Threshold for the expected measure of the convex hull of random points with independent coordinates

Minas Pafis

Abstract

Let μ be an even Borel probability measure on $\mathbb{R}$. For every $N > n$ consider N independent random vectors $\vec{X}_1, \dots, \vec{X}_N$ in $\mathbb{R}^n$, with independent coordinates having distribution μ. We establish a sharp threshold for the product measure μ_n of the random polytope $K_N := \mathrm{conv}\{\vec{X}_1, \dots, \vec{X}_N\}$ in $\mathbb{R}^n$ under the assumption that the Legendre transform Λ_μ^* of the logarithmic moment generating function of μ satisfies the condition

$$\lim_{x \uparrow x^*} \frac{-\ln \mu([x,\infty))}{\Lambda_\mu^*(x)} = 1,$$

where $x^* = \sup\{x \in \mathbb{R} \colon \mu([x,\infty)) > 0\}$. An application is a sharp threshold for the case of the product measure $\nu_p^n = \nu_p^{\otimes n}$, $p \geqslant 1$ with density $(2\gamma_p)^{-n}\exp(-\|x\|_p^p)$, where $\|\cdot\|_p$ is the ℓ_p^n-norm and $\gamma_p = \Gamma(1+1/p)$.

1 Introduction

Let μ be an even Borel probability measure on the real line and let $X_1, \dots, X_n$ be independent and identically distributed random variables, defined on some probability space $(\Omega, \mathcal{F}, P)$, each with distribution μ, i.e., $\mu(B) := P(X_i \in B)$ for all $i \leqslant n$ and all B in the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. Consider the random vector $\vec{X} = (X_1, \dots, X_n)$ and, for a fixed N satisfying $N > n$, consider N independent copies $\vec{X}_1, \dots, \vec{X}_N$ of $\vec{X}$. The distribution of $\vec{X}$ is $\mu_n := \mu \otimes \cdots \otimes \mu$ (n times) and the distribution of $(\vec{X}_1, \dots, \vec{X}_N)$ is $\mu_n^N := \mu_n \otimes \cdots \otimes \mu_n$ (N times). Our aim is to obtain a sharp threshold for the expected μ_n-measure of the random polytope

$$K_N := \mathrm{conv}\{\vec{X}_1, \dots, \vec{X}_N\}.$$

In order to make the notion of a sharp threshold precise, for any $n \geqslant 1$ and $\delta \in \left(0, \frac{1}{2}\right)$ we define the upper threshold

$$\varrho_1(\mu_n, \delta) = \sup\{\varrho_1 : \sup\{\mathbb{E}_{\mu_n^N}[\mu_n(K_N)] : N \leqslant \exp(\varrho_1 n)\} \leqslant \delta\} \tag{1.1}$$

and the lower threshold

$$\varrho_2(\mu_n, \delta) = \inf\{\varrho_2 : \inf\{\mathbb{E}_{\mu_n^N}[\mu_n(K_N)] : N \geqslant \exp(\varrho_2 n)\} \geqslant 1 - \delta\}. \tag{1.2}$$

Then, we say that $\{\mu_n\}_{n=1}^\infty$ exhibits a sharp threshold if

$$\varrho(\mu_n, \delta) := \varrho_2(\mu_n, \delta) - \varrho_1(\mu_n, \delta) \longrightarrow 0$$

as $n \to \infty$, for any fixed $\delta \in \left(0, \frac{1}{2}\right)$.

A threshold of this form was first established in the classical work of Dyer, Füredi and McDiarmid [10] for the case of the uniform measure μ on $[-1,1]$. We apply the general approach that was proposed in [5] and obtain an affirmative answer for a general even probability measure μ on $\mathbb{R}$ that satisfies some additional assumptions, which we briefly explain (see Section 2 for more details). We assume that μ is non-degenerate, i.e. $\mathrm{Var}(X) > 0$. Let

$$x^* = x^*(\mu) := \sup\{x \in \mathbb{R} \colon \mu([x,\infty)) > 0\}$$

be the right endpoint of the support of μ and set $I_\mu = (-x^*, x^*)$. Note that since μ is non-degenerate and even, we have that $x^* > 0$. Let

$$g(t) := \mathbb{E}\big(e^{tX}\big) := \int_{\mathbb{R}} e^{tx}\, d\mu(x), \qquad t \in \mathbb{R}$$

denote the moment generating function of X, and let $\Lambda_\mu(t) := \ln g(t)$ be its logarithmic moment generating function. By Hölder's inequality, Λ_μ is a convex function on $\mathbb{R}$. Consider the Legendre transform $\Lambda_\mu^* : I_\mu \to \mathbb{R}$ of Λ_μ; this is the function

$$\Lambda_\mu^*(x) := \sup\{tx - \Lambda_\mu(t) \colon t \in \mathbb{R}\}.$$

One can show (see Proposition 2.6) that Λ_μ^* has finite moments of all orders.

We say that μ is *admissible* if it is non-degenerate, i.e. $\mathrm{Var}_\mu(X) > 0$, and satisfies the following conditions:

(i) There exists $r > 0$ such that $\mathbb{E}\big(e^{tX}\big) < \infty$ for all $t \in (-r, r)$; in particular, X has finite moments of all orders.

(ii) One of the following holds: (1) $x^* < +\infty$ and $P(X = x^*) = 0$, or (2) $x^* = +\infty$ and $\{\Lambda_\mu < \infty\} = \mathbb{R}$, or (3) $x^* = +\infty$, $\{\Lambda_\mu < \infty\}$ is bounded and μ is log-concave.

Finally, we say that μ satisfies *the Λ^*-condition* if

$$\lim_{x \uparrow x^*} \frac{-\ln \mu([x, \infty))}{\Lambda_\mu^*(x)} = 1.$$

We often express this condition in the form $-\ln \mu([x, \infty)) \sim \Lambda_\mu^*(x)$ as $x \uparrow x^*$, where "$a(x) \sim b(x)$ as $x \to A$" stands for "$\lim\limits_{x \to A} \frac{a(x)}{b(x)} = 1$". With these definitions, our main result is the following.

Theorem 1.1. *Let μ be an admissible even probability measure on $\mathbb{R}$ that satisfies the Λ^*-condition. Then, for any $\delta \in \big(0, \frac{1}{2}\big)$ and any $\varepsilon \in (0, 1)$ there exists $n_0(\mu, \delta, \varepsilon)$ such that*

$$\varrho_1(\mu_n, \delta) \geqslant (1 - \varepsilon)\mathbb{E}_\mu(\Lambda_\mu^*) \quad \textit{and} \quad \varrho_2(\mu_n, \delta) \leqslant (1 + \varepsilon)\mathbb{E}_\mu(\Lambda_\mu^*)$$

for every $n \geqslant n_0(\mu, \delta, \varepsilon)$. In particular, $\{\mu_n\}_{n=1}^\infty$ exhibits a sharp threshold, i.e. $\lim\limits_{n \to \infty} \varrho(\mu_n, \delta) = 0$, with "threshold constant" $\mathbb{E}_\mu(\Lambda_\mu^*)$.

In Section 4 we give an application of Theorem 1.1 to the case of the product p-measure $\nu_p^n := \nu_p^{\otimes n}$. For any $p \geqslant 1$ we denote by ν_p the probability distribution on $\mathbb{R}$ with density $(2\gamma_p)^{-1} \exp(-|x|^p)$, where $\gamma_p = \Gamma(1 + 1/p)$. We show that ν_p satisfies the Λ^*-condition.

Theorem 1.2. *For any $p \geqslant 1$ we have that*

$$\lim_{x \to \infty} \frac{-\ln(\nu_p[x, \infty))}{\Lambda_{\nu_p}^*(x)} = 1.$$

Note that the measure ν_p is admissible for all $1 \leqslant p < \infty$; it satisfies condition (ii-3) if $p = 1$ and condition (ii-2) for all $1 < p < \infty$. Therefore, Theorem 1.2 implies that if K_N is the convex hull of $N > n$ independent random vectors $\vec{X}_1, \ldots, \vec{X}_N$ with distribution ν_p^n then the expected measure $\mathbb{E}_{(\nu_p^n)^N}(\nu_p^n(K_N))$ exhibits a sharp threshold at $N = \exp((1 \pm \varepsilon)\mathbb{E}_{\nu_p}(\Lambda_{\nu_p}^*)n)$; for any $\delta \in \big(0, \frac{1}{2}\big)$ we have that $\lim\limits_{n \to \infty} \varrho(\nu_p^n, \delta) = 0$.

