Topics on the

Asymptotic Geometry of Random Polytopes

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Abstract

In this set of notes we review several central results on the geometry of random polytopes in the context of asymptotic geometric analysis, obtained within the last decade. In particular, we study the method of Dafnis, Giannopoulos and Tsolomitis [20] on the determination of the asymptotic shape, and the method of Klartag and Kozma [43] on the boundedness of the isotropic constant, of a polytope generated by a set of N vectors chosen independently and uniformly from an isotropic convex body K. Approximation of K by such a random polytope is also discussed. Most of the necessary background on notions and tools from asymptotic geometric analysis that get involved is collected in a first introductory section.

Contents

1 Notation and background from asymptotic convex geometry

1.1 Basics on convex bodies

We work in $\mathbb{R}^n$, equipped with the standard inner product $\langle \cdot, \cdot \rangle$ which induces the euclidean norm $\| \cdot \|_2$. By dx we denote integration with respect to the Lebesgue measure in the appropriate dimension. We write S^{n-1} and B_2^n for the euclidean unit sphere and closed unit ball of $\mathbb{R}^n$ respectively. We write $GL(n)$ for the family of all invertible linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$. The subclass $\{T \in GL(n) : \det(T) = 1\}$ is denoted by $SL(n)$, and $O(n)$ is the group of rotations. An *ellipsoid* in $\mathbb{R}^n$ can be always thought of as a linear image of B_2^n, that is $T(B_2^n)$ for some $T \in GL(n)$. By F. John's theorem [40], every convex body K contains an ellipsoid of maximal volume; this will be the *maximal volume ellipsoid of* K. Equivalently, K is contained in an ellipsoid of minimal volume. We say that K is in *John's position* if its maximal volume ellipsoid is B_2^n (respectively, K is in *Löwner's position* if B_2^n is the minimal ellipsoid of K). We denote by $\theta^\perp$ the central hyperplane that is orthogonal to $\theta \in S^{n-1}$. For an integer $1 \leqslant k \leqslant n-1$, $G_{n,k}$ stands for the set of all k-dimensional subspaces of $\mathbb{R}^n$, and if $A \subseteq \mathbb{R}^n$ then $P_F(A)$ is the orthogonal projection of A onto $F \in G_{n,k}$.

Moreover, σ will usually denote the unique rotation-invariant Haar probability measure on S^{n-1}. Avoiding formal details, one way to define this is as follows: $O(n)$ is a compact group, so it is naturally equipped with a translation invariant Haar probability measure, let us call it ν_n. Translation invariance amounts to $\nu_n(\mathcal{O}) = \nu_n(U(\mathcal{O}))$ for every measurable $\mathcal{O} \subseteq O(n)$ and $U \in O(n)$. Now fix an arbitrary $\theta_0 \in S^{n-1}$ and set

$$\sigma(A) := \nu_n \left(\{ U \in O(n) : U(\theta_0) \in A \} \right).$$

for every measurable $A \subseteq S^{n-1}$. It is immediate to check that that σ is a probability measure (because ν_n is), invariant under the action of $O(n)$ (due to the translation invariance of ν_n).

Similarly one constructs a probability measure on $G_{n,k}$; fix some arbitrary $F_0 \in G_{n,k}$ and define the measure $\nu_{n,k}$ by

$$\nu_{n,k}(\mathcal{G}) := \nu_n(\{U \in O(n) : U(F_0) \in \mathcal{G}\}),$$

for any measurable $\mathcal{G} \subseteq G_{n,k}$. This is also a rotation-invariant probability measure.

Asymptotic notation: In the context of asymptotic geometric analysis, the focus usually lies not on detailed numerical calculations, but actually understanding the dependence of different quantities on a varying set of parameters, most importantly the dimension of the base space n, as the latter tends to infinity. It is often the case that various absolute numerical constants get involved in our computations: We denote them by C, c, c_1 etc. Since the exact values of these constants are irrelative to the nature of the results presented, we rarely pursue to compute them explicitly and might treat them a bit carelessly, e.g. the same letter might refer to a different constant from line to line. Sometimes we might even relax our notation: $a \lesssim b$ will then mean "$a \leqslant cb$ for some (suitable) absolute constant $c > 0$", and $a \asymp b$ will stand for "$a \lesssim b \wedge a \gtrsim b$". If A, B are sets, $A \approx B$ will similarly state that $c_1 A \subseteq B \subseteq c_2 A$ for some absolute constants $c_1, c_2 > 0$.

Convex bodies and volume

A *convex body* in $\mathbb{R}^n$ is a compact and convex subset of $\mathbb{R}^n$ with non-empty interior. We say that a convex body K is (origin) *symmetric* if $K = -K$ (i.e. $x \in K \Leftrightarrow -x \in K$), and *centered* if its barycenter lies at the origin, that is

$$\int_K \langle x, \theta \rangle \, dx = 0$$

for all $\theta \in S^{n-1}$.

Note that there is a one to one correspondence between the class of origin symmetric convex bodies in $\mathbb{R}^n$ and the class of n-dimensional Banach spaces. This is because, on the one hand, due to compactness and symmetry, the Minkowski functional of a convex body K is actually a norm on $\mathbb{R}^n$. We will denote this by

$\|\cdot\|_K$. Conversely, if $(X, \|\cdot\|_X)$ is any n-dimensional normed space then the closed unit ball B_X of X is the unique convex body with $\|\cdot\|_{B_X} = \|\cdot\|_X$. This very correspondence gave rise to a fascinating and ongoing interplay between convex geometry and functional analysis.

We refer to the n-dimensional Lebesgue measure of a convex body K in $\mathbb{R}^n$ as its *volume*, which we denote $\mathrm{vol}_n(K)$. The quantity $\mathrm{vol}_n(K)^{1/n}$ is called the *volume radius*[1] of the convex body K. We often abbreviate $\omega_n := \mathrm{vol}_n(B_2^n)$. Using polar coordinates, one can compute the volume of B_2^n, that is

$$\omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Since, by Stirling's approximation formula, $\Gamma\left(\frac{n}{2}+1\right) \asymp \sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{(n+1)/2}$, we get the estimate

$$\omega_n^{1/n} \asymp \frac{1}{\sqrt{n}}$$

that gets frequently involved in our calculations.

Remark 1.1. Since $\mathrm{vol}_n(B_\infty^n) = 2^n$, note that $\omega_n/\mathrm{vol}_n(B_\infty^n) \to 0$ as $n \to \infty$; although the euclidean ball is the maximal volume ellipsoid inside the cube, it captures only a tiny bit of its volume, that tends to zero superexponentially as the dimension grows. Informally, this indicates that volume in the cube is not distributed "uniformly", but rather tends to concentrate "near its corners". Although distribution of volume is not our main focus in this text, the above provides a first hint at how our lower-dimensional intuition might lead us astray when it comes to high-dimensional phenomena.

Note that, if K is a convex body in $\mathbb{R}^n$, normalization by the volume induces a probability measure on K: We can actually define μ_K by

$$\mu_K(A) := \frac{\mathrm{vol}_n(A \cap K)}{\mathrm{vol}_n(K)}$$

for any measurable $A \subseteq \mathbb{R}^n$. This is what we call the *uniform measure* on K. The probability measure σ on S^{n-1} that we mentioned earlier can be equivalently defined via the Lebesgue measure as follows: If A is a measurable subset of S^{n-1}, consider the "cone" $C(A) = \{t\theta : \theta \in A, t \in [0,1]\}$ and let

$$\sigma(A) := \frac{\mathrm{vol}_n(C(A))}{\omega_n}.$$

The fact that the two definitions are equivalent is due to the uniqueness of the Haar measure.

Support function, radii and mean width

Given a convex body K in $\mathbb{R}^n$, we define its *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ by

$$h_K(u) = \max_{x \in K} \langle x, u \rangle.$$

There is a clear geometric interpretation of the support function: If we choose a direction $\theta \in S^{n-1}$, then $h_K(\theta)$ is actually the (signed) distance of the supporting hyperplane of K in the direction θ from the origin. We collect some elementary properties of the support function in the next Lemma.

Lemma 1.2 (Properties of the support function)**.** *Let K, L two convex bodies in $\mathbb{R}^n$. Then,*

[1] We use this terminology for convenience throughout the text. Normally the volume radius of K is defined as $\left(\frac{\mathrm{vol}_n(K)}{\omega_n}\right)^{1/n}$, i.e. the radius of the euclidean ball that has volume equal to that of K.

(a) h_K *is positively homogeneous and subadditive, i.e.*

$$h_K(tx) = th_K(x),$$

for all $t \geqslant 0$ *and* $x \in \mathbb{R}^n$*, and*

$$h_K(x+y) \leqslant h_K(x) + h_K(y)$$

for all $x, y \in \mathbb{R}^n$*.*

(b) h_K *uniquely determines* K*, that is* $K \subseteq L$ *if and only if* $h_K \leqslant h_L$*.*

(c) *Any positively homogeneous and subadditive function* $h : \mathbb{R}^n \to \mathbb{R}$ *is the support function of some unique convex body* K*.*

(d) K *is symmetric if and only if* $h_K(\theta) = h_K(-\theta)$ *for all* $\theta \in S^{n-1}$*.*

(e) $0 \in K$ *if and only if* $h_K \geqslant 0$*.*

The justification of (most of) the above statements is no more than an easy exercise, nevertheless this is stuff that can be found everywhere in the literature, e.g. one can have a look at [24, Section 0.6], [60, Section 1.7.1], or [8, Appendix A.1] . Note that properties (a), (d) and (e) above imply that if K is a centrally symmetric convex body in $\mathbb{R}^n$, then h_K defines a norm in $\mathbb{R}^n$. The unit ball of this norm is called the *polar body* of K. Actually this set is defined even without the symmetry assumption: For any convex body K in $\mathbb{R}^n$ with $0 \in K$, we define

$$K^\circ := \left\{ y \in \mathbb{R}^n : \sup_{x \in K} \langle x, y \rangle \leqslant 1 \right\}.$$

Some intuition on what the polar body represents (in the centrally symmetric case, at least) might come from functional analysis: If X_K is the n-dimensional space whose closed unit ball is the body K, then K° is simply the closed unit ball of the dual space $(X_K)^*$. For example note that for an ℓ_p-ball, $1 \leqslant p \leqslant \infty$, $(B_p^n)^\circ = B_q^n$, where $\frac{1}{p} + \frac{1}{q} = 1$. One can easily check that $(K^\circ)^\circ = K$, and $h_K(\cdot) = \| \cdot \|_{K^\circ}$ for any centrally symmetric convex body K.

The *circumradius* $R(K)$ of a convex body K is the radius of the smallest euclidean ball enclosing K, that is

$$R(K) := \min\{r > 0 : K \subseteq rB_2^n\}$$

Obviously, $R(K) = \max_{x \in K} \|x\|_2$, and thus $R(K) = \max_{\theta \in S^{n-1}} h_K(\theta)$.

Respectively, the *inradius* $r(K)$ of K is the radius of the largest euclidean ball that lies inside K, or

$$r(K) := \max\{r > 0 : rB_2^n \subseteq K\}.$$

As with $R(K)$, one can check that $r(K) = \min_{\theta \in S^{n-1}} h_K(\theta)$.

By definition of the support function, we can see that $h_K(\theta) + h_K(-\theta)$ essentially measures the "width" of the body K on the direction $\theta \in S^{n-1}$. Taking expectation (and dividing by 2), we get what is defined as the *mean width* of a convex body K, $w(K)$:

$$w(K) := \int_{S^{n-1}} h_K(\theta)\, d\sigma(\theta).$$

More generally one can define, for every $q \in [-n, n]$, $q \neq 0$,

$$w_q(K) := \left(\int_{S^{n-1}} h_K(\theta)^q \, d\sigma(\theta) \right)^{1/q}.$$

These quantitites are usually refered to as the *mixed widths* of K. The behaviour of such q-th moments of h_K will turn out to be of interest in the sequel.

Quermassintegrals of a convex body

Let us also introduce the family of quermassintegrals of a convex body K. We will try to do this without mentioning the notion of mixed volume or getting into the deep related theory, for a detailed account of which we refer e.g. to [8, Appendix B] or [60, Chapter 5]. We may start from *Steiner's formula*, which asserts that, given a convex body K in $\mathbb{R}^n$, the volume of $K + tB_2^n$ can be expressed as a polynomial in t: There are non-negative coefficients $(W_k(K))_{k=0}^n$ such that

$$\mathrm{vol}_n(K + tB_2^n) = \sum_{k=0}^{n} \binom{n}{k} W_k(K) t^k. \tag{1.1}$$

We call $W_k(K)$ in the above expression the *k-th quermassintegral* of K. These quantities have a convenient integral representation through *Kubota's formula*:

$$W_k(K) = \frac{\omega_n}{\omega_{n-k}} \int_{G_{n,n-k}} \mathrm{vol}_{n-k}(P_F(K))\, d\nu_{n,n-k}(F). \tag{1.2}$$

Remark 1.3. (a) An application of (1.2) for $k = n-1$ gives us

$$W_{n-1}(K) = \omega_n w(K),$$

and, obviously, $W_0(K) = \mathrm{vol}_n(K)$, $W_n(K) = \omega_n$.
(b) It is a corollary of the Alexandrov-Fenchel inequality on mixed volumes [8, Theorem B.2.1], that

$$\left(\frac{W_k(K)}{\omega_n}\right)^{\frac{1}{n-k}} \geqslant \left(\frac{W_j(K)}{\omega_n}\right)^{\frac{1}{n-j}},$$

for every $0 \leqslant j < k < n$.

The contents of Remark 1.3 motivate us to consider a different normalization. We define, for every $1 \leqslant k \leqslant n$,

$$Q_k(K) := \left(\frac{W_{n-k}(K)}{\omega_n}\right)^{\frac{1}{k}}.$$

We refer to $Q_k(K)$ as the *normalized k-th quermassintegral* of K. Due to Kubota's formula and the facts stated in Remark 1.3, we have the following:

Lemma 1.4. *The normalized quermassintegrals $Q_k(K)$ of a convex body K in $\mathbb{R}^n$, $1 \leqslant k \leqslant n$, satisfy the following*

(a) *The integral representation*

$$Q_k(K) = \left(\frac{1}{\omega_k} \int_{G_{n,k}} \mathrm{vol}_k(P_F(K))\, d\nu_{n,k}(F)\right)^{1/k}$$

is valid for every $1 \leqslant k \leqslant n$.

(b) $Q_1(K) = w(K)$ *and* $Q_n(K) = \left(\frac{\mathrm{vol}_n(K)}{\omega_n}\right)^{1/n}$

(c) *The sequence $(Q_k)_{k \leqslant n}$ is decreasing in k.*

Geometric inequalities

We close this introductory section stating a number of classical geometric inequalities for convex bodies that will be of use in the sequel. We start with a fundamental result in classical convexity.

Theorem 1.5 (Brunn-Minkowski inequality)**.** *Let K and L be two non-empty compact subsets of $\mathbb{R}^n$. Then,*

$$\mathrm{vol}_n(K+L)^{1/n} \geqslant \mathrm{vol}_n(K)^{1/n} + \mathrm{vol}_n(L)^{1/n}. \tag{1.3}$$

If we further suppose that K and L are convex bodies, then equality in (1.3) *holds if and only if K and L are homothetical.*

The Brunn-Minkowski inequality relates volume to Minkowski addition. It is often encountered in two different (in the end, equivalent) forms: For any $\lambda \in (0,1)$, and non-empty, compact $K, L \subseteq \mathbb{R}^n$,

$$\mathrm{vol}_n(\lambda K + (1-\lambda)L)^{1/n} \geqslant \lambda \mathrm{vol}_n(K)^{1/n} + (1-\lambda)\mathrm{vol}_n(L)^{1/n}, \tag{1.4}$$

or (using the arithmetic-geometric means inequality),

$$\mathrm{vol}_n(\lambda K + (1-\lambda)L) \geqslant \mathrm{vol}_n(K)^{\lambda}\mathrm{vol}_n(L)^{1-\lambda}. \tag{1.5}$$

The latter shows that volume is a log-concave function with respect to Minkowski addition. In particular, if K is a convex body then the induced measure μ_K is a log-concave probability measure on $\mathbb{R}^n$.

A classical inequality can be derived as a consequence of the Brunn-Minkowski inequality and Steiner symmetrization (for a proof, see e.g. [8, Theorem 1.5.11]). This is originally due to Urysohn.

Theorem 1.6 (Urysohn's inequality)**.** *Let K be a convex body in $\mathbb{R}^n$. Then*

$$w(K) \geqslant \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B_2^n)}\right)^{1/n}.$$

This inequality will let us estimate the mean width of K once we get a lower bound on the volume radius of K (and vice versa).

The Blaschke-Santaló inequality essentially states that the volume product $\mathrm{vol}_n(K)\mathrm{vol}_n(K^\circ)$ is maximized when K is an ellipsoid. This allows one to estimate the volume of K in terms of the volume of its polar body.

