Ψ_α-estimates for marginals of log-concave probability measures

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Abstract

We show that a random marginal $\pi_F(\mu)$ of an isotropic log-concave probability measure μ on $\mathbb{R}^n$ exhibits better ψ_α-behavior. For a natural variant ψ'_α of the standard ψ_α-norm we show the following:

(i) If $k \leq \sqrt{n}$, then for a random $F \in G_{n,k}$ we have that $\pi_F(\mu)$ is a ψ'_2-measure. We complement this result by showing that a random $\pi_F(\mu)$ is, at the same time, supergaussian.

(ii) If $k = n^\delta$, $\frac{1}{2} < \delta < 1$, then for a random $F \in G_{n,k}$ we have that $\pi_F(\mu)$ is a $\psi'_{\alpha(\delta)}$-measure, where $\alpha(\delta) = \frac{2\delta}{3\delta-1}$.

1 Introduction

The purpose of this note is to provide estimates on the ψ_α-behavior of random marginals of log-concave probability measures. We show that random k–dimensional projections of a high-dimensional measure of the log-concave class have better tail properties than the original measure. We give precise quantitative estimates for every $1 \leq k < n$. A typical k-dimensional marginal is ψ_2 as long as $k \leq \sqrt{n}$; after this critical value we still have non-trivial information (α is always greater than a simple function of $\frac{\log n}{\log k}$) in full generality. This observation may be viewed as a continuation of the ideas and the tools that were developed in [17]. It is also parallel to the philosophy behind Klartag's proof of the central limit theorem for convex bodies in [7] and [8] (see also [5] and [4]). A main ingredient in these works is the fact that appropriate marginals of log-concave measures in power-type dimensions ($k \simeq n^\epsilon$ for some $\epsilon > 0$) are approximately spherically-symmetric. As Klartag proves in [9] this phenomenon appears for a much wider class of probability measures and constitutes the measure analogue of Dvoretzky's theorem on approximately Euclidean sections of high-dimensional convex bodies. Actually, Dvoretzky's theorem plays a crucial role in all these works, as well as in the present note.

Recall that a probability measure μ on $\mathbb{R}^n$ is called log-concave if for any Borel sets A, B in $\mathbb{R}^n$ and any $\lambda \in (0, 1)$,

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}. \tag{1.1}$$

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It is known (see [2]) that if μ is log-concave and if $\mu(H) < 1$ for every hyperplane H, then μ has a density $f = f_\mu$, with respect to the Lebesgue measure, which is log-concave: $\log f$ is concave on its support $\{f > 0\}$.

We say that μ is isotropic if it is centered, i.e.

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle f(x)\, dx = 0, \tag{1.2}$$

and satisfies the isotropic condition

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 f(x)\, dx = 1 \tag{1.3}$$

for all $\theta \in S^{n-1}$. Then, the isotropic constant of μ is defined by $L_\mu := f(0)^{1/n}$.

Let $1 \leq \alpha \leq 2$. We say that a direction $\theta \in S^{n-1}$ is a ψ_α-direction for μ with constant $r > 0$ if

$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leq r \|\langle \cdot, \theta \rangle\|_2, \tag{1.4}$$

where

$$\|u\|_{\psi_\alpha} = \inf \left\{ t > 0 : \int_{\mathbb{R}^n} \exp\left((|u(x)|/t)^\alpha \right) f(x)\, dx \leq 2 \right\}. \tag{1.5}$$

We say that μ is a ψ_α measure with constant $r > 0$ if (1.4) holds true for every $\theta \in S^{n-1}$. It is well known that there exists an absolute constant $C > 0$ such that every log-concave probability measure μ is ψ_1 with constant C.

We study the ψ_α-behavior of marginals of μ. For every integer $1 \leq k < n$ and any $F \in G_{n,k}$, we consider the measure $\pi_F(\mu)$ with density

$$\pi_F(f)(x) = \int_{x + F^\perp} f(y)\, dy. \tag{1.6}$$

By the Prékopa–Leindler inequality (see [20]), $\pi_F(\mu)$ is a log-concave probability measure on F. As a simple consequence of Fubini's theorem, one can check that if μ is isotropic then $\pi_F(\mu)$ is also isotropic.

For the study of marginals, we need a variant of the ψ_α norm. We start with the well-known fact that $\|u\|_{\psi_\alpha} \simeq \sup\left\{ \frac{\|u\|_q}{q^{1/\alpha}} : q \geq \alpha \right\}$ and recall that if μ is the Lebesgue measure μ_K on an isotropic convex body K in $\mathbb{R}^n$ and if u is a linear functional, then

$$\|u\|_{\psi_\alpha} \simeq \sup_{q \geq \alpha} \frac{\|u\|_q}{q^{1/\alpha}} \simeq \sup_{\alpha \leq q \leq n} \frac{\|u\|_q}{q^{1/\alpha}}. \tag{1.7}$$

We define

$$\|u\|_{\psi'_\alpha} = \sup_{\alpha \leq q \leq n} \frac{\|u\|_q}{q^{1/\alpha}}. \tag{1.8}$$

It is clear that $\|u\|_{\psi'_\alpha} \le c\|u\|_{\psi_\alpha}$. In view of (1.7) this is a natural definition of a "ψ_α-norm" when one studies the behavior of linear functionals with respect to a log-concave measure on $\mathbb{R}^n$; see, for example, the applications in Section 4.