We close this introductory section with a brief review of the history of the problem that we study and related results. A variant of the question, in which $\mu_n(K_N)$ is replaced by the volume of K_N, has been studied in the case where μ is compactly supported. Define

$$\kappa = \kappa(\mu) := \frac{1}{2x^*} \int_{-x^*}^{x^*} \Lambda_\mu^*(x)dx.$$

In [14] the following threshold for the expected volume of K_N was established for a large class of compactly supported distributions μ: For every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup \left\{ (2x^*)^{-n}\mathbb{E}(|K_N|) \colon N \leqslant \exp((\kappa - \varepsilon)n) \right\} = 0 \tag{1.3}$$

and

$$\lim_{n\to\infty} \inf \left\{ (2x^*)^{-n}\mathbb{E}(|K_N|) \colon N \geqslant \exp((\kappa + \varepsilon)n) \right\} = 1. \tag{1.4}$$

This result generalized the work of Dyer, Füredi and McDiarmid [10] who studied the following two cases:

(i) If $\mu(\{1\}) = \mu(\{-1\}) = \frac{1}{2}$ then $\Lambda_\mu(t) = \ln(\cosh t)$ and $\Lambda_\mu^* : (-1,1) \to \mathbb{R}$ is given by

$$\Lambda_\mu^*(x) = \tfrac{1}{2}(1+x)\ln(1+x) + \tfrac{1}{2}(1-x)\ln(1-x),$$

and the result holds with $\kappa = \ln 2 - \frac{1}{2}$. This is the case of ± 1 polytopes.

(ii) If μ is the uniform distribution on $[-1,1]$, then $\Lambda_\mu(t) = \ln(\sinh t/t)$, and the result holds with

$$\kappa = \int_0^\infty \left(\frac{1}{u} - \frac{1}{e^u - 1} \right)^2 du.$$

The generalization from [14] states that if μ is an even, compactly supported, Borel probability measure on the real line and $0 < \kappa(\mu) < \infty$, then (1.3) holds for every $\varepsilon > 0$, and (1.4) holds for every $\varepsilon > 0$ provided that the distribution μ satisfies the Λ^*-condition.

Further sharp thresholds for the volume of various classes of random polytopes appear in [20] and [2], [3] where the same question is addressed for a number of cases where $\vec{X}_i$ have rotationally invariant densities. Exponential in the dimension upper and lower thresholds are obtained in [12] for the case where $\vec{X}_i$ are uniformly distributed in a simplex. General upper and lower thresholds have been obtained by Chakraborti, Tkocz and Vritsiou in [7] for some general families of distributions; see also [4].

2 Background and auxiliary results

As stated in the introduction, we consider an even Borel probability measure μ on the real line and a random variable X, on some probability space $(\Omega, \mathcal{F}, P)$, with distribution μ. In order to avoid trivialities we assume that $\mathrm{Var}_\mu(X) > 0$, and in particular that $p_\mu := \max\{P(X = x) : x \in \mathbb{R}\} < 1$. Recall that μ is even if $\mu(-B) = \mu(B)$ for every Borel subset B of $\mathbb{R}$.

For the proof of our main result we have to make a number of additional assumptions on μ. The first one is that there exists $r > 0$ such that

$$\mathbb{E}\big(e^{tX}\big) := \int_{\mathbb{R}} e^{tx}\, d\mu(x) < \infty \tag{2.1}$$

for all $t \in (-r, r)$. This assumption ensures that X has finite moments of all orders.

We define $x^* := \sup\{x \in \mathbb{R} \colon \mu([x,\infty)) > 0\}$ and $I_\mu := (-x^*, x^*)$. Note that we may have $x^* = \infty$. Our second assumption is that if $x^* < \infty$ then

$$P(X = x^*) = \mu(\{x^*\}) = 0. \tag{2.2}$$

Let $g(t) := \mathbb{E}\big(e^{tX}\big)$ for $t \in \mathbb{R}$ and $\Lambda_\mu(t) := \ln g(t)$. One can easily check that Λ_μ is an even convex function and $\Lambda_\mu(0) = 0$, therefore, Λ_μ is a non-negative function. The assumption (2.1) implies that the interval $J_\mu := \{\Lambda_\mu < \infty\}$ is a non-degenerate symmetric interval, possibly the whole real line. We define $t^* = \sup J_\mu$. Then, Λ_μ is C^∞ and strictly convex on $(-t^*, t^*)$ (for the first assertion see [21, Section 1.3] or [13, Section 2]; the strict convexity of Λ_μ follows from the fact that Λ_μ' is strictly increasing on $(-t^*, t^*)$, as explained below).

For every $t \in (-t^*, t^*)$ we define the probability measure P_t on $(\Omega, \mathcal{F})$ by

$$P_t(A) := \mathbb{E}\big(e^{tX-\Lambda_\mu(t)}\mathbb{1}_A\big), \qquad A \in \mathcal{F}.$$

Define also $\mu_t(B) := P_t(X \in B)$ for any Borel subset B of $\mathbb{R}$. Since $dP_t = e^{tX-\Lambda_\mu(t)}dP$ and $\mathbb{E}_\mu(X^k e^{tX}) < +\infty$ for all $k \geqslant 1$ and $t \in J_\mu$, we see that μ_t has finite moments of all orders. Also, differentiating twice Λ_μ and taking into account the definition of P_t, we check that

$$\mathbb{E}_t(X) = \Lambda'_\mu(t) \quad \text{and} \quad \mathrm{Var}_t(X) = \Lambda''_\mu(t), \tag{2.3}$$

where $\mathbb{E}_t$ and Var_t denote expectation and variance with respect to P_t. Notice that $P_0 = P$ and $\mu_0 = \mu$. Since μ is non-degenerate we have that $\mu_t(\{c\}) \neq 1$ for all $c \in \mathbb{R}$ and $t \in (-t^*, t^*)$, which implies that $\Lambda''_\mu(t) > 0$ for all $t \in (-t^*, t^*)$. It follows that Λ'_μ is strictly increasing and since $\Lambda'_\mu(0) = 0$ we conclude that Λ_μ is strictly increasing on $[0, t^*)$.

Let $m : [0, x^*) \to [0, \infty)$ be defined by

$$m(x) = -\ln \mu([x, \infty)).$$

It is clear that m is non-decreasing. Observe that, from Markov's inequality, for any $x \in (0, x^*)$ and any $t \geqslant 0$, we have $\mathbb{E}\big(e^{tX}\big) \geqslant e^{tx}\mu([x, \infty))$, and hence,

$$\Lambda_\mu(t) \geqslant tx - m(x). \tag{2.4}$$

An important case where (2.1) is satisfied is when μ is log-concave. Recall that a Borel measure μ on $\mathbb{R}$ is called log-concave if $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for all compact subsets A and B of $\mathbb{R}$ and any $\lambda \in (0, 1)$. A function $f : \mathbb{R} \to [0, \infty)$ is called log-concave if its support $\{f > 0\}$ is an interval in $\mathbb{R}$ and the restriction of $\ln f$ to it is concave. Any non-degenerate log-concave probability measure μ on $\mathbb{R}$ has a log-concave density $f := f_\mu$. Since f has finite positive integral, one can check that there exist constants $A, B > 0$ such that $f(x) \leqslant Ae^{-B|x|}$ for all $x \in \mathbb{R}$ (see [6, Lemma 2.2.1]). In particular, f has finite moments of all orders. We refer to [6] for more information on log-concave probability measures.

The next lemma describes the behavior of Λ_μ at the endpoints of J_μ for a log-concave probability measure with unbounded support on $\mathbb{R}$.

Lemma 2.1. *Let μ be an even log-concave probability measure on $\mathbb{R}$ with*

$$x^* = \sup\{x \in \mathbb{R} \colon \mu([x, \infty)) > 0\} = +\infty.$$

If J_μ is a bounded interval, then $J_\mu = (-t^*, t^*)$ for some $t^* > 0$ and $\lim_{t \uparrow t^*} \Lambda_\mu(t) = +\infty$.

Proof. Let f denote the density of μ. Since $x^* = +\infty$, we have that $\mathrm{supp}(\mu) = \mathbb{R}$, and hence, f can be written as $f = e^{-q}$, where $q : \mathbb{R} \to \mathbb{R}$ is an even convex function. By symmetry, it is enough to consider the convergence of $\Lambda_\mu(t)$ for $t > 0$.

Note that, since q is even and convex on $\mathbb{R}$, we have $\lim_{x\to+\infty} q(x) = +\infty$ and the function $u(x) = \frac{q(x)-q(0)}{x}$ is increasing on $(0, \infty)$. First we observe that we cannot have $\lim_{x\to\infty} u(x) = \infty$. If this was the case then we would have $\lim_{x\to\infty} \frac{q(x)}{x} = \infty$, and hence

$$\int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{x\left(t - \frac{q(x)}{x}\right)}dx < \infty$$

for all $t > 0$, i.e. $\Lambda_\mu(t) < \infty$ for all $t > 0$, which is not our case.

Therefore, since u is increasing, there exists $t^* > 0$ such that

$$\lim_{x\to\infty} u(x) = \lim_{x\to\infty} \frac{q(x) - q(0)}{x} = t^*.$$

Assume that $0 < t < t^*$. If $\varepsilon > 0$ satisfies $t + \varepsilon < t^*$ then there exists $M > 0$ such that $u(x) - t > \varepsilon$ for all $x \geqslant M$ and then

$$\int_0^\infty e^{tx} f(x)dx = e^{-q(0)} \int_0^\infty e^{-x(u(x)-t)}dx < \infty,$$

which shows that $t \in J_\mu$, and hence $(-t^*, t^*) \subseteq J_\mu$.