Theorem 1.7 (Blaschke-Santaló inequality)**.** *Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Then*

$$\mathrm{vol}_n(K)\mathrm{vol}_n(K^\circ) \leqslant \omega_n^2,$$

with equality if and only if K is an ellipsoid.

Note that we chose to state the result for symmetric convex bodies, although it actually holds in greater generality. As with Urysohn's inequality, a proof can be given using Steiner symmetrization and the Brunn-Minkowski inequality [8, Section 1.5.4].

On to another consequence of the Brunn-Minkowski inequality, the next result was proved by C. Borell [12], and is often referred to as "Borell's Lemma". It can be viewed as a primitive "concentration of volume" inequality.

Theorem 1.8 (Borell's Lemma)**.** *Let K be a convex body of volume 1 in $\mathbb{R}^n$, and $A \subseteq \mathbb{R}^n$ closed, convex and symmetric, with $\mathrm{vol}(K\cap A) = \delta > \frac{1}{2}$. Then for every $t \geqslant 1$,*

$$\mathrm{vol}(K \cap (tA)^c) \leqslant \delta \left(\frac{1-\delta}{\delta}\right)^{\frac{t+1}{2}}.$$

Indication of proof. Show that $A^c \supseteq \frac{2}{t+1}(tA)^c + \frac{t-1}{t+1}A$, then take intersection with K and apply the Brunn-Minkowski inequality. □

A useful corollary of Borell's lemma is the validity of reverse Hölder inequalities for general seminorms on $\mathbb{R}^n$.

Corollary 1.9. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. If $f:\mathbb{R}^n \to \mathbb{R}$ is a seminorm, then for every $1 \leqslant p < q$ we have*

$$\left(\int_K |f(x)|^p\,dx\right)^{1/p} \leqslant \left(\int_K |f(x)|^q\,dx\right)^{1/q} \leqslant c\frac{q}{p}\left(\int_K |f(x)|^p\,dx\right)^{1/p},$$

where $c > 0$ is an absolute constant

Proof. We have to prove only the right-hand side inequality, since the left-hand is simply Hölder's. We apply Theorem 1.8 for the set

$$A = \{x \in \mathbb{R}^n : |f(x)| \leqslant 3\|f\|_p\},$$

which is closed, symmetric and convex. By Markov's inequality we can see that $\mathrm{vol}(K\cap A) \geqslant 1 - 3^{-p} > 1/2$. Note that for $\delta > 1/2$ one has

$$\delta\left(\frac{1-\delta}{\delta}\right)^{\frac{t+1}{2}} < \frac{(1-\delta)^{\frac{t-1}{2}}}{\delta^{\frac{t-1}{2}}} = \left(\frac{1}{\delta} - 1\right)^{\frac{t-1}{2}},$$

and for $\delta = 1 - 3^{-p}$, $\frac{1}{\delta} - 1 = \frac{3^{-p}}{1-3^{-p}} \leqslant e^{-p/2}$. Theorem 1.8 then yields

$$\mathrm{vol}(\{x \in K : |f(x)| \geqslant 3t\|f\|_p\}) \leqslant e^{-c_1 p(t-1)}$$

for any $t > 1$, with $c_1 = 1/4$. Now we write

$$\begin{aligned}\int_K |f(x)|^q\,dx &= \int_0^\infty q s^{q-1}\mathrm{vol}(\{x \in K : |f(x)| \geqslant s\})\,ds \\ &\leqslant (3\|f\|_p)^q + (3\|f\|_p)^q \int_1^\infty q t^{q-1} e^{-c_1 p(t-1)}\,dt \\ &\leqslant (3\|f\|_p)^q + e^{c_1 p}(3\|f\|_p)^q \int_1^\infty q t^{q-1} e^{-c_1 p t}\,dt \\ &\leqslant (3\|f\|_p)^q + e^{c_1 p}\left(\frac{3\|f\|_p}{c_1 p}\right)^q \Gamma(q+1).\end{aligned}$$

The wanted statement is proved applying Stirling's approximation and the fact that $(a+b)^{1/q} \leqslant a^{1/q} + b^{1/q}$ for all $a, b > 0$ and $q \geqslant 1$. □

Last, we quote a result of Grünbaum [35] according to which a hyperplane passing through the barycenter of a convex body K divides the body in two parts of more or less the same volume.

Theorem 1.10 (Grünbaum's Lemma)**.** *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For every $\theta \in S^{n-1}$ we have*

$$\frac{1}{e} \leqslant \mathrm{vol}_n(\{x \in K : \langle x, \theta\rangle \geqslant 0\}) \leqslant 1 - \frac{1}{e}.$$

We remark that Theorems 1.8 (thus also Corollary 1.9) and 1.10 actually hold in greater generality: We can replace the uniform measure on K by any centered log-concave probability measure in $\mathbb{R}^n$.

1.2 Isotropic position and the slicing problem

Let us now get introduced to the notion of isotropicity, as well as the main open problem in the field.

Definition 1.11 (Isotropic convex body)**.** A convex body K in $\mathbb{R}^n$ is called *isotropic* if it is centered, $\mathrm{vol}_n(K) = 1$, and satisfies the *isotropic condition*, that is: There exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 \, dx = L_K^2, \tag{1.6}$$

for all $\theta \in S^{n-1}$. We call the constant L_K the *isotropic constant* of K.

Note that if K satisfies the isotropic condition (1.6), then

$$\int_K \|x\|_2^2 \, dx = \sum_{i=1}^{n} \int_K \langle x, e_i \rangle^2 \, dx = nL_K^2, \tag{1.7}$$

where $\{e_j\}_{j \leqslant n}$ is the usual (or any) orthonormal basis of $\mathbb{R}^n$

In probabilistic terms, the isotropic condition says that the variance of all the 1-dimensional marginals in K is the same. So, roughly speaking, an isotropic convex body is equally "spread" in all directions $\theta \in S^{n-1}$, and the isotropic constant L_K measures this "spread". Note that if K is an isotropic convex body, then $U(K)$ is also isotropic, for every $U \in O(n)$.

Although isotropicity seems to be a nice and special property, it is actually the case that any centered convex body has a *position* (i.e. linear image) that is isotropic.

Proposition 1.12. *Let K be a centered convex body in $\mathbb{R}^n$. Then there exists $T \in GL(n)$ such that $T(K)$ is isotropic.*

Proof. If we define the linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ with $A(y) = \int_K \langle x, y \rangle x \, dx$, one can see that A is symmetric and positive definite; therefore it has a symmetric and positive definite square root S (i.e. S is such that $S^2 = A$). If we consider the linear image $\tilde{K} = S^{-1}(K)$ of K, then for every $\theta \in S^{n-1}$,

$$\begin{aligned}
\int_{S^{-1}(K)} \langle x, \theta \rangle^2 \, dx &= |\det S|^{-1} \int_K \langle S^{-1}x, \theta \rangle^2 \, dx \\
&= |\det S|^{-1} \int_K \langle x, S^{-1}\theta \rangle^2 \, dx \\
&= |\det S|^{-1} \left\langle \int_K \langle x, S^{-1}\theta \rangle x \, dx, s^{-1}\theta \right\rangle \\
&= |\det S|^{-1} \langle AS^{-1}\theta, S^{-1}\theta \rangle = |\det S|^{-1} \|\theta\|_2^2 = |\det S|^{-1}.
\end{aligned}$$

Normalizing by $\mathrm{vol}_n(\tilde{K})$ we get the result. □

Actually more is true: The isotropic position of a convex body is uniquely determined (up to orthogonal transformations) and arises as a solution of a specific minimization problem. Specifically, if K is a centered convex body of volume 1 in $\mathbb{R}^n$, then $\tilde{K} = T_0(K)$, $T_0 \in SL(n)$, is an isotropic position of K if and only if T_0 minimizes the quantity

$$\int_{TK} \|x\|_2^2 \, dx$$

over $T \in SL(n)$ (see Theorem 2.3.4 in [17]). We can thus give the following general definition.

Definition 1.13 (Isotropic constant, general definition)**.** For every centered convex body K in $\mathbb{R}^n$, we define the isotropic constant of K by

$$L_K^2 := \frac{1}{n} \min_{T \in GL(n)} \left\{ \frac{1}{\mathrm{vol}_n(T(K))^{1+\frac{2}{n}}} \int_{T(K)} \|x\|_2^2 \, dx \right\}$$

Note that, in view of the above, if $\tilde{K}$ is an isotropic position of K, then for all $\theta \in S^{n-1}$ we have

$$\int_{\tilde{K}} \langle x, \theta \rangle^2 = L_K^2.$$

It is not hard to see that the isotropic constant of any convex body can be bounded below by an absolute constant. Specifically,

Proposition 1.14. *The Euclidean ball B_2^n minimizes the isotropic constant, that is, for any convex body K in $\mathbb{R}^n$, one has $L_K \geqslant L_{B_2^n} \geqslant c$, where $c > 0$ is an absolute constant.*

Proof. Set $r_n = \omega_n^{-1/n}$, so that $\mathrm{vol}_n(r_n B_2^n) = 1$ and $r_n B_2^n$ is isotropic. Note that $\mathrm{vol}_n(K \setminus r_n B_2^n) = \mathrm{vol}_n(r_n B_2^n \setminus K)$, since $\mathrm{vol}_n(K) = \mathrm{vol}_n(r_n B_2^n) = 1$. Moreover, $\|x\|_2 > r_n$ on $K \setminus r_n B_2^n$ and $\|x\|_2 \leqslant r_n$ on $r_n B_2^n \setminus K$, so we can write

$$\begin{aligned} nL_K^2 &= \int_K \|x\|_2^2 \, dx = \int_{K \cap r_n B_2^n} \|x\|_2^2 \, dx + \int_{K \setminus r_n B_2^n} \|x\|_2^2 \, dx \\ &\geqslant \int_{K \cap r_n B_2^n} \|x\|_2^2 \, dx + \int_{r_n B_2^n \setminus K} \|x\|_2^2 \, dx = \int_{r_n B_2^n} \|x\|_2^2 \, dx = nL_{B_2^n}^2. \end{aligned}$$

Now using polar coordinates and the fact that $\omega_n^{1/n} \asymp n^{-1/2}$ we get

$$L_{B_2^n}^2 = \frac{1}{n} \int_{r_n B_2^n} \|x\|_2^2 \, dx = \frac{1}{n} \frac{n\omega_n}{n+2} r_n^{n+2} = \frac{\omega_n^{-2/n}}{n+2} \geqslant c,$$

for some absolute constant $c > 0$. $\square$

On the other hand, in 1986 J. Bourgain [13] (see also [51]) formulated the conjecture that a uniform upper bound on the isotropic constant of all convex bodies (of any dimension) should hold. This has become a topic of ongoing studies and is still the central open problem in the field of asymptotic geometric analysis.

Conjecture 1.15 (Isotropic constant conjecture)**.** *There exists an absolute constant $C > 0$ such that*

$$L_K \leqslant C$$

for every $n \in \mathbb{N}$ and for every convex body K in $\mathbb{R}^n$.

Using a result of Hensley [39] (see also [51, Corollary 3.2]) which shows that for any $n \in \mathbb{N}$, any isotropic convex body K in $\mathbb{R}^n$ and any $\theta \in S^{n-1}$,

$$c_1 \frac{1}{L_K} \leqslant \mathrm{vol}_{n-1}(K \cap \theta^\perp) \leqslant c_2 \frac{1}{L_K} \tag{1.8}$$

for some absolute constants $c_1, c_2 > 0$, one can show (e.g. [17, pp. 107-108]) that the Isotropic constant conjecture is equivalent to the famous slicing problem, or hyperplane conjecture:

Conjecture 1.16 (Hyperplane conjecture)**.** *There exists an absolute constant $c > 0$ such that for any $n \in \mathbb{N}$ and every centered convex body K in $\mathbb{R}^n$ of volume 1,*

$$\max_{\theta \in S^{n-1}} \mathrm{vol}_{n-1}(K \cap \theta^\perp) \geqslant c.$$

In view of (1.8), we can give a simple upper bound on L_K. Let us state a lemma first, that compares the inradius and circumradius of K to L_K.

Lemma 1.17. *If K is an isotropic convex body in $\mathbb{R}^n$, then*

$$c_1 L_K \leqslant r(K) \leqslant R(K) \leqslant c_2 n L_K,$$

where $c_1, c_2 > 0$ are absolute constants.

Proof. For the right hand side inequality, let $\theta \in S^{n-1}$ and consider $x_\theta \in K$ such that $h_K(\theta) = \langle x_\theta, \theta\rangle$. If $C(\theta)$ is the cone $\mathrm{conv}\{K \cap \theta^\perp, x_\theta\}$, then $C(\theta) \subseteq K$, and hence

$$1 = \mathrm{vol}_n(K) \geqslant \mathrm{vol}_n(C(\theta)) = \frac{\mathrm{vol}_{n-1}(K \cap \theta^\perp) h_K(\theta)}{n}$$

By (1.8), it follows that $h_K(\theta) \leqslant c_2 n L_K$, but $\theta \in S^{n-1}$ was arbitrary, so $R(K) \leqslant c_2 n L_K$.

For the left hand side inequality, we fix again some $\theta \in S^{n-1}$ and use Grünbaum's Lemma (Lemma 1.10):

$$\frac{1}{e} \leqslant \mathrm{vol}_n(\{x \in K : \langle x, \theta\rangle \geqslant 0\}) \leqslant h_K(\theta) \max_{0 \leqslant t \leqslant h_K(\theta)} \mathrm{vol}_{n-1}(\{x \in K : \langle x, \theta\rangle = t\}).$$

If we define $f_{K,\theta}(t) := \mathrm{vol}_{n-1}(\{x \in K : \langle x, \theta\rangle = t\})$, we have, by a result of Fradelizi [23] that $\|f_{K,\theta}\|_\infty \leqslant e f_{K,\theta}(0)$, so that

$$\frac{1}{e} \leqslant h_K(\theta) \cdot e \, \mathrm{vol}_{n-1}(K \cap \theta^\perp).$$

Now (1.8) gives $h_K(\theta) \geqslant c_1 L_K$, and having proved this for every $\theta \in S^{n-1}$ we have $r(K) \geqslant c_1 L_K$. □

Remark 1.18. (a) Note that if K is symmetric, we actually have $r(K) \geqslant L_K$, because $|\langle x, \theta\rangle| \leqslant h_K(\theta)$ for all $x \in K$, and hence

$$h_K(\theta) \geqslant \left(\int_K \langle x, \theta\rangle^2 \, dx\right)^{1/2} = L_K$$

for all $\theta \in S^{n-1}$.
(b) For the bound $R(K) \leqslant c n L_K$, we also have the more precise bound $R(K) \leqslant (n+1) L_K$, by an argument of Kannan, Lovász and Simonovits [41] (see also [17, Theorem 3.2.1]).

Now the inequality $r(K) \geqslant c L_K$ lets us see that $L_K \lesssim \sqrt{n}$, for every K.

Proposition 1.19 (Simple upper bound on L_K)**.** *For every isotropic convex body K in $\mathbb{R}^n$,*

$$L_K \leqslant c\sqrt{n},$$

where $c > 0$ is an absolute constant.

Proof. Since $r(K) \geqslant c_1 L_K$, by 1.17, it follows that $c_1 L_K B_2^n \subseteq r(K) B_2^n \subseteq K$. Taking volumes we get

$$(c_1 L_K)^n \omega_n \leqslant r(K)^n \omega_n = \mathrm{vol}_n(r(K) B_2^n) \leqslant \mathrm{vol}_n(K) = 1.$$

This proves that $L_K \leqslant c_1^{-1} \omega_n^{-1/n} \leqslant c\sqrt{n}$, for some absolute constant $c > 0$. □

In the case that K is symmetric, the $O(\sqrt{n})$ bound on L_K can be also easily obtained as a consequence of John's theorem [40]. However improving on this "trivial" bound turned out to be a difficult task. The first argument of Bourgain [14], whose presentation is of course beyond the scope of these notes, gave the estimate $L_K \lesssim \sqrt[4]{n} \log n$. Since then, absolute uniform bounds were given for L_K in the case that K lies in some special class of bodies[2], e.g. unconditional convex bodies, zonoids and duals of zonoids, bodies with bounded outer volume ratio or unit balls of the Schatten p-classes. The general problem however remains open; the only improvement upon Bourgain's bound has only been up to the $\log n$ factor. This was done in 2006 by Klartag [42], who established the estimate $L_K \lesssim \sqrt[4]{n}$.

The basic step to the proof of the $\sqrt[4]{n}$ bound was the solution of what was known as the "isomorphic slicing problem". Specifically, Klartag proved the following.

[2] For detailed references on each special case, we direct the reader to [8, Section 10.7] and for even more details to [17, Chapter 4].

Theorem 1.20 (Klartag, [42]). *Let K be a convex body in $\mathbb{R}^n$. For every $\varepsilon \in (0,1)$ there exists a centered convex body T in $\mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that*

$$\frac{1}{1+\varepsilon} T \subseteq K + x \subseteq (1+\varepsilon)T$$

and

$$L_T \leqslant \frac{C}{\sqrt{\varepsilon}},$$

for some absolute constant $C > 0$.

