, and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then, for every $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$ and $S\subseteq E_2^s$ with $|S|\leqslant e^{c\min\{n/\varrho^2,d(K)\}}$, a random $U\in O(n)$ satisfies*

$$\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j\in U(C)\ \textit{for all } j\ \textit{and} \Big\|\sum_{j=1}^s\epsilon_jt_jx_j\Big\|_K\leqslant c_1L_C\sqrt{n}M(K)\,\|\mathbf{t}\|_2\ \textit{for some } \epsilon\in S\Big\}\Big)\leqslant e^{-c_2\min\{n/\varrho^2,d(K)\}}$$

with probability greater than $1-e^{-c_2\min\{n/\varrho^2,d(K)\}}$. In particular, for any centrally symmetric convex body K in $\mathbb{R}^n$ and any $S\subseteq E_2^s$ with $|S|\leqslant e^{cd(K)}$ we have

$$\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j\in \overline{B_2^n}\ \textit{for all } j\ \textit{and} \ \Big\|\sum_{j=1}^s\epsilon_jt_jx_j\Big\|_K\leqslant c_1L_C\sqrt{n}M(K)\,\|\mathbf{t}\|_2\ \textit{for some } \epsilon\in S\Big\}\Big)\leqslant e^{-c_2d(K)}.$$

Remark 7.5. Choosing $t_1=\cdots=t_s=1$, one may view the previous results as lower bounds for a "randomized" version of the parameter $\beta_s(C,K)$. A general lower bound for $\beta_n(C,K)$ was proved by Banaszczyk; in [2] he showed that if C and K are centrally symmetric convex bodies in $\mathbb{R}^n$ then

$$\beta_n(C,K)\geqslant c\sqrt{n}(\mathrm{vol}_n(C)/\mathrm{vol}_n(K))^{1/n} \tag{7.5}$$

for an absolute constant $c>0$. An alternative proof of this lower bound can be deduced from a more general result of Gluskin and V. Milman in [17]: If $\mathrm{vol}_n(K)=\mathrm{vol}_n(C)$ then, for any $0<u<1$ one has

$$\mathrm{vol}_{n^2}\Big(\Big\{(x_j)_{j=1}^n : x_j\in C \ \text{ for all } j \text{ and } \Big\|\sum_{j=1}^n t_jx_j\Big\|_K\leqslant u\|\mathbf{t}\|_2\Big\}\Big)\leqslant u^n e^{\frac{(1-u^2)n}{2}},$$

which implies that, for each $\mathbf{t}\in\mathbb{R}^n$, with probability greater than $1-e^{-n}$ with respect to $(x_1,\dots,x_n)$ we have

$$\min_{\epsilon\in E_2^n}\Big\|\sum_{j=1}^n\epsilon_jt_jx_j\Big\|_K\geqslant\frac{1}{10}\|\mathbf{t}\|_2.$$

Banaszczyk's theorem corresponds to the case $s=n$ and $\mathbf{t}=(1,1,\dots,1)$. Starting from this observation, the first and third authors of this article proved in [12] several results in the spirit of Theorem 7.3 and Corollary 7.4. For example, they showed that if K is a centrally symmetric convex body in $\mathbb{R}^n$ and $S\subseteq E_2^n$ then

$$\mathrm{vol}_{n^2}\Big(\Big\{(x_j)_{j=1}^n\subseteq B_2^n : \Big\|\sum_{j=1}^n\epsilon_jx_j\Big\|_K\leqslant c\delta\sqrt{n}M(K), \text{ for some } \epsilon\in S\Big\}\Big)\leqslant|S|\cdot\gamma_n(\delta\sqrt{n}M(K)\,K)+e^{-n}.$$