On the other hand, if $t = t^*$ then using the fact that $u(x) \leqslant t^*$ for all $x > 0$ we get

$$\int_0^\infty e^{t^* x} f(x)dx = e^{-q(0)} \int_0^\infty e^{x(t^* - u(x))}dx = +\infty.$$

This shows that $J_\mu = (-t^*, t^*)$.

Finally, if we consider a strictly increasing sequence $t_n \to t^*$ then by the monotone convergence theorem we get

$$e^{\Lambda_\mu(t_n)} = \int_0^\infty e^{t_n x} f(x)dx \longrightarrow \int_0^\infty e^{t^* x} f(x)dx = +\infty,$$

which shows that $\lim_{t\uparrow t^*} \Lambda_\mu(t) = +\infty$. □

Definition 2.2. Let μ be an even probability measure on $\mathbb{R}$. We will call μ *admissible* if it satisfies (2.1) and (2.2), as well as one of the following conditions:

(i) μ is compactly supported, i.e. $x^* < +\infty$.

(ii) $x^* = +\infty$ and $\{\Lambda_\mu < \infty\} = \mathbb{R}$.

(iii) $x^* = +\infty$, $\{\Lambda_\mu < \infty\}$ is bounded and μ is log-concave.

Note that if $x^* < +\infty$ then $\{\Lambda_\mu < \infty\} = \mathbb{R}$. Taking also into account Lemma 2.1 we see that, in all the cases that we consider, the interval $J_\mu = \{\Lambda_\mu < \infty\}$ is open, i.e. $J_\mu = (-t^*, t^*)$ where $t^* = \sup J_\mu$.

The next lemma describes the behavior of Λ_μ' for an admissible measure μ. The first case was treated in [14].

Lemma 2.3. *Let μ be an admissible even Borel probability measure on the real line. Then, $\Lambda_\mu' : J_\mu \to I_\mu$ is strictly increasing and surjective. In particular,*

$$\lim_{t\to\pm t^*} \Lambda_\mu'(t) = \pm x^*.$$

Proof. We have already explained that, since $(\Lambda_\mu')'(t) = \Lambda_\mu''(t) = \mathrm{Var}_t(X) > 0$, the function Λ_μ' is strictly increasing. Now, we consider the three cases of the lemma separately.

(i) From the inequality $-x^* e^{tX} \leqslant X e^{tX} \leqslant x^* e^{tX}$, which holds with probability 1 for each fixed t, and the formula $\Lambda_\mu'(t) = \mathbb{E}\big(Xe^{tX}\big)/\mathbb{E}\big(e^{tX}\big)$, we easily check that $\Lambda_\mu'(t) \in (-x^*, x^*)$ for every $t \in \mathbb{R}$.

It remains to show that Λ_μ' is onto I_μ. Let $x \in (0, x^*)$ and $y \in (x, x^*)$. Since $\Lambda_\mu(t) \geqslant ty - m(y)$ for all $t \geqslant 0$, we have that $\Lambda_\mu\big(m(y)/(y-x)\big) \geqslant xm(y)/(y-x)$. It follows that if we consider the function $q_x(t) := tx - \Lambda_\mu(t)$, then $q_x(0) = 0$ and $q_x\big(m(y)/(y-x)\big) \leqslant 0$. Since q_x is concave and $q_x'(0) = x > 0$, this shows that q_x attains its maximum at some point in the open interval $\big(0, m(y)/(y-x)\big)$, and hence, $\Lambda_\mu'(t) = x$ for some t in this interval. The same argument applies for all $x \in (-x^*, 0)$. Finally, for $x = 0$ we have that $\Lambda_\mu'(0) = x$.

(ii) We apply the same argument as in (i).

(iii) Assume that Λ_μ' is bounded from above. Then, there exists $x > 0$ such that $\Lambda_\mu'(t) < x$ for all $t \in J_\mu$. We consider the function $q_x : J_\mu \to \mathbb{R}$ with $q_x(t) = tx - \Lambda_\mu(t)$. Then, q_x is strictly increasing. However, $\lim_{t\uparrow t^*} q_x(t) = -\infty$ because $\lim_{t\uparrow t^*} \Lambda_\mu(t) = +\infty$ by Lemma 2.1, which leads to a contradiction. □

Let μ be an admissible even Borel probability measure on the real line. Lemma 2.3 allows us to define $h\colon I_\mu \to J_\mu$ by $h := (\Lambda'_\mu)^{-1}$. Observe that h is a strictly increasing C^∞ function and

$$h'(x) = \frac{1}{\Lambda''_\mu(h(x))}. \tag{2.5}$$

Next, consider the Legendre transform of Λ_μ. This is the function

$$\Lambda^*_\mu(x) := \sup\left\{tx - \Lambda_\mu(t)\colon t \in \mathbb{R}\right\}, \qquad x \in \mathbb{R}.$$

In fact, since $tx - \Lambda_\mu(t) < 0$ for $t < 0$ when $x \in [0, x^*)$, we have that $\Lambda^*_\mu(x) = \sup\{tx - \Lambda_\mu(t)\colon t \geqslant 0\}$ in this case, and similarly $\Lambda^*_\mu(x) := \sup\{tx - \Lambda_\mu(t)\colon t \leqslant 0\}$ when $x \in (-x^*, 0]$.

The basic properties of Λ^*_μ are described in the next lemma (for a proof, see e.g. [13, Proposition 2.12]).

Lemma 2.4. *Let μ be an admissible even probability measure on $\mathbb{R}$. Then,*

(i) $\Lambda^*_\mu \geqslant 0$, $\Lambda^*_\mu(0) = 0$ *and* $\Lambda^*_\mu(x) = \infty$ *for* $x \in \mathbb{R} \setminus [-x^*, x^*]$.

(ii) *For every $x \in I_\mu$ we have $\Lambda^*_\mu(x) = tx - \Lambda_\mu(t)$ if and only if $\Lambda'_\mu(t) = x$; hence*

$$\Lambda^*_\mu(x) = xh(x) - \Lambda_\mu(h(x)) \qquad \text{for } x \in I_\mu.$$

(iii) Λ^*_μ *is a strictly convex C^∞ function on I_μ, and*

$$(\Lambda^*_\mu)'(x) = h(x).$$

(iv) Λ^*_μ *attains its unique minimum on I_μ at $x = 0$.*

(v) $\Lambda^*_\mu(x) \leqslant m(x)$ *for all $x \in [0, x^*)$; this is a direct consequence of* (2.4).

Corollary 2.5. *We have that* $\lim_{x\uparrow x^*} \Lambda^*_\mu(x) = +\infty$.

Proof. If $x^* = +\infty$ then the convexity of Λ^*_μ and the fact that $(\Lambda^*_\mu)'(x) > 0$ for all $x > 0$ (which is a consequence of Lemma 2.4 (iv) and of the fact that $(\Lambda^*_\mu)'' = h' > 0$) imply that $\lim_{x\uparrow x^*} \Lambda^*_\mu(x) = +\infty$.

Next, assume that $x^* < +\infty$. Since $\Lambda'_\mu(t) \leqslant x^*$ for all t, the function $t \mapsto tx^* - \Lambda_\mu(t)$ is non-decreasing. Therefore,

$$\Lambda^*_\mu(x^*) = \sup_{t\in\mathbb{R}}[tx^* - \Lambda_\mu(t)] = \lim_{t\uparrow\infty}[tx^* - \Lambda_\mu(t)].$$

However,

$$\lim_{t\uparrow\infty} e^{-(tx^* - \Lambda_\mu(t))} = \lim_{t\uparrow\infty} e^{-tx^*} g(t) = \lim_{t\uparrow\infty} \mathbb{E}\big(e^{t(X-x^*)}\big) = \mathbb{E}\Big(\lim_{t\uparrow\infty} e^{t(X-x^*)}\Big) = P(X = x^*),$$

the third equality being a consequence of the dominated convergence theorem. It follows that $\Lambda^*_\mu(x^*) = -\ln P(X = x^*) = +\infty$. Since Λ^*_μ is lower semi-continuous on $\mathbb{R}$ as the pointwise supremum of the continuous functions $x \mapsto tx - \Lambda_\mu(t)$, $t \in \mathbb{R}$, it follows that $\lim_{x\uparrow x^*} \Lambda^*_\mu(x) = +\infty$. □

The next result generalizes an observation from [5] which states that Λ^*_μ has finite moments of all orders in the case where μ is absolutely continuous with respect to Lebesgue measure. The more general statement of the next proposition can be found as an exercise in [9].

Proposition 2.6. *Let μ be an even probability measure on $\mathbb{R}$. Then,*

$$\int_{I_\mu} e^{\Lambda^*_\mu(x)/2} d\mu(x) \leqslant 4.$$

In particular, for all $p \geqslant 1$ we have that $\int_{I_\mu} (\Lambda^*_\mu(x))^p \, d\mu(x) < +\infty$.

