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RESEARCH ARTICLE

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On the mean width ratio of convex bodies

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Abstract

Given two convex bodies K, L in $\mathbb{R}^n$. The *mean width ratio* of L and K is defined by $\mathrm{wr}(L,K) = \inf \frac{w(TL)}{w(K)}$, where the infimum is over all linear transformations T of $\mathbb{R}^n$ for which $K \subseteq T(L)$. For L symmetric and K containing the origin not necessarily symmetric convex bodies we show that $\mathrm{wr}(L,K) \leqslant c\sqrt{n}\log n$, where $c > 0$ is an absolute constant.

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1 | INTRODUCTION AND MAIN RESULT

Let L and K be two convex bodies in $\mathbb{R}^n$. The *mean width ratio* of L and K is defined by

$$\mathrm{wr}(L,K) := \inf\left\{\frac{w(TL)}{w(K)} : T \in GL(n),\ K \subseteq T(L)\right\}. \tag{1.1}$$

(by $w(K)$ we denote the mean width of a convex body K)

Böröczky and Schneider in [3] proved that the minimal mean width of all simplices circumscribed about a convex body of given mean width attains its maximum precisely if the body is a ball, thus the mean width ratio of a simplex and a convex body has upper bound $c\sqrt{n}\sqrt{\log n}$, where $c > 0$ is an absolute constant.

Schechtman and Schmuckenschläger in [11] showed that among all convex symmetric convex bodies C with maximal volume ellipsoid B_2^n the unit cube $Q_n = [-1,1]^n$ has the largest mean width. It is straightforward to check that

$$\mathrm{wr}(C, B_2^n) \leqslant \sqrt{n}.$$

In this note, we will discuss the following question: what is the upper bound for the mean width ratio for every symmetric convex body L and K containing origin not necessarily symmetric convex body in $\mathbb{R}^n$? Our main result provides an almost sharp affirmative answer.

Theorem 1.1. *Let K be a convex body containing origin (not necessarily symmetric) and L be a symmetric convex body in $\mathbb{R}^n$. Then, we have*

$$\mathrm{wr}(L,K) \leqslant c\sqrt{n}\log(1+d_L),$$

where $c>0$ is an absolute constant and d_L is the Banach–Mazur distance $d(X_L,\ell_2^n)$. (See also Section 2 for notation and background.)

The example of the ball and the cube shows that this estimate is optimal up to the logarithmic term.

Our approach is presented in Section 3, the proof of the Theorem 1.1 is based on the method of random orthogonal factorizations which was used for the first time by Jaegermann in [13] and developed later by Benyamini and Gordon in [1] to estimate the Banach–Mazur distance between two n-dimensional normed spaces. The cornerstone of the method is a special case of Chevet's inequality (see [14] §43) from Gaussian stochastic process theory.

In Section 4, we use additional information that one has when L is an ellipsoid or parallelepiped (centrally symmetric), and we obtain the following results:

(i) Let $\mathcal{E}$ be an ellipsoid and K be a convex body in $\mathbb{R}^n$. Then, we have

$$\mathrm{wr}(\mathcal{E},K) \leqslant c\sqrt{n},$$

where $c>0$ is an absolute constant.

(ii) Let $\mathcal{P}$ be a parallelepiped and K be a convex body in $\mathbb{R}^n$. Then, we have

$$\mathrm{wr}(\mathcal{P},K) \leqslant c\sqrt{n\log n},$$

where $c>0$ is an absolute constant.

Background information is provided in Section 2.

2 | NOTATION AND PRELIMINARIES

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}, and ν for the Haar probability measure on the orthogonal group $O(n)$.

The letters c, c', c_1, c_2, etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$.

A convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that K is symmetric or centrally symmetric if $x \in K$ implies that $-x \in K$. The support function of a convex body K is defined by $h_K(y) = \max\{\langle x, y\rangle : x \in K\}$, and the mean width of K is given by

$$w(K) = \int_{S^{n-1}} h_K(u)\, d\sigma(u).$$

The radius of K is the quantity $R(K) = \max\{\|x\|_2 : x \in K\}$, that is, the smallest $R > 0$ for which $K \subseteq RB_2^n$. We define the polar body K° of K by $K^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\}$.

