Inequalities for sections and projections of log-concave functions

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Abstract

We provide extensions of geometric inequalities about sections and projections of convex bodies to the setting of integrable log-concave functions. Namely, we consider suitable generalizations of the affine and dual affine quermassintegrals of a log-concave function f and obtain upper and lower estimates for them in terms of the integral $\|f\|_1$ of f, we give estimates for sections and projections of log-concave functions in the spirit of the lower dimensional Busemann-Petty and Shephard problem, and we extend to log-concave functions the affirmative answer to a variant of the Busemann-Petty and Shephard problems, proposed by V. Milman. The main goal of this article is to show that the assumption of log-concavity leads to inequalities in which the constants are of the same order as that of the constants in the original corresponding geometric inequalities.

1 Introduction

In this article we present extensions of some geometric inequalities about sections and projections of convex bodies to the setting of integrable log-concave functions. Various extensions of this type have been given for even more general classes of functions but our aim is to show that the assumption of log-concavity leads to much better constants, namely of the order of the constants in the corresponding geometric inequalities. We say that a function $f:\mathbb{R}^n\to[0,+\infty)$ is log-concave if $f=e^{-\varphi}$ where $\varphi:\mathbb{R}^n\to(-\infty,\infty]$ is convex and lower semicontinuous. We also say that f is a geometric log-concave function if it is log-concave and $\|f\|_\infty=f(0)=1$. We denote by $\mathcal{F}(\mathbb{R}^n)$ the class of log-concave integrable functions on $\mathbb{R}^n$ and by $\mathcal{F}_0(\mathbb{R}^n)$ the class of geometric log-concave integrable functions. Let $f\in\mathcal{F}(\mathbb{R}^n)$. Given $E\in G_{n,k}$, where $G_{n,k}$ is the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$, the "section" of f with E is the restriction $f\big|_E$ of f onto E and the "projection" or "shadow" of f onto E is the function

$$P_Ef(x):=\max\{f(y):y\in x+E^\perp\},\quad x\in E$$

where $E^\perp$ is the orthogonal subspace of E.

We start with a very brief description of our results. First we consider suitable generalizations of the affine and dual affine quermassintegrals of a log-concave function f and provide upper and lower estimates for them in terms of the integral

$$\|f\|_1=\int_{\mathbb{R}^n}f(x)\,dx$$

of f. Then, we extend to log-concave functions the affirmative answer to a variant of the Busemann-Petty and Shephard problems, proposed by V. Milman. We also provide estimates for sections of log-concave functions in the spirit of the lower dimensional Busemann-Petty problem, and estimates for projections of log-concave functions in the spirit of the lower dimensional Shephard problem. More information about each of these questions, and about the classical geometric counterparts of our results, is provided separately in the corresponding sections of the article.

In Section 3 we consider the affine and dual affine quermassintegrals of log-concave integrable functions. Recall that for every convex body K or, more generally, for any bounded Borel set in $\mathbb{R}^n$, and for any $1 \leqslant k \leqslant n-1$, the k-th dual affine quermassintegral of K is defined by

$$\Psi_k(K) := \left(\int_{G_{n,k}} |K \cap E^{\perp}|^n \, d\nu_{n,k}(E) \right)^{\frac{1}{kn}},$$

where $|\cdot|$ denotes volume in the appropriate dimension and $\nu_{n,k}$ is the Haar probability measure on $G_{n,k}$. We consider the following natural generalization of Ψ_k for a log-concave integrable function:

$$\Psi_k(f) := \left(\int_{G_{n,k}} \|f\,|_{E^{\perp}}\|_1^n \, d\nu_{n,k}(E) \right)^{\frac{1}{kn}}.$$

Our estimates for $\Psi_k(f)$ are the following.