The solution of the isomorphic slicing, although closely related to the isotropic constant conjecture, could still by itself yield no improvement on the $\sqrt[4]{n}\log n$ bound of Bourgain for L_K. The missing ingredient was found in a -now famous- large deviation inequality on the distribution of volume on convex bodies, proved earlier by Paouris.

Theorem 1.21 (Paouris, [57]). *Let K be an isotropic convex body in $\mathbb{R}^n$. Then*

$$\mathrm{vol}_n(\{x \in K : \|x\|_2 \geqslant ct\sqrt{n}L_K\}) \leqslant e^{-t\sqrt{n}}, \tag{1.9}$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant.

Paouris' inequality essentially states that the volume of a convex body K is practically concentrated in a euclidean ball of radius of the order of $\sqrt{n}L_K$: what is left out of it tends to zero exponentially as $n \to \infty$. Theorem 1.21 let actually Klartag prove that a choice of $\varepsilon = 1/\sqrt{n}$ in Theorem 1.20 would yield $L_K \lesssim L_T \lesssim \sqrt[4]{n}$. We omit the details, which can be found in [17, Lemma 7.3.4].

1.3 Moments of $\|\cdot\|_2$ and the L_q-centroid bodies

Let $q \neq 0$, K be an isotropic convex body in $\mathbb{R}^n$, and define

$$I_q(K) := \left(\int_K \|x\|_2^q \, dx\right)^{1/q}.$$

Note that $I_2(K) = \sqrt{n}L_K$, since K is isotropic. The line of thinking to the proof of 1.21 was as follows: For any $q \geqslant 2$, Markov's inequality yields

$$\mathrm{vol}_n(\{x \in K : \|x\|_2 \geqslant e^3 I_q(K)\}) = \mathrm{vol}_n(\{x \in K : \|x\|_2^q \geqslant e^{3q} I_q(K)^q\}) \leqslant e^{-3q}.$$

Then applying Borell's lemma (Theorem 1.8) for $A = \{x : \|x\|_2 < e^3 I_q(K)\}$, we have that for any $t \geqslant 1$,

$$\mathrm{vol}_n(\{x \in K : \|x\|_2 \geqslant e^3 t I_q(K)\}) \leqslant (1 - e^{-3q}) \left(\frac{e^{-3q}}{1 - e^{-3q}}\right)^{(t+1)/2} \leqslant e^{-cqt}.$$

Note now that any bound of the form $I_q(K) \leqslant A \cdot I_2(K) = A\sqrt{n}L_K$ would give

$$\mathrm{vol}_n(\{x \in K : \|x\|_2 \geqslant e^3 At\sqrt{n}L_K\}) \leqslant \mathrm{vol}_n(\{x \in K : \|x\|_2 \geqslant e^3 t I_q(K)\}) \leqslant e^{-cqt},$$

so we would like A to be independent of n and q (which can practically go up to n) to get the Theorem.

By Corollary 1.9 applied for $f(x) = \|x\|_2$, we have, for any $1 \leqslant p < q$,

$$I_p(K) \leqslant I_q(K) \leqslant c\frac{q}{p} I_p(K),$$

in particular $I_q(K) \leqslant c_1 q I_2(K)$, for any $q \geqslant 2$, and even the estimate $I_q(K) \leqslant c_2\sqrt{q} I_2(K)$ was known for all $2 \leqslant q \leqslant n$ (by the result of [2], see also [17, Theorem 3.2.15]), however this was not sufficient. The contribution of Paouris was the proof of the following.

Theorem 1.22. *There exist absolute constants $c_3, c_4 > 0$ such that, for any isotropic convex body K in $\mathbb{R}^n$,*

$$I_q(K) \leqslant c_3 I_2(K), \tag{1.10}$$

if $1 \leqslant q \leqslant c_4\sqrt{n}$.

From the discussion above, it is now clear that (1.10) implies Theorem 1.21.

Paouris' starting point for the study of the means $I_q(K)$ was a known formula at the time, which we prove for completeness.

Lemma 1.23. *Let K be a convex body of volume 1 in $\mathbb{R}^n$, and $q \geqslant 1$. Then*

$$I_q(K) \asymp \sqrt{\frac{n}{q}} \left(\int_{S^{n-1}} \int_K |\langle x, \theta\rangle|^q \, dx \, d\sigma(\theta) \right)^{1/q}. \tag{1.11}$$

Proof. Let $q \geqslant 1$ and $x \in \mathbb{R}^n$. We will prove that

$$\left(\int_{S^{n-1}} |\langle x, \theta\rangle|^q \, d\sigma(\theta) \right)^{1/q} \asymp \sqrt{\frac{q}{n+q}} \|x\|_2, \tag{1.12}$$

which gives the statement of the Lemma, applying Fubini's theorem.

First, use polar coordinates to compute

$$\int_{B_2^n} |\langle x, y\rangle|^q \, dy = n\omega_n \int_0^1 r^{n+q-1} \, dr \int_{S^{n-1}} |\langle x, \theta\rangle|^q \, d\sigma(\theta) = \frac{n\omega_n}{n+q} \int_{S^{n-1}} |\langle x, \theta\rangle|^q \, d\sigma(\theta).$$

On the other hand, due to rotational invariance of the Lebesgue measure, we also have

$$\begin{aligned} \int_{B_2^n} |\langle x, y\rangle|^q \, dy &= \|x\|_2^q \int_{B_2^n} \left| \left\langle \frac{x}{\|x\|_2}, y \right\rangle \right|^q dy = \|x\|_2^q \int_{B_2^n} |\langle e_1, y\rangle|^q \, dy \\ &= 2\omega_{n-1} \|x\|_2^q \int_0^1 t^q (1-t^2)^{\frac{n-1}{2}} \, dt = \omega_{n-1} \|x\|_2^q \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+q+2}{2}\right)}. \end{aligned}$$

Using Stirling's approximation we then check the validity of (1.12). □

A crucial observation is that the function $\|\langle \cdot, \theta\rangle\|_q = \left(\int_K |\langle x, \theta\rangle|^q \, dx\right)^{1/q}$ that appears in (1.11) above was known to be a norm, and thus the support function of a convex body.

The centroid bodies and their geometry

The notion of L_q-centroid bodies was introduced, under a different normalization, by E. Lutwak and G. Zhang in [47], and studied by Lutwak, Yang and Zhang in [48]. Paouris was the first to exploit their properties from an asymptotic point of view, originally towards the proof of Theorem 1.21. Here we will review their definition and some basic facts that will be of use in our framework.

Definition 1.24 (L_q-centroid body)**.** Let K be a convex body in $\mathbb{R}^n$ with $\mathrm{vol}_n(K) = 1$, and let $q \geqslant 1$. We define the L_q-centroid body of K, denoted $Z_q(K)$, via its support function

$$h_{Z_q(K)}(\theta) := \|\langle \cdot, \theta\rangle\|_{L_q(K)} = \left(\int_K |\langle x, \theta\rangle|^q \, dx \right)^{1/q}, \quad \theta \in S^{n-1}.$$

For $q = +\infty$, we define $Z_\infty(K) := \mathrm{conv}\{K, -K\}$.

The contents of the following remark are easy for one to see, coming straight from the definition of $Z_q(K)$ and the properties of the support function (Lemma 1.2).

Remark 1.25. (a) The function $h_{Z_q(K)}$ as defined above is positively homogeneous and, by Minkowski's inequality, subadditive. Therefore there exists a unique convex body having $h_{Z_q(K)}$ as its support function.
(b) For any $q \geqslant 1$, $Z_q(K)$ is an origin symmetric convex body.
(c) A centered convex body K of volume 1 in $\mathbb{R}^n$ is isotropic if and only if $Z_2(K) = L_K B_2^n$.
(d) By the definition of $h_{Z_q(K)}$, note that Lemma 1.23 can now be restated as

$$w_q(Z_q(K)) \asymp \sqrt{\frac{q}{n}} I_q(K), \tag{1.13}$$

for every $q \geqslant 1$.

Next we collect several basic properties of the centroid bodies. The proofs are more or less standard, so we will not examine them in detail, but provide the appropriate references instead.

Lemma 1.26. *Let K be a convex body in $\mathbb{R}^n$ with $\mathrm{vol}_n(K) = 1$. Then:*

(a) *If K is isotropic, then $w(Z_2(K)) = L_K$.*

(b) *For all $1 \leqslant p < q \leqslant \infty$,*
$$Z_p(K) \subseteq Z_q(K) \subseteq c_1 \frac{q}{p} Z_p(K),$$
where $c_1 > 0$ is an absolute constant.

(c) *If K is centered, then*
$$Z_q(K) \supseteq c_2 Z_\infty(K),$$
for every $q \geqslant n$, where $c_2 > 0$ is an absolute constant.

Indication of proofs. Part (a) is immediate from the fact that $Z_2(K) = L_K B_2^n$. The inclusions in (b) are an immediate application of Corollary 1.9 for the seminorm $f(x) = |\langle x, \theta\rangle|$. As for part (c), it is due to the fact that $\|\langle \cdot, \theta\rangle\|_n \asymp \max\{h_K(\theta), h_K(-\theta)\}$, observed by Paouris in [53], so the rest of the claim follows also from Corollary 1.9. □

Remark 1.27. Lemma 1.26 (c) shows that the family $(Z_q(K))_{q\geqslant 1}$ "stabilizes" after $q = n$. In particular if K is symmetric, we have that $Z_q(K) \subseteq K$ for all $q \geqslant 1$, and $Z_q(K) \approx K$ for all $q \geqslant n$.

We close this section recording the known estimates on the volume of the L_q-centroid bodies. Although they are crucial for the subsequent results presented, proofs of these deep facts are hard to fit in these notes so we limit ourselves to providing exact references.

Theorem 1.28. *If K is an isotropic convex body in $\mathbb{R}^n$, then*

(a) *For every $1 \leqslant q \leqslant n$,*
$$\mathrm{vol}_n(Z_q(K))^{1/n} \gtrsim \sqrt{\frac{q}{n}}. \tag{1.14}$$

(b) *If $q \leqslant c\sqrt{n}$ for an appropriate absolute constant $c > 0$, the estimate of* (a) *above can be strengthened to*
$$\mathrm{vol}_n(Z_q(K))^{1/n} \gtrsim \sqrt{\frac{q}{n}} L_K. \tag{1.15}$$

(c) *On the other hand, the estimate*
$$\mathrm{vol}_n(Z_q(K))^{1/n} \lesssim \sqrt{\frac{q}{n}} L_K \tag{1.16}$$
holds for every $1 \leqslant q \leqslant n$.

The lower bound $\sqrt{q/n}$ on the volume radius of $Z_q(K)$ is due to Lutwak-Yang-Zhang [48]. Later, the work of Klartag-E. Milman [44] led to the stronger $\sqrt{q/n}L_K$, for $q \leqslant \sqrt{n}$. The upper bound (1.16) is due to Paouris [57] (see also [17, Theorem 5.1.17]).

As a first application of the above, we can provide a convenient estimate for the mean width of $Z_q(K)$, when q is up to the order of $\sqrt{n}$.

Lemma 1.29. *Let K be an isotropic convex body in $\mathbb{R}^n$. There is an absolute constant $c > 0$ such that, for every $1 \leqslant q \leqslant c\sqrt{n}$,*

$$w(Z_q(K)) \asymp \sqrt{q}L_K. \tag{1.17}$$

Proof. In view of 1.13, using Theorem 1.22 (recall that $I_q(K) \asymp I_2(K)$ anyway, if $1 \leqslant q \leqslant 2$), and ignoring the absolute constants involved, we have that

$$w_q(Z_q(K)) \asymp \sqrt{\frac{q}{n}}I_q(K) \asymp \sqrt{\frac{q}{n}}I_2(K) = \sqrt{n}L_K.$$

Now $w(Z_q(K)) \leqslant w_q(Z_q(K))$ holds, by Hölder's inequality. The lower bound can be justified immediately using the lower bound on the volume of $Z_q(K)$ (Theorem 1.28 (b)), and Urysohn's inequality:

$$w(Z_q(K)) \geqslant \left(\frac{\mathrm{vol}_n(Z_q(K))}{\omega_n}\right)^{1/n} \gtrsim \sqrt{n}\sqrt{\frac{q}{n}}L_K.$$

□

Negative moments and a small ball probability estimate

In the sequel, we will also need results on the behaviour of negative moments of $\|\cdot\|_2$, as well as mixed widths for $q < 0$. We have already explained that $I_q(K) \asymp \sqrt{n/q}w_q(Z_q(K))$ for any convex body K of volume 1 in $\mathbb{R}^n$ and $q \geqslant 1$ (recall Lemma 1.29). A similar identity also holds for negative values of q.

Theorem 1.30. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every $1 \leqslant q < n$,*

$$w_{-q}(Z_q(K)) \asymp \sqrt{\frac{q}{n}}I_{-q}(K). \tag{1.18}$$

The proof is slightly more involved than that of Lemma 1.23, so we chose to omit it and refer the reader to [17, Section 5.3.3]. It is also explained there how, using (1.18), one can deduce the following extension of Theorem 1.22.

Theorem 1.31. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then*

$$I_{-q}(K) \asymp I_q(K),$$

for every $1 \leqslant q \leqslant c\sqrt{n}$, where $c > 0$ is an absolute constant.

To see that Theorem 1.31 implies (1.10), note that $I_{-q}(K) \leqslant I_1(K)$, for any $q \geqslant 1$: Write

$$1 = \int_K \|x\|_2^{\frac{q}{q+1}} \|x\|_2^{-\frac{q}{q+1}}\, dx$$

and then apply Hölder's inequality for $p = (q+1)/q$.

Let us remark that while, as explained in the beginning of Section 1.3, Theorem 1.22 immediately implies the large deviation estimate (1.9), a direct corollary of the stronger statement Theorem 1.31 is a small-ball type inequality for isotropic convex bodies:

Theorem 1.32. *If K is isotropic in $\mathbb{R}^n$, then*

$$\mathrm{vol}_n(\{x \in K : \|x\|_2 \leqslant \varepsilon\sqrt{n}L_K\}) \leqslant \varepsilon^{c_2\sqrt{n}} \tag{1.19}$$

for every $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0, c_2 > 0$ are absolute constants.

Deduction of Theorem 1.32 from Theorem 1.31. Let $k \leqslant c\sqrt{n}$, where $c > 0$ is the constant in Theorem 1.32. Applying Theorem 1.32 and Markov's inequality we get that, for some absolute constant $c_1 > 0$,

$$\begin{aligned}\mathrm{vol}_n(\{x \in K : \|x\|_2 \leqslant \varepsilon I_2(K)\}) &\leqslant \mathrm{vol}_n(\{x \in K : \|x\|_2 \leqslant c_1\varepsilon I_{-k}(K)\}) \\ &= \mathrm{vol}_n(\{x \in K : \|x\|_2^{-k} \geqslant (c_1\varepsilon)^{-k} I_{-k}(K)^{-k}\}) \leqslant (c_1\varepsilon)^k.\end{aligned}$$

Since $(c_1\varepsilon)^k \leqslant \varepsilon^{k/2}$ if $0 < \varepsilon < \varepsilon_0 := c_1^{-2}$, the Theorem follows with $c_2 = c/2$. □

Theorem 1.32 is due to Paouris [58], where also Theorems 1.30 and 1.31 are proved, see also [17, Chapter 5.3].

1.4 Polytopes generated by random points in convex bodies

We finally introduce our central object of study in these notes. Generally speaking, by a *random polytope* in $\mathbb{R}^n$ we mean the convex hull of $N > n$ points which are chosen idependently according to a common probability distribution μ. Specific cases of interest might be the following:

- μ is taken to be the Gauss measure in $\mathbb{R}^n$, that is the rotationally invariant probability measure in $\mathbb{R}^n$ with density $f_n(x) = (2\pi)^{-n/2}e^{-\|x\|_2^2/2}$. If we choose $X_1, \dots, X_N$ points in $\mathbb{R}^n$ independently according to μ, the resulting polytope $\mathrm{conv}\{X_1, \dots, X_N\}$ is called a *Gaussian polytope* (and every X_i is a *gaussian random vector*, that is $X_i = (g_1^i, \dots, g_n^i)$ with $g_1^i, \dots, g_n^i$ independent standard gaussians).
- μ is taken to be the uniform measure on the discrete cube $\{-1, 1\}^n$, and we form the polytope $\mathrm{conv}\{X_1, \dots, X_N\}$ choosing $X_1, \dots, X_N$ independently according to μ. Then for every $i \in [N]$, X_i is a Bernoulli random vector i.e. $X_i = (x_1^i, \dots, x_n^i)$ with $x_1^i, \dots, x_n^i$ independent ± 1 Bernoulli random variables.
- We have explained how every convex body K in $\mathbb{R}^n$ induces a natural probability measure μ_K on $\mathbb{R}^n$. A μ_K-random polytope constructed as above will be our main object of interest in what follows. Obviously, in this instance the vertices $X_1, \dots, X_N$ are points chosen uniformly and independently in K.