Sketch of the proof. We define $F(x) = \mu((-\infty, x])$ and for any fixed $z > 0$ we set $\alpha(x) = F(x) - F(z)$ and $\beta(x) = \exp(I(x)/2)$ where $I(x) = 0$ if $x \leqslant 0$ and $I(x) = \Lambda_\mu^*(x)$ if $x > 0$. Note that α is right continuous and increasing, and β is increasing. Applying [15, Theorem 21.67 (iv)] we write

$$\int_0^z \beta(x)d\alpha(x) + \int_0^z \alpha(x-)d\beta(x) = \alpha(z)e^{I(z+)/2} - \alpha(0-)e^{I(0-)/2},$$

where, for a function f, we denote $f(x+) = \lim_{y\to x^+} f(y)$ and $f(x-) = \lim_{y\to x^-} f(y)$. It follows that, for every $0 < z < x^*$,

$$\begin{aligned}\int_0^z e^{\Lambda_\mu^*(x)/2}d\mu(x) &= \int_0^z \beta(x)d\alpha(x) = -\int_0^z \alpha(x-)d\beta(x) + \alpha(z)e^{I(z+)/2} - \alpha(0-)e^{I(0-)/2}\\ &\leqslant \int_0^z e^{-I(x)}d\beta(x) + 1,\end{aligned}$$

where we have used the fact that $-\alpha(x-) = \mu([x,z]) \leqslant e^{-\Lambda_\mu^*(x)}$ and $I(0-) = 0$, $-\alpha(0-) \leqslant 1$. Finally, we note that

$$\int_0^z e^{-I(x)}d\beta(x) + 1 = \int_0^z \beta(x)^{-2}d\beta(x) + 1 \leqslant \int_1^\infty t^{-2}dt + 1 = 2,$$

because β is strictly increasing and continuous on $[0,z]$ and $\beta(0) = 1$. The result follows by symmetry. □

We close this section by recalling the Λ^*-condition that was already mentioned in the introduction.

Definition 2.7. Let μ be an admissible even Borel probability measure on the real line. Recall that $\Lambda_\mu^*(x) \leqslant m(x)$ for all $x \in [0, x^*)$. We shall say that μ *satisfies the* Λ^**-condition* if

$$\lim_{x\uparrow x^*} \frac{m(x)}{\Lambda_\mu^*(x)} = 1.$$

3 Proof of the main theorem

Let μ be an admissible even Borel probability measure on the real line. Recall that $\mu_n = \mu \otimes \cdots \otimes \mu$ (n times), and hence the support of μ_n is $I_{\mu_n} = I_\mu^n$. The logarithmic Laplace transform of μ_n is defined by

$$\Lambda_{\mu_n}(\xi) = \ln\Big(\int_{\mathbb{R}^n} e^{\langle \xi, z\rangle} d\mu_n(z)\Big), \qquad \xi \in \mathbb{R}^n$$

and the Cramer transform of μ_n is the Legendre transform of Λ_{μ_n}, defined by

$$\Lambda_{\mu_n}^*(x) = \sup_{\xi\in\mathbb{R}^n} \left\{\langle x, \xi\rangle - \Lambda_{\mu_n}(\xi)\right\}, \qquad x \in \mathbb{R}^n.$$

Since μ_n is a product measure, we can easily check that $\Lambda_{\mu_n}^*(x) = \sum_{i=1}^n \Lambda_\mu^*(x_i)$ for all $x = (x_1, \dots, x_n) \in I_{\mu_n}$, which implies that

$$\int_{I_{\mu_n}} e^{\Lambda_{\mu_n}^*(x)/2} d\mu_n(x) = \prod_{i=1}^n \left(\int_{I_\mu} e^{\Lambda_\mu^*(x_i)/2} d\mu(x_i)\right) < +\infty.$$

In particular, for all $p \geqslant 1$ we have that $\int_{I_{\mu_n}} (\Lambda_{\mu_n}^*(x))^p \, d\mu_n(x) < +\infty$. We also define the parameter

(3.1) $$\beta(\mu_n) = \frac{\mathrm{Var}_{\mu_n}(\Lambda_{\mu_n}^*)}{(\mathbb{E}_{\mu_n}(\Lambda_{\mu_n}^*))^2}.$$

Since $\mu_n = \mu \otimes \cdots \otimes \mu$, we have $\mathrm{Var}_{\mu_n}(\Lambda^*_{\mu_n}) = n\mathrm{Var}_\mu(\Lambda^*_\mu)$ and $\mathbb{E}_{\mu_n}(\Lambda^*_\mu) = n\mathbb{E}_\mu(\Lambda^*_\mu)$. Therefore,

$$\beta(\mu_n) = \frac{\mathrm{Var}_{\mu_n}(\Lambda^*_{\mu_n})}{(\mathbb{E}_{\mu_n}(\Lambda^*_{\mu_n}))^2} = \frac{\beta(\mu)}{n},$$

where $\beta(\mu)$ is a finite positive constant which is independent of n. In particular, $\beta(\mu_n) \to 0$ as $n \to \infty$.

In order to estimate $\varrho_i(\mu_n, \delta)$, $i = 1, 2$, we shall follow the approach of [5]. For every $r > 0$ we define

$$B_r(\mu_n) := \{x \in \mathbb{R}^n : \Lambda^*_{\mu_n}(x) \leqslant r\}.$$

Note that, since $\Lambda^*_{\mu_n}(x) = \sum_{i=1}^n \Lambda^*_\mu(x_i)$ for all $x = (x_1, \ldots, x_n)$ and $\Lambda^*_\mu(y)$ increases to $+\infty$ as $y \uparrow x^*$, for every $r > 0$ there exists $0 < M_r < x^*$ such that $B_r(\mu_n) \subseteq [-M_r, M_r]^n \subseteq I^n_\mu$, and hence $B_r(\mu_n)$ is a compact subset of I^n_μ.

For any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x. Then we define

$$\varphi_{\mu_n}(x) = \inf\{\mu_n(H) : H \in \mathcal{H}(x)\}.$$

The function φ_{μ_n} is called Tukey's half-space depth. We refer the reader to the survey article of Nagy, Schütt and Werner [18] for a comprehensive account and references. We start with the upper threshold. Note that the Λ^*-condition is not required for this result.

Theorem 3.1. *Let μ be an even probability measure on $\mathbb{R}$. Then, for any $\delta \in \left(0, \frac{1}{2}\right)$ there exist $c(\mu, \delta) > 0$ and $n_0(\mu, \delta) \in \mathbb{N}$ such that*

$$\varrho_1(\mu_n, \delta) \geqslant \left(1 - \frac{c(\mu, \delta)}{\sqrt{n}}\right) \mathbb{E}_\mu(\Lambda^*_\mu).$$

Proof. The standard approach towards an upper threshold is based on the next fact which holds true in general, for any Borel probability measure on $\mathbb{R}^n$. For every $r > 0$ and every $N > n$ we have

$$\mathbb{E}_{\mu_n^N}(\mu_n(K_N)) \leqslant \mu_n(B_r(\mu_n)) + N \exp(-r). \tag{3.2}$$

This estimate appeared originally in [10] and follows from the observation that (by the definition of φ_{μ_n}, Markov's inequality and the definition of $\Lambda^*_{\mu_n}$) for every $x \in \mathbb{R}^n$ we have

$$\varphi_{\mu_n}(x) \leqslant \exp(-\Lambda^*_{\mu_n}(x)). \tag{3.3}$$

We use (3.2) in the following way. Let $T_1 := \mathbb{E}_\mu(\Lambda^*_\mu)$ and $T_n := \mathbb{E}_{\mu_n}(\Lambda^*_{\mu_n}) = T_1 n$. Then, for all $\zeta \in (0, 1)$, from Chebyshev's inequality we have that

$$\mu_n(\{\Lambda^*_{\mu_n} \leqslant T_n - \zeta T_n\}) \leqslant \mu_n(\{|\Lambda^*_{\mu_n} - T_n| \geqslant \zeta T_n\}) \leqslant \frac{\mathbb{E}_{\mu_n}|\Lambda^*_{\mu_n} - T_n|^2}{\zeta^2 T_n^2} = \frac{\beta(\mu_n)}{\zeta^2} = \frac{\beta(\mu)}{\zeta^2 n}.$$

Equivalently,

$$\mu_n(B_{(1-\zeta)T_n}(\mu_n)) \leqslant \frac{\beta(\mu)}{\zeta^2 n}.$$

Let $\delta \in \left(0, \frac{1}{2}\right)$. We may find $n_0(\mu, \delta)$ such that if $n \geqslant n_0(\mu, \delta)$ then $8\beta(\mu)/n < \delta < 1/2$. We choose $\zeta = \sqrt{2\beta(\mu)/n\delta}$, which implies that

$$\mu(B_{(1-\zeta)T_n}(\mu_n)) \leqslant \frac{\delta}{2}.$$

From (3.2) we see that

$$\begin{aligned} \sup\{\mathbb{E}_{\mu_n^N}(\mu_n(K_N)) : N \leqslant e^{(1-2\zeta)T_n}\} &\leqslant \mu_n(B_{(1-\zeta)T_n}(\mu_n)) + e^{(1-2\zeta)T_n} e^{-(1-\zeta)T_n} \\ &\leqslant \frac{\delta}{2} + e^{-\zeta T_n} \leqslant \delta, \end{aligned}$$

provided that $\zeta T_n \geqslant \ln(2/\delta)$. Since $T_n = T_1 n$, the last condition takes the form $T_1 n \geqslant c_1 \ln(2/\delta)\sqrt{\delta n/\beta(\mu)}$ and it is certainly satisfied if $n \geqslant n_0(\mu)$, where $n_0(\mu)$ depends only on $\beta(\mu)$ because $\sqrt{\delta}\ln(2/\delta)$ is bounded on $\left(0, \frac{1}{2}\right)$. By the choice of ζ we conclude that

$$\varrho_1(\mu_n, \delta) \geqslant \left(1 - \sqrt{8\beta(\mu)/n\delta}\right) \mathbb{E}_\mu(\Lambda_\mu^*)$$

as claimed. $\square$

For the proof of the lower threshold we need a basic fact that plays a main role in the proof of all the lower thresholds that have been obtained so far. For a proof see [14, Lemma 4.1].