A concrete application of this fact is that, for every $1\leqslant p\leqslant\log n$ and any $S\subseteq E_2^n$ with $|S|\leqslant 2^{c_pn}$, a random n-tuple of points in B_2^n satisfies, with probability greater than $1-e^{-n}$,

$$\Big\|\sum_{j=1}^n\epsilon_jx_j\Big\|_p\geqslant c\sqrt{p}\sqrt{n}\big(\mathrm{vol}_n(B_2^n)/\mathrm{vol}_n(B_p^n)\big)^{1/n}$$

for all $\epsilon=(\epsilon_1,\dots,\epsilon_n)\in S$, while in the case $p>\log n$ one can deduce that for any $0<\delta<1$ and $S\subseteq E_2^n$ with $|S|\leqslant 2^{n^{1-\delta}}$, a random n-tuple of points in B_2^n satisfies, with probability greater than $1-e^{-n}$,

$$\Big\|\sum_{j=1}^n\epsilon_jx_j\Big\|_p\geqslant c(\delta)\sqrt{\log n}\sqrt{n}\big(\mathrm{vol}_n(B_2^n)/\mathrm{vol}_n(B_p^n)\big)^{1/n}$$

for all $\epsilon=(\epsilon_1,\dots,\epsilon_n)\in S$. We can obtain (in fact, more direct) variants and generalizations of these bounds from Corollary 7.4 and the available information on $d(B_p^n)$.

Following the proof of Theorem 7.3 we can also obtain upper bounds for the $\|\cdot\|_K$-norm of signed sums of random points from an isotropic body C.

Theorem 7.6. *Let C be an isotropic centrally symmetric convex body in $\mathbb{R}^n$ and K be a centrally symmetric convex body in $\mathbb{R}^n$. Then, for every $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$ and $S\subseteq E_2^n$ with $|S|\leqslant e^{p(\mathbf{t})}$, a random $U\in O(n)$ satisfies*

$$\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j\in U(C) \ \textit{for all } j \textit{ and } \Big\|\sum_{j=1}^s\epsilon_jt_jx_j\Big\|_K\geqslant cL_C\sqrt{n}M(K)\,\|\mathbf{t}\|_2 \ \textit{for some } \epsilon\in S\Big\}\Big)\leqslant e^{-p(\mathbf{t})}$$

with probability greater than $1-e^{-2p(\mathbf{t})}$, where

$$p(\mathbf{t}):=\min\{q_*(\mu_{\mathbf{t}}),k(K)\}.$$

Proof. We may assume that $\|\mathbf{t}\|_2=1$. We start by writing

$$\int_C\cdots\int_C\Big\|\sum_{j=1}^s t_jx_j\Big\|_K^{p(\mathbf{t})}dx_1\cdots dx_s=\int_{\mathbb{R}^n}\|x\|_K^{p(\mathbf{t})}d\nu_{\mathbf{t}}(x)=L_C^{p(\mathbf{t})}\int_{\mathbb{R}^n}\|x\|_K^{p(\mathbf{t})}d\mu_{\mathbf{t}}(x).$$

It follows that

$$\int_{O(n)}\int_C\cdots\int_C\Big\|\sum_{j=1}^s t_jx_j\Big\|_{U(K)}^{p(\mathbf{t})}dx_1\cdots dx_s\,d\nu(U)=L_C^{p(\mathbf{t})}\int_{\mathbb{R}^n}\int_{O(n)}\|x\|_{U(K)}^{p(\mathbf{t})}d\nu(U)\,d\mu_{\mathbf{t}}(x)$$
$$=L_C^{p(\mathbf{t})}M_{p(\mathbf{t})}^{p(\mathbf{t})}(K)\int_{\mathbb{R}^n}\|x\|_2^{p(\mathbf{t})}d\mu_{\mathbf{t}}(x)$$
$$=L_C^{p(\mathbf{t})}I_{p(\mathbf{t})}^{p(\mathbf{t})}(\mu_{\mathbf{t}})M_{p(\mathbf{t})}^{p(\mathbf{t})}(K).$$

Then, we proceed as in the proof of Theorem 7.3 using Markov's inequality, and then (7.2) and (7.4). $\square$

We can also obtain an analogue of Corollary 7.4 under the assumption that C is a ψ_2-body with constant ϱ. In particular, we have:

Corollary 7.7. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Then, for every $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$ and any $S\subseteq E_2^s$ with $|S|\leqslant e^{ck(K)}$ we have*

$$\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j\in\overline{B_2^n}\ \textit{for all } j \textit{ and } \Big\|\sum_{j=1}^s\epsilon_jt_jx_j\Big\|_K\geqslant cL_C\sqrt{n}M(K)\,\|\mathbf{t}\|_2\ \textit{for some }\epsilon\in S\Big\}\Big)$$
$$\leqslant e^{-ck(K)}.$$

Finally, we briefly describe the proof of Theorem 1.7. Recall that for any centrally symmetric convex body K in $\mathbb{R}^n$ and any $\delta\in(0,1)$ the parameter $\beta_{\delta,s}^{(R)}(K,K)$ is defined by

$$\beta_{\delta,s}^{(R)}(K,K):=\min\Big\{r>0:\mathrm{vol}_{ns}\Big(\Big\{(x_j)_{j=1}^s : x_j\in K \text{ for all } j \text{ and } \min_{\epsilon\in E_2^s}\Big\|\sum_{j=1}^s\epsilon_jx_j\Big\|_K\leqslant r\Big\}\Big)\geqslant 1-\delta\Big\}.$$

Proof of Theorem 1.7. Our starting point is Lemma 2.1; applied for the vector $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^n$, it shows that for any centrally symmetric convex body K in $\mathbb{R}^n$,

$$\Big(\mathbb{E}_{K^n}\Big\|\sum_{j=1}^n x_j\Big\|_K^q\Big)^{1/q}\leqslant cq\,\|\mathbf{1}\|_{K^n,K},\tag{7.6}$$

where $c>0$ is an absolute constant. On the other hand, by the symmetry of K we have that, for any $q\geqslant 1$,

$$\mathbb{E}_{K^n}\Big\|\sum_{j=1}^n x_j\Big\|_K^q\,dx_1\cdots dx_n=\mathbb{E}_{K^n}\Big(\mathbb{E}_\epsilon\Big\|\sum_{j=1}^n\epsilon_jx_j\Big\|_K^q\Big).\tag{7.7}$$

Combining the above we have, in particular,

$$\Big(\mathbb{E}_{K^n}\min_{\epsilon\in E_2^n}\Big\|\sum_{j=1}^n\epsilon_jx_j\Big\|_K^q\Big)^{1/q}\leqslant c_1q\,\|\mathbf{1}\|_{K^n,K}.\tag{7.8}$$

It follows that a random n-tuple $(x_1,\dots,x_n)\in K^n$ satisfies

$$\min_{\epsilon\in E_2^n}\Big\|\sum_{j=1}^n\epsilon_jx_j\Big\|_K\leqslant c_2q\,\|\mathbf{1}\|_{K^n,K}$$

with probability greater than $1-e^{-q}$. Choosing $q=\log(2/\delta)$ we see that

$$\beta_{\delta,n}^{(R)}(K,K)\leqslant c_2\log(2/\delta)\,\|\mathbf{1}\|_{K^n,K}.\tag{7.9}$$

Inserting our upper bounds for $\|\mathbf{1}\|_{K^n,K}$ into (7.9) we conclude the proof. $\square$

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References

[1] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Vol. I*, Mathematical Surveys and Monographs **202**, Amer. Math. Society (2015).

[2] W. Banaszczyk, *Balancing vectors and convex bodies*, Studia Math. **106** (1993), 93–100.

[3] I. Bárány and V. S. Grinberg, *On some combinatorial questions in finite dimensional spaces*, Linear Algebra Appl. **41** (1981), 1–9.

[4] F. Barthe, O. Guédon, S. Mendelson and A. Naor, *A probabilistic approach to the geometry of the* ℓ_p^n*-ball*, Ann. Prob. **33** (2005), 480–513.

[5] S. G. Bobkov and M. Madiman, *The entropy per coordinate of a random vector is highly constrained under convexity conditions*, IEEE Trans. Inform. Theory **57** (2011), 4940–4954.