Given a convex body K in $\mathbb{R}^n$, $N \geqslant n$ and N independent random points $X_1, \dots, X_N$ uniformly distributed in K, we denote

$$K_N := \mathrm{conv}\{\pm X_1, \dots, \pm X_N\}.$$

This is a symmetric random polytope with $2N$ vertices. Although most of these notes deals with this symmetric case, we might also refer to the non-symmetric one: If $N > n$, let

$$C_N := \mathrm{conv}\{X_1, \dots, X_N\}$$

be the non-symmetric analogue.

In what follows, we will get a glimpse on the study of such random polytopes in connection with the theory of isotropic convex bodies. In Section 2 we describe a technique which gives meaningful estimates on various geometric characteristics that determine what we call the "asymptotic shape" of a random polytope. In Section 3 we outline a general method, variations of which can provide absolute upper bounds on the isotropic constant of several classes of random polytopes.

2 Asymptotic shape of random polytopes

Determining the "asymptotic shape" of a random polytope P amounts to getting sharp estimates on various geometric parameters associated with P. Our goal in sections 2.1 and 2.2 is to exploit a method introduced by N. Dafnis, A. Giannopoulos and A. Tsolomitis [20] that leads to estimates on the mean width and volume radius, but also of the whole sequence of quermassintegrals of the symmetric random polytope K_N formed by vertices chosen uniformly at random from an isotropic convex body K. We can sum up the results in the following statement [3].

Theorem 2.1 (Dafnis-Giannopoulos-Tsolomitis [20], [21]). *Let $n, N \in \mathbb{N}$, and K be an isotropic convex body in $\mathbb{R}^n$.*

(a) *If $n \lesssim N \leqslant e^{\sqrt{n}}$, then*

$$\mathrm{vol}_n(K_N)^{1/n} \gtrsim \sqrt{\frac{\log(2N/n)}{n}} L_K$$

with probability greater than $1 - \exp(-c\sqrt{N})$ for some absolute constant $c > 0$.
On the other hand, for every $n^2 \lesssim N \leqslant e^n$,

$$\mathrm{vol}_n(K_N)^{1/n} \lesssim \sqrt{\frac{\log(2N/n)}{n}} L_K,$$

with probability greater than $1 - \frac{1}{N^2}$.

(b) *If $n \lesssim N \leqslant e^{\sqrt{n}}$, then*

$$\sqrt{\log(2N/n)} L_K \lesssim \mathbb{E}(w(K_N)) \lesssim \sqrt{\log N} L_K.$$

(c) *If $n \lesssim N \leqslant e^{\sqrt{n}}$, then for every $1 \leqslant k \leqslant n$ we have*

$$\sqrt{\log(2N/n)} L_K \lesssim \mathbb{E}(Q_k(K_N)) \lesssim \sqrt{\log N} L_K.$$

Roughly speaking, the central idea in [20] is to try to compare K_N to the class of the L_q-centroid bodies of K, whose geometry was already well studied in conjuction with the distribution of volume in convex bodies and the isotropic constant conjecture. The machinery developed in section 1.3 will then lead to the estimates of Theorem 2.1. This idea has its origins in the earlier study [26] on random polytopes generated by vertices of the discrete cube $\{-1, 1\}^n$, as explained below.

In section 2.3 we exploit a variation of the method of [20] to attack a different problem on the geometry of random polytopes in convex bodies. That is wellness of approximation, i.e. how well does K_N, or its non-symmetric analogue C_N, "fit" inside K. This is actually an old problem, studied in many aspects, see e.g. the surveys [18], [32]. In the case that one is interested in approximation of K by a polytope with "few" (that is $O(n)$) number of vertices, the following result was obtained in [11]: if K is an origin symmetric convex body in $\mathbb{R}^n$, then for any $d > 1$ there exist $N \leqslant dn$ points $x_1, \ldots, x_N \in K$ such that

$$K \subseteq \gamma_d \sqrt{n}\, \mathrm{conv}\{\pm x_1, \ldots, \pm x_N\},$$

where $\gamma_d := \frac{\sqrt{d}+1}{\sqrt{d}-1}$. Later, in [15] a non-symmetric analogue of this statement was proved, namely that there exist $x_1, \ldots, x_N \in K$, $N \leqslant an$, such that $K \subseteq cn^{3/2}\, \mathrm{conv}\{x_1, \ldots, x_N\}$, for some absolute constants $a, c > 0$. This motivated us to study the random aspect of this instant: can one, for linear number of points N chosen independently and uniformly in K, get a better dependence on n with high probability? The following result is proved in [16].

[3] In the statement of the Theorem, we assume N to be greater than a constant multiple of n. Nevertheless, similar bounds also hold in in the linear regime $n \leqslant N \leqslant an$, see [29], [59]

Theorem 2.2. *There exists an absolute constant $a > 1$ with the following property: If K is a centered convex body in $\mathbb{R}^n$ and $N = \lceil an \rceil$, then with probability greater than $1 - e^{-c_1 n}$ we have*

$$K \subseteq c_2 n C_N,$$

where $c_1, c_2 > 0$ are absolute constants.

In section 2.3 we present the proof of the above theorem (actually a more general result is obtained), adjusting the argument of [20] to the non-symmetric setting of our problem.

2.1 Comparison with the centroid bodies

The idea of comparing the random polytope K_N to the class of L_q-centroid bodies for suitable values of $q \geqslant 1$, has its origins in the study of Giannopoulos and Hartzoulaki [26] on the behaviour of symmetric random ± 1-polytopes. These are polytopes

$$K_{n,N}^{\pm 1} = \mathrm{conv}\{\pm X_1, \ldots, \pm X_N\}$$

formed by $N > n$ independent random points $X_1, \ldots, X_N$ chosen according to the uniform probability measure on $\{-1, 1\}^n$. It was proved in [26] that for every $n \geqslant n_0$ and $N \geqslant n(\log n)^2$, the inclusion

$$K_{n,N}^{\pm 1} \supseteq c\left(\sqrt{\log(N/n)}B_2^n \cap B_\infty^n\right) \tag{2.1}$$

holds with probability greater than $1 - e^{-n}$, where $c > 0$ is an absolute constant. By John's theorem $aB_2^n \cap B_\infty^n \supseteq \frac{a}{\sqrt{n}}B_\infty^n$, so (2.1) states that, with high probability, $K_{n,N}^{\pm 1}$ contains a cube with edge length $\sqrt{\log(N/n)/n}$.

The result of [26] was then improved, and the assumptions relaxed, in a work of Litvak, Pajor, Rudelson and Tomczak-Jaegermann [46] who obtained an inclusion similar to (2.1) for a large class of random polytopes K_N that includes the previous Bernoulli model, and the Gaussian (and sub-gaussian) model as well: The polytopes considered in [46] arise as absolute convex hulls of the rows of the random matrix $\Gamma_{n,N} = (\xi_{ij})_{(i,j)\in[N]\times[n]}$, with certain mild assumptions imposed on the random variables ξ_{ij}. The proofs in [46] rely on a lower bound of the order of $\sqrt{N}$ for the smallest singular value of $\Gamma_{n,N}$.

A key observation, first stated explicitly in [20], is that (2.1) can be rewritten in terms of the L_q-centroid bodies of the body $\frac{1}{2}B_\infty^n$. To be more specific, one starts from the fact that for any $a \geqslant 1$,

$$h_{aB_2^n \cap B_\infty^n}(\theta) = K_{1,2}(\theta, a)$$

for every $\theta \in S^{n-1}$, where

$$K_{1,2}(x,t) = \inf\{\|y\|_1 + t\|x - y\|_2 : y \in \mathbb{R}^n\}$$

(to see this, recall that $h_{aB_2^n \cap B_\infty^n} = \|\cdot\|_{a^{-1}B_2^n \cup B_1^n}$). If we write $(x_j^*)_{j\leqslant n}$ for the decreasing rearrangement of $(|x_j|)_{j\leqslant n}$, then Holmsted's approximation formula (see [36, Theorem 4.1]) yields

$$K_{1,2}(x,t) \asymp \sum_{j=1}^{\lceil t^2\rceil} x_j^* + t\left(\sum_{j=\lceil t^2\rceil+1}^{n} (x_j^*)^2\right)^{1/2}.$$

On the other hand, one also has

$$\|\langle\cdot,\theta\rangle\|_{L_q(\frac{1}{2}B_\infty^n)} \asymp \sum_{j\leqslant q} \theta_j^* + \sqrt{q}\left(\sum_{j=q+1}^{n} (\theta_j^*)^2\right)^{1/2}$$

for every $q \geqslant 1$ and $\theta \in S^{n-1}$ (see [10, Lemma 6, Proposition 7]). It is thus the case that $h_{\sqrt{q}B_2^n \cap B_\infty^n} \asymp h_{Z_q(\frac{1}{2}B_\infty^n)}$, which shows that

$$\sqrt{q}B_2^n \cap B_\infty^n \approx Z_q\left(\frac{1}{2}B_\infty^n\right).$$

In view of all this, (2.1) essentially states that

$$K_{n,N}^{\pm 1} \supseteq c_1 Z_{\log(N/n)}\left(\frac{1}{2}B_\infty^n\right).$$

The above observation gave rise to the idea of comparing a random polytope K_N inside a convex body K to the body $Z_{\log(N/n)}(K)$. This was carried out succesfully in [20]. We will review the proof of this central theorem which follows a modification of an idea of [46].

Theorem 2.3 ([20])**.** *Let $\beta \in (0, \frac{1}{2}]$, $\gamma > 1$. If $N \geqslant c\gamma n$ where $c > 0$ is an absolute constant, then for every isotropic convex body K in $\mathbb{R}^n$ we have that*

$$K_N \supseteq c_1 Z_q(K)$$

for all $q \leqslant c_2 \beta \log(N/n)$, with probability greater than $1 - \exp(-c_0\gamma\sqrt{N})$.

Proof. We will prove the desired inclusion via the support function: We actually need to prove that

$$h_{K_N}(z) \geqslant c_1 \|\langle \cdot, z\rangle\|_q$$

holds for all $z \in \mathbb{R}^n$ and for as large a q as possible, with probability greater than $1 - \exp(-f(n,\gamma))$, where f is a suitable function of $N \geqslant \gamma n$ and n.

Let $\Gamma : \ell_2^n \to \ell_2^N$ be the random operator defined by

$$\Gamma(y) = (\langle X_1, y\rangle, \ldots, \langle X_N, y\rangle),$$

and define $m := \lfloor 8(N/n)^{2\beta} \rfloor$ and $k := \lfloor N/m \rfloor$. We can then fix a partition $\sigma_1, \ldots, \sigma_k$ of $[N]$ with $m \leqslant |\sigma_i|$ for all $i \in [k]$, and define the norm

$$\|u\|_0 := \frac{1}{k}\sum_{i=1}^{k} \|P_{\sigma_i}(u)\|_\infty, \quad u \in \mathbb{R}^N.$$

Since

$$h_{K_N}(z) = \max_{1 \leqslant j \leqslant N} |\langle X_j, z\rangle| \geqslant \max_{j \in \sigma_i} |\langle X_j, z\rangle| = \|P_{\sigma_i}\Gamma(z)\|_\infty$$

for all $z \in S^{n-1}$ and $i \in [k]$, we have $h_{K_N}(z) \geqslant \|\Gamma(z)\|_0$ for all $z \in S^{n-1}$. We thus need to prove that an estimate of the form $\|\Gamma(z)\|_0 \geqslant c\|\langle \cdot, z\rangle\|_q$ holds with high probability, for all $z \in S^{n-1}$. We will first estimate this probability for a fixed $z \in S^{n-1}$, and then use a net argument to pass from a finite collection of points to the whole space.

Let $z \in S^{n-1}$ and suppose that $\|\Gamma(z)\|_0 < \frac{1}{4}\|\langle \cdot, z\rangle\|_q$. Since $\|\cdot\|_0$ is the expected value of k random variables, by Markov's inequality we get that there exists $I \subseteq [k]$ with $|I| \geqslant k/2$ such that $\|P_{\sigma_i}\Gamma(z)\|_\infty < \frac{1}{2}\|\langle \cdot, z\rangle\|_q$ for all $i \in I$. Using a union bound and independence we get

$$\begin{aligned} \mathbb{P}\left(\|\Gamma(z)\|_0 < \frac{1}{4}\|\langle \cdot, z\rangle\|_q\right) &\leqslant \sum_{\substack{I \subseteq [k] \\ |I| = \lfloor (k+1)/2 \rfloor}} \mathbb{P}\left(\|P_{\sigma_i}\Gamma(z)\|_\infty < \frac{1}{2}\|\langle \cdot, z\rangle\|_q, \forall i \in I\right) \\ &= \sum_{\substack{I \subseteq [k] \\ |I| = \lfloor (k+1)/2 \rfloor}} \prod_{i \in I} \mathbb{P}\left(\|P_{\sigma_i}\Gamma(z)\|_\infty < \frac{1}{2}\|\langle \cdot, z\rangle\|_q\right). \end{aligned} \tag{2.2}$$

To estimate the latter probability, we use the next lemma, which is a consequence of the Paley-Zygmund inequality.

Lemma 2.4. *For every $\sigma \subseteq [N]$, independent random vectors $X_1, \dots, X_N$ in $\mathbb{R}^n$ and any $\theta \in S^{n-1}$, $q \geqslant 1$,*

$$\mathbb{P}\left(\max_{j\in\sigma} |\langle X_j, \theta\rangle| \leqslant \frac{1}{2}\|\langle\cdot,\theta\rangle\|_q\right) \leqslant \exp(-|\sigma|/(4C^q)),$$

where $C > 0$ is an absolute constant.

Proof. We will first treat each X_j independently. For this we recall the Paley-Zygmund inequality: If $Z \geqslant 0$ is a random variable and $t \in [0,1]$, then

$$\mathbb{P}\left(Z \geqslant t\mathbb{E}(Z)\right) \geqslant (1-t)^2 \frac{\mathbb{E}(Z)^2}{\mathbb{E}(Z^2)}. \tag{2.3}$$

Apply this inequality for $t = 1/2^q$, $Z = |\langle X, \theta\rangle|^q$, where X is a random vector in $\mathbb{R}^n$, and then use the fact that, by the reverse Hölder's inequality (Corollary 1.9) for $f := |\langle\cdot,\theta\rangle|$,

$$\mathbb{E}(|\langle X,\theta\rangle|^{2q}) = \|\langle\cdot,\theta\rangle\|_{2q}^{2q} \leqslant \left(c\frac{2q}{q}\|\langle\cdot,\theta\rangle\|_q\right)^{2q} = C^q(\|\langle\cdot,\theta\rangle\|_q^q)^2 = C^q(\mathbb{E}|\langle X,\theta\rangle|^q)^2,$$

to get

$$\begin{aligned}\mathbb{P}\left(|\langle X,\theta\rangle| \geqslant \frac{1}{2}\|\langle\cdot,\theta\rangle\|_q\right) &= \mathbb{P}\left(|\langle X,\theta\rangle|^q \geqslant \frac{1}{2^q}\mathbb{E}(|\langle X,\theta\rangle|^q)\right)\\ &\geqslant \left(1-\frac{1}{2^q}\right)^2 \frac{(\mathbb{E}|\langle X,\theta\rangle|^q)^2}{\mathbb{E}(|\langle X,\theta\rangle|^{2q})}\\ &\geqslant \frac{1}{4C^q},\end{aligned} \tag{2.4}$$

since $q \geqslant 1$. To finish the proof, we use the independence of the X_j's and the estimate above:

$$\begin{aligned}\mathbb{P}\left(\max_{j\in\sigma}|\langle X_j,\theta\rangle| \leqslant \frac{1}{2}\|\langle\cdot,\theta\rangle\|_q\right) &= \prod_{j\in\sigma}\mathbb{P}\left(|\langle X_j,\theta\rangle| \leqslant \frac{1}{2}\|\langle\cdot,\theta\rangle\|_q\right)\\ &\leqslant \left(1-\frac{1}{4C^q}\right)^{|\sigma|}\\ &\leqslant \exp(-|\sigma|/(4C^q)),\end{aligned}$$

by the inequality $1 - v \leqslant e^{-v}$ for every $v > 0$. □

Back to the proof of the Theorem, in view of Lemma 2.4, (2.2) states that

$$\begin{aligned}\mathbb{P}\left(\|\Gamma(z)\|_0 < \frac{1}{4}\|\langle\cdot,z\rangle\|_q\right) &\leqslant \binom{k}{\lfloor(k+1)/2\rfloor}\exp(-c_1km/C^q)\\ &\leqslant \exp(k\log 2 - c_1km/C^q),\end{aligned} \tag{2.5}$$

where $c_1 = 1/8$ (and we have used the trivial bound $\binom{k}{l} \leqslant 2^k$ for $l \leqslant k$). The choice

$$q = \frac{\beta}{\log C}\log(N/n)$$

will then give us

$$\mathbb{P}\left(\|\Gamma(z)\|_0 < \frac{1}{4}\|\langle\cdot,z\rangle\|_q\right) \leqslant \exp(-c_2N^{1-\beta}n^\beta).$$