Our first result provides estimates on the ψ'_α-behavior of random marginals of μ.

Theorem 1.1. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$.*

(i) *If $k \le \sqrt{n}$ then there exists $A_k \subseteq G_{n,k}$ with measure $\nu_{n,k}(A_k) > 1 - \exp(-c\sqrt{n})$ such that, for every $F \in A_k$, $\pi_F(\mu)$ is a ψ'_2-measure with constant C, where $C > 0$ is an absolute constant.*

(ii) *If $k = n^\delta$, $\frac{1}{2} < \delta < 1$ then there exists $A_k \subseteq G_{n,k}$ with measure $\nu_{n,k}(A_k) > 1 - \exp(-ck)$ such that, for every $F \in A_k$, $\pi_F(\mu)$ is a $\psi'_{\alpha(\delta)}$-measure with constant C, where $\alpha(\delta) = \frac{2\delta}{3\delta-1}$ and $C > 0$ is an absolute constant.*

We next consider the question whether, in the case $1 \le k \le \sqrt{n}$, random marginals $\pi_F(\mu)$ of an isotropic log-concave probability measure μ on $\mathbb{R}^n$ are supergaussian (in the terminology of [19]). If ν is an isotropic log-concave probability measure on $\mathbb{R}^k$, a direction $\theta \in S^{k-1}$ is called supergaussian for ν with constant $r > 0$ if, for all $1 \le t \le \frac{\sqrt{k}}{r}$,

$$\nu\left(\{x : |\langle x, \theta\rangle| \ge t\}\right) \ge e^{-r^2t^2}. \tag{1.9}$$

The minimum of the set of $r > 0$ for which (1.9) holds true is called the supergaussian constant of ν in the direction of θ and is denoted by $\overline{sg}_\nu(\theta)$. It was proved in [19] that if K is an isotropic convex body in $\mathbb{R}^k$, then a random direction is supergaussian for ν_K with a constant $O(L_K)$ (the same question had been considered by Pivovarov [21] for the class of 1-unconditional bodies). We prove the following.

Theorem 1.2. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. If $k \le \sqrt{n}$, then there exists $B_k \subseteq G_{n,k}$ with measure $\nu_{n,k}(B_k) > 1 - \exp(-c\sqrt{n})$ such that, for every $F \in B_k$, $\pi_F(\mu)$ is a supergaussian measure with constant c, where $c > 0$ is an absolute constant: this means that*

$$\inf_{\theta \in S_F} \overline{sg}_{\pi_F(\mu)}(\theta) \ge c. \tag{1.10}$$

The paper is organized as follows. In Section 2 we introduce background material on L_q–centroid bodies; these play a central role in our approach. The proof of the two main results is presented in Section 3. Generalizations, applications and further remarks are collected in Section 4.

Notation and Preliminaries. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. We also write $\widetilde{A}$ for the homothetic image of volume 1 of a compact set $A \subseteq \mathbb{R}^n$, i.e. $\widetilde{A} := \frac{A}{|A|^{1/n}}$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. We refer to [14], [6] and [18] for information on isotropic convex bodies and to the books [15] and [20] for the asymptotic theory of finite dimensional normed spaces.

A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if $\int_C \langle x, \theta \rangle \, dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$. For each $-\infty < p < \infty$, $p \neq 0$, we define the p-mean width of C by

$$w_p(C) = \left(\int_{S^{n-1}} h_C^p(\theta) \sigma(d\theta) \right)^{1/p}. \tag{1.11}$$

Note that $w(C) := w_1(C)$ is the mean width of C. The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C, the polar body of C is $C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in C\}$. If K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that the measure μ_K with density $\mathbf{1}_{\widetilde{K}}$ is log-concave. The usual definition of an isotropic convex body is the following: a convex body K of volume 1 in $\mathbb{R}^n$ is called isotropic if it has center of mass at the origin and $Z_2(K) = L_K B_2^n$ for some constant $L_K > 0$ (the definition of the L_q-centroid bodies $Z_q(K)$ is given in the next section). One can check that K is isotropic if and only if the log-concave measure $L_K^n \mu_{L_K^{-1} K}$ is isotropic.

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2 Basic formulas

2.1. Let μ be a log-concave probability measure on $\mathbb{R}^n$ with a log-concave density f. For every $q \geq 1$ and $y \in \mathbb{R}^n$ we define

$$h_{Z_q(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y \rangle|^q f(x) \, dx \right)^{1/q}. \tag{2.1}$$

The integral is finite for every $q \geq 1$, by the log-concavity of μ. We define the L_q-centroid body $Z_q(\mu)$ of μ to be the centrally symmetric convex set with support function $h_{Z_q(\mu)}$.