Theorem 1.1. *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a geometric log-concave integrable function. Then,*

$$\frac{c}{\psi_{n,k}} \|f\|_1^{\frac{n-k}{kn}} \leqslant \Psi_k(f) \leqslant \sqrt{e} \|f\|_1^{\frac{n-k}{kn}}$$

for every $1 \leqslant k \leqslant n-1$, where $\psi_{n,k} := \min\{L_n, \sqrt{n/k}\,\sqrt{\log(en/k)}\}$ and $c > 0$ is an absolute constant.

In the statement above, $L_n = \max\{L_f : f \in \mathcal{F}(\mathbb{R}^n)\}$ where L_f is the isotropic constant of f, and it is known that $L_n = O(\sqrt{\log n})$ (see Section 2 for background information).

Next, recall that for every convex body K or, more generally, for any bounded Borel set in $\mathbb{R}^n$, and for any $1 \leqslant k \leqslant n-1$, the k-th affine quermassintegral of K is defined by

$$\Phi_k(K) := \left(\int_{G_{n,k}} |P_E(K)|^{-n} \, d\nu_{n,k}(E) \right)^{-\frac{1}{kn}}.$$

We generalize the definition of Φ_k as follows: for any log-concave integrable function we set

$$\Phi_k(f) = \left(\int_0^{\infty} (\Phi_k(R_t(f)))^n \, dt \right)^{1/n}, \tag{1.1}$$

where $R_t(f) = \{x : f(x) \geqslant t\}$, $t > 0$. We note at this point that if $f = \mathbb{1}_K$ is the indicator function of a convex body K in $\mathbb{R}^n$ then $\Phi_k(\mathbb{1}_K) = \Phi_k(K)$ for all $1 \leqslant k \leqslant n-1$ (as we explain in Section 3, this follows from the observation that $P_E(R_t(f)) = R_t(P_E \mathbb{1}_K) = P_E(K)$ for all $0 \leqslant t < 1$ and $R_t(P_E \mathbb{1}_K) = \varnothing$ for all $t \geqslant 1$). Therefore, our definition of $\Phi_k(f)$ in (1.1) generalizes the definition of the k-th affine quermassintegral of a convex body. We obtain the following estimates.

Theorem 1.2. *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a geometric log-concave integrable function. Then,*

$$c_1 \sqrt{n/k}\, \|f\|_1^{\frac{1}{n}} \leqslant \Phi_k(f) \leqslant c_2 \sqrt{n/k}\, \phi_{n,k} \, \|f\|_1^{\frac{1}{n}}$$

for every $1 \leqslant k \leqslant n-1$, where $\phi_{n,k} = \min\left\{\log n, \frac{n}{k}\sqrt{\log(en/k)}\right\}$ and $c_1, c_2 > 0$ are absolute constants.

In Section 4 we study a variant of the Busemann-Petty and Shephard problems, proposed by V. Milman in the setting of convex bodies: If K and T are origin-symmetric convex bodies in $\mathbb{R}^n$ that satisfy $|P_{\xi^\perp}(K)| \leqslant |T \cap \xi^\perp|$ for all $\xi \in S^{n-1}$, does it follow that $|K| \leqslant |T|$? Giannopoulos and Koldobsky proved in [21] that the answer to this question (and in fact to the lower dimensional analogue of the question) is affirmative. Moreover, one can relax the symmetry assumption and even the assumption of convexity for T: If K is a convex body in $\mathbb{R}^n$ and T is a bounded Borel subset of $\mathbb{R}^n$ such that $|P_E(K)| \leqslant |T \cap E|$ for some $1 \leqslant k \leqslant n-1$ and for all $E \in G_{n,n-k}$, then $|K| \leqslant |T|$.

We extend this result to log-concave integrable functions.