Now comes the net argument: We let $S = \{z \in S^{n-1} : \|\langle \cdot, z\rangle\|_q/2 = 1\}$, and consider a δ-net $\mathcal{N}_\delta \subset S$ with respect to $\|\langle \cdot, z\rangle\|_q/2$, that is, for every $z \in S$ there exists some $u \in \mathcal{N}_\delta$ such that $\frac{1}{2}\|\langle \cdot, z-u\rangle\|_q < \delta$. We can assume that $|\mathcal{N}_\delta| \leqslant (3/\delta)^n$. By the discussion above, it is clear that for every $u \in \mathcal{N}_\delta$ we have

$$\mathbb{P}\left(\|\Gamma(u)\|_0 < \frac{1}{2}\right) \leqslant \exp(-c_2 N^{1-\beta} n^\beta)$$

and hence,

$$\mathbb{P}\left(\bigcup_{u\in\mathcal{N}_\delta} \|\Gamma(u)\|_0 < \frac{1}{2}\right) \leqslant \exp(n\log(3/\delta) - c_2 N^{1-\beta} n^\beta). \tag{2.6}$$

For $\gamma > 1$ we define

$$\Omega_\gamma = \{\Gamma : \|\Gamma : \ell_2^n \to \ell_2^N\| \leqslant \gamma L_K \sqrt{N}\}.$$

Since $Z_q(K) \supseteq L_K B_2^n$ for all $q \geqslant 2$ (Lemma 1.26 (b)), we have, for all $z \in \mathbb{R}^n$ and every $\Gamma \in \Omega_\gamma$,

$$\|\Gamma(z)\|_0 \leqslant \frac{1}{\sqrt{k}}\|\Gamma(z)\|_2 \leqslant \gamma L_K \sqrt{N/k}\|z\|_2 \leqslant \gamma\sqrt{N/k}\|\langle\cdot, z\rangle\|_q.$$

Let $z \in S$. There exists $u \in \mathcal{N}_\delta$ such that $\frac{1}{2}\|\langle \cdot, z-u\rangle\|_q < \delta$, which implies that

$$\|\Gamma(u)\|_0 \leqslant \|\Gamma(z)\|_0 + 2\gamma\delta\sqrt{N/k}$$

for every $\Gamma \in \Omega_\gamma$. The choice $\delta = \sqrt{k/N}/(8\gamma)$ yields

$$\begin{aligned} \mathbb{P}\left(\exists z \in S^{n-1} : \|\Gamma(z)\|_0 \leqslant \|\langle\cdot, z\rangle\|_q/8\right) &= \mathbb{P}\left(\exists z \in S : \|\Gamma(z)\|_0 \leqslant 1/4\right) \\ &\leqslant \mathbb{P}\left(\exists u \in \mathcal{N}_\delta : \|\Gamma(u)\|_0 \leqslant 1/2\right) \\ &\overset{(2.6)}{\leqslant} \exp\left(n\log(24\gamma\sqrt{N/k}) - c_2 N^{1-\beta} n^\beta\right). \end{aligned} \tag{2.7}$$

We now impose the restriction on N to assure that the last estimate is at most $\exp(-c_3 N^{1-\beta} n^\beta)$: First recall the definitions of k and m, and note that, since $\beta \leqslant 1/2$, it is sufficient that

$$\log\left(c\gamma\left(\frac{N}{n}\right)^\beta\right) \leqslant \frac{c_2}{2}\sqrt{\frac{N}{n}},$$

for some absolute constant $c > 0$. Now if we take $N \geqslant c_4\gamma n$, the desired bound is achieved if

$$\log\left(c'\left(\frac{N}{n}\right)^{1+\beta}\right) \leqslant \frac{c_2}{2}\sqrt{\frac{N}{n}},$$

where $c' = c/c_4$. Using again the fact that $\beta \leqslant 1/2$, we can see that the left hand side above is always smaller than a constant multiple of $\log(N/n)$. It is thus sufficient to have

$$\log\left(\frac{N}{n}\right) \lesssim \sqrt{\frac{N}{n}},$$

which in turn is achieved if $N \geqslant c_5 n$ for a suitable absolute constant $c_5 > 0$. Taking $N \geqslant \max\{c_4\gamma, c_5\}n$ ensures then that the desired probability is less than $\exp(-c_3 N^{1-\beta} n^\beta)$.

Summing up all of the above, and remembering that $h_{K_N}(z) \geqslant \|\Gamma(z)\|_0$ for all $z \in \mathbb{R}^n$, we have that, for sufficiently large N, the probability that $K_N \supseteq cZ_q(K)$ is greater than

$$1 - \exp(-c_3 N^{1-\beta} n^\beta) - \mathbb{P}(\Omega_\gamma^c).$$

The last step is to estimate $\mathbb{P}(\Omega_\gamma)$. In [20], this was done using results of Mendelson and Pajor [49] and Guédon and Rudelson [34]. One can achieve a better bound using a result of Adamczak-Litvak-Pajor and Tomczak Jaegermann [1] (see also [17, Corollary 10.1.6]), which gives, as a special case, that if $N \geqslant \gamma n$ then

$$\mathbb{P}\left(\|\Gamma : \ell_2^n \to \ell_2^N\| > c\gamma\sqrt{N}L_K\right) \leqslant \exp(-c\gamma\sqrt{N}),$$

for some absolute constant $c > 0$. We thus have that $\mathbb{P}(\Omega_\gamma^c) < e^{-c\gamma\sqrt{N}}$ for every $N \geqslant \gamma n$, and given that $\beta \leqslant 1/2$, we get the assertion of the Theorem. □

Remark 2.5. Regarding the sharpness of Theorem 2.3, actually we cannot expect an inclusion of the form

$$K_N \subseteq CZ_q(K)$$

for some absolute constant $C > 0$ with probability close to 1, unless we choose $q \asymp n$. This is because

$$\begin{aligned}\mathbb{P}\left(K_N \subseteq CZ_q(K)\right) &= \mathbb{P}\left(X_1, \dots, X_N \in CZ_q(K)\right) \\ &= \mathbb{P}\left(X_1 \in CZ_q(K)\right)^N = \mathrm{vol}_n(CZ_q(K))^N \leqslant \left(C'\sqrt{\frac{q}{n}}L_K\right)^{nN},\end{aligned}$$

by independence and 1.28 (c). So, if we assume that L_K is bounded, q has to be of the order of n for the right hand side to remain close to 1. Nevertheless, in the next section we will see that a weaker type of "reverse inclusion" holds, that can still yield upper bounds on the expected mean width of K_N, as well as $\mathrm{vol}_n(K_N)^{1/n}$, with high probability.

Proofs of the lower bounds

The lower bounds in Theorem 2.1 (a), (b) and (c) are now easy to prove: The bound on the volume radius is immediate, and then Urysohn's inequality will lower bound $w(K_N)$.

Proof of the lower bounds in Theorem 2.1. Simply choose $q \asymp \log(N/n)$ in Theorem 2.3 to get that $K_N \supset cZ_q(K)$ with probability greater than $1 - \exp(-c_1\sqrt{N})$. It is then the case that

$$\mathrm{vol}_n(K_N)^{1/n} \gtrsim \mathrm{vol}_n(Z_q(K))^{1/n} \asymp \mathrm{vol}_n(Z_{\log\frac{N}{n}}(K))^{1/n} \asymp \sqrt{\frac{\log(N/n)}{n}}L_K,$$

if $N \leqslant e^{\sqrt{n}}$, where we have used Lemma 1.26 (b) and (1.15). In the regime $e^{\sqrt{n}} \leqslant N \leqslant e^n$, we can only apply the weaker bound $\mathrm{vol}_n(Z_q(K))^{1/n} \gtrsim \sqrt{\frac{q}{n}}$ of (1.14), that leads to the weaker estimate $\mathrm{vol}_n(K_N)^{1/n} \gtrsim \sqrt{\frac{\log(2N/n)}{n}}$.

The lower estimate on $\mathbb{E}w((K_N))$ is now straight from Urysohn's inequality (Theorem 1.6). Denote $I := \{K_N \supseteq cZ_q(K)\}$ the "good" event. Then $\mathbb{P}(I) \geqslant 1 - 1/e$ for every $n \in \mathbb{N}$, and by the fact that $\omega_n^{1/n} \asymp 1/\sqrt{n}$ we get

$$\mathbb{E}\left(w(K_N)\right) \geqslant \mathbb{E}\left(\left(\frac{\mathrm{vol}_n(K_N)}{\mathrm{vol}_n(B_2^n)}\right)^{1/n}\right) \gtrsim \sqrt{n}\mathbb{E}\left(\mathrm{vol}_n(K_N)^{1/n}\right) \gtrsim \sqrt{n}\mathbb{P}(I)\sqrt{\frac{\log(N/n)}{n}}L_K \gtrsim \sqrt{\log\frac{N}{n}}L_K,$$

as in the statement of the Theorem.

Regarding $Q_k(K_N)$, note that, by Lemma 1.4,

$$\mathbb{E}(Q_k(K_N)) \geqslant \mathbb{E}(Q_n(K_N)) = \mathbb{E}\left(\frac{\mathrm{vol}_n(K)}{\omega_n}\right)^{1/n},$$

so the lower bound follows again by (a). Of course, this gives an alternate way to justify the estimate for the mean width, since $Q_1(K_N) = w(K_N)$. □

2.2 Weak reverse inclusion

As we have pointed out in Remark 2.5, one cannot hope that $K_N \subseteq cZ_q(K)$ with high probability for any meaningful value of q. What we can still prove however is that, for any $q \geqslant 1$ and $c > 1$, there are, on average, only a "few" directions $\theta \in S^{n-1}$ on which K_N "exceeds" $cZ_q(K)$. We can state the result more accurately as follows.

Lemma 2.6. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. For any $q \geqslant 1$ and $a > 1$ one has*

$$\mathbb{E}\left(\sigma(\{\theta \in S^{n-1} : h_{K_N}(\theta) \geqslant a h_{Z_q(K)}(\theta)\})\right) \leqslant N a^{-q}.$$

Proof. Let $\theta \in S^{n-1}$. If X is a random vector uniformly distributed in K, then by Markov's inequality,

$$\mathbb{P}(|\langle X, \theta\rangle| \geqslant a\|\langle \cdot, \theta\rangle\|_q) \leqslant a^{-q}.$$

A union bound gives then

$$\mathbb{P}(h_{K_N}(\theta) \geqslant a h_{Z_q(K)}(\theta)) = \mathbb{P}\left(\max_{j\in[N]} |\langle X_j, \theta\rangle| \geqslant a\|\langle \cdot, \theta\rangle\|_q\right) \leqslant N a^{-q}.$$

Note finally that

$$\mathbb{E}\left(\sigma(\{\theta \in S^{n-1} : h_{K_N}(\theta) \geqslant a h_{Z_q(K)}(\theta)\})\right) = \int_{S^{n-1}} \mathbb{P}(h_{K_N}(\theta) \geqslant a h_{Z_q(K)}(\theta))\, d\sigma(\theta),$$

by Fubini's theorem □

Upper bound for the average mean width and quermassintegrals

We will see how the "weak inclusion" of Lemma 2.6 yields the $\sqrt{\log N} L_K$ bound in Theorem 2.1 (b). Towards the proof, we first stress out how Lemma 2.6 provides a bound on the average mean width of K_N in terms of the mean width of $Z_{\log N}(K)$.

Proposition 2.7. *Let $n \leqslant N \leqslant e^n$ and K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\mathbb{E}(w(K_N)) \leqslant c w(Z_{\log N}(K))$$

for some absolute constant $c > 0$.

Proof. Let $q \geqslant 1$ and define

$$A_N := \{\theta \in S^{n-1} : h_{K_N}(\theta) \leqslant e h_{Z_q(K)}(\theta)\}.$$

Using the simple fact that $h_K(\theta) \leqslant R(K)$ for every $\theta \in S^{n-1}$, Lemma 1.17 and the definition of A_N, we write

$$\begin{aligned} w(K_N) &\leqslant \int_{A_N} h_{K_N}(\theta)\, d\sigma(\theta) + \sigma(A_N^c) R(K) \\ &\leqslant e \int_{A_N} h_{Z_q(K)}(\theta)\, d\sigma(\theta) + c\sigma(A_N^c) n L_K \\ &\leqslant e w(Z_q(K)) + c\sigma(A_N^c) n L_K. \end{aligned}$$

Now taking expectation we can use Lemma 2.6 to bound $\mathbb{E}(\sigma(A_N^c))$, and using $w(Z_q(K)) \geqslant w(Z_2(K)) = L_K$ which is valid for any $q \geqslant 2$ (Lemma 1.26) we get

$$\begin{aligned} \mathbb{E}(w(K_N)) &\leqslant e w(Z_q(K)) + cNe^{-q} n L_K \\ &\leqslant (e + cNe^{-q} n) w(Z_q(K)). \end{aligned}$$

Now any choice $q \geqslant 2\log N$ makes the factor $(e + cNe^{-q}n)$ independent of N and n (since $n \leqslant N$), so one can choose $q = 2\log N$ and then the fact that $Z_{2\log N}(K) \subseteq c_1 Z_{\log N}(K)$ for some absolute constant $c_1 > 0$ to get the required statement. □

Proof of the upper bounds in Theorem 2.1 (b), (c). The statement of the Theorem is an immediate consequence of Proposition 2.7 and (1.17), which imposes the constraint $N \leqslant e^{\sqrt{n}}$. For the quermassintegrals, note that $Q_k(K_N) \leqslant Q_1(K_N) = w(K_N)$ for every $1 \leqslant k \leqslant n$. □

Remark 2.8 (Further reading)**.** (a) A different proof of $\mathbb{E}(w(K_N)) \asymp \sqrt{\log N} L_K$ in the low regime $n \leqslant N \leqslant n^2$, but not for all the quermassintegrals, was given later by Alonso-Gutiérrez and Prochno [7].
(b) In [21], the authors also proved that the asymptotic formula $Q_k(K_N) \asymp \sqrt{\log N} L_K$ for the range $n^2 \leqslant N \leqslant e^{\sqrt{n}}$ holds with high probability, and obtained estimates on the regularity of covering numbers and the volume of random projections of K_N. These results were later (almost) extended to the regime $e^{\sqrt{n}} \leqslant N \leqslant e^n$ in [28]. This was possible due to the recent result of E. Milman [50] on the mean width of the centroid bodies: We now know that $w(Z_q(K)) \lesssim \sqrt{q}\log(1+q)^2 L_K$, for all $q \in [\sqrt{n}, n]$ (recall that on average, $Q_k(K_N)$ is controlled by $w(Z_{\log N}(K))$).

Upper bound for the volume radius

We now turn to proving upper bounds for the volume radius of K_N. A bound of the order of $\sqrt{(\log N)/n} L_K$ for $\mathbb{E}(\mathrm{vol}_n(K)^{1/n})$ in the regime $n \leqslant N \leqslant e^{\sqrt{n}}$ is immediate by the upper bound in Theorem 2.1 (b), and Urysohn's inequality.

Proposition 2.9. *Let K be an isotropic convex body in $\mathbb{R}^n$, and $n \leqslant N \leqslant e^{\sqrt{n}}$. Then*

$$\mathbb{E}(\mathrm{vol}_n(K_N)^{1/n}) \leqslant c\sqrt{\frac{\log N}{n}} L_K,$$

where $c > 0$ is an absolute constant.

However it is the case that one can prove much more: Lemma 2.6 can provide a bound on $\mathrm{vol}_n(K_N)^{1/n}$ of the same order but with high probability, for every $N \leqslant e^n$. The exact statement is the following.

Theorem 2.10. *For every K isotropic body in $\mathbb{R}^n$ and $n^2 \leqslant N \leqslant e^n$, one has*

$$\mathrm{vol}_n(K_N)^{1/n} \leqslant c\sqrt{\frac{\log(2N/n)}{n}} L_K,$$

with probability greater than $1 - \frac{1}{4N^2}$, where $c > 0$ is an absolute constant.

For the proof of this fact, we will need the estimate of Paouris [58] on the negative moments of h_K (Theorem 1.30). We first state a general lemma that upper bounds the volume radius of a convex body in terms of the negative moments $w_{-q}(K)$.

Lemma 2.11. *For any symmetric convex body K in $\mathbb{R}^n$ and any $1 \leqslant q \leqslant n$,*

$$\mathrm{vol}_n(K)^{1/n} \leqslant c_1 \frac{w_{-q}(K)}{\sqrt{n}}$$

for some absolute constant $c_1 > 0$.