L_q–centroid bodies were introduced in [11]. The normalization and notation was different (see also [12] where an L_q affine isoperimetric inequality was proved). We follow the normalization and notation of [17]. If K is a convex body of volume 1, we also write $Z_q(K)$ instead of $Z_q(\mu_K)$.

It is a simple consequence of Hölder's inequality that $Z_p(\mu) \subseteq Z_q(\mu)$ for all $1 \leq p \leq q < \infty$. On the other hand, Borell's lemma (see [15]) implies that

$$Z_q(\mu) \subseteq \bar{c}_0 \frac{q}{p} Z_p(\mu) \tag{2.2}$$

for all $1 \leq p < q < \infty$, where $\bar{c}_0 \geq 1$ is an absolute constant. For additional information on L_q–centroid bodies, we refer to [17] and [18].

2.2. Let μ be a log-concave probability measure on $\mathbb{R}^n$ with a log-concave density f, and let $1 \leq k \leq n$ and $F \in G_{n,k}$. Fubini's theorem shows that, for every $q \geq 1$ and $\theta \in S_F$,

$$\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^q d\mu(x) = \int_F |\langle x, \theta \rangle|^q d\pi_F(\mu)(x). \tag{2.3}$$

Since $h_{P_F(Z_q(\mu))}(\theta) = h_{Z_q(\mu)}(\theta)$ for all $\theta \in S_F$, it follows that

$$P_F(Z_q(\mu)) = Z_q(\pi_F(\mu)). \tag{2.4}$$

2.3. Let μ be a log-concave centered probability measure on $\mathbb{R}^n$. For every $q > -n$, $q \neq 0$, we define the quantities $I_q(\mu)$ by

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q \, d\mu(x) \right)^{1/q}. \tag{2.5}$$

The following fact is proved in [18]: For every $1 \leq q \leq n/2$,

$$I_{-q}(\mu) \simeq \sqrt{n/q}\, w_{-q}(Z_q(\mu)) \tag{2.6}$$

and

$$I_q(\mu) \simeq \sqrt{n/q}\, w_q(Z_q(\mu)). \tag{2.7}$$

2.4. Let C be a symmetric convex body in $\mathbb{R}^n$. Define $k_*(C)$ as the largest positive integer $k \leq n$ for which a random k-dimensional projection of C is 4-Euclidean: this can be made precise if we ask, for example, that the measure of the set of $F \in G_{n,k}$ which satisfy

$$\frac{1}{2} W(C)(B_2^n \cap F) \subseteq P_F(C) \subseteq 2W(C)(B_2^n \cap F) \tag{2.8}$$

is greater than $\frac{n}{n+k}$. The parameter $k_*(C)$ is determined by the parameters $w(C)$ and $R(C)$: There exist absolute constants $c_1, c_2 > 0$ such that

$$c_1 n \frac{w(C)^2}{R(C)^2} \leq k_*(C) \leq c_2 n \frac{w(C)^2}{R(C)^2} \tag{2.9}$$

for every symmetric convex body C in $\mathbb{R}^n$. The lower bound appears in Milman's proof of Dvoretzky's theorem (see [13]) and the upper bound was proved in [16]. The following Lemma is proved in [10]:

Lemma 2.1. *Let C be a symmetric convex body in $\mathbb{R}^n$. Then,*

(i) $w_q(C) \simeq w(C)$ *for all* $q \leq k_*(C)$.

(ii) $w_q(C) \simeq \sqrt{q/n}\, R(C)$ *for all* $k_*(C) \leq q \leq n$.

(iii) $w_q(C) \simeq R(C)$ *for all* $q \geq n$.

2.5. We define

$$q_*(\mu) := \max\{k \leq n : k_*(Z_k(\mu)) \geq k\}. \tag{2.10}$$

Then, the main result of [18] states that, for every centered log-concave probability measure μ on $\mathbb{R}^n$, one has

$$I_{-q}(\mu) \simeq I_q(\mu) \tag{2.11}$$

for every $1 \leq q \leq q_*(\mu)$. In particular, for all $q \leq q_*(\mu)$ one has $I_q(\mu) \leq CI_2(\mu)$, where $C > 0$ is an absolute constant.

Assuming that μ is isotropic, one can check that $q_*(\mu) \geq c\sqrt{n}$, where $c > 0$ is an absolute constant (for a proof, see [17]). Thus, using (2.7), one has

$$I_q(\mu) \leq CI_2(\mu) \ \text{ for every } q \leq \sqrt{n}. \tag{2.12}$$

3 Ψ_α-estimates for marginals

Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. We first prove Theorem 1.1(i) and Theorem 1.2.

3.1. The case $k \leq \sqrt{n}$**.** From (2.2) we see that $Z_q(\mu) \subseteq cqZ_2(\mu)$ for all $q \geq 2$. Since μ is isotropic, we have $Z_2(\mu) = B_2^n$, and hence, $R(Z_q(\mu)) \leq cq$ for all $q \geq 1$.

