REMARKS ON THE RÉNYI ENTROPY OF A SUM OF IID RANDOM VARIABLES

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Abstract. In this note we study a conjecture of Madiman and Wang [MW] which predicted that the generalized Gaussian distribution minimizes the Rényi entropy of the sum of independent random variables. Through a variational analysis, we show that the generalized Gaussian fails to be a minimizer for the problem.

1. Introduction

For $p > 1$, the $p$-Rényi entropy $h_p(X)$ of a (continuous) random variable $X$ in $\mathbb{R}^d$ distributed with density $f$ is defined by

$$h_p(X) = -\frac{1}{p-1} \log \int_{\mathbb{R}^d} f^p dm_d,$$

where $m_d$ denotes the $d$-dimensional Lebesgue measure. As $p \to 1^+$, $h_p(X)$ converges to the Shannon entropy $h(X) = -\int_{\mathbb{R}^d} f \log f dm_d$, provided that the density of $X$ is regular enough to justify passage of the limit. See Principe [Pr] for more information about where the Rényi entropy arises; see also Bobkov, Marsiglietti [BM] for a related discussion.

In this note we make some elementary remarks on the following basic mathematical question: Over all random variables $X$ with $h_p(X)$ some fixed quantity, what are the minimizers of the entropy $h_p(X + X')$, where $X'$ is an independent copy of $X$?

We learnt about this question from the papers of Madiman, Melbourne, Xu, and Wang [MW, MMX], who studied unifying entropy power inequalities for the Rényi entropy, which, in the limit $p \to 1^+$ recover the statement that, over all probability distributions with $h(X)$ fixed, $h(X + X')$ is minimized if (and only if) $X$ is a Gaussian, see e.g. [DCT].

Following [LYZ, MW, MMX], for $\beta > 0$, consider the Generalized Gaussian

$$G_\beta(x) = \alpha (1 - \beta |x|^2)_+^{1/(p-1)},$$
where $\alpha$ is chosen so that $\int_{\mathbb{R}^d} G_\beta dm_d = 1$. The generalized Gaussian is the distribution with the smallest $p$-th moment with a given Rényi entropy, see [LYZ]. Madiman and Wang conjectured (Conjecture IV.3 in [MW]) that if $X_j, j = 1, \ldots, n$, are independent random variables with densities $f_j$, and $Z_j$ are independent random variables distributed with respect to $G_{\beta_j}$ where $\beta_j$ is chosen so that

$$h_p(X_j) = h_p(Z_j),$$

then $h_p(X_1 + \cdots + X_n) \geq h_p(Z_1 + \cdots + Z_n)$.

In this note we will show that unfortunately this conjecture does not hold in the special case when $d = 1, p = 2, n = 2$ and $X_1$ and $X_2$ are identically distributed, see Section 4. However, we do suspect that a minimizing distribution is a relatively small perturbation of the generalized Gaussian.

Throughout this note we only consider the case where $X_1, \ldots, X_n$ are independent copies of a random variable $X$ with density $f$. The question of finding the minimizer of $h_p(X_1 + \cdots X_n)$ with $h_p(X)$ fixed can then be rephrased as a constrained maximization problem, which we introduce in Section 2. Subsequently, in Section 3 we take the first variation of this maximization problem. We have not been able to develop a satisfactory theory of the associated Euler-Lagrange equation (3.1), but we show in Section 4 that the generalized Gaussian is not a solution to (3.1), and so fails to be a maximizer of the extremal problem. We conclude the paper with some elementary remarks and speculation.

