REMARKS ON AN INEQUALITY OF ROGERS AND SHEPHARD

APOSTOLOS GIANNOPoulos, ELEFTHERIOS MARKESSINIS, AND ANTONIS TSOLOMITIS

Abstract. A classical inequality of Rogers and Shephard states that if $K$ is a centered convex body of volume 1 in $\mathbb{R}^n$ then

$$1 \leq g(K, k; F) := (\text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp))^{1/k} \leq \left(\frac{n}{k}\right)^{1/k} \leq cn^k$$

for every $F \in G_{n,k}$, where $c > 0$ is an absolute constant. We show that if $K$ is origin symmetric and isotropic then, for every $1 \leq k \leq n-1$, a random $F \in G_{n,k}$ satisfies

$$c_1 L_K^{-1} \sqrt{n/k} \leq g(K, k; F) \leq c_2 n \sqrt{n/k} (\log n)^2 L_K$$

with probability greater than $1 - e^{-k}$, where $L_K$ is the isotropic constant of $K$ and $c_1, c_2 > 0$ are absolute constants.

1. Introduction

Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$ with $0 \in \text{int}(K)$. For every $1 \leq k \leq n-1$ and any $F \in G_{n,k}$ we define

$$(1.1) \quad g(K, k; F) := \left(\text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp)\right)^{1/k},$$

where $F^\perp$ denotes the orthogonal subspace of $F$ in $\mathbb{R}^n$. A classical inequality of Rogers and Shephard [13] (see also Chakerian [5]) states that if $K$ is origin symmetric then

$$(1.2) \quad 1 \leq g(K, k; F) \leq \left(\frac{n}{k}\right)^{1/k} \leq c_0 n^k,$$

where $c_0 > 0$ is an absolute constant. The right-hand side inequality holds true under the more general assumption that $0 \in \text{int}(K)$. On the other hand, Spingarn [15] showed that the lower bound remains valid if we assume that $K$ is centered, i.e. that the barycenter of $K$ is at the origin.

Both estimates are sharp: let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and set $F = \text{span}\{e_1, \ldots, e_k\}$. Consider a convex body $A \subset F$ and a convex body $B \subset F^\perp$ with $0 \in \text{int}(A) \cap \text{int}(B)$. One can check that if $K = A \times B = \{a + b : a \in A, b \in B\}$ then $P_F(K) = A$, $K \cap F^\perp = B$ and $\text{vol}_n(K) = \text{vol}_k(A)\text{vol}_{n-k}(B)$. On the other hand, if we consider the convex body $K' = \text{conv}(A \cup B) = \{(1-t)a + tb :\}$

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Let \( 1 \leq k \leq n-1 \) then \( P_F(K') = A, K' \cap F^\perp = B \) and \( \Vol_n(K') = \binom{n}{k} \Vol_k(A) \Vol_{n-k}(B) \).

Our starting point is the observation that the behavior of \( g(\mathcal{E}, k; F) \) lies “in the middle” when \( \mathcal{E} \) is an ellipsoid.

**Proposition 1.1.** For every ellipsoid \( \mathcal{E} \) in \( \mathbb{R}^n \) and for all \( 1 \leq k \leq n-1 \) and \( F \in G_{n,k} \) the product \( \Vol_k(P_F(\mathcal{E})) \Vol_{n-k}(\mathcal{E} \cap F^\perp) \) is independent of the subspace \( F \). More precisely, we have

\[
\Vol_k(P_F(\mathcal{E})) \Vol_{n-k}(\mathcal{E} \cap F^\perp) = \frac{\Vol_k(B_2^n) \Vol_{n-k}(B_2^{n-k})}{\Vol_n(B_2^n)} \Vol_n(\mathcal{E}).
\]

Therefore,

\[
\left( \frac{cn}{k} \right)^{k/2} \Vol_n(\mathcal{E}) \leq \Vol_k(P_F(\mathcal{E})) \Vol_{n-k}(\mathcal{E} \cap F^\perp) \leq \left( \frac{c_2 n}{k} \right)^{k/2} \Vol_n(\mathcal{E}),
\]

where \( c_1, c_2 > 0 \) are absolute constants.