Lemma 3.2. *For every Borel subset A of $\mathbb{R}^n$ we have that*

$$1 - \mu_n^N(K_N \supseteq A) \leqslant \binom{N}{n} p_\mu^{N-n} + 2\binom{N}{n}\left(1 - \inf_{x\in A} \varphi_{\mu_n}(x)\right)^{N-n}.$$

where $p_\mu = \max\{P(X = x) : x \in \mathbb{R}\} < 1$. Therefore,

$$\mathbb{E}_{\mu_n^N}\left[\mu_n(K_N)\right] \geqslant \mu_n(A)\left(1 - \binom{N}{n} p_\mu^{N-n} - 2\binom{N}{n}\left(1 - \inf_{x\in A} \varphi_{\mu_n}(x)\right)^{N-n}\right). \tag{3.4}$$

We are going to apply Lemma 3.2 with $A = B_{(1+\varepsilon)T_n}(\mu_n)$, using Chebyshev's inequality exactly as in the proof of Theorem 3.1. From (3.4) it is clear that we will also need a lower bound for $\inf_{x\in B_{(1+\varepsilon)T_n}(\mu_n)} \varphi_{\mu_n}(x)$ which will imply that

$$2\binom{N}{n}\left(1 - \inf_{x\in B_{(1+\varepsilon)T_n}(\mu_n)} \varphi_{\mu_n}(x)\right)^{N-n} = o_n(1).$$

The main technical step is to obtain the next inequality.

Theorem 3.3. *Let μ be an admissible even probability measure on $\mathbb{R}$ that satisfies the Λ^*-condition, i.e. $m(x) \sim \Lambda_\mu^*(x)$ as $x \uparrow x^*$. Then, for every $\zeta > 0$, there exists $n_0(\mu, \zeta) \in \mathbb{N}$, depending only on ζ and μ, such that for all $r > 0$ and all $n \geqslant n_0(\mu, \zeta)$ we have that*

$$\inf_{x\in B_r(\mu_n)} \varphi_{\mu_n}(x) \geqslant \exp(-(1+\zeta)r - 2\zeta n).$$

Proof. Let $x \in B_r(\mu_n)$ and H_1 be a closed half-space with $x \in \partial H_1$. There exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $H_1 = \{y \in \mathbb{R}^n : \langle v, y - x\rangle \geqslant 0\}$. Consider the function $q : B_r(\mu_n) \to \mathbb{R}$, $q(w) = \langle v, w\rangle$. Since q is continuous and $B_r(\mu_n)$ is compact, q attains its maximum at some point $z \in B_r(\mu_n)$. Define $H = \{y \in \mathbb{R}^n : \langle v, y - z\rangle \geqslant 0\}$. Then, $z \in \partial(H)$ and for every $w \in B_r(\mu_n)$ we have $\langle v, w\rangle \leqslant \langle v, z\rangle$, which shows that $\partial(H)$ supports $B_r(\mu_n)$ at z. Moreover, $H \subseteq H_1$ and hence $P(\vec{X} \in H) \leqslant P(\vec{X} \in H_1)$. This shows that $\inf\{\varphi_{\mu_n}(x) : x \in B_r(\mu_n)\}$ is attained for some closed half-space H whose bounding hyperplane supports $B_r(\mu_n)$. Therefore, for the proof of the theorem it suffices to show that given $\zeta > 0$ we may find $n_0(\mu, \zeta)$ so that if $n \geqslant n_0(\mu, \zeta)$ then

$$P\big(\vec{X} \in H\big) \geqslant \exp(-(1+\zeta)r - 2\zeta n) \tag{3.5}$$

for any closed half-space H whose bounding hyperplane supports $B_r(\mu_n)$.

Let H be such a half-space. Then, there exists $x \in \partial(B_r(\mu_n))$ such that

$$P\big(\vec{X} \in H\big) = P\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right),$$

where $t_i = h(x_i)$, because the normal vector to H is $\nabla\Lambda_{\mu_n}^*(x)$ and $(\Lambda_\mu^*)' = h$ by Lemma 2.4 (iii). We fix this x for the rest of the proof. By symmetry and independence we may assume that $x_i \geqslant 0$ for all $1 \leqslant i \leqslant n$. Recall that $\Lambda_\mu^*(0) = 0$ and that μ satisfies the Λ^*-condition: we have $m(x) \sim \Lambda_\mu^*(x)$ as $x \uparrow x^*$. Therefore, we can find $M > \tau > 0$ with the following properties:

(i) If $0 \leqslant x \leqslant \tau$ then $0 \leqslant \Lambda_\mu^*(x) \leqslant \zeta$.

(ii) If $M < x < x^*$ then $P(X \geqslant x) \geqslant \exp(-\Lambda_\mu^*(x)(1+\zeta))$.

Set $[n] = \{1, \ldots, n\}$. We consider the sets of indices

$$\begin{aligned} A_1 = A_1(x) &:= \{i \in [n] : x_i < \tau\} \\ A_2 = A_2(x) &:= \{i \in [n] : \tau \leqslant x_i \leqslant M\}, \\ A_3 = A_3(x) &:= \{i \in [n] : x_i > M\} \end{aligned}$$

and the probabilities

$$P_j = P_j(x) := P\left(\sum_{i \in A_j} t_i(X_i - x_i) \geqslant 0\right) \qquad j = 1, 2, 3.$$

By independence we have that

$$P\big(\vec{X} \in H\big) = P\left(\sum_{i=1}^n t_i(X_i - x_i) \geqslant 0\right) \geqslant P_1 P_2 P_3.$$

We will give lower bounds for P_1, P_2 and P_3 separately.

Lemma 3.4. *We have that*

$$P_1 \geqslant \exp\left(-\sum_{i \in A_1} (\Lambda_\mu^*(x_i) + \zeta) - c_1 \ln|A_1| - c_2\right),$$

where $c_1, c_2 > 0$ depend only on ζ and μ.

Proof. We write

$$P_1 = P\left(\sum_{i \in A_1} t_i(X_i - x_i) \geqslant 0\right) \geqslant P\left(\sum_{i \in A_1} t_i(X_i - \tau) \geqslant 0\right), \tag{3.6}$$

and use the following fact (see [14, Lemma 4.3]): For every $\tau \in (0, x^*)$, there exists $c(\tau) > 0$ depending only on τ and μ, such that for any $k \in \mathbb{N}$ and any $v_1, \ldots, v_k \in \mathbb{R}$ with $\sum_{i=1}^k v_i > 0$ we have that

$$P\left(\sum_{i=1}^k v_i(X_i - \tau) \geqslant 0\right) \geqslant c(\tau)\, k^{-3/2}\, e^{-k\Lambda_\mu^*(\tau)}.$$

Combining the above with (3.6) and using the simple bound $\Lambda_\mu^*(\tau) \leqslant \zeta \leqslant \Lambda_\mu^*(x) + \zeta$ for x in $[0, \tau]$, we conclude the proof of the lemma. □

Lemma 3.5. *We have that*

$$P_3 \geqslant \exp\left(-(1+\zeta) \sum_{i \in A_3} \Lambda_\mu^*(x_i)\right).$$

Proof. By independence, we can write

$$P_3 = P\left(\sum_{i \in A_3} t_i(X_i - x_i) \geqslant 0\right) \geqslant \prod_{i \in A_3} P(X_i \geqslant x_i).$$

By the choice of M we see that

$$P(X_i \geqslant x_i) \geqslant e^{-\Lambda_\mu^*(x_i)(1+\zeta)}$$

for all $i \in A_3$, and this immediately gives the lemma. □

Lemma 3.6. *There exist* $c_3, c_4 > 0$ *depending only on* ζ, M *and* μ, *such that*

$$P\left(\sum_{i\in A_2} t_i(X_i - x_i) \geqslant 0\right) \geqslant \exp\left(-\sum_{i\in A_2} \Lambda_\mu^*(x_i) - c_3\sqrt{|A_2|} - c_4\right).$$