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2. The constrained maximization problem

Denote by $C_n(f)$ the $(n - 1)$-fold convolution of a given function $f$ with itself, that is, $C_n(f) = f \ast f \ast \cdots \ast f$, where there are $n$ factors of $f$ (and $n - 1$ convolutions). Then $C_1(f) = f$, and it will be convenient to set $C_0(f) = \delta_0$, the Dirac delta measure.
Throughout the text, we fix $M > 0$, $n \in \mathbb{N}$ and $p \in (1, \infty)$. We set
\[ \mathcal{F} = \{ f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), f \geq 0, \|f\|_p^p = M, \|f\|_1 = 1 \} \]
and consider the extremal problem
\[
\begin{aligned}
\text{(2.1)} & \quad \left\{ \begin{array}{ll}
\text{Maximize} & I(f) \overset{\text{def}}{=} \int_{\mathbb{R}^d} C_n(f)^p dm_d \\
\text{subject to} & f \in \mathcal{F}.
\end{array} \right.
\end{aligned}
\]
Put
\[
\Lambda = \Lambda(p, M) = \sup \{ I(f) : f \in \mathcal{F} \}.
\]
We begin with a simple scaling lemma, which we will use often in what follows.

**Lemma 2.1.** Suppose that $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ is non-negative, and $\|f\|_1 > 0$. The function
\[
\tilde{f} = \frac{1}{\lambda^d \|f\|_1^p} f\left(\frac{\cdot}{\lambda}\right), \quad \text{with} \quad \lambda = \left(\frac{\|f\|_p^p}{M \|f\|_1^p}\right)^{\frac{1}{d(p-1)}},
\]
belongs to $\mathcal{F}$, and
\[
I(\tilde{f}) = \frac{1}{\lambda^d \|f\|_1^p \|f\|_1^{p(n-1)}} I(f).
\]

**Proof.** Observe that, for any $r \in [1, \infty)$,
\[
\|\tilde{f}\|_r^r = \frac{1}{\lambda^{d(r-1)} \|f\|_1^r} \|f\|_r^r.
\]
Plugging in $r = 1$ and $r = p$ (and recalling the definition of $\lambda$) we see that $\tilde{f} \in \mathcal{F}$. Next, observe that
\[
C_n(\tilde{f})(x) = \frac{1}{\|f\|_1^p \lambda^d} C_n(f)\left(\frac{x}{\lambda}\right) \text{ for any } x \in \mathbb{R}^d.
\]
Whence,
\[
I(\tilde{f}) = \frac{1}{\lambda^d \|f\|_1^{p(n-1)}} I(f),
\]
and the proof is complete by recalling the definition of $\lambda$. \qed

We next prove that (2.1) has a maximizer. A radial function $f$ on $\mathbb{R}^d$ is called decreasing if $f(y) \leq f(x)$ whenever $|y| \geq |x|$.

**Proposition 2.2.** The problem (2.1) has a lower-semicontinuous, radially decreasing, maximizer $Q$.

**Proof.** We begin with two observations.
(1) Repeated application of Young’s convolution inequality \[ \text{LL} \] yields that, with \( p' = p/(p - 1) \),
\[
I(f) \leq \|f\|_p^{n p}_{\langle np'\rangle'},
\]
where \( (np')' \) is the Hölder conjugate of \( np' \). Since \( n > 1 \), we have that \( (np')' \in (1, p) \).

(2) By iterating Riesz’s rearrangement inequality \[ \text{LL, Theorem 3.7} \]
we have that
\[
I(f) \leq I(f^*),
\]
where \( f^* \) is the symmetric rearrangement of \( f \); see \[ \text{B, Section 3.4} \] for related multiple convolution rearrangement inequalities and their equality cases.

Now take non-negative functions \( f_j \in \mathcal{F} \) such that \( \Lambda = \lim_{j \to \infty} I(f_j) \) (recall \( \Lambda \) from \( (2.2) \)). From the second observation we may assume that \( f_j \) are radial and decreasing. Passing to a subsequence if necessary, we may in addition assume that \( f_j \to f \) weakly in \( L^p(\mathbb{R}^d) \). Consequently, \( f \) is radial, decreasing, \( f \geq 0 \), and \( \|f\|_p^p \leq M \). By modifying \( f \) on a set of measure zero if necessary, we may assume that \( f \) is lower semi-continuous\(^1\).

Our primary goal will be to show that \( f_j \to f \) strongly in \( L^{(np')'} \) as \( j \to \infty \). From this the first observation above would yield that \( I(f) = \Lambda \).