Let $d(q) = n\frac{w^2(Z_q(\mu))}{R^2(Z_q(\mu))}$ and $D(\mu) = \{q \geq 2 : q \leq d(q)\}$. Let q_0 be the maximum of the set of $q \geq 2$ for which $[2, q] \subseteq D(\mu)$. Then, by the continuity of $d(q)$, we have $q_0 = d(q_0)$. In particular, from Lemma 2.1 and (2.7) we have

$$w(Z_{q_0}(\mu)) \simeq w_{q_0}(Z_{q_0}(\mu)) \simeq \sqrt{q_0/n}\, I_{q_0}(\mu) \geq c_1\sqrt{q_0}. \tag{3.1}$$

It follows that

$$q_0 = n\frac{w^2(Z_{q_0}(\mu))}{R^2(Z_{q_0}(\mu))} \geq \frac{c_1^2 n q_0}{q_0^2} = \frac{c_1^2 n}{q_0}, \tag{3.2}$$

and hence $q_0 \geq c_1\sqrt{n}$. By the definition of q_0, for all $q \leq c\sqrt{n}$ we have $q \leq d(q)$, and the previous argument, applied for q, shows that

$$w(Z_q(\mu)) \geq c_2\sqrt{q} \ \text{ and } \ k_*(Z_q(\mu)) \geq c_2 n/q. \tag{3.3}$$

Now, let $k \leq \sqrt{n}$. From (2.12) we see that for every $1 \leq q \leq k$ we have $I_q(\mu) \leq CI_2(\mu) = C\sqrt{n}$, and hence, by (2.7),

$$w(Z_q(\mu)) \leq w_q(Z_q(\mu)) \leq C\sqrt{q}. \tag{3.4}$$

Then, if we fix $q \leq k$, Dvoretzky's theorem (see [15]) shows that

$$\frac{1}{2}w(Z_q(\mu))(B_2^n \cap F) \subseteq P_F(Z_q(\mu)) \subseteq 2w(Z_q(\mu))(B_2^n \cap F) \tag{3.5}$$

for all F in a subset $B_{k,q}$ of $G_{n,k}$ of measure

$$\nu_{n,k}(B_{k,q}) \geq 1 - e^{-c_3 k_*(Z_q(\mu))} \geq 1 - e^{-c_4\sqrt{n}}. \tag{3.6}$$

Applying this argument for $q = 2^i$, $i = 1, \dots \log_2 k$, and taking into account the fact that, from (2.2), $Z_p(\mu) \subseteq Z_q(\mu) \subseteq 2\overline{c}_0 Z_p(\mu)$ if $p < q \leq 2p$, we conclude that there exists $B_k \subset G_{n,k}$ with $\nu_{n,k}(B_k) \geq 1 - e^{-c_5\sqrt{n}}$ such that, for every $F \in B_k$ and every $1 \leq q \leq k$,

$$\frac{1}{2}w(Z_q(\mu))(B_2^n \cap F) \subseteq Z_q(\pi_F(\mu)) = P_F(Z_q(\mu)) \subseteq 2w(Z_q(\mu))(B_2^n \cap F). \tag{3.7}$$

From (3.3) and (3.4) we have $w(Z_q(\mu)) \simeq \sqrt{q}$ for all $q \leq \sqrt{n}$. Therefore, the last formula can be written in the form

$$h_{Z_q(\pi_F(\mu))}(\theta) \simeq \sqrt{q} \tag{3.8}$$

for all $F \in B_k$, $\theta \in S_F$ and $1 \leq q \leq k$.

From the inequality

$$\sup_{1 \leq q \leq k} \frac{\|\langle \cdot, \theta\rangle\|_{L_q(\pi_F(\mu))}}{\sqrt{q}} = \sup_{1 \leq q \leq k} \frac{h_{Z_q(\pi_F(\mu))}(\theta)}{\sqrt{q}} \leq C, \quad \theta \in S_F \tag{3.9}$$

we immediately get Theorem 1.1(i).

Next, we give the proof of Theorem 1.2, following an argument which essentially appears in [19]. Using the fact that

$$h_{Z_{2q}(\pi_F(\mu))}(\theta) \leq 2\overline{c}_0 h_{Z_q(\pi_F(\mu))}(\theta), \tag{3.10}$$

and applying the Paley-Zygmund inequality $\mathbb{P}\,(g(x) \geq t^q \mathbb{E}\,(g)) \geq (1 - t^q)^2 \frac{[\mathbb{E}\,(g)]^2}{\mathbb{E}\,(g^2)}$ for the function $g(x) = |\langle x, \theta\rangle|^q$, we see that, for every $q \geq 1$ and every $\theta \in S_F$,

$$[\pi_F(\mu)]\left(\left\{x \in F : |\langle x, \theta\rangle| \geq \frac{1}{2}h_{Z_q(\pi_F(\mu))}(\theta)\right\}\right) \geq e^{-c_6 q}. \tag{3.11}$$

Then, (3.8) gives

$$[\pi_F(\mu)]\,(\{x \in F : |\langle x, \theta\rangle| \geq c_7\sqrt{q}\}) \geq e^{-c_6 q} \tag{3.12}$$

for every $1 \leq q \leq k$ and every $\theta \in S_F$.