[6] S. G. Bobkov and F. L. Nazarov, *On convex bodies and log-concave probability measures with unconditional basis*, Geom. Aspects of Funct. Analysis (Milman-Schechtman eds.), Lecture Notes in Math. **1807** (2003), 53–69.

[7] S. G. Bobkov and F. L. Nazarov, *Large deviations of typical linear functionals on a convex body with unconditional basis*, Stochastic Inequalities and Applications, Progr. Probab. **56**, Birkhäuser, Basel (2003), 3–13.

[8] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Lecture Notes in Mathematics **1469**, Springer, Berlin (1991), 127–137.

[9] J. Bourgain, M. Meyer, V. D. Milman and A. Pajor, *On a geometric inequality*, Geometric aspects of functional analysis (1986-'87), Lecture Notes in Math., 1317, Springer, Berlin (1988), 271–282.

[10] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^n$, Invent. Math. **88** (1987), 319–340.

[11] S. Brazitikos, A. Giannopoulos, P. Valettas and B-H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs **196**, American Mathematical Society, Providence, RI, 2014.

[12] G. Chasapis and N. Skarmogiannis, *A note on norms of signed sums of vectors*, Adv. in Geometry (to appear).

[13] A. Dembo, T. Cover, and J. Thomas, *Information-theoretic inequalities*, IEEE Trans. Inform. Theory **37** (1991), 1501–1518.

[14] A. Giannopoulos, M. Hartzoulaki and A. Tsolomitis, *Random points in isotropic unconditional convex bodies*, J. London Math. Soc. **72** (2005), 779–798.

[15] A. Giannopoulos and E. Milman, M*-estimates for isotropic convex bodies and their* L_q*-centroid bodies*, in Geom. Aspects of Funct. Analysis, Lecture Notes in Mathematics **2116** (2014), 159–182.

[16] A. Giannopoulos, P. Stavrakakis, A. Tsolomitis and B-H. Vritsiou, *Geometry of the* L_q*-centroid bodies of an isotropic log-concave measure*, Trans. Amer. Math. Soc. **367** (2015), 4569-4593.

[17] E. D. Gluskin and V. D. Milman, *Geometric probability and random cotype* 2, Geometric aspects of functional analysis, 123–138, Lecture Notes in Math., 1850, Springer, Berlin, 2004.

[18] M. I. Kadets and V. M. Kadets, *Series in Banach spaces*, Operator Theory Advances and Applications, vol. 94, Birkhäuser Verlag, 1997.

[19] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), 1274–1290.

[20] B. Klartag and E. Milman, *On volume distribution in 2-convex bodies*, Israel J. Math. **164** (2008), 221–249.

[21] B. Klartag and R. Vershynin, *Small ball probability and Dvoretzky theorem*, Israel J. Math. **157** (2007), 193–207.

[22] E. H. Lieb, *Proof of an entropy conjecture of Wehrl*, Comm. Math. Phys. **62** (1978), 35–41.

[23] A. Litvak, V.D. Milman and G. Schechtman, *Averages of norms and quasi-norms*, Math. Ann. **312** (1998), 95–124.

[24] E. Milman, *Dual mixed volumes and the slicing problem*, Adv. Math. **207** (2006), 566–598.

[25] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Lecture Notes in Mathematics **1376**, Springer, Berlin (1989), 64–104.

[26] G. Paouris, *Concentration of mass in convex bodies*, Geom. Funct. Analysis **16** (2006), 1021–1049.

[27] G. Paouris, *Small ball probability estimates for log-concave measures*, Trans. Amer. Math. Soc. **364** (2012), 287–308.

[28] J. Spencer, *Six standard deviations suffice*, Trans. Amer. Math. Soc. **289** (1985), 679–706.

[29] A. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Information and Control **2** (1959), 101–112.

[30] M. Talagrand, *Regularity of Gaussian processes*, Acta Math. **159** (1987), 99–147.

[31] R. Vershynin, *High-Dimensional Probability: An Introduction with Applications in Data Science*, Cambridge Series in Statistical and Probabilistic Mathematics **47**, Cambridge University Press, Cambridge, 2018.

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