For the reader’s convenience we include a proof of this observation in Section 3. Assuming that \( \Vol_n(\mathcal{E}) = 1 \), from Proposition 1.1 we see that

\[
g(\mathcal{E}, k; F) \simeq \sqrt{n/k}
\]

for all \( 1 \leq k \leq n-1 \) and \( F \in G_{n,k} \). The question that we discuss in this note is if this is the typical (with respect to \( F \in G_{n,k} \)) behavior of \( g(K, k; F) \) for any symmetric (or, more generally, centered) convex body \( K \) of volume 1 in \( \mathbb{R}^n \). Our main result provides an (almost sharp) affirmative answer if we assume that \( K \) is in isotropic position.

**Theorem 1.2.** Let \( K \) be an origin symmetric isotropic convex body in \( \mathbb{R}^n \). For every \( 1 \leq k \leq n-1 \) a random \( F \in G_{n,k} \) satisfies

\[
ce_1 L_K^{-1} \sqrt{n/k} \leq g(K, k; F) \leq c_2 \sqrt{n/k}(\log n)^2 L_K
\]

with probability greater than \( 1 - e^{-k} \), where \( c_1, c_2 > 0 \) are absolute constants.

Our approach is presented in Section 4 and leads to some general lower and upper bounds that might be useful for other classical positions of \( K \), such as the minimal surface area position or minimal mean width position or John position. In Section 5 we use the additional information that one has when \( K \) is isotropic, and obtain the bounds of Theorem 1.2. The left hand side inequality in (1.6) remains valid for any isotropic convex body \( K \) in \( \mathbb{R}^n \). For the right hand side inequality we employ a recent result of E. Milman on the mean width of origin symmetric isotropic convex bodies, see [8]; this forces the assumption of symmetry in Theorem 1.2. Background information is provided in Section 2 and in the beginning of Section 5.

2. **Notation and background information**

We work in \( \mathbb{R}^n \), which is equipped with a Euclidean structure \( \langle \cdot, \cdot \rangle \). We denote by \( \| \cdot \|_2 \) the corresponding Euclidean norm, and write \( B_2^n \) for the Euclidean unit ball, and \( S_2^{n-1} \) for the unit sphere. The volume of an \( s \)-dimensional set \( A \) is denoted by \( \Vol_s(A) \). We write \( \omega_n \) for the volume of \( B_2^n \) and \( \sigma_n \) for the rotationally invariant probability measure on \( S_2^{n-1} \). The Grassmann manifold \( G_{n,k} \) of \( k \)-dimensional subspaces of \( \mathbb{R}^n \) is equipped with the Haar probability measure \( \nu_{n,k} \). Let \( 1 \leq k \leq n-1 \) and \( F \in G_{n,k} \). We write \( F^\perp \) for the orthogonal subspace of \( F \) in \( \mathbb{R}^n \).
We will denote the orthogonal projection from $\mathbb{R}^n$ onto $F$ by $P_F$. We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters $c, c', c_1, c_2$ etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Similarly, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$. We also write $\mathcal{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$, i.e. $\mathcal{A} := \text{vol}_n(A)^{-1/n} A$.

A convex body is a compact convex subset $K$ of $\mathbb{R}^n$ with non-empty interior. We say that $K$ is origin symmetric if $-x \in K$ whenever $x \in K$. We say that $K$ is centered if it has barycenter at the origin, i.e. $\int_K \langle x, \theta \rangle d\theta = 0$ for every $\theta \in S^{n-1}$. The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of $K$ is defined by $h_K(x) = \max\{\langle x, y \rangle : y \in K\}$. The radius of $K$ is defined as $R(K) = \max\{\|x\|_2 : x \in K\}$ and, if the origin is an interior point of $K$, the polar body $K^\circ$ of $K$ is

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.$$  

We will use the fact that

$$c^n \text{vol}_n(B_2^n)^2 \leq \text{vol}_n(K) \text{vol}_n(K^\circ) \leq \text{vol}_n(B_2^n)^2$$

for every centered convex body $K$ in $\mathbb{R}^n$. The right-hand side inequality is the Blaschke-Santaló inequality, while the left-hand side inequality is due to Bourgain and V. Milman [3] and holds true if we just assume that $0 \in \text{int}(K)$.