Proof. For any convex body L in $\mathbb{R}^n$, using integration in polar coordinates and Fubini we have

$$\mathrm{vol}_n(L) = \int_{\mathbb{R}^n} \mathbb{1}_L(x)\,dx = n\omega_n \int_0^\infty \int_{S^{n-1}} \mathbb{1}_L(t\theta) t^{n-1}\,d\sigma(\theta)\,dt = n\omega_n \int_{S^{n-1}} \int_0^\infty \mathbb{1}_L(t\theta) t^{n-1}\,dt\,d\sigma(\theta).$$

Next note that, for every $\theta \in S^{n-1}$, $t\theta \in L$ if and only if $t \leqslant \|\theta\|_L^{-1}$. This results in

$$\mathrm{vol}_n(L) = n\omega_n \int_{S^{n-1}} \int_0^{\|\theta\|_L^{-1}} t^{n-1}\,dt\,d\sigma(\theta) = \omega_n \int_{S^{n-1}} \|\theta\|_L^{-n}\,d\sigma(\theta).$$

Now if we take $L = K^\circ$ (remember that $\|\cdot\|_{K^\circ} = h_K$), the last identity and Hölder's inequality yield

$$\left(\frac{\mathrm{vol}_n(K^\circ)}{\omega_n}\right)^{1/n} = \left(\int_{S^{n-1}} h_K(\theta)^{-n}\, d\sigma(\theta)\right)^{1/n} \geqslant \left(\int_{S^{n-1}} h_K(\theta)^{-q}\, d\sigma(\theta)\right)^{1/q} = \frac{1}{w_{-q}(K)}.$$

Apply the Blaschke-Santaló inequality and the fact that $\omega_n^{1/n} \asymp 1/\sqrt{n}$ to get

$$\mathrm{vol}_n(K)^{1/n} \leqslant \omega_n^{2/n}\mathrm{vol}_n(K^\circ)^{-1/n} \leqslant \omega_n^{1/n} w_{-q}(K) \leqslant c_1 \frac{w_{-q}(K)}{\sqrt{n}},$$

for some absolute constant $c_1 > 0$. □

Now the next step is to connect $w_{-q}(K)$ to $w_{-q}(Z_q(K))$ for the appropriate q, that is, of the order $\log N$. This is where Lemma 2.6 comes in handy.

Lemma 2.12. *Let K be an isotropic convex body in $\mathbb{R}^n$, and $n \leqslant N \leqslant e^n$. Then*

$$w_{-2\log(2N)}(K_N) \lesssim w_{-\log(2N)}(Z_{2\log(2N)}(K)),$$

with probability greater than $1 - \frac{1}{4N^2}$.

Proof. We start using Hölder's inequality, to write

$$\begin{aligned}(w_{-q/2}(Z_q(K)))^{-q} &= \left(\int_{S^{n-1}} \frac{1}{h_{Z_q(K)}(\theta)^{q/2}}\, d\sigma(\theta)\right)^2 \\ &\leqslant \left(\int_{S^{n-1}} \frac{1}{h_{K_N}(\theta)^q}\, d\sigma(\theta)\right)\left(\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta)\right). \qquad (2.8)\end{aligned}$$

Since $K_N \subseteq c_1 n L_K B_2^n$ (recall Lemma 1.17) and $Z_q(K) \supseteq Z_2(K) = L_K B_2^n$, we have $h_{K_N}(\theta) \leqslant c_1 n h_{Z_q(K)}(\theta)$ for every $\theta \in S^{n-1}$. Therefore

$$\int_{S^{n-1}} \left(\frac{h_{K_N}(\theta)}{h_{Z_q(K)}(\theta)}\right)^q d\sigma(\theta) = \int_0^{c_1 n} q t^{q-1} \sigma\left(\{\theta \in S^{n-1} : h_{K_N}(\theta) \geqslant t h_{Z_q(K)}(\theta)\}\right) dt.$$

Taking expectations and using Lemma 2.6 we get, for every $\alpha > 1$,

$$\mathbb{E}\left(\int_{S^{n-1}} \frac{h_{K_N}(\theta)^q}{h_{Z_q(K)}(\theta)^q}\, d\sigma(\theta)\right) \leqslant \alpha^q + \int_\alpha^{c_1 n} q t^{q-1} N t^{-q}\, dt = \alpha^q + qN\log\left(\frac{c_1 n}{\alpha}\right).$$

Note that the choice $q := 2\log(2N)$ implies $e^q = (2N)^2 \gtrsim qN\log\left(\frac{c_1 n}{2e}\right)$, so applying the above for $\alpha = 2e$ implies

$$\mathbb{E}\left(\int_{S^{n-1}} \frac{h_{K_N}(\theta)^{2\log(2N)}}{h_{Z_{2\log(2N)}(K)}(\theta)^{2\log(2N)}}\, d\sigma(\theta)\right) \leqslant c_2^q,$$

where $c_2 > 0$ is an absolute constant. Then by Markov's inequality we get that

$$\int_{S^{n-1}} \frac{h_{K_N}(\theta)^{2\log(2N)}}{h_{Z_{2\log(2N)}(K)}(\theta)^{2\log(2N)}}\, d\sigma(\theta) \leqslant (c_2 e)^q$$

with probability greater than $1 - e^{-q} = 1 - \frac{1}{4N^2}$. Plugging this last estimate into (2.8) we get the assertion of the lemma. □

We can now give a proof of Theorem 2.10: Remember that Lemma 2.11 provides us with an estimate of $\mathrm{vol}_n(K_N)^{1/n}$ in terms of the negative moments $w_{-q}(K)$. By means of Lemma 2.12, one can then use the results of Paouris stated in section 1.3 to get the statement.

Proof of Theorem 2.10. Combining Lemma 2.11 and Lemma 2.12, we have that

$$\mathrm{vol}_n(K_N)^{1/n} \lesssim \frac{1}{\sqrt{n}} w_{-q/2}(Z_q(K))$$

for $q = 2\log(2N)$, with probability greater than $1 - e^{-q}$. Due to $Z_q(K) \subseteq cZ_{q/2}(K)$ (Lemma 1.26 (b)) and Theorem 1.30, we have

$$\mathrm{vol}_n(K_N)^{1/n} \lesssim \frac{\sqrt{q}}{n} I_{-q/2}(K).$$

Finally, since $I_{-q/2} \leqslant I_2(K) = \sqrt{n}L_K$ is valid for any $q \leqslant n$, and using $n^2 \leqslant N$, we get

$$\mathrm{vol}_n(K_N)^{1/n} \lesssim \frac{\sqrt{q}}{\sqrt{n}} L_K \asymp \frac{\sqrt{\log 2N}}{\sqrt{n}} L_K \lesssim \frac{\sqrt{\log(2N/n)}}{\sqrt{n}} L_K,$$

with probability greater than $1 - e^{-q} = 1 - \frac{1}{4N^2}$. □

2.3 A variation of the method: Random approximation of a convex body

We now turn to the problem of approximating a convex body K by the random polytope C_N. The result we will prove in this section is the following.

Theorem 2.13. *For every $\beta \in (0,1)$ there exist a constant $a = a(\beta) > 1$ (depending only on β), and an absolute constant $c > 0$ with the following property: If K is a centered convex body in $\mathbb{R}^n$ and $an \leqslant N \leqslant e^n$, then*

$$C_N \supseteq c_1 \beta \frac{\log(N/n)}{n} K,$$

with probability greater than $1 - e^{-N^{1-\beta} n^{\beta}}$.

Note that an application of the above for $\beta = 1/2$ and $N = \lceil an \rceil$ yields immediately the statement of Theorem 2.2. On the other hand, in the case that N is exponential in n, a similar result was proved in [30]: For every $\delta \in (0,1)$ there exists $n_0 = n_0(\delta)$ such that for every $n \geqslant n_0$ if $C\log(n)/n \leqslant \gamma \leqslant 1$ for some absolute constant $C > 0$, then if $N = e^{\gamma n}$ the inclusion $C_N \supseteq c(\delta)\gamma K$ holds with probability greater than $1 - \delta$, where $c(\delta) > 0$ is a constant depending only on δ. See also the recent paper of Naszodi [52], where the above results are reproved via an entirely different method.

One-sided centroid bodies

Due to non-symmetricity of C_N, using a "non-symmetric" generalization of the centroid bodies seems more appropriate. The one sided centroid bodies, introduced below, were used by Guédon and E. Milman in [33]. A similar definition was used by Haberl [38]. In what follows, $a_+ := \max\{a, 0\}$.

Definition 2.14. Let K be an isotropic convex body in $\mathbb{R}^n$. The one-sided L_q-centroid bodies $Z_q^+(K)$ of K, $q \geqslant 1$, are the bodies with support function

$$h_{Z_q^+(K)}(\theta) = \left(2\int_K \langle x, \theta\rangle_+^q \, dx\right)^{1/q},$$

for every $\theta \in S^{n-1}$.

It is immediate to see that when K is symmetric $Z_q^+(K) = Z_q(K)$, and generally we obviously have the inclusion

$$Z_q^+(K) \subseteq 2^{1/q} Z_q(K).$$

Note that $Z_q^+(K) \subseteq 2^{1/q} K$ for all $q \geqslant 1$. We also have an analouge of Lemma 1.26.

Lemma 2.15. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then*

(a) $Z_2^+(K) \supseteq c_0 L_K B_2^n$, *for some absolute constant $c_0 > 0$.*

(b) *If $1 \leqslant q < r < \infty$, then*

$$\left(\frac{2}{e}\right)^{\frac{1}{q}-\frac{1}{r}} Z_q^+(K) \subseteq Z_r^+(K) \subseteq C\frac{r}{q}\left(\frac{2e-2}{e}\right)^{\frac{1}{q}-\frac{1}{r}} Z_q^+(K),$$

where $C > 0$ is an absolute constant.

Indication of proof. Part (a) appears as Lemma A.4 in [33]. Part (b) can be recovered in the same way as Lemma 1.26 (b), using Grünbaum's lemma (Theorem 1.10). □

We will also need to approach K by $Z_q^+(K)$ from the outside, so we prove the next lemma.

Lemma 2.16. *If K ia a centered convex body of volume 1 in $\mathbb{R}^n$, then*

$$\left(\frac{2}{e^2}\right)^{1/q}\left(\frac{\Gamma(n)\Gamma(q+1)}{\Gamma(n+q+1)}\right)^{1/q} h_K(\theta) \leqslant h_{Z_q^+(K)}(\theta).$$

for every $\theta \in S^{n-1}$.

Proof. Let $H_\theta^+ = \{x \in \mathbb{R}^n : \langle x, \theta\rangle \geqslant 0\}$, $H_\theta(t) = \{x \in \mathbb{R}^n : \langle x, \theta\rangle = t\}$, and

$$f_\theta(t) = \mathrm{vol}_{n-1}(K \cap H_\theta(t)).$$

Note that $f_\theta^{\frac{1}{n-1}}$ is a concave function of θ. This is due to the Brunn-Minkowski inequality. As a consequence, we have

$$f_\theta(t) \geqslant \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f_\theta(0)$$

for all $t \in [0, h_K(\theta)]$. Using this and the definition of f_θ we can write

$$\begin{aligned} h^q_{Z_q^+(K)}(\theta) &= 2\int_0^{h_K(\theta)} t^q f_\theta(t)\,dt \geqslant 2\int_0^{h_K(\theta)} t^q \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f_\theta(0) \\ &= 2f_\theta(0) h_K(\theta)^{q+1}\int_0^1 s^q(1-s)^{n-1}ds = \frac{\Gamma(n)\Gamma(q+1)}{\Gamma(n+q+1)} 2f_\theta(0)h_K(\theta)^{q+1}. \end{aligned}$$

We finish as in the proof of Lemma 1.17: observe that $\mathrm{vol}_n(K \cap H_\theta^+) \leqslant h_K(\theta)\|f_\theta\|_\infty$. Using $\|f_\theta\|_\infty \leqslant e f_\theta(0)$ (due to Fradelizi [23], or see [17, Theorem 2.2.2]) and Grünbaum's Lemma, we get the statement of the Lemma. □

Comparison with $Z_q^+(K)$ and proof of the theorem

Here we prove Theorem 2.13. The central feature is an inclusion theorem in the spirit of Theorem 2.3, which in a sense generalizes the approach of [20] to the non-symmetric case.

Theorem 2.17. *Let $\beta \in (0,1)$. There exist a constant $a = a(\beta) > 1$ and absolute constants $c_1, c_2 > 0$ with the following property: If K is a centered convex body of volume 1 in $\mathbb{R}^n$ and $N \geqslant an$, then for $q = c_1\beta\log(N/n)$ the inclusion*

$$C_N \supseteq c_2 Z_q^+(K)$$

holds, with probability greater than $1 - e^{-N^{1-\beta}n^\beta}$.

For the proof we will require an estimate similar to that of Lemma 2.4. This is again obtained using the Paley-Zygmund inequality.

Lemma 2.18. *There exists an absolute constant $C>0$ such that for every $n\in\mathbb{N}$, every centered convex body K of volume 1 in $\mathbb{R}^n$ and every $q\geqslant 2$, the bound*

$$\mathrm{vol}_n\left(\left\{x\in K:\langle x,\theta\rangle>\frac{1}{2}h_{Z_q^+(K)}(\theta)\right\}\right)\geqslant C^{-q}$$

holds for every $\theta\in S^{n-1}$.

Proof. Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $q\geqslant 2$ and $\theta\in S^{n-1}$. Consider the non-negative random variable

$$g_\theta(x)=2\langle x,\theta\rangle_+^q$$

on K. By Lemma 2.15 (b) for $r=2q$, we see that

$$\mathbb{E}(g_\theta^2)=h^{2q}_{Z^+_{2q}(K)}(\theta)\leqslant C_1^q h^{2q}_{Z_q^+(K)}(\theta)=C_1^q[\mathbb{E}(g_\theta)]^2,$$

where $C_1>0$ is an absolute constant. Now, for every $t\in(0,2^{-1/q})$,

$$\begin{aligned}\mathrm{vol}_n\left(\left\{x\in K:\langle x,\theta\rangle>th_{Z_q^+(K)}\right\}\right)&=\mathrm{vol}_n\left(\left\{x\in K:\langle x,\theta\rangle_+>t[\mathbb{E}(g_\theta)]^{1/q}\right\}\right)\\&=\mathrm{vol}_n\left(\left\{x\in K:\langle x,\theta\rangle_+^q>t^q\mathbb{E}(g_\theta)\right\}\right)\\&=\mathrm{vol}_n\left(\{x\in K:g_\theta(x)>2t^q\mathbb{E}(g_\theta)\}\right)\\&\geqslant(1-2t^q)^2\frac{[\mathbb{E}(g_\theta)]^2}{\mathbb{E}(g_\theta^2)}\geqslant\frac{(1-2t^q)^2}{c_1^q},\end{aligned}$$

where in the penultimate step we have used the Paley-Zygmund inequality (2.3) for g_θ. Choosing $t=1/2$ we get the lemma with $C=4C_1$. □

The proof of Theorem 2.17 is based on the method of Dyer-Füredi-McDiarmid [22]. This idea has been used numerous times since, see e.g. [9], [27], [25].

Proof of Theorem 2.17. Let $q\geqslant 2$ and consider the random polytope $C_N:=\mathrm{conv}\{x_1,\dots,x_N\}$. With probability equal to one, C_N has non-empty interior and, for every $J=\{j_1,\dots,j_n\}\subset\{1,\dots,N\}$, the points $x_{j_1},\dots,x_{j_n}$ are affinely independent. Write H_J for the affine subspace determined by $x_{j_1},\dots,x_{j_n}$ and H_J^+, H_J^- for the two closed halfspaces whose bounding hyperplane is H_J.