The proof of this estimate requires some preparation. Without loss of generality, we may assume that $A_2 = \{1, \dots, k\}$ for some $k \leqslant n$. Recall that $t_i = h(x_i)$ for each i, and that this is equivalent to having $x_i = \Lambda_\mu'(t_i)$ for each i (see Lemma 2.4 (ii)). Define the probability measure $P_{x_1,\dots,x_k}$ on $(\Omega, \mathcal{F})$, by

$$P_{x_1,\dots,x_k}(A) := \mathbb{E}\left[\mathbb{1}_A \cdot \exp\left(\sum_{i=1}^k (t_i X_i - \Lambda_\mu(t_i))\right)\right]$$

for $A \in \mathcal{F}$. Direct computation shows that, under $P_{x_1,\dots,x_k}$, the random variables $t_1X_1, \dots, t_kX_k$ are independent, with mean, variance and absolute central third moment given by

$$\begin{aligned}
\mathbb{E}_{x_1,\dots,x_k}(t_iX_i) &= t_i\Lambda_\mu'(t_i) = t_ix_i, \\
\mathbb{E}_{x_1,\dots,x_k}\left(|t_i(X_i - x_i)|^2\right) &= t_i^2\Lambda_\mu''(t_i), \\
\mathbb{E}_{x_1,\dots,x_k}\left(|t_i(X_i - x_i)|^3\right) &= |t_i|^3\,\mathbb{E}_{t_i}\left(\left|X - \Lambda_\mu'(t_i)\right|^3\right),
\end{aligned}$$

respectively. Set $\sigma_i^2 := t_i^2\Lambda_\mu''(t_i)$,

$$s_k^2 := \sum_{i=1}^k \mathbb{E}_{x_1,\dots,x_k}\left(|t_i(X_i - x_i)|^2\right) = \sum_{i=1}^k t_i^2\Lambda_\mu''(t_i) = \sum_{i=1}^k \sigma_i^2$$

and

$$S_k := \sum_{i=1}^k t_i(X_i - x_i),$$

and let $F_k\colon \mathbb{R} \to \mathbb{R}$ denote the cumulative distribution function of the random variable S_k/s_k under the probability law $P_{x_1,\dots,x_k}$: $F_k(x) := P_{x_1,\dots,x_k}(S_k \leqslant xs_k)$ $(x \in \mathbb{R})$. Write also ν_k for the probability measure on $\mathbb{R}$ defined by $\nu_k(-\infty, x] := F_k(x)$ $(x \in \mathbb{R})$. Notice that $\mathbb{E}_{x_1,\dots,x_k}(S_k/s_k) = 0$ and $\mathrm{Var}_{x_1,\dots,x_k}(S_k/s_k) = 1$.

Lemma 3.7. *The following identity holds*:

$$P\left(\sum_{i=1}^k t_i(X_i - x_i) \geqslant 0\right) = \left(\int_{[0,\infty)} e^{-s_k u}\, d\nu_k(u)\right) \exp\left(-\sum_{i=1}^k \Lambda_\mu^*(x_i)\right).$$

Proof. By definition of the measure $P_{x_1,\dots,x_k}$, we have that

$$P\left(\sum_{i=1}^k t_i(X_i - x_i) \geqslant 0\right) = P(S_k \geqslant 0) = \mathbb{E}_{x_1,\dots,x_k}\left[\mathbb{1}_{[0,\infty)}(S_k) \cdot \exp\left(-\sum_{i=1}^k (t_iX_i - \Lambda_\mu(t_i))\right)\right].$$

It follows that

$$P\left(\sum_{i=1}^k t_i(X_i - x_i) \geqslant 0\right) = \int_{[0,\infty)} e^{-s_k u}\, d\nu_k(u) \cdot \exp\left(\sum_{i=1}^k (\Lambda_\mu(t_i) - t_ix_i)\right),$$

and the lemma now follows from Lemma 2.4 (ii). ☐

We will also use the following consequence of the Berry-Esseen theorem (cf. [11], p. 544).

Lemma 3.8. *For any $a, b > 0$, there exist $k_0 \in \mathbb{N}$ and $\theta > 0$ with the following property: If $k \geqslant k_0$, and if $Y_1, \ldots, Y_k$ are independent random variables with*

$$\mathbb{E}(Y_i) = 0, \quad \sigma_i^2 := \mathbb{E}(Y_i^2) \geqslant a, \quad \mathbb{E}(|Y_i|^3) \leqslant b,$$

then

$$\mathbb{P}\left(0 \leqslant \sum_{i=1}^{k} Y_i \leqslant \sigma\right) \geqslant \theta,$$

where $\sigma^2 = \sigma_1^2 + \cdots + \sigma_k^2$.

Proof of Lemma 3.6. Consider the random variables $Y_i := t_i(X_i - x_i)$, $i \in A_2 = \{1, \ldots, k\}$, which are independent with respect to $P_{x_1,\ldots,x_k}$ and satisfy $\mathbb{E}_{x_1,\ldots,x_k}(Y_i) = 0$ for all $1 \leqslant i \leqslant k$. Set $J_\mu^* = (\Lambda_\mu')^{-1}([\tau, M])$. Since $\tau \leqslant x_i \leqslant M$ for all $1 \leqslant i \leqslant k$, we see that

$$\sigma_i^2 = \mathbb{E}_{x_1,\ldots,x_k}\left(Y_i^2\right) = t_i^2 \Lambda_\mu''(t_i) \geqslant \min_{t \in J_\mu^*} t^2 \Lambda_\mu''(t) =: a_1 > 0$$

and

$$\mathbb{E}_{x_1,\ldots,x_k}\left(|Y_i|^3\right) = |t_i|^3\, \mathbb{E}_{t_i}\left(\left|X - \Lambda_\mu'(t_i)\right|^3\right) \leqslant \max_{t \in J_\mu^*} |t|^3\, \mathbb{E}_t\left(\left|X - \Lambda_\mu'(t)\right|^3\right) =: b_1 < +\infty$$

for all $1 \leqslant i \leqslant k$. Applying Lemma 3.8 we find $\theta > 0$ and $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$ then

$$\mathbb{P}_{x_1,\ldots,x_k}\left(0 \leqslant \sum_{i=1}^{k} Y_i \leqslant s_k\right) \geqslant \theta. \tag{3.7}$$

Now, we distinguish two cases:

Case 1: Assume that $k < k_0$. Then, working as for A_3, we see that

$$P\left(\sum_{i \in A_2} t_i(X_i - x_i) \geqslant 0\right) \geqslant \prod_{i \in A_2} P(X_i \geqslant x_i) \geqslant \prod_{i \in A_2} P(X_i \geqslant M) = e^{-m(M)k} \geqslant e^{-m(M)k_0}.$$

Case 2: Assume that $k \geqslant k_0$. From Lemma 3.7 we have

$$\begin{aligned} P\left(\sum_{i \in A_2} t_i(X_i - x_i) \geqslant 0\right) &= \left(\int_{[0,\infty)} e^{-s_k u}\, d\nu_k(u)\right) \exp\left(-\sum_{i=1}^{k} \Lambda_\mu^*(x_i)\right) \\ &\geqslant e^{-s_k} \nu_k([0,1]) \exp\left(-\sum_{i \in A_2} \Lambda_\mu^*(x_i)\right). \end{aligned} \tag{3.8}$$

From (3.7) we see that

$$\nu_k([0,1]) = P_{x_1,\ldots,x_k}(0 \leqslant S_k \leqslant s_k) = \mathbb{P}\left(0 \leqslant \sum_{i=1}^{k} Y_i \leqslant s_k\right) \geqslant \theta.$$

Moreover, $s_k \leqslant c\sqrt{k}$. Combining the two cases we get the estimate of Lemma 3.6 for P_2. $\square$

We can now complete the proof of Theorem 3.3. Collecting the estimates from Lemma 3.4, Lemma 3.5

and Lemma 3.6, we may write

$$\begin{aligned}P\left(\sum_{i=1}^{n} t_i(X_i - x_i) \geqslant 0\right) &\geqslant \prod_{j=1}^{3} P\left(\sum_{i\in A_j} t_i(X_i - x_i) \geqslant 0\right)\\ &\geqslant \exp\left(-\sum_{i=1}^{n}\Lambda_\mu^*(x_i)\right)\\ &\quad\times \exp\left(-\zeta\,|A_1| - c_1\ln|A_1| - c_2 - \zeta\sum_{i\in A_3}\Lambda_\mu^*(x_i) - c_3\sqrt{|A_2|} - c_4\right)\\ &\geqslant \exp\left(-\sum_{i=1}^{n}\Lambda_\mu^*(x_i) - \zeta\sum_{i=1}^{n}\Lambda_\mu^*(x_i) - 2\zeta n\right),\end{aligned}$$

provided $n \geqslant n(\mu,\zeta)$ for an appropriate $n(\mu,\zeta)\in\mathbb{N}$ depending only on ζ and μ. This proves (3.5). □

We are now able to provide an upper bound for $\varrho_2(\mu_n,\delta)$.

Theorem 3.9. *Let μ be an admissible even probability measure on $\mathbb{R}$ that satisfies the Λ^*-condition, i.e. $m(x)\sim\Lambda_\mu^*(x)$ as $x\uparrow x^*$. Then, for any $\delta\in\left(0,\frac{1}{2}\right)$ and $\varepsilon\in(0,1)$ we can find $n_0(\mu,\delta,\varepsilon)$ such that*

$$\varrho_2(\mu_n,\delta) \leqslant (1+\varepsilon)\,\mathbb{E}_\mu(\Lambda_\mu^*)$$

for all $n\geqslant n_0(\mu,\delta,\varepsilon)$.