**Claim 2.3.** As \( j \to \infty \), \( f_j \to f \) \( m_d \)-almost everywhere.

**Proof.** For \( r > 0 \), define \( v_j(r) = f_j(x) \) whenever \( |x| = r \). Then since \( f_j \) converges weakly to \( f \) in \( L^p(\mathbb{R}^d) \), we have that whenever \( I \) is a closed interval of finite measure in \( (0, \infty) \),
\[
\lim_{j \to \infty} \int_I v_j dm_1 = \int_I v dm_1.
\]
Recall that almost every point \( r \) of a function \( v \in L^1_{\text{loc}}((0, \infty)) \) is a \textit{Lebesgue point}, that is,
\[
v(r) = \lim_{|I| \to 0, r \in I} \frac{1}{|I|} \int_I v dm_1,
\]
where the limit is taken over any sequence of intervals \( I \) containing \( r \) that shrink to \( r \) (not necessarily centered at \( r \)). But then if \( r > 0 \) is a Lebesgue point, and \( I_k = [r - 2^{-k}, r] \), then
\[
v(r) = \lim_{k \to \infty} \frac{1}{2^{-k}} \int_{I_k} v dm_1 = \lim_{k \to \infty} \lim_{j \to \infty} \frac{1}{2^{-k}} \int_{I_k} v_j dm_1
\]
\(^1\)If \( f \) is discontinuous at \( x \in \mathbb{R}^d \), then define \( f(x) = \sup_{|y|>|x|} f(y) \) (i.e. the one-sided radial limit from the right). Then \( \{ f > \lambda \} \) is open for every \( \lambda > 0 \).
but since $v_j$ is decreasing we have that $v_j \geq v_j(r)$ on $I_k$. Thus
\[ v(r) \geq \limsup_{j \to \infty} v_j(r). \]

Arguing similarly with intervals whose left end-point is $r$, we also have that
\[ v(r) \leq \liminf_{j \to \infty} v_j(r). \]

Thus $\lim_{j \to \infty} v_j = v$ at every Lebesgue point. From this we readily deduce the claim, since if $E$ is a Lebesgue null set in $(0, \infty)$, then $E \times \mathbb{S}^{d-1}$ is a Lebesgue null set in $\mathbb{R}^d$. 

Notice that, as a consequence of this claim, Fatou’s Lemma ensures that $\|f\|_1 \leq 1$. Our next claim is

**Claim 2.4.** If $1 < q < p$, then $f_j \to f$ strongly in $L^q(\mathbb{R}^d)$ as $j \to \infty$.

The proof of this claim is a variant of the Vitali convergence theorem, but observe that it does not necessarily hold if one was to remove the radially decreasing property of the functions $f_j$ (just consider a sequence of translates of a fixed function).

**Proof.** Fix $\varepsilon > 0, \delta > 0$. Insofar as the functions $f_j$ and $f$ are radially decreasing,
\[ \bigcup_j \{|f_j| \geq \frac{\delta}{2}\} \cup \{|f| \geq \frac{\delta}{2}\} \subset B, \]

where $B$ is the closed ball centred at 0 of radius $(\frac{2}{m_d(B(0,1))})^{1/d}$. (Otherwise we would have $\|f_j\|_1 > 1$ for some $j$, or $\|f\|_1 > 1$.)

On $\mathbb{R}^d \setminus B$, we have $|f_j| < \delta/2$ for every $j$, and $|f| < \delta/2$, whence
\[ \int_{\mathbb{R}^d \setminus B} |f_j - f|^q dm_d \leq \delta^{q-1} (\|f_j\|_1 + \|f\|_1) \leq 2\delta^{q-1} \leq \frac{\varepsilon}{3} \]

provided $\delta > 0$ is chosen sufficiently small.