For each $p > -n, p \neq 0$, we set

$$I_p(K) := \left( \int_K \|x\|_2^p dx \right)^{1/p}$$

and for each $-\infty < p < \infty, p \neq 0$, we define the $p$-mean width of $K$ by

$$w_p(K) := \left( \int_{S^{n-1}} h_K^p(\theta) d\sigma_n(\theta) \right)^{1/p}.$$  

From Hölder’s inequality, both are increasing functions of $p$. The mean width of $K$ is the quantity $w(K) = w_1(K)$. Note that

$$w_{-n}(K) = \left( \frac{\text{vol}_n(B_2^n)}{\text{vol}_n(K^\circ)} \right)^{\frac{1}{n}}.$$  

This is immediate if we express $\text{vol}_n(K^\circ)$ in polar coordinates. If $K$ is an origin symmetric convex body in $\mathbb{R}^n$ and $\| \cdot \|_K$ is the norm induced to $\mathbb{R}^n$ by $K$, we set

$$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_n(x)$$

and write $b(K)$ for the smallest positive constant $b$ with the property $\|x\|_K \leq b\|x\|_2$ for all $x \in \mathbb{R}^n$. From V. Milman’s proof of Dvoretzky’s theorem (see [10]) we know that if $k \leq cn(M(K)/b(K))^2$ then for most $F \in G_{n,k}$ we have $K \cap F \simeq \frac{1}{M(K)} B_F$.

For every convex body $K$ in $\mathbb{R}^n$ and for every $1 \leq k \leq n - 1$ we define the normalized $k$-th quermassintegral of $K$ by

$$Q_k(K) = \left( \frac{1}{\omega_k} \int_{G_{n,k}} \text{vol}_k(P_F(K)) d\nu_{n,k}(F) \right)^{1/k}.$$
Note that $Q_1(K) = w(K)$. From the Aleksandrov-Fenchel inequality (see [14]) it follows that $Q_k(K)$ is a decreasing function of $k$. In particular,

$$\left( \int_{G_{n,k}} \text{vol}_k(P_F(K)) \, d\nu_{n,k}(F) \right)^{1/k} \leq c_1 w(K) \sqrt{k}.$$  

where $c_1 > 0$ is an absolute constant. We refer to the books [14] and [10] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

The next two functionals will play an essential role in our argument.

(i) $p$-mean projection function. For every $1 \leq k \leq n - 1$ and for every $p \neq 0$ we define the $p$-mean projection function $W_{[k,p]}(K)$ by

$$W_{[k,p]}(K) := \left( \int_{G_{n,k}} \text{vol}_k(P_F(K))^p \, d\nu_{n,k}(F) \right)^{1/p}.$$  

We also set $W_{[n]}(K) := \text{vol}_n(K)^{1/n}$.

(ii) $p$-mean section function. For every $1 \leq k \leq n - 1$ and for every $p \neq 0$ we define the $p$-mean section function $\tilde{W}_{[k,p]}(K)$ by

$$\tilde{W}_{[k,p]}(K) = \left( \int_{G_{n,k}} \text{vol}_{n-k}(K \cap F^\perp)^p \, d\nu_{n,k}(F) \right)^{1/p}.$$  

The normalized dual $k$-th quermassintegral of $K$ is the quantity $\tilde{W}_{[k]}(K) := \tilde{W}_{[k,1]}(K)$.

3. Ellipsoids

We start with the proof of Proposition 1.1. We will use the classical fact that Steiner symmetrization transforms an ellipsoid to an ellipsoid (see for example [2]). Here we state it as a lemma and include its proof for the sake of completeness.