Theorem 1.3. *Let $f, g : \mathbb{R}^n \to [0, +\infty)$ be geometric log-concave integrable functions such that, for some $1 \leqslant k \leqslant n-1$, we have that*

$$\|P_E f\|_1 \leqslant \|g\big|_E\|_1 \qquad \text{for all } E \in G_{n,n-k}.$$

Then,

$$\|f\|_1 \leqslant \frac{n!}{[(n-k)!]^{\frac{n}{n-k}}} \|g\|_1. \tag{1.2}$$

Using Stirling's formula we check that the constant $n!/[(n-k)!]^{\frac{n}{n-k}}$ that appears in (1.2) is of the order of $C^{\frac{kn}{n-k}}$.

In Section 5 we extend to log-concave integrable functions a number of inequalities related to the Busemann-Petty problem and the slicing problem. They all follow from the next general inequality about the Radon transform on convex sets.

Theorem 1.4. *Let $f, g : \mathbb{R}^n \to [0, \infty)$ be non-negative integrable functions such that f is log-concave with $f(0) > 0$ and $\|g\|_\infty = g(0) = 1$. If K is a convex body in $\mathbb{R}^n$ and T is a compact subset of $\mathbb{R}^n$ with $0 \in K \cap T$, then*

$$\left(\frac{f(0)^{-1} \int_K f(x)\, dx}{\int_T g(x)\, dx} \right)^{\frac{n-k}{n}} \leqslant (c\beta_{n,k})^k \max_{E \in G_{n,n-k}} \frac{f(0)^{-1} \int_{K \cap E} f(x)\, dx}{\int_{T \cap E} g(x)\, dx}, \tag{1.3}$$

where $\beta_{n,k} = \min\left\{ L_n, \sqrt{n/k}\, (\log(en/k))^{\frac{3}{2}} \right\}$ and $c > 0$ is an absolute constant.

We discuss several consequences of Theorem 1.4 in Section 5. In particular, we obtain a lower estimate for the sup-norm of the Radon transform which sharpens Koldobsky's slicing inequality for arbitrary functions (see [30]) under the log-concavity assumption.

Theorem 1.5. *Let $f \in \mathcal{F}_0(\mathbb{R}^n)$. Then, for every $1 \leqslant k \leqslant n-1$ we have that*

$$\int_K f(x)\, dx \leqslant (c\beta_{n,k})^k |K|^{\frac{k}{n}} \max_{E \in G_{n,n-k}} \int_{K \cap E} f(x)\, dx.$$

In Section 6 we obtain a Shephard-type estimate for log-concave integrable functions. Let $1 \leqslant k \leqslant n-1$ and let $S_{n,k}$ be the smallest constant $S > 0$ with the following property: For every pair of convex bodies K and T in $\mathbb{R}^n$ that satisfy $|P_E(K)| \leqslant |P_E(T)|$ for all $E \in G_{n,n-k}$, one has that $|K|^{\frac{n-k}{n}} \leqslant S^k\, |T|^{\frac{n-k}{n}}$. The final (negative) answer to the classical Shephard's problem is the fact that $S_{n,1} \approx \sqrt{n}$ (see Section 6 for more details and references). General estimates for $S_{n,k}$ were obtained in [21] where it was shown that if

$$\tilde{S}_{n,k} = \min\left\{ \sqrt{\tfrac{n}{n-k}} \ln\left(\tfrac{en}{n-k}\right), \ln n \right\}$$

then $S_{n,k} \leqslant (c_1\tilde{S}_{n,k})^{\frac{n-k}{k}}$. In particular, $S_{n,k} \leqslant C^{\frac{n-k}{k}}$ if $\frac{k}{n-k}$ is bounded. We provide a third estimate, namely that $S_{n,k} \leqslant \sqrt{n}$ for all n and k, which is consistent with the estimate for $S_{n,1}$ and better than the previous ones if $k \ll n/\log n$. Our main goal is to obtain an analogue of the above estimates for $S_{n,k}$ for geometric log-concave integrable functions.