Let K a symmetric convex body in $\mathbb{R}^n$, then the Minkowski functional

$$\|x\|_K = \inf\{\lambda \geqslant 0 : x \in \lambda K\}$$

is a norm on $\mathbb{R}^n$ and K is the unit ball of the normed space $(\mathbb{R}^n, \|\cdot\|_K)$.

If X_K and X_L are two n-dimensional normed spaces their Banach–Mazur distance $d(X_K, X_L)$ is defined by

$$d(X_K, X_L) = \inf\ ||T : X_K \to X_L|| \cdot ||T^{-1} : X_L \to X_K||.$$

We write d_K for the Banach–Mazur distance $d(X_K, \ell_2^n)$.

Uryshon's inequality (see [10], p. 6) states that

$$\left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}} \leqslant w(K)$$

with equality if and only if K is a ball.

The expectation of the norm of a convex body C (on the sphere S^{n-1}) is denoted by $M(C)$:

$$M(C) = \int_{S^{n-1}} \|x\|_C\, d\sigma(x).$$

We write γ_n for the Gaussian measure in $\mathbb{R}^n$ which has the density function $x \mapsto (\sqrt{2\pi})^{-n} e^{-\|x\|_2^2/2}$, $x \in \mathbb{R}^n$ with respect to Lebesgue measure.

Let $L(\ell_2^n, X_C)$ denote the space of all linear operators from ℓ_2^n to X_C. We define the ℓ-norm of an operator $T \in L(\ell_2^n, X_C)$ by

$$\ell(T) = \ell(T, C) := \left(\int_{\mathbb{R}^n} \|T(x)\|_C^2 d\gamma_n(x)\right)^{1/2},$$

where γ_n is the standard Gaussian measure on $\mathbb{R}^n$. We also write $\ell(T^{-1}(C)) := \ell(T, C)$.

We say that $T(C)$ is in the ℓ-position if $\ell(T(C))\ell((T(C))^\circ)$ is minimal over all $T \in GL(n)$ and is equivalent to say that $T(C)$ is in the ℓ-position if $M(T(C))w((T(C))^\circ)$ is minimal over all $T \in GL(n)$ (see Lemma 3.1).

3 | PROOF OF THE THEOREM

Our first tool is a simple fact about the ℓ-functional $L \mapsto \ell(L)$.

Lemma 3.1. *Let L be a symmetric convex body in $\mathbb{R}^n$. Then*

$$w(L^\circ) \leqslant \frac{c}{\sqrt{n}} \ell(L) \tag{3.1}$$

Proof. We apply the Cauchy–Schwartz inequality to get

$$\int_{\mathbb{R}^n} \|x\|_L \, \gamma_n(dx) \leqslant \ell(\breve{\ })$$

and then use integration in polar coordinates. □

Our second tool is Chevet's inequality in the spirit of Benyamini and Gordon (see [1]).

Lemma 3.2. *Let L and K be two symmetric convex bodies in $\mathbb{R}^n$. Then,*

$$\begin{aligned}\int_{O(n)} \|U : X_K \to X_L\| \, d\nu(U) &\leqslant c_o(R(K)M(L) + R(L^\circ)M(K^\circ)) \\ &\leqslant \frac{c}{\sqrt{n}}(R(K)\ell(L) + R(L^\circ)\ell(K^\circ)),\end{aligned}$$

where $c_o, c > 0$ are absolute constants.

We will also use the following lemma:

Lemma 3.3. *For any symmetric convex body K in $\mathbb{R}^n$, we have*

$$R(K) \leqslant c_1 \sqrt{n} w(K), \tag{3.2}$$

where $c_1 > 0$ is an absolute constant.