**Lemma 3.1.** For every $u \in S^{n-1}$ and for every ellipsoid $E$ the Steiner symmetral $S_u(E)$ of $E$ with respect to $u$ is an ellipsoid.

**Proof.** Assume without loss of generality that the ellipsoid is centered at the origin. Consider a positive definite map $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$E = \{ x \in \mathbb{R}^n : \langle Tx, x \rangle \leq 1 \}.$$  

By the definition of Steiner symmetrization, a point $y \in \mathbb{R}^n$ belongs to $S_u(E)$ if the line $L = \{ y + \lambda u : \lambda \in \mathbb{R} \}$ intersects $E$ and

$$|\langle y, u \rangle| \leq \frac{1}{2} \text{length}(E \cap L).$$  

The assumption that $L$ intersects $E$ means that there exists $\lambda \in \mathbb{R}$ so that $\langle T(y + \lambda u), (y + \lambda u) \rangle \leq 1$. The left-hand side is a quadratic function of $\lambda$, so its discriminant is non-negative, that is

$$\langle Ty, u \rangle^2 + \langle Tu, u \rangle - \langle Tu, u \rangle \langle Ty, y \rangle \geq 0.$$  

In this case the length in (3.1) equals

$$\frac{2\sqrt{\langle Ty, u \rangle^2 - \langle Tu, u \rangle \langle Ty, y \rangle}}{\langle Tu, u \rangle}.$$
Substituting in (3.1) we get that

\[ S_u(E) = \left\{ y \in \mathbb{R}^n : \langle Tu, u \rangle^2 y^2 \leq \langle Ty, u \rangle^2 - \langle Tu, u \rangle \left( \langle Ty, y \rangle - 1 \right) \right\}. \]

This set is clearly an ellipsoid (it is defined by a quadratic form). \( \square \)

**Note.** In fact, it is known that Lemma 3.1 characterizes ellipsoids in the following sense: if \( K \) is a convex body with the property that all its Steiner symmetrals \( S_u(K) \) are affine images of \( K \), then \( K \) is an ellipsoid (see e.g. [7]).

**Proof of Proposition 1.1.** Assume without loss of generality that \( E \) is centered at the origin. We first prove (1.3). We distinguish two cases.

**Case 1:** \( F \) is generated by the unit vectors of \( k \) semiaxes of \( E \). In this case if \( \lambda_1, \ldots, \lambda_n \) are the positive lengths of the ellipsoid’s semiaxes then obviously

\[
\vol_k(\mathcal{P}_F(E)) \vol_{n-k}(E \cap F^\perp) = \left( \prod_{j=1}^n \lambda_j \right) \frac{\vol_k(B_2^k) \vol_{n-k}(B_2^{n-k})}{\vol_n(B_2^k)} \vol_n(E).
\]

**Case 2:** \( F \) is any element of \( G_{n,k} \). Let \( u_1, \ldots, u_k \) be any orthonormal basis of \( F \). We write \( E' = S_{u_1} \ldots (S_{u_k}(E) \ldots) \) for the ellipsoid obtained by successive Steiner symmetrizations of \( E \) in the directions \( u_1, \ldots, u_k \). By the properties of Steiner symmetrization we have that

\[
\vol_k(\mathcal{P}_F(E)) = \vol_k(\mathcal{P}_F(E')) \quad \text{and} \quad \vol_{n-k}(E \cap F^\perp) = \vol_{n-k}(E' \cap F^\perp).
\]

From Lemma 3.1 it follows that \( E' \) is an ellipsoid which in addition has the same volume as \( E \). Moreover, observe that Case 1 applies now to the ellipsoid \( E' \) and the subspace \( F \). Thus, we get

\[
\vol_k(\mathcal{P}_F(E)) \vol_{n-k}(E \cap F^\perp) = \vol_k(\mathcal{P}_F(E')) \vol_{n-k}(E' \cap F^\perp)
= \frac{\vol_k(B_2^k) \vol_{n-k}(B_2^{n-k})}{\vol_n(B_2^k)} \vol_n(E'),
= \frac{\vol_k(B_2^k) \vol_{n-k}(B_2^{n-k})}{\vol_n(B_2^k)} \vol_n(E),
\]

completing the proof of (1.3).