There exists an absolute constant $c_8 > 0$ such that if $1 \leq t \leq c_8\sqrt{k}$ we can write t in the form $t := c_7\sqrt{q}$ for some $q \leq k$. Then, a direct application of (3.12) gives

$$[\pi_F(\mu)](\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geq t\}| \geq e^{-c_9 t^2} \tag{3.13}$$

for all $\theta \in S_F$. This implies that $\overline{sg}_{\pi_F(\mu)}(\theta) \geq c$ for all $\theta \in S_F$, where $c > 0$ is an absolute constant. $\square$

It remains to prove Theorem 1.1(ii).

3.2. The case $k > \sqrt{n}$. Fix $k = n^\delta$, where $\delta \in \left(\frac{1}{2}, 1\right)$, and let $1 \leq q \leq k$. We first prove the following Lemma.

Lemma 3.1. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. For every $1 \leq k \leq n$ and $1 \leq q \leq k$,*

$$\left(\int_{G_{n,k}} R^k(Z_q(\pi_F(\mu)))\, d\nu_{n,k}(F)\right)^{1/k} \simeq w_k(Z_q(\mu)). \tag{3.14}$$

Proof. Using Lemma 2.1 and the identity (2.4), we see that, for every $F \in G_{n,k}$,

$$R(Z_q(\pi_F(\mu)) \simeq w_k(Z_q(\pi_F(\mu))) = w_k(P_F(Z_q(\mu))). \tag{3.15}$$

Therefore,

$$\begin{aligned}\left(\int_{G_{n,k}} R^k(Z_q(\pi_F(\mu)))\, d\nu_{n,k}(F)\right)^{1/k} &\simeq \left(\int_{G_{n,k}} w_k^k(P_F(Z_q(\mu)))\, d\nu_{n,k}(F)\right)^{1/k} \\ &= \left(\int_{G_{n,k}} \int_{S_F} h^k_{P_F(Z_q(\mu))}(\theta)\, d\sigma_F(\theta)\, d\nu_{n,k}(F)\right)^{1/k},\end{aligned}$$

where σ_F is the rotationally invariant probability measure on the sphere $S_F := S^{n-1} \cap F$. Since

$$h_{P_F(Z_q(\mu))}(\theta) = h_{Z_q(\mu)}(\theta), \qquad \theta \in S_F, \tag{3.16}$$

and

$$\int_{G_{n,k}} \int_{S_F} h^k_{Z_q(\mu)}(\theta)\, d\sigma_F(\theta)\, d\nu_{n,k}(F) = \int_{S^{n-1}} h^k_{Z_q(\mu)}(\theta)\, d\sigma(\theta) = w_k^k(Z_q(\mu)), \tag{3.17}$$

we get the result. $\square$

The next Lemma gives some bounds for $w_k(Z_q(\mu))$.

Lemma 3.2. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. If $k = n^\delta$, $\delta \in \left(\frac{1}{2}, 1\right)$ and $1 \leq q \leq k$, then*

$$w_k(Z_q(\mu)) \leq c_3 q^{1/\alpha(\delta)}, \tag{3.18}$$

where $\alpha(\delta) = \frac{2\delta}{3\delta - 1}$.

Proof. Let $1 \le q \le k$. We distinguish two cases:
(i) Assume that $k \le n/q$. Then, we have $q \le n/q$ and (3.3) shows that $k_*(Z_q(\mu)) \ge cn/q$. Therefore, $k \le ck_*(Z_q(\mu))$ and, taking into account (2.2), one can check that

$$w_k(Z_q(\mu)) \simeq w(Z_q(\mu)) \le w_q(Z_q(\mu)) \simeq \sqrt{q}. \tag{3.19}$$

(ii) Assume that $k > n/q$. From Lemma 2.1 we have that $w_k(Z_q(\mu)) \simeq w(Z_q(\mu))$ if $k \le k_*(Z_q(\mu))$ and $w_k(Z_q(\mu)) \simeq \sqrt{k/n}R(Z_q(K))$ if $k \ge k_*(Z_q(\mu))$. Since $q \le k$, using (3.4) we get that $w_k(Z_q(K)) \le f(q,k)$, where $f(q,k) \le c\sqrt{q}$ if $q \le k \le k_*(Z_q(\mu))$ and $f(q,k) \le cq\sqrt{k/n}$ if $k \ge k_*(Z_q(\mu))$. Note that $k_*(Z_q(\mu)) \ge n/k$. So we get that

$$f(q,k) \le c\sqrt{q} \text{ if } q \le n/k \text{ and } f(q,k) \le q\sqrt{k/n} \text{ if } n/k \le q. \tag{3.20}$$

We want $q^{1-\frac{1}{\alpha}}\sqrt{k} \le C\sqrt{n}$ for all $q \le k$. This will be true if $k^{\frac{3}{2}-\frac{1}{\alpha}} \simeq n^{1/2}$. Since $k = n^\delta$, the optimal value of α is

$$\alpha(\delta) = \frac{2\delta}{3\delta - 1}. \tag{3.21}$$

From (i) we check that (3.18) holds true for $k \le n/q$ as well. This proves the Lemma. $\square$