Now fix $\varkappa > 0$. Observe that,
\[ \int_{B \cap \{|f_j - f| < \varkappa\}} |f_j - f|^q dm_d \leq m_d(B) \varkappa^q < \frac{\varepsilon}{3} \]

if $\varkappa$ is chosen sufficiently small. On the other hand, inasmuch as $B$ has finite measure, we have that $f_j \to f$ in measure on $B$ as $j \to \infty$. From the inequalities
\[ \int_{B \cap \{|f_j - f| \geq \varkappa\}} |f_n - f|^q dm_d \leq m_d(B \cap \{|f_j - f| \geq \varkappa\})^{1-q/p} \|f_j - f\|_p^q \]
\[ \leq 2^q M^{q/p} m_d(B \cap \{|f_j - f| \geq \varkappa\})^{1-p/q}, \]

where $M_d$ is the maximal operator.

\[ \varepsilon \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2^q M^{q/p} \varkappa^{1-p/q} \]

which is possible for $\varkappa$ chosen sufficiently small.
we infer that there exists \( N \in \mathbb{N} \) such that
\[
\int_{B \cap \{|f_j - f|^q \geq \epsilon\}} |f_j - f|^q dm < \frac{\epsilon}{3} \text{ for all } j \geq N.
\]
Bringing these estimates together, it follows that \( \|f_j - f\|_q^q < \epsilon \) for every \( j \geq N \).

In particular, since \( (np')' \in (1,p) \), this second claim ensures that \( f_j \to f \) in \( L^{(np')'} \) as \( j \to \infty \). Thus by the remarks preceding Claim 2.3, we have \( \mathcal{I}(f) = \Lambda \) (so \( f \) is not identically zero). It remains to show that \( f \in \mathcal{F} \). To this end, we apply Lemma 2.1. Consider the function
\[
\tilde{f} = \frac{1}{\|f\|_1} f \left( \frac{\cdot}{\lambda} \right), \text{ with } \lambda = \left( \frac{\|f\|_p^p}{M \|f\|_1^p} \right)^{\frac{1}{p-1}}.
\]
Then \( \tilde{f} \in \mathcal{F} \) and \( \mathcal{I}(\tilde{f}) = \frac{M}{\|f\|_p^p} \|f\|_1^{p-1} \Lambda \). Consequently, if \( \|f\|_p < M \) or \( \|f\|_1 < 1 \), then \( \mathcal{I}(\tilde{f}) > \Lambda \), which is absurd. Thus \( f \in \mathcal{F} \) and the proof of the proposition is complete.

3. The First Variation

With the existence of a maximizer proved, we now wish to analyze it analytically. We shall derive the following criterion.

**Proposition 3.1.** A radial non-negative lower-semicontinuous function \( Q \) is a maximizer of the problem (2.1) if and only if

\[
C_{n-1}(Q) \ast [C_n(Q)]^{p-1} = \frac{\Lambda}{Mn} Q^{p-1} + \frac{\Lambda(n-1)}{n} \text{ on } \{Q > 0\}.
\]

**Proof.** The sufficiency is easy to show. Integrating both sides of (3.1) against \( Q \), and recalling that \( Q \in \mathcal{F} \), yields
\[
\mathcal{I}(Q) = \int_{\mathbb{R}^d} Q \cdot (C_{n-1}(Q) \ast [C_n(Q)]^{p-1}) dm = \Lambda.
\]
But using Tonelli’s theorem, the left hand side equals \( \mathcal{I}(Q) \). This is just the fact that, for even functions \( f, g \) and \( h \), \( \int f(g \ast h) dm = \int (f \ast g)h dm \).

Conversely, consider a bounded function \( \varphi \) compactly supported in the open set \( \{Q > 0\} \). Since \( Q \) is lower-semicontinuous, \( \inf_{\text{supp}(\varphi)} Q > 0 \). Therefore, (insofar as \( \varphi \) is bounded) there exists a constant \( C > 0 \) such that
\[
|\varphi| \leq CQ \text{ on } \mathbb{R}^d,
\]
so in particular, there exists $t_0 > 0$ such that for $|t| \leq t_0$ it follows that $Q_t \overset{\text{def}}{=} Q + t\varphi$ is non-negative. In the notation of Lemma 2.1 with $f = Q_t$, we consider the function

$$
\tilde{Q}_t = \frac{1}{\lambda^d} \frac{(Q + t\varphi)(\frac{1}{\lambda})}{\|Q + t\varphi\|_1},
$$