Proof. Let $\varepsilon\in(0,1)$ and define $\zeta = T_1\varepsilon/(3T_1+4)$. Note that if $T_n := \mathbb{E}_{\mu_n}(\Lambda_{\mu_n}^*)$ then, as in the proof of Theorem 3.1, Chebyshev's inequality implies that

$$\mu_n(\{\Lambda_{\mu_n}^* \geqslant T_n + \zeta T_n\}) \leqslant \mu_n(\{|\Lambda_{\mu_n}^* - T_n| \geqslant \zeta T_n\}) \leqslant \frac{\beta(\mu)}{\zeta^2 n}.$$

Since ζ depends only on ε and μ we can find $n_0(\mu,\delta,\varepsilon)$ such that

$$\frac{\beta(\mu)}{\zeta^2 n} \leqslant \frac{\delta}{2}$$

and hence

$$\mu_n(B_{(1+\zeta)T_n}(\mu_n)) \geqslant 1 - \frac{\delta}{2}$$

for all $n\geqslant n_0(\mu,\delta,\varepsilon)$. Assume that $N\geqslant \exp((1+\varepsilon)T_n) = \exp((1+3\zeta)T_n + 4\zeta n)$. Applying (3.4) with $A = B_{(1+\zeta)T_n}(\mu_n)$ and using the estimate of Theorem 3.3 we get

$$\mathbb{E}_{\mu_n^N}\left[\mu_n(K_N)\right] \geqslant \mu_n(B_{(1+\zeta)T_n}(\mu_n))\left(1 - \binom{N}{n}p_\mu^{N-n} - 2\binom{N}{n}\left(1-\exp(-(1+\zeta)^2T_n - 2\zeta n)\right)^{N-n}\right). \tag{3.9}$$

Therefore, taking into account the fact that $(1+\zeta)^2 < 1+3\zeta$ for $\zeta<1$, we will have that

$$\varrho_2(\mu_n,\delta) \leqslant (1+\varepsilon)T_1$$

if we check that

$$\binom{N}{n}p_\mu^{N-n} + 2\binom{N}{n}\left(1-\exp(-(1+3\zeta)T_1 n - 2\zeta n)\right)^{N-n} \leqslant \frac{\delta}{2}.$$

We first claim that there exists $n_1(\mu,\delta)$ such that

$$\binom{N}{n}p_\mu^{N-n} < \frac{\delta}{4}$$

for all $n \geqslant n_1(\mu, \delta)$. Indeed, since $\binom{N}{n} \leqslant (eN/n)^n$, it suffices to check that

$$1 + \ln\left(\frac{N}{n}\right) + \frac{N-n}{n}\,\ln p_\mu < \frac{1}{n}\ln(\delta/4). \tag{3.10}$$

Set $x := N/n$. Then, (3.10) is equivalent to

$$(x-1)\ln(1/p_\mu) - \ln x > 1 + \frac{1}{n}\ln(4/\delta).$$

The claim follows from the facts that the function on the left-hand side increases to infinity as $x \to \infty$, and $x = N/n \geqslant \exp((1+3\zeta)T_1 n + 4\zeta n)/n \geqslant e^{4\zeta n}/n \to \infty$ when $n \to \infty$.

Next we check that there exists $n_2(\mu, \delta, \varepsilon)$ such that

$$2\binom{N}{n}\big[1 - \exp(-(1+3\zeta)T_1 n - 2\zeta n)\big]^{N-n} < \frac{\delta}{4}$$

for all $n \geqslant n_2(\mu, \delta, \varepsilon)$. Since $1 - y \leqslant e^{-y}$, it suffices to check that

$$\left(\frac{2eN}{n}\right)^n \exp(-(N-n)\exp(-(1+3\zeta)T_1 n - 2\zeta n)) < \frac{\delta}{4} \tag{3.11}$$

for all $n \geqslant n_2$. Setting $x := N/n$, we see that this inequality is equivalent to

$$\exp((1+3\zeta)T_1 n + 2\zeta n) < \frac{x-1}{\ln x + \ln(2e) + n^{-1}\ln(4/\delta)}.$$

Since $N \geqslant \exp((1+3\zeta)T_1 n + 4\zeta n)$, we easily check that the right-hand side exceeds $\exp((1+3\zeta)T_1 n + 3\zeta n)$ when $n \geqslant n_2(\mu, \zeta, \delta) = n_2(\mu, \varepsilon, \delta)$, and hence we get (3.11). Combining the above we conclude that

$$\varrho_2(\mu_n, \delta) \leqslant (1+\varepsilon)\,T_1$$

for all $n \geqslant n_0$, where $n_0 = n_0(\mu, \delta, \varepsilon)$ depends only on μ, δ and ε. $\square$

Proof of Theorem 1.1. Let $\delta \in \left(0, \frac{1}{2}\right)$ and $\varepsilon \in (0,1)$. From the estimates of Theorem 3.1 and Theorem 3.9 we see that there exists $n_0(\mu, \delta, \varepsilon)$ such that if $n \geqslant n_0$ then $\frac{c(\mu,\delta)}{\sqrt{n}} < \varepsilon$ (where $c(\mu, \delta)$ is the constant in Theorem 3.1) and

$$\varrho_1(\mu_n, \delta) \geqslant \left(1 - \frac{c(\mu,\delta)}{\sqrt{n}}\right)\mathbb{E}_\mu(\Lambda_\mu^*)$$

as well as

$$\varrho_2(\mu_n, \delta) \leqslant (1+\varepsilon)\,\mathbb{E}_\mu(\Lambda_\mu^*).$$

Therefore,

$$\varrho(\mu_n, \delta) \leqslant 2\varepsilon\mathbb{E}_\mu(\Lambda_\mu^*)$$

for all $n \geqslant n_0$. Since $\varepsilon \in (0,1)$ was arbitrary, we see that $\lim\limits_{n\to\infty} \varrho(\mu_n, \delta) \to 0$, as claimed in Theorem 1.1. $\square$

4 Threshold for the p-measures

We write ν for the symmetric exponential distribution on $\mathbb{R}$; thus, ν is the probability measure with density $\frac{1}{2}\exp(-|x|)$. More generally, for any $p \geqslant 1$ we denote by ν_p the probability distribution on $\mathbb{R}$ with density $(2\gamma_p)^{-1}\exp(-|x|^p)$, where $\gamma_p = \Gamma(1+1/p)$. Note that $\nu_1 = \nu$. The product measure $\nu_p^n = \nu_p^{\otimes n}$ has density $(2\gamma_p)^{-n}\exp(-\|x\|_p^p)$, where $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ is the ℓ_p^n-norm.

Our aim in this section is to show that ν_p satisfies the Λ^*-condition.

Theorem 4.1. *For any $p \geqslant 1$ we have that $-\ln(\nu_p[x,\infty)) \sim \Lambda^*_{\nu_p}(x)$ as $x \to \infty$. In other words,*

$$\lim_{x\to+\infty} \frac{-\ln(\nu_p[x,\infty))}{\Lambda^*_{\nu_p}(x)} = 1. \tag{4.1}$$

Proof of the case $p = 1$. We start with the case $p = 1$ which is simple because Λ^*_ν can be computed explicitly. A direct calculation shows that

$$\Lambda^*_\nu(x) = \sqrt{1+x^2} - 1 - \ln\left(\frac{\sqrt{1+x^2}+1}{2}\right), \qquad x \in \mathbb{R}.$$

It follows that $\Lambda^*_\nu(x) \sim x$ as $x \to \infty$. On the other hand, $\nu([x,\infty)) = \frac{1}{2}e^{-x}$ for all $x > 0$, which shows that $-\ln(\nu([x,\infty)) = x + \ln 2$, and hence $-\ln(\nu[x,\infty)) \sim x$ as $x \to \infty$. Combining the above we immediately see that (4.1) is satisfied for $p = 1$. □

For the rest of this section we fix $p > 1$. Following [1] we say that a non-negative function $f : \mathbb{R} \to \mathbb{R}$ is regularly varying of index $s \in \mathbb{R}$, and write $f \in R_s$, if $\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^s$ for every $\lambda > 0$. It is proved in [1, Theorem 4.12.10] that if $f \in R_s$ for some $s > 0$ then

$$-\ln\left(\int_x^\infty e^{-f(t)}dt\right) \sim f(x)$$

as $x \to \infty$. Let $f_p(x) = |x|^p$, $x \geqslant 0$. It is clear that $f_p \in R_p$, and hence

$$-\ln(\nu_p[x,\infty)) = -\ln\left((2\gamma_p)^{-1}\int_x^\infty e^{-f_p(t)}dt\right) = \ln(2\gamma_p) - \ln\left(\int_x^\infty e^{-f_p(t)}dt\right) \sim f_p(x)$$

as $x \to \infty$. This proves the following.