Proof of Theorem 1.1(ii). We apply Markov's inequality for $q = 2^i$, $i = 1, \dots \log_2 k$ in Lemma 3.1, and taking into account the fact that $Z_p(\mu) \subseteq Z_q(\mu) \subseteq cZ_p(\mu)$ if $p < q \le 2p$, we conclude that

$$\sup_{1\le q\le k} \frac{R(Z_q(\pi_F(\mu)))}{w_k(Z_q(\mu))} \le C, \tag{3.22}$$

where $C > 0$ is an absolute constant, for all F in a subset A_k of $G_{n,k}$ with measure $\nu_{n,k}(A_k) \ge 1 - (\log_2 k)e^{-2k} \ge 1 - e^{-k}$.

Now, we are using the estimates from Lemma 3.2; for every $F \in A_k$ we have

$$\|\langle \cdot, \theta\rangle\|_{\psi'_{\alpha(\delta)}} = \sup_{1\le q\le k} \frac{\|\langle \cdot, \theta\rangle\|_{L_q(\pi_F(\mu))}}{q^{1/\alpha(\delta)}} \le C_1 \sup_{1\le q\le k} \frac{R(Z_q(\pi_F(\mu)))}{w_k(Z_q(\mu))} \le C_2 \tag{3.23}$$

for all $\theta \in S_F$, where $C_2 > 0$ is an absolute constant. $\square$

4 Further remarks and applications

4.1. Assume that μ is a ψ_β measure with constant $r > 0$ for some $\beta \in (1,2)$. Then, the argument of Section 3 leads to the following generalization of Theorem 1.1.

Theorem 4.1. *Let $\beta \in (1,2)$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ which is ψ_β with constant $r > 0$.*

(i) *If $k \leq n^{\frac{\beta}{2}}$ then there exists $A_k \subseteq G_{n,k}$ with measure $\nu_{n,k}(A_k) > 1 - \exp(-cn^{\frac{\beta}{2}})$ such that, for every $F \in A_k$, $\pi_F(\mu)$ is a ψ_2'-measure with constant $C(r)$.*

(ii) *If $k = n^{\delta}$, $\frac{\beta}{2} < \delta < 1$ then there exists $A_k \subseteq G_{n,k}$ with measure $\nu_{n,k}(A_k) > 1 - \exp(-ck)$ such that, for every $F \in A_k$, $\pi_F(\mu)$ is a $\psi'_{\alpha(\delta,\beta)}$-measure with constant $C(r)$, where $\alpha(\delta,\beta) = \frac{2\beta\delta}{(2\delta-\beta)+\beta\delta}$.*

4.2. The estimate for $\alpha(\delta,\beta)$ in Theorem 4.1 is optimal in the following sense: let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ which is ψ_β and has the property that there exists $\theta \in S^{n-1}$ such that $h_{Z_q(\mu)}(\theta) \simeq q^{\frac{1}{\beta}}$ for all $1 \leq q \leq n$. Then, Lemma 2.1 shows that for $k = n^{\delta} \geq n^{\frac{\beta}{2}}$ we have

$$w_k(Z_k(\mu)) \simeq \sqrt{k/n}\, R(Z_k(\mu)) \simeq \sqrt{k/n}\, k^{\frac{1}{\beta}} = k^{\frac{1}{\alpha(\beta,\delta)}}. \tag{4.1}$$

Then, using (3.17) and the Paley-Zygmund inequality we can check that there exists $A_k \subseteq G_{n,k}$ with measure $\nu_{n,k}(A_k) > \exp(-ck)$ such that, for every $F \in A_k$ there exists $\theta \in S_F$ such that $h_{Z_k(\mu)}(\theta) \geq ck^{\frac{1}{\alpha(\beta,\delta)}}$.

4.3. For every $p \geq 1$ we consider the convex body $K_p(\mu)$ (introduced by K. Ball in [1]) with gauge function

$$\|x\|_{K_p(\mu)} := \left(\frac{p}{f_\mu(0)}\int_0^\infty f_\mu(rx) r^{p-1}\,dr\right)^{-1/p}. \tag{4.2}$$

Let $1 \leq k < n$ and $F \in G_{n,k}$. For $\theta \in S_F$ we define

$$\|\theta\|_{B_{k+1}(\mu,F)} := \|\theta\|_{K_{k+1}(\pi_F(\mu))}. \tag{4.3}$$

For all $1 \leq q \leq k < n$ and $F \in G_{n,k}$, one has (see [18] and [3])

$$f_{\pi_F(\mu)}(0)^{\frac{1}{k}} Z_q(\pi_F(\mu)) \simeq f_\mu(0)^{\frac{1}{n}} Z_q(\widetilde{B_{k+1}}(\mu,F)). \tag{4.4}$$