Theorem 1.6. *Let $f, g \in \mathcal{F}_0(\mathbb{R}^n)$ and $1 \leqslant k \leqslant n-1$. Assume that $|R_t(P_E f)| \leqslant |R_t(P_E g)|$ for all $E \in G_{n,n-k}$ and $0 \leqslant t < 1$. Then,*

$$\|f\|_1^{\frac{n-k}{n}} \leqslant S_{n,k}^k \|g\|_1^{\frac{n-k}{n}}.$$

Theorem 1.6 extends the known Shephard-type estimates for convex bodies. To see this, we note that if $f = \mathbb{1}_K$ and $g = \mathbb{1}_T$ are the indicator functions of two convex bodies K and T in $\mathbb{R}^n$ then the assumption in Theorem 1.6 is equivalent to the assumption $|P_E(K)| \leqslant |P_E(T)|$ in the original Shephard's problem.

We close this introductory section with a few comments on some of the main tools that are used for the proofs of the above results. A very fruitful idea in order to study the geometric properties of a log-concave function f on $\mathbb{R}^n$ with $f(0) > 0$ is to use the family of bodies $K_p(f)$, introduced by K. Ball in [3]. In Section 2 we recall their definition: $K_p(f)$ is the star body with radial function

$$\rho_{K_p(f)}(x) = \left(\frac{1}{f(0)} \int_0^\infty p r^{p-1} f(rx)\, dr \right)^{1/p}$$

for $x \neq 0$. The log-concavity of f implies that the bodies $K_p(f)$ are convex. Direct computation shows that

$$|K_n(f)| = \frac{1}{f(0)} \int_{\mathbb{R}^n} f(x)\, dx$$

and

$$|K_{n-k}(f) \cap E| = \frac{1}{f(0)} \int_E f(x), dx$$

for any $1 \leqslant k \leqslant n-1$ and $E \in G_{n,n-k}$. Moreover (see Lemma 2.1 and Lemma 2.2), one can compare the volumes of $|K_{n-k}(f)|$ and $|K_n(f)|$. These facts allow us to translate computations involving sections of f to computations about the sections of the convex body $K_{n-k}(f)$ and exploit what is known in the setting of convex bodies. This idea is used in the proof of Theorem 1.1 and Theorem 1.4.

In order to study the projections of a log-concave function f on $\mathbb{R}^n$ we use mixed integrals of functions. If $f_1, \ldots, f_n : \mathbb{R}^n \to [0, \infty)$ are bounded integrable functions such that $R_t(f_j) = \{f_j \geqslant t\}$ is a compact convex set for all $1 \leqslant j \leqslant n$ and $t > 0$, their mixed integral (see [38]) is defined by

$$V(f_1, f_2, \ldots, f_n) := \int_0^\infty V(R_t(f_1), R_t(f_2), \ldots, R_t(f_n))\, dt,$$

where $V(R_t(f_1), R_t(f_2), \ldots, R_t(f_n))$ is the classical mixed volume of the sets $R_t(f_j)$. The k-th quermassintegral of f is defined as $W_k(f) = V((f, n-k), (\mathbb{1}_{B_2^n}, k))$. Direct computation shows that

$$V(f, \ldots, f) = \|f\|_1$$

and

$$W_k(f) = \frac{\omega_n}{\omega_{n-k}} \int_{G_{n,n-k}} \|P_E(f)\|_1\, d\nu_{n,n-k}(E).$$

Moreover, the quermassintegrals $W_k(f)$ can be compared via Aleksandrov-type inequalities (see (4.3)). These facts allow us to translate computations involving projections of f to computations about the quermassintegrals of the convex bodies $R_t(f)$ and exploit what is known in the setting of convex bodies. This idea is used in the proof of Theorem 1.2 and Theorem 1.3.

2 Background information and auxiliary results

We write $\langle\cdot,\cdot\rangle$ for the standard inner product in $\mathbb{R}^n$ and $|\cdot|$ for the Euclidean norm. Lebesgue measure in $\mathbb{R}^n$ is also denoted by $|\cdot|$. We denote the Euclidean unit ball and unit sphere by B_2^n and S^{n-1} respectively, and write σ for the rotationally invariant probability measure on S^{n-1}. We define $\omega_n = |B_2^n|$. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. The letters c, c', c_j, c_j' etc. denote absolute positive constants whose value may change from line to line.