Lemma 4.2. *For every $p \geqslant 1$ we have that $-\ln(\nu_p[x,\infty)) \sim x^p$ as $x \to \infty$.*

Lemma 4.2 shows that in order to complete the proof of the theorem we have to show that $\Lambda^*_{\nu_p}(x) \sim x^p$ as $x \to \infty$. Let $g_p(x) = x^2$ for $0 \leqslant x < 1$ and $g_p(x) = x^p$ for $x \geqslant 1$. It is shown in [16] that for any $p \geqslant 1$ and $x \in \mathbb{R}$ one has

$$\Lambda^*_{\nu_p}(x/c) \leqslant g_p(|x|) \leqslant \Lambda^*_{\nu_p}(cx)$$

where $c > 1$ is an absolute constant.

For the proof of $\Lambda^*_{\nu_p}(x) \sim x^p$ as $x \to \infty$ we shall apply the Laplace method; more precisely, we shall use the next version of Watson's lemma (see equation (2.34) in [17, Section 2.2]).

Proposition 4.3. *Let $S < a < T \leqslant \infty$ and $g, h : [S,T] \to \mathbb{R}$, where g is continuous with a Taylor series in a neighborhood of a, and h is twice continuously differentiable and has its maximum at a and satisfies $h'(a) = 0$ and $h''(a) < 0$. Assume also that the integral*

$$\int_S^T g(x)e^{th(x)}\,dx$$

converges for large values of t. Then,

$$\int_S^T g(x)e^{th(x)}\,dx \sim g(a)\left(-\frac{2\pi}{th''(a)}\right)^{1/2} e^{th(a)} + e^{th(a)}O(t^{-3/2})$$

as $t \to +\infty$.

We apply Proposition 4.3 to get the next asymptotic estimate.

Lemma 4.4. *Let $p > 1$ and q be the conjugate exponent of p. Then, setting $y = t^q$ we have that*

$$I(t) := \int_0^\infty e^{tx-x^p}\, dx \sim y^{\frac{1}{p}} e^{yh(a)} \left[\left(-\frac{2\pi}{yh''(a)} \right)^{1/2} + O(y^{-3/2}) \right]$$

as $t \to +\infty$, where $h(s) = s - s^p$ on $[0, \infty)$ and $a = p^{-q/p}$.

Proof. We set $x = \lambda s$ and $t = \lambda^{p-1}$. Then,

$$I(t) = I(\lambda^{p-1}) = \lambda \int_0^\infty e^{\lambda^p (s-s^p)}\, ds.$$

Now, set $y = \lambda^p = t^q$. Then,

$$I(t) = y^{1/p} \int_0^\infty e^{y(s-s^p)} ds.$$

We have $h'(s) = 1 - ps^{p-1}$, therefore h attains its maximum at $a = (1/p)^{\frac{1}{p-1}} = p^{-q/p}$. Now, applying Proposition 4.3 with $g \equiv 1$ we see that

$$\int_0^\infty e^{yh(s)} ds \sim e^{yh(a)} \left[\left(-\frac{2\pi}{yh''(a)} \right)^{1/2} + O(y^{-3/2}) \right],$$

and the lemma follows. □

We proceed to study the asymptotic behavior of $\Lambda_{\nu_p}(t)$. Recall that

$$\Lambda_{\nu_p}(t) = \ln \left(c_p \int_{-\infty}^\infty e^{tx-|x|^p}\, dx \right),$$

where $c_p = (2\Gamma(1 + 1/p))^{-1}$. By the dominated convergence theorem,

$$\int_{-\infty}^0 e^{tx-|x|^p}\, dx \longrightarrow 0$$

as $t \to +\infty$. Therefore, from Lemma 4.4,

$$c_p \int_{-\infty}^\infty e^{tx-|x|^p}\, dx \sim c_p \int_0^\infty e^{tx-x^p}\, dx \sim c_p y^{\frac{1}{p}} e^{yh(a)} \left[\left(-\frac{2\pi}{yh''(a)} \right)^{1/2} + O(y^{-3/2}) \right],$$

where $h(s) = s - s^p$ on $[0, \infty)$, $a = p^{-q/p}$ and $y = t^q$. Now,

$$\ln \left(c_p y^{\frac{1}{p}} e^{yh(a)} \left[\left(-\frac{2\pi}{yh''(a)} \right)^{1/2} + O(y^{-3/2}) \right] \right) = \ln c_p + \frac{1}{p} \ln y + yh(a) + O(\ln y) \sim yh(a).$$

It follows that $\Lambda_{\nu_p}(t) \sim yh(a) = (p^{-q/p} - p^{-q})t^q$, where q is the conjugate exponent of p. We rewrite this as follows.

Lemma 4.5. *Let $p > 1$ and q be the conjugate exponent of p. Then,*

$$\Lambda_{\nu_p}(t) \sim \frac{p-1}{p^q} t^q \quad \textit{as} \ \ t \to +\infty.$$

Lemma 4.5 allows us to determine the asymptotic behavior of $\Lambda^*_{\nu_p}(x)$ as $x \to \infty$. We need a lemma which appears in [8] and [19].

Lemma 4.6. *Let $q \geqslant 1$, $a > 0$ and $f : [a, \infty) \to \mathbb{R}$ be a continuously differentiable function such that f' is increasing on $[a, \infty)$ and $f(t) \sim t^q$ as $t \to +\infty$. Then, $f'(t) \sim qt^{q-1}$ as $t \to +\infty$.*

Sketch of the proof. Let $\varepsilon \in (0, 1)$. There exists $b > a$ and $\eta : [b, \infty) \to \mathbb{R}$ such that $|\eta(t)| \leqslant \varepsilon$ and $f(t) = t^q(1 + \eta(t))$ for all $t > b$. Since f' is increasing, for any $s > 0$ we have that

$$\begin{aligned} sf'(t) &\leqslant \int_t^{t+s} f'(u)\,du = f(t+s) - f(t) = \big((t+s)^q - t^q\big) + \big((t+s)^q\eta(t+s) - t^q\eta(t)\big) \\ &\leqslant sq(t+s)^{q-1} + 2\varepsilon(t+s)^q. \end{aligned}$$

We set $s = \sqrt{\varepsilon}t$. Then,

$$f'(t) \leqslant qt^{q-1}\big((1+\sqrt{\varepsilon})^{q-1} + 2q^{-1}\sqrt{\varepsilon}(1+\sqrt{\varepsilon})^q\big)$$

for all $t > b$. In the same way we see that

$$f'(t) \geqslant qt^{q-1}\big((1-\sqrt{\varepsilon})^{q-1} - 2q^{-1}\sqrt{\varepsilon}\big)$$

for all $t > b/(1-\sqrt{\varepsilon})$, and the lemma follows. □

We also need the next simple lemma.

Lemma 4.7. *Let $a > 0$ and $f : [a, +\infty) \to \mathbb{R}$ be a strictly increasing function. Assume that for some $C > 0$ and $p > 1$ we have $f(x) \sim Cx^p$ as $x \to +\infty$, and that $\lim_{y\to+\infty} f^{-1}(y) = +\infty$. Then, $f^{-1}(y) \sim (y/C)^{1/p}$ as $y \to +\infty$.*

Proof. We may write $f(x) = Cx^p g(x)$ for some function $g : [a, +\infty) \to \mathbb{R}$ with $\lim_{x\to+\infty} g(x) = 1$. Then, for sufficiently large x we have that $x = \left(\frac{f(x)}{C} \cdot \frac{1}{g(x)}\right)^{1/p}$. It follows that, for sufficiently large y,

$$f^{-1}(y) = \left(\frac{y}{C}\frac{1}{g(f^{-1}(y))}\right)^{1/p},$$

and the lemma follows because $\lim_{y\to+\infty} f^{-1}(y) = +\infty$ and $\lim_{x\to+\infty} g(x) = 1$. □

Proof of the case $p > 1$ in Theorem 4.1. Now, we can show that

$$\Lambda^*_{\nu_p}(x) \sim x^p \tag{4.2}$$

as $x \to \infty$. We know that $\Lambda^*_{\nu_p}(x) = xh(x) - \Lambda_{\nu_p}(h(x))$ where $h(x) = (\Lambda'_{\nu_p})^{-1}(x)$. From Lemma 4.5 and Lemma 4.6 we see that $\Lambda'_{\nu_p}(t) \sim p^{-(q-1)}t^{q-1}$, and Lemma 4.7 implies that

$$h(x) \sim px^{\frac{1}{q-1}} = px^{p-1},$$

using also the fact that $(p-1)(q-1) = 1$. It follows that

$$\frac{\Lambda^*_{\nu_p}(x)}{x^p} = \frac{h(x)}{x^{p-1}} - \frac{\Lambda_{\nu_p}(h(x))}{x^p} = \frac{h(x)}{x^{p-1}} - \frac{\Lambda_{\nu_p}(h(x))}{h(x)^{\frac{p}{p-1}}}\left(\frac{h(x)^{\frac{1}{p-1}}}{x}\right)^p \longrightarrow p - \frac{p-1}{p^q}\cdot p^q = 1$$

as $x \to \infty$. This proves (4.2) and completes the proof of the theorem. □

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MINAS PAFIS: Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis 157-84, Athens, Greece.

E-mail: mipafis@math.uoa.gr