Proof. Let $x_0 \in K$ such that $\|x_0\|_2 = R(K)$ and note that

$$\begin{aligned}w(K) &= \int_{S^{n-1}} \max_{x \in K} |\langle x, \theta \rangle| \, d\sigma(\theta) \geqslant \int_{S^{n-1}} |\langle x_0, \theta \rangle| \, d\sigma(\theta) \\ &= \|x_0\|_2 \int_{S^{n-1}} |\langle e_1, \theta \rangle| \, d\sigma(\theta).\end{aligned}$$

Then, (3.2) follows from the fact that $\int |\langle e_1, \theta \rangle| \, d\sigma(\theta) \simeq n^{-1/2}$. □

The first step for the proof of the Theorem 1.1 is the following proposition:

Proposition 3.4. *Let L and K be two symmetric convex bodies in $\mathbb{R}^n$. Then, we have*

$$\mathrm{wr}(L, K) \leqslant c\sqrt{n} M(TL) w(TL),$$

where $c > 0$ is an absolute constant and TL is in ℓ-position.

Proof. Recall that

$$\mathrm{wr}(L,K) := \inf\left\{\frac{w(TL)}{w(K)} : T \in GL(n),\ K \subseteq T(L)\right\}.$$

Let $T \in GL(n)$ such that TL is in ℓ-position. We apply Lemma 3.2 with TL instead of L:

$$\int_{O(n)} \|U : X_K \to X_{TL}\|\, d\nu(U) \leqslant c_2(R(K)M(TL) + R((TL)^\circ)M(K^\circ))$$

$$\leqslant c_3\sqrt{n}(w(K)M(TL) + w((TL)^\circ)M(K^\circ)),$$

where we have also used (3.2). Taking into account the fact that $w((TL)^\circ) = M(TL)$ and $M(K^\circ) = w(K)$, we get

$$\int_{O(n)} \|U : X_K \to X_{TL}\|\, d\nu(U) \leqslant 2c_3\sqrt{n}M(TL)w(K).$$

Thus, there exists $U \in O(n)$ such that $U(K) \subseteq \alpha T(L)$, where

$$\alpha = 2c_3\sqrt{n}M(TL)w(K).$$

Consider the operator $S := \alpha U^{-1}T \in GL(n)$. Then, we have $K \subseteq S(L)$ and in addition

$$\mathrm{wr}(L,K) \leqslant \frac{w(SL)}{w(K)} = \alpha\frac{w(TL)}{w(K)}$$

$$= c_4\sqrt{n}M(TL)w(TL).$$

Our next step is to extend Proposition 3.4 in the nonsymmetric case for K. □

Proposition 3.5. *Let K be a convex body containing origin (not necessarily symmetric) and L be a symmetric convex body in $\mathbb{R}^n$. Then we have*

$$\mathrm{wr}(L,K) \leqslant c\sqrt{n}M(TL)w(TL),$$

where $c > 0$ is an absolute constant and TL is in ℓ-position.

Proof. Let K be a convex body containing origin (not necessarily symmetric) and L be a symmetric convex body in $\mathbb{R}^n$. It is easy to check that $w(K-K) = 2w(K)$. By Proposition 3.4, there exists $T_1 \in GL(n)$ such that $K - K \subseteq T_1(L)$ and

$$\frac{w(T_1(L))}{w(K-K)} \leqslant c\sqrt{n}M(TL)w(TL) \tag{3.3}$$

where TL is in ℓ-position. Apparently, $K \subseteq K - K \subseteq T_1(L)$, therefore,

$$\mathrm{wr}(L,K) \leqslant \frac{w(K-K)}{w(K)}\frac{w(T_1(L))}{w(K-K)} \leqslant 2c\sqrt{n}M(TL)w(TL).$$ □

In order to prove Theorem 1.1 by Proposition 3.5, we need an upper bound of the quantity $M(TL)w(TL)$ when TL is in ℓ-position.

Figiel and Tomczak in [4] using a general result of Lewis [7] about trace dual norms of operators, proved that for every C symmetric convex body there exists $S \in L(\ell_2^n, X_C)$ such that

$$\ell(S(C))\ell((S(C))^\circ) \leqslant n\,K(X_C), \tag{3.4}$$

where $K(X_C)$ is the K-convexity constant of X_C (see [10], p. 20) on the other hand, an important inequality of Pisier [9] states that

$$K(X_C) \leqslant c_1 \log(d_C + 1) \tag{3.5}$$

for every C, where c_1 is an absolute constant. An immediate outcome of (3.4) and (3.5) is the following lemma:

Lemma 3.6. *Let L be a symmetric convex body in $\mathbb{R}^n$. There exists an invertible linear image $S(L)$ such that*

$$\ell(S(L))\ell((S(L))^\circ) \leqslant cn \log(1 + d_L), \tag{3.6}$$

where $c > 0$ is an absolute constant.