A convex body in $\mathbb{R}^n$ is a compact convex set $K \subset \mathbb{R}^n$ with non-empty interior. We say that K is centrally symmetric if $-K = K$ and that K is centered if the barycenter $\mathrm{bar}(K) = \frac{1}{|K|}\int_K x\,dx$ of K is at the origin. If K is a convex body with $0 \in \mathrm{int}(K)$ then the radial function ϱ_K of K is defined for all $x \neq 0$ by $\varrho_K(x) = \sup\{t > 0 : tx \in K\}$ and the support function of K is defined by $h_K(x) = \sup\{\langle x, y\rangle : y \in K\}$ for all $x \in \mathbb{R}^n$. The polar body K° of a convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$ is the convex body

$$K^\circ := \big\{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in K\big\}.$$

A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, and its inertia matrix is a multiple of the identity matrix. This is equivalent to the fact that there exists a constant $L_K > 0$, the isotropic constant of K, such that

$$\|\langle\cdot,\xi\rangle\|_{L_2(K)}^2 := \int_K \langle x,\xi\rangle^2 dx = L_K^2$$

for all $\xi \in S^{n-1}$.

A Borel measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any pair of compact sets A and B in $\mathbb{R}^n$ and any $\lambda \in (0,1)$. A function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if its support $\{f > 0\}$ is a convex set in $\mathbb{R}^n$ and the restriction of $\ln f$ to it is concave. If f has finite positive integral then there exist constants $A, B > 0$ such that $f(x) \leqslant Ae^{-B|x|}$ for all $x \in \mathbb{R}^n$ (see [9, Lemma 2.2.1]). In particular, f has finite moments of all orders. A result of Borell [6] shows that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H in $\mathbb{R}^n$, then μ has a log-concave density f_μ. We say that μ is even if $\mu(-B) = \mu(B)$ for every Borel subset B of $\mathbb{R}^n$ and that μ is centered if

$$\mathrm{bar}(\mu) := \int_{\mathbb{R}^n} \langle x,\xi\rangle d\mu(x) = \int_{\mathbb{R}^n} \langle x,\xi\rangle f_\mu(x) dx = 0$$

for all $\xi \in S^{n-1}$. Fradelizi has shown in [17] that if μ is a centered log-concave probability measure on $\mathbb{R}^k$ then

$$\|f_\mu\|_\infty \leqslant e^k f_\mu(0). \tag{2.1}$$

Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that the indicator function $\mathbb{1}_K$ of K is the density of a log-concave measure, the Lebesgue measure on K.

Let μ and ν be two Borel measures on $\mathbb{R}^n$. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function which is defined ν-almost everywhere and satisfies

$$\mu(B) = \nu(T^{-1}(B))$$

for every Borel subset B of $\mathbb{R}^n$ then we say that T pushes forward ν to μ and write $T_*\nu = \mu$. It is easy to see that $T_*\nu = \mu$ if and only if for every bounded Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} g(x) d\mu(x) = \int_{\mathbb{R}^n} g(T(y)) d\nu(y).$$

If μ is a log-concave measure on $\mathbb{R}^n$ with density f_μ, we define the isotropic constant of μ by

$$L_\mu := \left(\frac{\sup_{x\in\mathbb{R}^n} f_\mu(x)}{\int_{\mathbb{R}^n} f_\mu(x)dx} \right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} - \frac{\int_{\mathbb{R}^n} x_i f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx} \frac{\int_{\mathbb{R}^n} x_j f_\mu(x)\, dx}{\int_{\mathbb{R}^n} f_\mu(x)\, dx}.$$