with the corresponding $\lambda > 0$ satisfying $\|\tilde{Q}_t\|_p^p = \|Q\|_p^p = M$. Of course we also have $\int_{\mathbb{R}} \tilde{Q}_t \, dm_d = 1$ regardless of $\lambda$ for $|t| < t_0$. We conclude that $\tilde{Q}_t$ belongs to $\mathcal{F}$ – and therefore $I(\tilde{Q}_t) \leq I(Q) = \Lambda$ – for all $|t| < t_0$. Moreover, as in Lemma 2.1,

$$
I(\tilde{Q}_t) = \frac{1}{\lambda^{d(p-1)}\|Q + t\varphi\|_1^n} \int_{\mathbb{R}^d} C_n(Q + t\varphi)^p \, dm_d.
$$

For $|t| < t_0$, we calculate, using commutativity and associativity of the convolution operator,

$$
\frac{d}{dt} C_n(Q + t\varphi)^p = pm[\varphi \ast C_{n-1}(Q + t\varphi)][C_n(Q + t\varphi)]^{p-1},
$$

and

$$
\frac{d^2}{dt^2} C_n(Q + t\varphi)^p = pm(n-1)\varphi \ast \varphi \ast C_{n-2}(Q + t\varphi)[C_n(Q + t\varphi)]^{p-1}
$$

$$
+ n^2 p (p-1)[\varphi \ast C_{n-1}(Q + t\varphi)]^2[C_n(Q + t\varphi)]^{p-2}.
$$

Crudely employing the bound (3.2) in (3.4), we infer that there is a constant $C > 0$ such that for all $|t| < t_0$,

$$
\left| \frac{d^2}{dt^2} C_n(Q + t\varphi)^p \right| \leq CC_n(Q)^p.
$$

Whence, the second order Taylor formula yields that

$$
|C_n(Q + t\varphi)^p - C_n(Q)^p - npt[\varphi \ast C_{n-1}(Q)][C_n(Q)]^{p-1}| \leq C t^2 C_n(Q)^p,
$$

for $|t| < t_0$. Integrating the pointwise inequality (3.5) yields

$$
\int_{\mathbb{R}^d} C_n(Q + t\varphi)^p \, dm_d = \Lambda + npt \int_{\mathbb{R}^d} [\varphi \ast C_{n-1}(Q)][C_n(Q)]^{p-1} \, dm_d + O(t^2)
$$

as $t \to 0$. 
Now, recalling the definition of $\lambda$, we calculate
\begin{equation}
\lambda^{d(p-1)}\|Q + t\varphi\|_1^{np} = \left(1 + \frac{pt}{M} \int_{\mathbb{R}^d} \varphi Q^{p-1} dm_d + O(t^2)\right) \left(1 + t(n - 1)p \int \varphi dm_d + O(t^2)\right),
\end{equation}
where in the expansion of $\|Q + t\varphi\|_p^p$ we have again used the inequality (3.2) to obtain the $O(t^2)$ term.

Plugging the two expansions (3.7) and (3.6) into (3.3) yields that, as $t \to 0$,
\begin{equation}
\mathcal{I}(\tilde{Q}_t) = \Lambda + pt\left\{n \int_{\mathbb{R}^d} [\varphi * C_{n-1}(Q)] [C_n(Q)]^{p-1} dm_d - \frac{\Lambda}{M} \int_{\mathbb{R}^d} \varphi Q^{p-1} dm_d - (n - 1)\Lambda \int \varphi dm_d\right\} + O(t^2).
\end{equation}

Since $\lim_{t \to 0} \mathcal{I}(\tilde{Q}_t) - \mathcal{I}(Q) = 0$, the second term in the above expansion must vanish. Therefore, we get that
\begin{equation}
\int_{\mathbb{R}^d} \varphi \left\{n[C_{n-1}(Q)] * [C_n(Q)]^{p-1} - \frac{\Lambda}{M} Q^{p-1} - (n - 1)\Lambda\right\} dm_d = 0.
\end{equation}

Since $\varphi$ was any bounded function compactly supported in $\{Q > 0\}$, we conclude that (3.1) holds. \hfill \Box

4. On the Madiman-Wang conjecture

Proposition 4.1. The generalized Gaussian is not necessarily the extremizer for problem (2.1).