Since \( \vol_n(B_2^n) = \pi^{n/2}/\Gamma(1+n/2) \) it is elementary to check that (1.4) holds true as well. \( \square \)

4. **General bounds**

Let \( K \) be a centered convex body of volume 1 in \( \mathbb{R}^n \). In order to obtain a lower bound for \( g(K, k; F) \) we will estimate the expectation \( \mathbb{E}_{\nu_n,k} \left[ (g(K, k; F))^{-a} \right] \) for some \( a > 0 \). For any pair \((p, q)\) of conjugate exponents, using Hölder’s inequality
we write
\[ \int_{G_{n,k}} \frac{1}{\text{vol}_k(K \cap F^\perp)} d\nu_{n,k}(F) \]
\[ \leq \left( \int_{G_{n,k}} \frac{1}{\text{vol}_k(P_F(K))^p} d\nu_{n,k}(F) \right)^{1/p} \left( \int_{G_{n,k}} \frac{1}{\text{vol}_{n-k}(K \cap F^\perp)^q} d\nu_{n,k}(F) \right)^{1/q}. \]
\[ (4.1) \]

For the first integral in the right-hand side of (4.1) one may use the next lemma (from [6]) which relates it to the mixed widths of \( K \).

**Lemma 4.1.** Let \( K \) be a centered convex body of volume 1 in \( \mathbb{R}^n \). Then, for every \( 1 \leq k \leq n-1 \) and \( p \geq 1 \),
\[ W_{[k, -p]}(K) = \left( \int_{G_{n,k}} \text{vol}_k(P_F(K))^{-p} d\nu_{n,k}(F) \right)^{\frac{1}{p}} \geq c_1 \frac{w_{-kp}(K)}{\sqrt{k}}, \]
where \( c_1 > 0 \) is an absolute constant.

**Proof.** Using Hölder’s inequality, the Blaschke-Santaló and the reverse Santaló inequality, for every \( p \geq 1 \) we can write
\[ \left( \int_{G_{n,k}} \text{vol}_k(P_F(K))^{-p} d\nu_{n,k}(F) \right)^{\frac{1}{p}} \simeq \left( \int_{G_{n,k}} \frac{\text{vol}_k((P_F(K))^\circ)^p}{\omega_k^{2p}} d\nu_{n,k}(F) \right)^{\frac{1}{p}} \]
\[ \simeq \sqrt{k} \left( \int_{G_{n,k}} \left( \int_{S_F} \frac{1}{h_{P_F(K)}(\theta)} d\sigma_F(\theta) \right)^p d\nu_{n,k}(F) \right)^{\frac{1}{p}} \]
\[ \simeq \sqrt{k} \left( \int_{G_{n,k}} \left( \int_{S_F} \frac{1}{h_k(\theta)} d\sigma_F(\theta) \right)^p d\nu_{n,k}(F) \right)^{\frac{1}{p}} \]
\[ \leq c \sqrt{k} \left( \int_{G_{n,k}} \int_{S_F} \frac{1}{h_k(\theta)} d\sigma(\theta) d\nu_{n,k}(F) \right)^{\frac{1}{p}} \]
\[ = c \sqrt{k} w_{-k}^{-1}(F) \]
\[ = c \sqrt{k} w_{-k}^{-1}(K). \]

The lemma follows. \( \Box \)

We set \( p := n/k > 1 \). Then, from Lemma 4.1, (2.5) and (2.2) we get
\[ W_{[k, -n/k]}(K) \geq \frac{w_{-n}(K)}{c_1 \sqrt{k}} \simeq \frac{1}{c_1 \sqrt{k}} \left( \frac{\text{vol}_n(B_2^n)}{\text{vol}_n(K^*)} \right)^{1/n} \simeq \sqrt{n/k}. \]
\[ (4.3) \]
This gives:

**Lemma 4.2.** Let \( K \) be a centered convex body of volume 1 in \( \mathbb{R}^n \). Then, for every \( 1 \leq k \leq n-1 \),
\[ W_{[k, -n/k]}^{-1}(K) = \left( \int_{G_{n,k}} \text{vol}_k(P_F(K))^{-n/k} d\nu_{n,k}(F) \right)^{1/n} \leq c_2 \sqrt{k/n} \]
\[ (4.4) \]
where $c_2 > 0$ is an absolute constant.

Taking into account (4.1) we get the next general estimate.

**Proposition 4.3.** Let $K$ be a centered convex body of volume $1$ in $\mathbb{R}^n$. For any $1 \leq k \leq n - 1$ we have

\[
\int_{G_{n,k}} \frac{1}{\text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp)} d\nu_{n,k}(F)
\leq \left( c_1 \sqrt{\frac{k}{n}} \right)^k \left( \int_{G_{n,k}} \frac{1}{\text{vol}_{n-k}(K \cap F^\perp) \frac{n-k}{n}} d\nu_{n,k}(F) \right)^{\frac{n-k}{n}},
\]

where $c_1 > 0$ is an absolute constant.

We turn to the upper bound. The next proposition shows that the normalized dual quermassintegrals $\tilde{W}_{[k]}(K)$ are strongly related to the quantities $I_p(K)$.

**Lemma 4.4.** Let $K$ be a convex body of volume $1$ in $\mathbb{R}^n$ and let $1 \leq k \leq n - 1$. Then,

\[
\tilde{W}_{[k]}(K) I_{-k}(K) = \left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} = \tilde{W}_{[k]}(B_n^2) I_{-k}(B_n^2).
\]

Direct computation shows that $\left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \simeq \sqrt{n}$.

**Proof.** We integrate in polar coordinates:

\[
I_{-k}(K) = \frac{n\omega_n}{n-k} \int_{S^{n-1}} \frac{1}{||x||^{n+k}} d\sigma(x)
\]

\[
= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} \omega_{n-k} \int_{S_F} \frac{1}{||\theta||^{n-k}} d\sigma(\theta) d\nu_{n,n-k}(F)
\]

\[
= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} \text{vol}_{n-k}(K \cap F) d\nu_{n,n-k}(F)
\]

\[
= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,k}} \text{vol}_{n-k}(K \cap F^\perp) d\nu_{n,k}(F),
\]

and the result follows from the definition of $\tilde{W}_{[k]}(K)$. \hfill \Box

It was proved in [12] that if $K$ is a centered convex body of volume $1$ in $\mathbb{R}^n$ then for any $p > -n$ we have

\[
I_p(K) \geq I_p(B_n^2).
\]

One can also check that $\tilde{W}_{[k]}(B_n^2) \simeq 1$ for all $1 \leq k \leq n - 1$. Then, Lemma 4.4 immediately gives:

**Lemma 4.5.** Let $K$ be a centered convex body of volume $1$ in $\mathbb{R}^n$. Then, for every $1 \leq k \leq n - 1$,

\[
\tilde{W}_{[k]}(K) \leq \tilde{W}_{[k]}(B_n^2) \simeq 1.
\]
Now we write

\[
\int_{G_{n,k}} \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/2} d\nu_{n,k}(F) \leq \left( \int_{G_{n,k}} \text{vol}_k(P_F(K)) d\nu_{n,k}(F) \right)^{1/2} \left( \int_{G_{n,k}} \text{vol}_{n-k}(K \cap F^\perp) d\nu_{n,k}(F) \right)^{1/2},
\]

and taking into account Lemma 4.5 we get the next general estimate.

**Proposition 4.6.** Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. For any $1 \leq k \leq n - 1$ we have

\[
\int_{G_{n,k}} \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/2} d\nu_{n,k}(F) \leq c_2 \left( \int_{G_{n,k}} \text{vol}_k(P_F(K)) d\nu_{n,k}(F) \right)^{1/2},
\]

where $c_2 > 0$ is an absolute constant.