Proof of Theorem 1.1. By Proposition 3.5, we have

$$\mathrm{wr}(L, K) \leqslant c\sqrt{n}M(TL)w(TL),$$

where $c > 0$ is an absolute constant and TL is in ℓ-position. Lemma 3.6 combined with Lemma 3.1 and the definition of ℓ-position gives the following estimation

$$M(TL)w(TL) \leqslant M(SL)w(SL) \leqslant c \log(1 + d_L),$$

thus

$$\mathrm{wr}(L, K) \leqslant c\sqrt{n} \log(1 + d_L). \qquad \square$$

Remark 3.7. For every symmetric convex body C in $\mathbb{R}^n$, we have $d_C \leqslant \sqrt{n}$ by John's Theorem [6], therefore Theorem 1.1 implies that

$$\mathrm{wr}(L, K) \leqslant c\sqrt{n} \log n.$$

Remark 3.8. An analogue of Theorem 1.1 for volume ratio was established in [5]: If K and L are two convex bodies in $\mathbb{R}^n$ then

$$\mathrm{vr}(L, K) := \inf \left(\frac{|L|}{|T(K)|} \right)^{1/n} \leqslant c\sqrt{n} \log n, \tag{3.7}$$

where the infimum is taken over all affine transformations T of $\mathbb{R}^n$ for which $T(K) \subseteq L$ and $c > 0$ is an absolute constant.

The width ratio as defined in (1.1) is not invariant under $T \in GL(n)$ as volume ratio in (3.7) (for affine transformations); it depends on the position of K. We discuss an alternative definition for the mean width ratio at the end of Section 4.

4 | SPECIAL CASES OF THE MEAN WIDTH RATIO

In this section, we study some special cases for the mean width ratio and we examine the sharpness of Propositions 3.4, 3.5, and Theorem 1.1.

4.1 | Mean width ratio when $L = \mathcal{E}$ is an ellipsoid

We deduce from Proposition 3.5 the following consequence:

Corollary 4.0.1. *Let $\mathcal{E}$ be an ellipsoid (centrally symmetric) and K be convex body in $\mathbb{R}^n$ containing the origin. Then, we have*

$$\mathrm{wr}(\mathcal{E}, K) \leqslant c\sqrt{n},$$

where $c > 0$ is an absolute constant.

Proof. By Proposition 3.5, we have

$$\mathrm{wr}(\mathcal{E}, K) \leqslant c\sqrt{n} M(T\mathcal{E}) w(T\mathcal{E}), \tag{4.1}$$

where $c > 0$ is an absolute constant and $T\mathcal{E}$ is the ℓ-position of the ellipsoid $\mathcal{E}$, thus by the definition of the ℓ-position note that

$$M(T\mathcal{E}) w(T\mathcal{E}) \leqslant M\left(B_2^n\right) w\left(B_2^n\right) = 1. \tag{4.2}$$

(4.1) and (4.2) show that

$$\mathrm{wr}(\mathcal{E}, K) \leqslant c\sqrt{n}.$$

□

Remark 4.1. The example of the inscribed regular simplex Δ_n in the Euclidean unit ball B_2^n shows that this estimate is optimal up to the $\sqrt{\log n}$. According to Böröczky in [2], we have $w(\Delta_n) \sim 4\sqrt{\frac{2\log n}{n}}$ as $n \to \infty$ and hence,

$$\mathrm{wr}(B_2^n, \Delta_n) \leqslant \frac{w(B_2^n)}{w(\Delta_n)} \leqslant c\sqrt{\frac{n}{\log n}},$$

where $c > 0$ is an absolute constant. Note that if we replace Δ_n with B_1^n the inscribed regular cross-polytope in the Euclidean unit ball B_2^n we get the same upper bound.