Proof. Consider the simplest case $d = 1$, $p = 2$, and $n = 2$. We shall show that the function $G(x) = \alpha(1 - |x|^2)_+$ does not satisfy the equation
\begin{equation}
C_3(f) = af + b \text{ on } [-1, 1] \text{ with } a, b > 0,
\end{equation}
and so no function of the form $\frac{c}{\lambda}G(\frac{x}{\lambda})$, with $c, \lambda > 0$, satisfies (3.1), for any value of $\Lambda$. In fact, we shall show that $C_3(G) = G * G * G$ is not a quadratic polynomial near 0.

For this, observe:
\begin{equation}
G'' = 2\alpha(\delta_{-1} - \chi_{[-1,1]} + \delta_1).
\end{equation}
Thus, $(G * G * G)''' = (G'' * G'' * G'')$ is the threefold convolution of the above measure. The threefold convolution of $-2\chi_{[-1,1]}$ equals $-8(3 - |x|^2)_+$, and no other term in the convolution $G'' * G'' * G''$ is
quadratic in $|x|$. Therefore, $G \ast G \ast G$ has non-vanishing sixth derivative at 0, but $a + bG$ does have vanishing sixth derivative at 0. □

5. ANY RADIALLY DECREASING SOLUTION OF (3.1) IS COMPACTLY SUPPORTED

In this section, we discuss the following

Proposition 5.1. Decreasing radial solutions of (3.1) are compactly supported.

Proof. Suppose that $\int_{\mathbb{R}^d} Q \, dm_d = 1$ and $Q$ is not compactly supported. Since $Q$ is non-negative and radially decreasing, its support is $\mathbb{R}^d$.

Since $\|C_{n-1}(Q) * (C_n(Q))^{p-1}\|_1 < \infty$ and $Q$ (along with any multiple convolution of $Q$) is radially decreasing, we have that $\lim_{|x| \to \infty} |C_{n-1}(Q) * (C_n(Q))^{p-1}(x)| = 0$. Since also $Q^{p-1}(x) \to 0$ as $|x| \to \infty$, we have that $\Lambda = 0$ in the equation (3.1). But, on the other hand, $\Lambda > 0$. □

6. REMARKS

In this section we make some remarks that suggest that although the generalized Gaussian is not an optimal distribution for the problem (2.1), a reasonably small perturbation of the generalized Gaussian could well be.

Beginning with $f_0(x) = 1_{[-1,1]}$, consider the following iteration for $j \geq 1$

$$f_j(x) = \frac{C_3(f_{j-1})(x) - C_3(f_{j-1})(1)}{C_3(f_{j-1})(0) - C_3(f_{j-1})(1)}.$$

Numerically, this iteration converges pointwise to a solution of the equation (4.1) for some $a, b > 0$ satisfying the constraints $f(0) = 1$ and $f(1) = 0$ (so the support of $f$ is $[-1,1]$). The resulting function $f$ can then be re-scaled via the transformation $\frac{\xi}{\lambda} f(\frac{\cdot}{\lambda})(c, \lambda > 0)$ to have any given positive integral and $L^2$-norm. We do not know if the solution of $C_3(f) = af + b$ is unique (modulo natural invariants in the problem), so we cannot say that this function $f$ corresponds to a solution of the constrained maximization problem (2.1).

We provide the graphs of $f_1, f_2, f_3$ and $f_4$ (see Figure 1 below), and the algebraic expressions for $f_1, f_2$ and $f_3$ on $[-1,1]$.

$$f_1(x) = 1 - x^2, \quad f_2(x) = 1 - \frac{6x^2}{5} + \frac{x^4}{5},$$
$$f_3(x) = 1 - \frac{62325x^2}{50521} + \frac{12810x^4}{50521} - \frac{1050x^6}{50521} + \frac{45x^8}{50521} - \frac{x^{10}}{50521}.$$
Figure 1. The graphs of $f_1, \ldots, f_4$ on $[-1,1]$.

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