Taking into account (2.7) we see that

\[
\int_{G_{n,k}} \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/2} d\nu_{n,k}(F) \leq \left( c_3 w(K) \sqrt{k} \right)^{k/2},
\]

where $c_3 > 0$ is an absolute constant. Then, Markov’s inequality implies the following.

**Proposition 4.7.** Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. For any $1 \leq k \leq n - 1$ we have that a random $F \in G_{n,k}$ satisfies

\[
g(K, k; F) = \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/k} \leq \frac{c_4 w(K)}{\sqrt{k}}
\]

with probability greater than $1 - e^{-k}$, where $c_4 > 0$ is an absolute constant.

5. The isotropic case

Recall that a convex body $K$ in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and its inertia matrix is a multiple of the identity: there exists a constant $L_K > 0$ such that

\[
\int_K \langle x, \theta \rangle^2 dx = L_K^2
\]

for every $\theta$ in the Euclidean unit sphere $S^{n-1}$. More generally, a log-concave probability measure $\mu$ on $\mathbb{R}^n$ is called isotropic if its barycenter is at the origin and its inertia matrix is the identity; in this case, the isotropic constant of $\mu$ is defined as

\[
L_\mu := \sup_{x \in \mathbb{R}^n} \left( f_\mu(x) \right)^{1/n},
\]

where $f_\mu$ is the density of $\mu$ with respect to the Lebesgue measure. Note that a centered convex body $K$ of volume 1 in $\mathbb{R}^n$ is isotropic if and only if the log-concave probability measure $\mu_K$ with density $x \mapsto L_K^2 \mathbf{1}_{L_K^2/(L_K^2)}(x)$ is isotropic. The reader may find a detailed and updated exposition of the theory of isotropic log-concave measures in the book [4].
Let $\mu$ be a probability measure on $\mathbb{R}^n$ with density $f_\mu$ with respect to the Lebesgue measure. For every $1 \leq k \leq n-1$ and every $E \in G_{n,k}$, the marginal of $\mu$ with respect to $E$ is the probability measure with density

$$f_{\pi_E \mu}(x) = \int_{x+E^\perp} f_\mu(y) dy.$$  

It is easily checked that if $\mu$ is centered, isotropic or log-concave, then $\pi_E \mu$ is also centered, isotropic or log-concave, respectively. For every log-concave probability measure $\mu$ on $\mathbb{R}^n$ and any $p > 0$ we define the set $K_p(\mu)$ as follows:

$$K_p(\mu) = \left\{ x \in \mathbb{R}^n : \int_0^\infty f_\mu(rx)^{p-1} dr \geq \frac{f_\mu(0)}{p} \right\}.$$

The bodies $K_p(\mu)$ were introduced by K. Ball [1] who showed that they are convex.

The next proposition is a generalization of a result of Ball from the same work (see also [9], and [4] for the precise statement below); it gives a very useful expression for the volume of central sections of an isotropic convex body.

**Proposition 5.1.** Let $K$ be an isotropic convex body in $\mathbb{R}^n$. We denote by $\mu_K$ the isotropic log-concave measure with density $L_n^{1_n} \cdot 1_{K^{-1}}$. Then, for every $1 \leq k \leq n-1$ and $F \in G_{n,k}$, the body $K_{k+1}(\pi_F(\mu_K))$ satisfies

$$\text{vol}_{n-k}(K \cap F^\perp)^{1/k} \simeq \frac{L_{K_{k+1}(\pi_F(\mu_K))}}{L_K}.$$