If $\frac12 Z_q^+(K)\not\subseteq C_N$, then there exists $x\in\frac12 Z_q^+(K)\setminus C_N$, and hence, there is a facet of C_N defining some affine subspace H_J as above that satisfies the following: either $x\in H_J^-$ and $C_N\subset H_J^+$, or $x\in H_J^+$ and $C_N\subset H_J^-$. Observe that, for every J, the probability of each of these two events is bounded by

$$\left(\sup_{\theta\in S^{n-1}}\mu_K\left(\left\{x:\langle x,\theta\rangle\leqslant\tfrac12 h_{Z_q^+(K)}(\theta)\right\}\right)\right)^{N-n}\leqslant\left(1-C^{-q}\right)^{N-n},$$

where $C>0$ is the constant in Lemma 2.18. It follows that

$$\mathbb{P}\left(\tfrac12 Z_q^+(K)\not\subseteq C_N\right)\leqslant 2\binom{N}{n}(1-C^{-q})^{N-n}.$$

Since $\binom{N}{n}\leqslant\left(\frac{eN}{n}\right)^n$, this probability is smaller than $\exp(-N^{1-\beta}n^\beta)$ if

$$\left(\frac{2eN}{n}\right)^n(1-C^{-q})^{N-n}<\left(\frac{2eN}{n}\right)^n e^{-C^{-q}(N-n)}<\exp(-N^{1-\beta}n^\beta),$$

and the second inequality is satisfied if

$$\frac{N}{n} - 1 > C^q \left[\left(\frac{N}{n}\right)^{1-\beta} + \log\left(\frac{2eN}{n}\right) \right]. \tag{2.9}$$

We choose $q = \frac{\beta}{2\log C}\log\left(\frac{N}{n}\right)$ and $\alpha_1(\beta) := C^{4/\beta}$. Note that if $N \geqslant \alpha_1(\beta)n$ then $q \geqslant 2$ and that (2.9) becomes

$$\frac{N}{n} - 1 > \left(\frac{N}{n}\right)^{1-\frac{\beta}{2}} + \left(\frac{N}{n}\right)^{\frac{\beta}{2}} \log\left(\frac{2eN}{n}\right). \tag{2.10}$$

Since

$$\lim_{t\to+\infty} \left[t - 1 - t^{1-\frac{\beta}{2}} - t^{\frac{\beta}{2}}\log(2et)\right] = +\infty,$$

we may find $\alpha_2(\beta)$ such that (2.10) is satisfied for all $N \geqslant \alpha_2(\beta)n$. Setting $\alpha = \max\{\alpha_1(\beta), \alpha_2(\beta)\}$ we see that the assertion of the theorem is satisfied with probability greater that $1 - e^{-N^{1-\beta}n^{\beta}}$ for all $N \geqslant \alpha n$, with $c_1 = \frac{1}{2\log C}$ and $c_2 = \frac{1}{2}$. □

For the deduction of Theorem 2.13 we will apply Theorem 2.17 plus the fact that $Z_n^+(K) \supseteq cK$. This comes from Lemma 2.16.

Proof of Theorem 2.13. Let $\beta \in (0,1)$ and $a = a(\beta)$ be the constant from Theorem 2.17. Suppose $an \leqslant N \leqslant e^n$ and that $X_1, \ldots, X_N$ are independent random points uniformly distributed in K. Lemma 2.16 for $q = n$ gives us

$$Z_n^+(K) \supseteq c_1 K,$$

for some absolute constant $c_1 > 0$. From Theorem 2.17, the choice $q = c_2\beta\log(N/n)$ implies

$$C_N \supseteq c_3 Z_q^+(K)$$

with probability greater than $1 - e^{-N^{1-\beta}n^{\beta}}$, where $c_2, c_3 > 0$ are absolute constants. The right inclusion in Lemma 2.15 (b) gives us then that

$$Z_n^+(K) \subseteq c_4 \frac{n}{q}\left(\frac{2e-2}{e}\right)^{\frac{1}{q}-\frac{1}{n}} Z_q^+(K) \subseteq 2c_4\frac{n}{q} Z_q^+(K),$$

where $c_4 > 0$ is an absolute constant. Combining all of the above, we get

$$C_N \supseteq c_5 \frac{q}{n} K \supseteq c_6 \frac{\beta\log(N/n)}{n} K$$

with probability greater than $1 - e^{-N^{1-\beta}n^{\beta}}$, where $c_5, c_6 > 0$ are absolute constants. □

3 The isotropic constant of random polytopes

Random polytopes have been studied in connection to the Isotropic constant conjecture essentially since the formulation of the problem. Already in [51] the authors established a connection between the volume of a random simplex in a convex body K and its isotropic constant L_K, namely

$$L_K^{2n} = n! \int_K \ldots \int_K \mathrm{vol}_n(\mathrm{conv}\{o, x_1, \ldots, x_n\})^2 \, dx_1 \ldots dx_n$$

(see Proposition 5.6 in [51], but also [17, Section 3.5.1], where the links with Sylvester's problem and the simplex conjecture are discussed).

On the other hand, random polytopes have a long history providing extremal examples in asymptotic geometric analysis. This has its roots at the seminal work of E. Gluskin [31], where it is proved that, with very high probability, two convex bodies in $\mathbb{R}^n$ picked uniformly from a certain class of symmetric random polytopes have maximal Banach-Mazur distance (that is, of the order of n). In view of this, as well as other similar results, random polytopes were initially considered as a potential counterexample to the Isotropic constant conjecture.

The first study on the isotropic constant of random polytopes was in the paper [43] of Klartag and Kozma. They proved that, with probability close to 1, the absolute convex hull of N standard gaussian vectors in $\mathbb{R}^n$ has an isotropic constant bounded by an absolute constant, and indicated that their method can be used to provide upper bounds for the isotropic constant of different types of random polytopes as well. Since then, the models that have been studied include random polytopes generated by random vertices in an unconditional convex body [19], the euclidean unit sphere S^{n-1} [3] as well as any ℓ_p-sphere [37], and more. Note that, in the mentioned works, the method is always a variation of the one in [43]. Non-probabilistic results for the isotropic constant of polytopes have also appeared in the literature: it is known that

$$L_{\tilde{K}_N} \lesssim \min\left\{\sqrt{N/n}, \log N\right\}$$

for every polytope $\tilde{K}_N$ in $\mathbb{R}^n$ with N vertices (see [4], [5]).

In this section we review the argument of [43] and apply it to give a $\sqrt{\log(2N/n)}$ bound for the isotropic constant of K_N when the vertices $x_1, \ldots, x_N$ are picked uniformly from a general isotropic convex body K. This result can be found in [6] and [28]. In the special case that K is a ψ_2-body (see below for the definition), the method yields an absolute upper bound on L_{K_N} (see also [17, Theorem 11.5.6]).

3.1 The method of Klartag and Kozma

Outline of the strategy

Let D be a convex body in $\mathbb{R}^n$. Our starting point is the fact that (recall the general definition of the isotropic constant L_D),

$$\mathrm{vol}_n(D)^{2/n} n L_D^2 \leqslant \frac{1}{\mathrm{vol}_n(D)} \int_D \|x\|_2^2 \, dx. \tag{3.1}$$

Using (3.1) for $D := K_N$, to prove that K_N has a bounded isotropic constant, with probability tending to 1, it suffices to give appropriate lower and upper bounds for $\mathrm{vol}_n(K_N)$ and $\mathbb{E}\|\cdot\|_2^2$ on K respectively. Since the problem is affinely invariant, we can assume that K is an isotropic convex body. The lower bound on $\mathrm{vol}_n(K_N)$ will be the one established in our study of the asymptotic shape of K_N, stated in Theorem 2.1 (a). The method that we will follow to provide an upper bound for $\mathbb{E}\|\cdot\|_2^2$ is the following: We first reduce the problem on bounding $\mathbb{E}\|\cdot\|_2^2$ on the facets F of K_N (this is done in Lemma 3.1 below). Next, in Proposition 3.2, we treat each facet $F = \mathrm{conv}\{y_1, \ldots, y_n\}$ seperately, providing an upper bound in terms of $\max_{\varepsilon_j = \pm 1} \|\varepsilon_1 y_1 + \ldots + \varepsilon_n y_n\|_2$. This leads us to try to bound the sum of the random variables $\langle \varepsilon_i y_i, \theta\rangle$, $i = 1, \ldots, n$, for every $\theta \in S^{n-1}$.

This is achieved using Bernstein's inequality. To treat the general case, as well as obtain an absolute bound for a wide class of probability measures in $\mathbb{R}^n$, we first give the definition of the ψ_α norm and ψ_α-directions in convex bodies. The property that every $\theta \in S^{n-1}$ is a ψ_α-direction for K (or, as we say, that K is a ψ_α-body) for some $\alpha > 1$ is equivalent to a certain assumption on the probability distribution μ_K. In particular, Bernstein's inequality yields improved bounds on the probability estimate for the boundedness of a sum of ψ_2 random variables than in the ψ_1 case. The result is that one can give an absolute bound for L_{K_N} if the vertices are chosen uniformly and independently from a ψ_2-body K. A weaker bound for the general (that is, ψ_1) case is also discussed.

Reduction to the facets

We denote by $\mathcal{F}(K_N)$ the family of facets of K_N. Note that, with probability equal to one, all the facets of K_N are simplices. Moreover, if $F = \mathrm{conv}\{y_1, \dots, y_n\}$ is a facet of K_N, then $y_j = \varepsilon_j x_{i_j}$ with $i_j \neq i_k$ for all $1 \leqslant j \neq k \leqslant n$, since the points x_i and $-x_i$ cannot lie in the same facet.

The reduction of the problem on estimating the mean value of $\|x\|_2^2$ on the facets of K_N is done in the following Lemma.

Lemma 3.1. *Let $F_1, \dots, F_M$ be the facets of K_N. Then*

$$\frac{1}{\mathrm{vol}_n(K_N)} \int_{K_N} \|x\|_2^2 \, dx \leqslant \frac{n}{n+2} \max_{1 \leqslant s \leqslant M} \frac{1}{\mathrm{vol}_{n-1}(F_s)} \int_{F_s} \|u\|_2^2 \, du.$$

Proof. Note that any $x \in K_N$ can be written in the form $x = \frac{t}{d_s} u$ for some $u \in F_s \in \mathcal{F}(K_N)$, $t \in [0, d_s]$, where $d_s := d(o, F_s)$ is the euclidean distance from the origin o to the affine subspace determined by F_s. Using this change of variables, we write

$$\begin{aligned}
\frac{1}{\mathrm{vol}_n(K_N)} \int_{K_N} \|x\|_2^2 \, dx &= \frac{1}{\mathrm{vol}_n(K_N)} \sum_{s=1}^{M} \int_{F_s} \int_0^{d_s} \left\| \frac{t}{d_s} u \right\|_2^2 \frac{t^{n-1}}{d_s^{n-1}} \, dt \, du \\
&= \frac{1}{\mathrm{vol}_n(K_N)} \sum_{s=1}^{M} \frac{1}{d_s^{n+1}} \left(\int_0^{d_s} t^{n+1} \, dt \right) \left(\int_{F_s} \|u\|_2^2 \, du \right) \\
&= \frac{1}{\mathrm{vol}_n(K_N)} \sum_{s=1}^{M} \frac{d(o, F_s)}{n+2} \int_{F_s} \|u\|_2^2 \, du
\end{aligned}$$

Combining the above with the general formula for the volume of a polytope

$$\mathrm{vol}_n(K_N) = \frac{1}{n} \sum_{s=1}^{M} d(o, F_s) \mathrm{vol}_{n-1}(F_s)$$

we get the assertion of the lemma. $\square$

Next, we bound the average 2-norm on the facets of K_N as follows.

Proposition 3.2. *Let $F = \mathrm{conv}\{y_1, \dots, y_n\}$ for some $y_1, \dots, y_n \in \mathbb{R}^n$. Then*

$$\frac{1}{\mathrm{vol}_{n-1}(F)} \int_F \|u\|_2^2 \, du \leqslant \frac{2}{n(n+1)} \max_{\varepsilon_j = \pm 1} \|\varepsilon_1 y_1 + \ldots + \varepsilon_n y_n\|_2^2.$$

Denote $\Delta^{n-1} := \mathrm{conv}\{e_1, \dots, e_n\}$. We will use the following fact for the $(n-1)$-simplex Δ^{n-1}. We include its elementary proof for completeness.

Lemma 3.3. *Let $u = (u_1, \dots, u_n)$ be a random vector uniformly distributed in Δ^{n-1}. Then for any $i, j = 1, \dots, n$,*

$$\frac{1}{\mathrm{vol}_n(\Delta^{n-1})} \int_{\Delta^{n-1}} u_i u_j \, du = \frac{1 + \delta_{ij}}{n(n+1)},$$

where δ_{ij} is the Kronecker delta.

Proof. Recall that $\mathrm{vol}_{n-1}(\Delta^{n-1}) = \frac{1}{(n-1)!}$. Assume that $j_1 = j_2 = 1$. Then

$$\begin{aligned}
\frac{1}{\mathrm{vol}_{n-1}(\Delta^{n-1})} \int_{\Delta^{n-1}} u_1^2 \, du &= (n-1)! \int_0^1 u_1^2 \left(\int_{\sum_{j=2}^n u_j \leqslant 1 - u_1} du_2 \ldots du_n \right) du_1 \\
&= (n-1)! \int_0^1 t^2 (1-t)^{n-2} \mathrm{vol}_{n-2}(\Delta^{n-2}) \, dt \\
&= \frac{(n-1)!}{(n-2)!} \frac{2\Gamma(n-1)}{\Gamma(n+2)} = \frac{2}{n(n+1)}.
\end{aligned}$$

If on the other hand $j_1 \neq j_2$, assume without loss of generality that $j_1 = 1, j_2 = 2$ and write

$$\begin{aligned}
\frac{1}{\mathrm{vol}_{n-1}(\Delta^{n-1})}\int_{\Delta^{n-1}} u_1u_2\,du &= (n-1)!\int_0^1\int_0^{1-u_1} u_1u_2\left(\int_{\sum_{j=3}^n u_j\leqslant 1-u_1-u_2} du_3\dots du_n\right)du_2\,du_1\\
&= (n-1)!\int_0^1\int_0^{1-t} ts(1-t-s)^{n-3}\mathrm{vol}_{n-3}(\Delta^{n-3})\,ds\,dt\\
&= \frac{(n-1)!}{(n-3)!}\cdot\frac{1}{(n-2)(n-1)n(n+1)} = \frac{1}{n(n+1)},
\end{aligned}$$

proving the claim. □

Proof of Proposition 3.2. If, $j = 1,\dots,n$, $y_j = (y_{j1},\dots,y_{jn})$, consider the matrix $T = (T_{ij})$ with $T_{ij} = y_{ji}$, so that $F = T(\Delta^{n-1})$. Assume that $\det T \neq 0$ (since $\mathbb{P}(\det T = 0) = 0$). Then

$$\begin{aligned}
\frac{1}{\mathrm{vol}_{n-1}(F)}\int_F \|u\|_2^2\,du &= \frac{1}{\mathrm{vol}_{n-1}(T(\Delta^{n-1}))}\int_{T(\Delta^{n-1})}\|u\|_2^2\,du\\
&= \frac{1}{\mathrm{vol}_{n-1}(\Delta^{n-1})}\int_{\Delta^{n-1}}\|Tu\|_2^2\,du\\
&= \frac{1}{\mathrm{vol}_{n-1}(\Delta^{n-1})}\int_{\Delta^{n-1}}\sum_{i=1}^n\left(\sum_{j=1}^n y_{ji}u_j\right)^2 du.
\end{aligned}$$

Note that, for every $i = 1,\dots,n$,

$$\left(\sum_{j=1}^n y_{ji}u_j\right)^2 = \sum_{j=1}^n\sum_{k=1}^n y_{ji}y_{ki}u_ju_k,$$

so applying Lemma 3.3 we get

$$\begin{aligned}
\frac{1}{\mathrm{vol}_{n-1}(\Delta^{n-1})}\int_{\Delta^{n-1}}\left(\sum_{j=1}^n y_{ji}u_j\right)^2 du &= \sum_{j=1}^n\sum_{k=1}^n y_{ji}y_{ki}\left(\frac{1}{\mathrm{vol}_{n-1}(\Delta^{n-1})}\int_{\Delta^{n-1}}u_ju_k\,du\right)\\
&= \sum_{j=1}^n\sum_{k=1}^n y_{ji}y_{ki}\left(\frac{1+\delta_{jk}}{n(n+1)}\right)\\
&= \frac{1}{n(n+1)}\left(\sum_{j=1}^n\sum_{k=1}^n y_{ji}y_{ki} + \sum_{j=1}^n y_{ji}^2\right)\\
&= \frac{1}{n(n+1)}\left(\left(\sum_{j=1}^n y_{ji}\right)^2 + \sum_{j=1}^n y_{ji}^2\right).
\end{aligned}$$

Ultimately, we have that

$$\frac{1}{\mathrm{vol}_{n-1}(F)}\int_F \|u\|_2^2\,du = \frac{1}{n(n+1)}\sum_{i=1}^n\left(\left(\sum_{j=1}^n y_{ji}\right)^2 + \sum_{j=1}^n y_{ji}^2\right). \tag{3.2}$$

Finally note that

$$\sum_{i=1}^n\left(\sum_{j=1}^n y_{ji}\right)^2 = \|y_1+\dots+y_n\|_2^2,$$

while, by the parallelogram law,

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} y_{ji}^2\right) = \mathrm{Ave}_{\varepsilon_j=\pm 1}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\varepsilon_j y_{ji}\right)^2 = \mathrm{Ave}_{\varepsilon_j=\pm 1}\|\varepsilon_1 y_1+\ldots+\varepsilon_n y_n\|_2^2$$

and so we bound the sum on the left hand side of (3.2) by $2\max_{\varepsilon_j=\pm 1}\|\varepsilon_1 y_1+\ldots+\varepsilon_n y_n\|_2^2$, which completes the proof. □

Combining the statements of Lemma 3.1 and Proposition 3.2, we have so far proved that[4]

$$\frac{1}{\mathrm{vol}_n(K_N)}\int_{K_N}\|x\|_2^2\,dx \leqslant \frac{2}{(n+1)(n+2)}\max_{\{i_1,\ldots,i_n\}\subseteq[2N]}\max_{\varepsilon_j=\pm 1}\|\varepsilon_1 x_{i_1}+\ldots+\varepsilon_n x_{i_n}\|_2^2, \tag{3.3}$$

and this is true for any isotropic convex body K. Next we will use Bernstein's inequality to bound the right hand side above. As we will see, we can get a better estimate if some extra assumptions are made for K (actually the underlying probability measure μ_K).