4.2 | Mean width ratio when $L = \mathcal{P}$ is a parallelepiped

Let $\mathcal{P}$ be a parallelepiped (centrally symmetric) and K be a convex body in $\mathbb{R}^n$ containing origin. Theorem 1.1 yields that

$$\mathrm{wr}(\mathcal{P}, K) \leqslant c\sqrt{n}\log(1 + d_{\mathcal{P}}) \leqslant c\sqrt{n}\log n, \tag{4.3}$$

where $c > 0$ is an absolute constant.

Remark 4.2. We can easily prove that the upper bound is optimal up to the logarithmic in the dimension term. It can actually be seen that $\mathrm{wr}(Q_n, B_2^n) \simeq \sqrt{n}$ where $Q_n = [-1,1]^n$ is the unit cube. It is straightforward to check that $\mathrm{wr}(Q_n, B_2^n) \leqslant \sqrt{n}$. For the lower bound note that for any $T \in GL(n)$ such that $B_2^n \subseteq T(Q_n)$, Uryshon's inequality gives

$$\frac{w(TQ_n)}{w(B_2^n)} \geqslant \left(\frac{|TQ_n|}{|B_2^n|}\right)^{1/n} \geqslant \mathrm{vr}(Q_n).$$

The result follows from the fact that

$$\mathrm{vr}(Q_n) := \inf\left\{\left(\frac{|TQ_n|}{|B_2^n|}\right)^{1/n} : T \in GL(n),\, T(Q_n) \supseteq B_2^n\right\} \simeq \sqrt{n}.$$

Note that by Proposition 3.5, we get the following corollary which improves (4.3) by a $\sqrt{\log n}$.

Corollary 4.2.1. *Let $\mathcal{P}$ be a parallelepiped and K be a convex body in $\mathbb{R}^n$. Then, we have*

$$\mathrm{wr}(\mathcal{P}, K) \leqslant c\sqrt{n\log n},$$

where $c > 0$ is an absolute constant.

Proof. By Proposition 3.5, we have

$$\mathrm{wr}(\mathcal{P}, K) \leqslant c\sqrt{n}M(T\mathcal{P})w(T\mathcal{P}), \tag{4.4}$$

where $c > 0$ is an absolute constant and $T\mathcal{P}$ is the ℓ-position of the parallelepiped $\mathcal{P}$, thus by the definition of the ℓ-position, we get

$$M(T\mathcal{P})w(T\mathcal{P}) \leqslant M(Q_n)w(Q_n) \leqslant c\sqrt{\log n}. \tag{4.5}$$

By (4.4) and (4.5) follows that

$$\mathrm{wr}(\mathcal{P}, K) \leqslant c\sqrt{n}\sqrt{\log n}.$$ □

Remark 4.3. The width ratio as defined in (1.1) is not invariant under $T \in GL(n)$; it depends on the position of K. An alternative definition would be to set

$$\mathrm{wr}'(C,K) = \inf\left\{ \frac{w(TC)}{w(SK)} \,:\, T,S \in GL(n),\ S(K) \subseteq T(C) \right\}.$$

However, one can easily check that then the definition is not interesting since

$$\mathrm{wr}'(Q_n, B_2^n) \simeq 1.$$

To see this, we first check that for every $T \in GL(n)$ one has

$$w(TB_2^n) \simeq \left(\frac{1}{n} \sum_{j=1}^{n} \|T(e_j)\|_2^2 \right)^{1/2},$$

while

$$w(TQ_n) \simeq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \|T(e_j)\|_2.$$

We define a diagonal operator $T = \mathrm{diag}(a_1, \dots, a_n) \in SL(n)$ with $a_i = 1/n$ for all $i = 1, \dots, n-1$ and $a_n = n^{n-1}$. Then,

$$\begin{aligned} \sum_{j=1}^{n} \|T(e_j)\|_2 &= \frac{1 + n^{n-1}}{\left((n-1)/n^2 + n^{2(n-1)}\right)^{1/2}} \left(\sum_{j=1}^{n} \|T(e_j)\|_2^2 \right)^{1/2} \\ &\leqslant \left(1 + \frac{1}{n^{n-1}}\right) \left(\sum_{j=1}^{n} \|T(e_j)\|_2^2 \right)^{1/2}. \end{aligned}$$

It follows that $\mathrm{wr}'(Q_n, B_2^n) \leqslant c$ for an absolute constant $c > 0$.

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