If μ is isotropic, then $\widetilde{B_{k+1}}(\mu,F)$ is an isotropic convex body in F. In particular, the case $q = 2$ of (4.4) shows that

$$f_{\pi_F(\mu)}(0)^{\frac{1}{k}} \simeq f_\mu(0)^{\frac{1}{n}} L_{B_{k+1}(\mu,F)}. \tag{4.5}$$

Since the ψ_α and ψ'_α norms are equivalent for convex bodies, as an immediate consequence of the above formulas we get the following version of Theorem 4.1:

Theorem 4.2. *Let $\beta \in (1,2)$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ which is ψ_β with constant $r > 0$.*

(i) *If $k \leq n^{\frac{\beta}{2}}$ then there exists $A_k \subseteq G_{n,k}$ with measure $\nu_{n,k}(A_k) > 1 - \exp(-cn^{\frac{\beta}{2}})$ such that, for every $F \in A_k$, $\widetilde{B_{k+1}}(\mu,F)$ is a ψ_2-body with constant $C(r)$.*

(ii) *If* $k = n^\delta$, $\frac{\beta}{2} < \delta < 1$ *then there exists* $A_k \subseteq G_{n,k}$ *with measure* $\nu_{n,k}(A_k) > 1 - \exp(-ck)$ *such that, for every* $F \in A_k$, $\widetilde{B_{k+1}}(\mu, F)$ *is a* $\psi_{\alpha(\delta,\beta)}$*-body with constant* $C(r)$*, where* $\alpha(\delta, \beta) = \frac{2\beta\delta}{(2\delta-\beta)+\beta\delta}$.

4.4. It was mentioned in §2.5 that if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$, then $I_{-q}(\mu) \simeq I_q(\mu)$ for every $1 \leq q \leq q_*(\mu)$. If μ is a ψ_β-measure, then $q_*(\mu) \geq cn^{\frac{\beta}{2}}$. This gives the lower bound

$$I_{-q}(\mu) \geq c\sqrt{n} \tag{4.6}$$

for all $q \leq n^{\frac{\beta}{2}}$. Using the results of this note, we can give some non-trivial lower bounds for $I_{-q}(\mu)$ when $q \gg n^{\frac{\beta}{2}}$. Let f be the density of μ. We start with a formula from [18, Proposition 4.6]: taking into account (4.5) we see that, for every $1 \leq k < n$,

$$I_{-k}(\mu) \simeq \sqrt{n} \left(\int_{G_{n,k}} L^k_{B_{k+1}(\mu,F)} d\nu_{n,k}(F) \right)^{-\frac{1}{k}}. \tag{4.7}$$

Then, what we need is an upper bound for the quantity

$$\int_{G_{n,k}} L^k_{B_{k+1}(\mu,F)} \, d\nu_{n,k}(F) \tag{4.8}$$

in the case $k = n^\delta$, $\delta \in \left(\frac{\beta}{2}, 1\right)$. We now use the following fact (see [6, Theorem 2.5.4]): If $\alpha \in (1, 2]$ and C is an isotropic convex body in $\mathbb{R}^k$ which is ψ_α with constant $r > 0$, then

$$L_C \leq cr^{\frac{\alpha}{2}} k^{\frac{2-\alpha}{4}} \log k. \tag{4.9}$$

From Lemma 3.2 we know that, for every $1 \leq q \leq k$, we have $w_k(Z_q(\mu)) \leq cq^{1/\alpha_*}$, where $\alpha_* = \frac{2\beta\delta}{(2\delta-\beta)+\beta\delta}$.

Then, the argument of Lemma 3.1 shows that the probability that $R(Z_q(B_{k+1}(\mu, F))) > csq^{1/\alpha_*}$ is less than s^{-k}. It follows that, for every $s \geq 1$ we have

$$\sup_{\theta \in S_F} \|\langle \cdot, \theta \rangle\|_{\psi_{\alpha_*}(B_{k+1}(\mu,F))} \leq c_1 s \tag{4.10}$$

on a subset $B_{k,s}$ of $G_{n,k}$ of measure $\nu_{n,k}(B_{k,s}) \geq 1 - s^{-k}$. Therefore,

$$L_{B_{k+1}(\mu,F)} \leq c_2 s^{\frac{\alpha_*}{2}} k^{\frac{2-\alpha_*}{4}} \log k \tag{4.11}$$

for all $F \in B_{k,s}$. Set $m(k) = c_2 k^{\frac{2-\alpha_*}{4}} \log k$. Then, we can estimate the integral