ψ_α random variables and directions in convex bodies

Given a random variable $f:\Omega\to\mathbb{R}$ on a probability space $(\Omega,\mathcal{A},\mu)$ and some $\alpha\geqslant 1$, recall the definition of the Orlicz norm,

$$\|f\|_{\psi_\alpha} := \|f\|_{L^{\psi_\alpha}(\mu)} = \inf\left\{t>0 : \int_\Omega \exp\left(\left(\frac{|f(x)|}{t}\right)^\alpha\right)d\mu(x)\leqslant 2\right\}.$$

The cases $\alpha\in\{1,2\}$ in the above definition are of particular interest: It is known that $f\in L^{\psi_2}(\mu)$ if and only if, for any $t>0$,

$$\mathbb{P}(|f|>t)\leqslant 2e^{-\frac{t^2}{b^2}},$$

where b is a constant multiple of $\|f\|_{\psi_2}$. This means that the tail decay of f is roughly similar to that of a standard Gaussian random variable, and explains the terminology *subgaussian* random variables, used normally to refer to the class $L^{\psi_2}(\mu)$. Examples of ψ_2 random variables include Gaussian, Bernoulli, and also any bounded random variable on $\mathbb{R}^n$.

Similarly, the class $L^{\psi_1}(\mu)$ coincides with the random variables $f:\mathbb{R}^n\to\mathbb{R}$ for which

$$\mathbb{P}(|f|>t)\leqslant 2e^{-\frac{t}{a}}$$

for any $t>0$, where a can be taken a constant multiple of $\|f\|_{\psi_1}$. Here we have a "slower" (sub-exponential) tail decay, so ψ_1 random variables are often called *subexponential*.

We will use the following Bernstein type inequalities for sums of random variables that satisfy either the ψ_1 or the ψ_2 condition.

Theorem 3.4 (Bernstein's inequality). *Let $f_1,\ldots,f_k$ be independent random variables with $\mathbb{E}(f_j)=0$ for every $1\leqslant j\leqslant k$, defined on some probability space (Ω,μ).*

(a) *If there is a constant $A>0$ such that $\max_{1\leqslant j\leqslant k}\|f_j\|_{\psi_1}\leqslant A$, then*

$$\mathbb{P}\left(\left|\sum_{j=1}^{k} f_j\right|\geqslant tk\right)\leqslant 2\exp\left(-c\min\left\{\frac{t^2k}{A^2},\frac{tk}{A}\right\}\right)$$

for every $t>0$.

[4] For brevity, notation is slightly abused here, relabelling $\{x_1,\ldots,x_{2N}\}:=\{\pm x_1,\ldots,\pm x_N\}$.

(b) *If there is a constant $B > 0$ such that* $\max_{1\leqslant j\leqslant k} \|f_j\|_{\psi_2} \leqslant B$*, then*

$$\mathbb{P}\left(\left| \sum_{j=1}^{k} f_j \right| \geqslant tk \right) \leqslant 2\exp\left(-\frac{t^2 k}{8B^2} \right)$$

for every $t > 0$.

Let us draw our attention to the class of probability measures associated to convex bodies in $\mathbb{R}^n$. We give the following definition.

Definition 3.5. Let μ be a log-concave[5] probability measure on $\mathbb{R}^n$ and $\alpha \geqslant 1$. We say that a direction $\theta \in S^{n-1}$ is a *ψ_α-direction for μ* if there is a constant $b_\alpha > 0$, depending only on α, such that

$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leqslant b_\alpha \|\langle \cdot, \theta \rangle\|_2.$$

Moreover, we say that μ is a *ψ_α-measure* if there is a constant $B_\alpha > 0$, depending only on α, such that

$$\sup_{\theta \in S^{n-1}} \frac{\|\langle \cdot, \theta \rangle\|_{\psi_\alpha}}{\|\langle \cdot, \theta \rangle\|_2} \leqslant B_\alpha.$$

In particular, if K is a centered convex body of volume 1 in $\mathbb{R}^n$, we say that K is a *ψ_α-body* if μ_K is a ψ_α-measure.

In the above setting where $f = \langle \cdot, \theta \rangle$, $\theta \in S^{n-1}$, the Orlicz norm has the following equivalent description.

Lemma 3.6. *Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then for every $\theta \in S^{n-1}$,*

$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \asymp \sup\left\{ \frac{\|\langle \cdot, \theta \rangle\|_q}{q^{1/\alpha}} : \alpha \leqslant q \leqslant \max\{n, \alpha\} \right\}.$$

A proof that the same estimate holds for a general f, but with the supremum on the right hand side taken over all $q \geqslant \alpha$ can be found in [17, Lemma 2.4.2]. The fact that for $f = \langle \cdot, \theta \rangle$ one can restrict the choices of q up to $\max\{n, \alpha\}$ is due to the result $\|\langle \cdot, \theta \rangle\|_n \asymp \max\{h_K(\theta), h_K(-\theta)\}$ of Paouris stated in Lemma 1.26 (c).

It is easy to see that every isotropic convex body is a ψ_1-body: Using Lemma 3.6 and the reverse Hölder inequalities of Corollary 1.9, it is clear that

$$\|\langle \cdot, \theta \rangle\|_{\psi_1} \asymp \sup_{1\leqslant q\leqslant n} \frac{\|\langle \cdot, \theta \rangle\|_q}{q} \leqslant C \|\langle \cdot, \theta \rangle\|_2$$

for some absolute constant $C > 0$. On the other hand, the ψ_2 behaviour of the directions $\theta \in S^{n-1}$ for a convex body K is a much more delicate matter: It is known that every ψ_2 body has a bounded isotropic constant, and the known bounds on L_K are actually based on estimates on the ψ_2 norm of $\langle \cdot, \theta \rangle$. However one can not expect that an arbitrary K is a ψ_2-body: It is actually an open problem whether any convex body K has at least one ψ_2-direction. We record the explicit connection of L_K to the ψ_α constant of a convex body K, proved by of Klartag and E. Milman in [44].

Theorem 3.7. *Let K be an isotropic convex body in $\mathbb{R}^n$. If K is a ψ_α body with constant B_α for some $\alpha \in [1, 2]$, then*

$$L_K \leqslant C B_\alpha^{\alpha/2} n^{(2-\alpha)/4},$$

where $C > 0$ is an absolute constant.

[5] Actually the definition makes sense for any probability measure on $\mathbb{R}^n$ that is absolutely continuous with respect to the Lebesgue measure.

Bounding the isotropic constant

We will now treat the case that K is assumed to be a ψ_2-body. As we shall see, in this case $L_{K_N} \leqslant C$ for some absolute constant $C > 0$. Remember that we need to bound the right hand side of (3.3). This is done using Theorem 3.4 for the functions $f_i = \langle \varepsilon_i y_i, \theta \rangle$.

Proposition 3.8. *Let K be an isotropic convex body in $\mathbb{R}^n$. Moreover, assume that $\|\langle \cdot, \theta \rangle\|_{\psi_2} \leqslant bL_K$ for some absolute constant $b > 0$. Then with probability greater than $1 - \exp(-cn \log(2N/n))$ for some absolute constant $c > 0$,*

$$\max_{\varepsilon_j = \pm 1} \|\varepsilon_1 x_{i_1} + \ldots + \varepsilon_n x_{i_n}\|_2 \leqslant CbL_K n \sqrt{\log(2N/n)}$$

for every $\{x_{i_1}, \ldots, x_{i_n}\} \subseteq \{\pm x_1, \ldots, \pm x_N\}$, where $C > 0$ is an absolute constant.

Proof. Fix some $\theta \in S^{n-1}$, some $\{y_1, \ldots, y_n\} \subseteq \{\pm x_1, \ldots, \pm x_N\}$, and a choice of $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$. We apply Theorem 3.4 (b) for the random variables $f_j(y_1, \ldots, y_n) = \langle \varepsilon_j y_j, \theta \rangle$ in $\Omega = K^n$: We check that $\mathbb{E} f_j = 0$ for every j, since K is centered, and by assumption, $\|f_j\|_{\psi_2} \leqslant bL_K$. Then for every $a > 0$ we have

$$\mathbb{P}\left(|\langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta \rangle| > abL_K n\right) \leqslant 2 \exp(-c_1 a^2 n) \tag{3.4}$$

for some absolute constant $c > 0$. Using a union bound over all $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$, we see that

$$\mathbb{P}\left(\exists \varepsilon \in \{-1,1\}^n : |\langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta \rangle| > abL_K n\right) \leqslant 2^{n+1} \exp(-c_1 a^2 n) \leqslant \exp((n+1)\log 2 - c_1 a^2 n),$$

so if $a > 0$ is such that $a^2 \geqslant \frac{2 \log 2}{c_1}$ we have that

$$\mathbb{P}\left(\exists \varepsilon \in \{-1,1\}^n : |\langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta \rangle| > abL_K n\right) \leqslant \exp(-c_2 a^2 n)$$

for some absolute constant $c_2 > 0$.

Now let $\mathcal{N}$ be a $\frac{1}{2}$-net in S^{n-1}, with $|\mathcal{N}| \leqslant 5^n$. Using once more the union bound as before, we can see that

$$\mathbb{P}\left(\exists \theta \in \mathcal{N}, \exists \varepsilon \in \{-1,1\}^n : |\langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta \rangle| > abL_K n\right) \leqslant \exp(-c_3 a^2 n)$$

for some absolute constant $c_3 > 0$, if we choose $a > 0$ such that $a^2 \geqslant \frac{2 \log 5}{c_2}$. By a standard approximation argument, any $\theta \in S^{n-1}$ can be written in the form $\theta = \sum_{j=1}^{\infty} \delta_j \theta_j$ for some $(\theta_j)_{j=1}^{\infty} \subseteq \mathcal{N}$ and $0 \leqslant \delta_j \leqslant 2^{-(j-1)}$ for every $j \in \mathbb{N}$. Now

$$\begin{aligned}
&\mathbb{P}\left(\exists \theta \in S^{n-1}, \exists \varepsilon \in \{-1,1\}^n : |\langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta \rangle| > 2abL_K n\right) \\
&\qquad = \mathbb{P}\left(\exists \theta \in S^{n-1}, \exists \varepsilon \in \{-1,1\}^n : \left|\sum_{j=1}^{\infty} \delta_j \langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta_j \rangle\right| > 2abL_K n\right) \\
&\qquad \leqslant \mathbb{P}\left(\exists \theta \in S^{n-1}, \exists \varepsilon \in \{-1,1\}^n : \sum_{j=1}^{\infty} \delta_j |\langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta_j \rangle| > 2abL_K n\right) \\
&\qquad \leqslant \mathbb{P}\left(\exists \theta_j \in \mathcal{N}, \exists \varepsilon \in \{-1,1\}^n : |\langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta_j \rangle| > abL_K n\right) \\
&\qquad \leqslant \exp(-c_3 a^2 n).
\end{aligned}$$

Up to this point, we have proved that if a is greater than an appropriate absolute constant, then

$$|\langle \varepsilon_1 y_1 + \ldots + \varepsilon_n y_n, \theta \rangle| \leqslant abL_K n$$

for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n$ and for every $\theta \in S^{n-1}$, and thus

$$\max_{\varepsilon_j = \pm 1} \|\varepsilon_1 y_1 + \ldots + \varepsilon_n y_n\|_2 \leqslant abL_K n$$

holds with probability greater than $1-\exp(-c_3a^2n)$, for given $\{y_1,\dots,y_n\}\subseteq\{\pm x_1,\dots,\pm x_N\}$.

It remains to take a union bound over all the n-tuplets of $\{\pm x_1,\dots,\pm x_N\}$. We have

$$\mathbb{P}\left(\bigcup_{\{i_1,\dots,i_n\}\subseteq[2N]}\max_{\varepsilon_j=\pm1}\|\varepsilon_1x_{i_1}+\ldots+\varepsilon_nx_{i_n}\|_2>abL_Kn\right)$$
$$\leqslant\binom{2N}{n}\mathbb{P}\left(\max_{\varepsilon_j=\pm1}\|\varepsilon_1x_{i_1}+\ldots+\varepsilon_nx_{i_n}\|_2>abL_Kn\right)$$
$$\leqslant\left(\frac{2eN}{n}\right)^n\exp(-c_3a^2n).$$

This is the very point that we need to choose $a=C\sqrt{\log(2N/n)}$ for some large enough absolute constant $C>0$, so that the resulting probability can be made less than $\exp(-c_4n\log(2N/n))$. □

We can now prove the desired result. We will use the fact that if K is a ψ_2-body, then it has a bounded isotropic constant.

Theorem 3.9. *Let K be an isotropic convex body in $\mathbb{R}^n$, which is a ψ_2-body with constant $b>0$, and some $n\leqslant N\leqslant e^n$. Then we have*

$$L_{K_N}\leqslant Cb^2,$$

with probability greater than $1-e^{-cn}$, for some absolute constants $C,c>0$.

Proof. Recall that

$$\mathrm{vol}_n(K_N)^{2/n}nL_{K_N}^2\leqslant\frac{1}{\mathrm{vol}_n(K_N)}\int_{K_N}\|x\|_2^2\,dx.$$

Combining (3.3) with the statement of Proposition 3.8 we have

$$\mathrm{vol}_n(K_N)^{2/n}nL_{K_N}^2\leqslant C_2b^2L_K^2\log(2N/n),$$

with probability greater than $1-e^{-cn\log(N/n)}$. On the other hand, note that Theorem 2.1 actually gives us two lower bounds on $\mathrm{vol}_n(K_N)^{1/n}$: If $n\leqslant N\leqslant e^{\sqrt{n}}$, we have

$$\mathrm{vol}_n(K_N)^{2/n}\gtrsim\frac{\log(2N/n)}{n}L_K^2$$

with probability greater than $1-e^{-c\sqrt{N}}$. This gives

$$L_{K_N}^2\leqslant C_3b^2,$$

and this holds with probability greater than $1-e^{-c'n}$ (the fact that the lower bound for $\mathrm{vol}_n(K_N)^{1/n}$ holds with this probability follows from 2.1 if $N\gtrsim n^2$ and from the results of Pivovarov [59] for smaller values of N).

In the regime $e^{\sqrt{n}}\leqslant N\leqslant e^n$, we only have

$$\mathrm{vol}_n(K_N)^{2/n}\gtrsim\frac{\log(2N/n)}{n},$$

so that $L_{K_N}^2\leqslant C_3b^2L_K^2$. However, since K is a ψ_2 body, Theorem 3.7 still guarantees that $L_{K_N}^2\leqslant C_4b^4$. □

Now let K be any isotropic convex body in $\mathbb{R}^n$. Since generally we only know that $\|\langle\cdot,\theta\rangle\|_{\psi_1}\leqslant cL_K$ for every $\theta\in S^{n-1}$, we can repeat the steps of the proof of Proposition 3.8, but application of the Bernstein inequality for ψ_1 random variables would give the weaker bound $\exp(-can)$ on the wanted probability. This causes no harm for the biggest part of the proof, since we can still get that for any fixed $y_1, \dots, y_n \in \{\pm x_1, \dots, \pm x_N\}$,

$$\max_{\varepsilon_j + \pm 1} \|\varepsilon_1 y_1 + \dots, + \varepsilon_n y_n\|_2 \leqslant c L_K a n$$

holds with probability greater than $1 - \exp(-c_2 a n)$, choosing a larger than an appropriate absolute constant. However, in the last step where we take the union bound over all n-tuplets of $2N$, we now need to choose $a \geqslant c_3^{-1} \log(2eN/n)$ (instead of $\sqrt{\log(2eN/n)}$) to keep the resulting probability estimate less than e^{-cn}. The cost is an extra $\sqrt{\log(2N/n)}$ factor in the final estimate for L_{K_N}.

Proposition 3.10. *Let $n \leqslant N \leqslant e^{\sqrt{n}}$, and K be an isotropic convex body in $\mathbb{R}^n$. Then we have*

$$L_{K_N} \leqslant C \sqrt{\log(2N/n)},$$

with probability greater than $1 - e^{-cn}$, for some absolute constants $C, c > 0$. In the case that $e^{\sqrt{n}} \leqslant N \leqslant e^n$ we only have the weaker bound

$$L_{K_N} \leqslant C \sqrt{\log(2N/n)} L_K.$$

Proof. By the discussion above, in the general case we have that

$$\mathrm{vol}_n(K_N)^{2/n} n L_K^2 \leqslant C L_K^2 \log(2N/n)^2$$

with probability greater than $1 - e^{-cn \log(2N/n)}$. The known lower bounds on $\mathrm{vol}_n(K_N)^{1/n}$ give us the stated results. □

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