Assume that $K$ is an isotropic convex body in $\mathbb{R}^n$. From Proposition 5.1 we know that, for every $1 \leq k \leq n-1$ and $F \in G_{n,k}$,

$$\text{vol}_{n-k}(K \cap F^\perp)^{-1/k} \simeq \frac{L_K}{L_{K_{k+1}(\pi_F(\mu_K))}} \leq c_2 L_K,$$

because $L_C \geq c$ for every convex body $C$, where $c > 0$ is an absolute constant (see for example Proposition 2.3.12 in [4]). Therefore, Proposition 4.3 gives

$$\int_{G_{n,k}} \frac{1}{\text{vol}_k(P_F(K))} \text{vol}_{n-k}(K \cap F^\perp) d\nu_{n,k}(F) \leq \left( c_1 \sqrt{k/n} \right)^k \left( c_2 L_K \right)^k \leq (c_3 \sqrt{k/n L_K})^k.$$

From Markov’s inequality we get:

**Proposition 5.2.** Let $K$ be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leq k \leq n-1$, a random $F \in G_{n,k}$ satisfies

$$g(K, k; F) := \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/k} \geq \frac{c_4 \sqrt{n/k}}{L_K}$$

with probability greater than $1 - e^{-k}$, where $c_4 > 0$ is an absolute constant.

For the upper bound we use (2.7) and a recent result of E. Milman [8]: if $K$ is isotropic, and if we make the additional assumption that $K$ is origin symmetric, then

$$w(K) \leq c_5 \sqrt{n} (\log n)^2 L_K.$$

Thus, applying directly Proposition 4.7 we get:
Proposition 5.3. Let $K$ be an origin symmetric isotropic convex body in $\mathbb{R}^n$. For every $1 \leq k \leq n - 1$ a random $F \in G_{n,k}$ satisfies

\begin{equation}
\tag{5.7}
g(K, k; F) := (\text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp))^\frac{1}{2} \leq c_0 \sqrt{n/k} (\log n)^2 L_K
\end{equation}

with probability greater than $1 - e^{-k}$.

Combining Proposition 5.2 and Proposition 5.3 we obtain Theorem 1.2.

Remark 5.4. (i) It is known that for every isotropic convex body $K$ in $\mathbb{R}^n$ we can find an origin-symmetric convex body $T$ with the property that $L_T = L_K$ (see [4, Proposition 2.5.10]): if we define a function $f$ supported on $K - \mathbb{R}$ by

$$f(x) = (1_K \ast 1_{-K})(x) = \int_{\mathbb{R}^n} 1_K(y)1_{-K}(x-y) \, dy = \text{vol}_n(K \cap (x + K))$$

then $f$ is an even isotropic log-concave density and one can check that $L_f = \sqrt{2} L_K$. It follows that the convex body $T = K_{n+2}(f)$ has the desired properties. From Proposition 4.6 we see that the upper bound in Theorem 1.2 remains valid for a not necessarily symmetric isotropic convex body $K$ and some $1 \leq k \leq n - 1$, provided that

$$\int_{G_{n,k}} \text{vol}_k(P_F(K)) \, d\nu_{n,k}(F) \leq C_k \int_{G_{n,k}} \text{vol}_k(P_F(T)) \, d\nu_{n,k}(F).$$

(ii) The logarithmic terms in (5.7) cannot be completely eliminated as long as the proof passes through estimates of the mean width of $K$. This is evident from the case of $K = B_1^n$, where $w(B_1^n) \simeq \sqrt{n \log(1 + n)}$. However, some of these terms may not be needed. For example, if the body is in the $\ell$-position (see [4, Section 1.11]) then the reverse Urysohn inequality $w(K) \leq c_0 \sqrt{n} \log n$ and Proposition 4.7 imply that $g(K, k; F) \leq c_0 \sqrt{n/k} \log n$ for a random $F \in G_{n,k}$.

References


**Department of Mathematics, University of Athens, Panepistimioupolis 157 84, Athens, Greece. E-mail: apgiannop@math.uoa.gr**

**Department of Mathematics, University of Athens, Panepistimioupolis 157 84, Athens, Greece. E-mail: lefteris128@yahoo.gr**

**Department of Mathematics, University of the Aegean, Karlovassi 832 00, Samos, Greece. E-mail: antonis.tsolomitis@gmail.com**