(4.7) as follows:

$$
\begin{aligned}
\int_{G_{n,k}} L^k_{B_{k+1}(\mu,F)}\, d\nu_{n,k}(F) &= \int_0^\infty kt^{k-1}\nu_{n,k}(F : L_{B_{k+1}(\mu,F)} \geq t)\, dt \\
&\leq m^k(k) + \int_{m(k)}^\infty kt^{k-1}\nu_{n,k}(F : L_{B_{k+1}(\mu,F)} \geq t)\, dt \\
&= m^k(k)\left[1 + \int_1^\infty ks^{\frac{(k-1)\alpha_*}{2}}\nu_{n,k}(F : L_{B_{k+1}(\mu,F)} \geq m(k)s^{\frac{\alpha_*}{2}})\, dt\right] \\
&\leq m^k(k)\left[1 + \frac{k\alpha_*}{2}\int_1^\infty s^{\frac{(k-1)\alpha_*}{2}} s^{\frac{\alpha_*}{2}-1} s^{-k} ds\right] \\
&= m^k(k)\left[1 + \frac{k\alpha_*}{2}\int_1^\infty s^{-1-k\left(1-\frac{\alpha_*}{2}\right)} ds\right] \\
&\simeq m^k(k) \simeq \left(k^{\frac{2-\alpha_*}{4}}\log k\right)^k.
\end{aligned}
$$

Inserting this information into (4.7) we get

$$
I_{-k}(\mu) \geq \frac{c\sqrt{n}}{k^{\frac{2-\alpha_*}{4}}\log k} \geq \frac{c}{\log n} n^{\frac{1}{2}-\frac{\delta(2-\alpha_*)}{4}}. \tag{4.12}
$$

Using our estimate for $\alpha_* = \alpha(\delta, \beta)$, we finally get the following:

Theorem 4.3. *Let $\beta \in [1, 2]$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$, which is a ψ_β-measure with constant $r > 0$.*

1. *If $k \leq n^{\frac{\beta}{2}}$, then $I_{-k}(\mu) \geq c(r)\sqrt{n}$.*
2. *If $k = n^\delta$ for some $\delta \in \left(\frac{\beta}{2}, 1\right)$, then*

$$
I_{-k}(\mu) \geq c(r)\frac{n^{\frac{(1-\delta)(2\delta-\beta)+\beta\delta}{2[(2\delta-\beta)+\beta\delta]}}}{\log n}. \tag{$*$}
$$

Final remark. In Theorem 4.3, we can actually obtain a stronger estimate. For an isotropic convex body C in $\mathbb{R}^s$, let $C_1 = C \cap (4\sqrt{s}L_C)B_2^s$ and $\overline{C} = \widetilde{C_1}$. For any $F \in G_{n,k}$ we consider the body $\overline{B_{k+1}(K,F)}$ and, using the estimates from Lemma 3.2, we observe that

(i) $h_{Z_q(\overline{B_{k+1}(K,F)})}(\theta) \leq c\sqrt{q}L_{B_{k+1}(K,F)}$, for $1 \leq q \leq \left(\frac{n}{k}\right)^\beta$,

(ii) $h_{Z_q(\overline{B_{k+1}(K,F)})}(\theta) \leq c\sqrt{\frac{k}{n}}q^{\frac{1}{\beta}}L_{B_{k+1}(K,F)}$, for $\left(\frac{n}{k}\right)^\beta \leq q \leq n^{\frac{\beta}{2}}$,

(iii) $h_{Z_q(\overline{B_{k+1}(K,F)})}(\theta) \leq c\sqrt{k}L_{B_{k+1}(K,F)}$, for $n^{\frac{\beta}{2}} \leq q \leq k$.

This implies that $\overline{B_{k+1}(K,F)}$ is a ψ_2-body with constant $O(n^{\frac{2\delta-\beta}{4}})$. Inserting this information in the proof of Theorem 4.3, and using the fact – proved in [3] – that if C is a ψ_2 body with constant r then $L_C \leq cr\sqrt{\log(er)}$ in the place of (4.9), one can prove the following fact: Let $\beta \in [1, 2]$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$, which is a ψ_β-measure with constant $r > 0$.

(i) If $k \leq n^{\frac{\beta}{2}}$, then $I_{-k}(\mu) \geq cr\sqrt{n}$.

(ii) If $k = n^\delta$ for some $\delta \in \left(\frac{\beta}{2}, 1\right)$, then

$$(**) \qquad I_{-k}(\mu) \geq cr \frac{n^{\frac{1}{2} - \frac{2\delta - \beta}{4}}}{\sqrt{\log\left((crn)^{\frac{2\delta-\beta}{4}}\right)}}.$$

Using this result, we can also slightly improve the small probability estimate

$$\mu\left(\{x \in \mathbb{R}^n : \|x\|_2 \leq c\varepsilon\sqrt{n}\}\right) \leq \varepsilon^{\sqrt{n}}$$

from [18]. Using $(**)$ one can show that if μ is an isotropic log-concave measure in $\mathbb{R}^n$ then, for every $\varepsilon \in (0,1)$,

$$(4.13) \qquad \mu\left(\{x \in \mathbb{R}^n : \|x\|_2 \leq c\varepsilon\sqrt{n}\}\right) \leq \varepsilon^{\sqrt{n}} \min\left\{1, \varepsilon^{n^{\delta(\varepsilon,n)}}\right\},$$

where $\delta(\varepsilon, n) = \frac{\log(\varepsilon^{-2})}{\log n} - \log\log n$. We omit the detailed proofs of these assertions; we would also like to mention that these estimates are optimal up to our current knowledge on L_K and a logarithmic in the dimension term.

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