We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered and $\mathrm{Cov}(\mu) = I_n$, where I_n is the identity $n \times n$ matrix. In this case, $L_\mu = \|f_\mu\|_\infty^{1/n}$. It is known that for every μ there exists an affine transformation T such that $T_*\mu$ is isotropic. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$L_n := \max\{L_\mu : \mu \text{ is an isotropic log-concave probability measure on } \mathbb{R}^n\} \leqslant C$$

for all $n \geqslant 1$. Bourgain proved in [8] that $L_n \leqslant c_1 \sqrt[4]{n} \log n$, and Klartag [26] improved this bound to $L_n \leqslant c_2 \sqrt[4]{n}$. After a breakthrough work of Chen [13] who proved that for any $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $L_n \leqslant n^\varepsilon$ for every $n \geqslant n_0(\varepsilon)$, and then polylogarithmic bounds for L_n obtained by Klartag and Lehec [28] and then by Jambulapati, Lee and Vempala in [25], the best to date known bound $L_n \leqslant c_5 \sqrt{\log n}$ is due to Klartag [27].

As mentioned in the introduction, a main tool in our work is the family of K_p-bodies of a function, introduced by K. Ball in [3]. Given a measurable function $f : \mathbb{R}^n \to [0, \infty)$ with $f(0) > 0$, for any $p > 0$ we define the set

$$K_p(f) = \left\{ x \in \mathbb{R}^n : \int_0^\infty f(rx) r^{p-1}\, dr \geqslant \frac{f(0)}{p} \right\}. \tag{2.2}$$

From the definition it follows that the radial function of $K_p(f)$ is given by

$$\rho_{K_p(f)}(x) = \left(\frac{1}{f(0)} \int_0^\infty p r^{p-1} f(rx)\, dr \right)^{1/p} \tag{2.3}$$

for $x \neq 0$. It was proved by K. Ball that if f is also log-concave and integrable then, for every $p > 0$, $K_p(f)$ is a convex set.

Let $0 < p < q$. The next two lemmas establish inclusions between $K_p(f)$ and $K_q(f)$.

Lemma 2.1. *Let $f : \mathbb{R}^n \to [0, \infty)$ be a bounded measurable function such that $f(0) > 0$. If $0 < p \leqslant q$, then*

$$K_p(f) \subseteq \left(\frac{\|f\|_\infty}{f(0)} \right)^{\frac{1}{p} - \frac{1}{q}} K_q(f). \tag{2.4}$$

Proof. For any $x \neq 0$ consider the function $f_x : [0, \infty) \to [0, \infty)$ with $f_x(r) = f(rx)$. It is known that the function

$$G(p) := \left(\frac{1}{\|f_x\|_\infty} \int_0^\infty p r^{p-1} f_x(r)\, dr \right)^{1/p}$$

is increasing on $(0, \infty)$. A proof may be found in [9, Lemma 2.2.4]; note that the log-concavity of f is not required for the argument. Applying this fact one can check (see [9, Proposition 2.5.7]) that

$$\rho_{K_q(f)}(x) \geqslant \left(\frac{\|f_x\|_\infty}{f(0)} \right)^{1/q - 1/p} \rho_{K_p(f)}(x)$$

for all $0 < p \leqslant q$, and the lemma follows if we also note that $\|f_x\|_\infty \leqslant \|f\|_\infty$. □

Lemma 2.2. *Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function such that $f(0) > 0$. If $0 < p \leqslant q$, then*

$$\frac{\Gamma(p+1)^{\frac{1}{p}}}{\Gamma(q+1)^{\frac{1}{q}}} K_q(f) \subseteq K_p(f). \tag{2.5}$$

Proof. For any $x \neq 0$ consider the log-concave function $f_x : [0, \infty) \to [0, \infty)$ with $f_x(r) = f(rx)$. It is known that the function

$$F(p) := \left(\frac{1}{f_x(0)\Gamma(p)} \int_0^\infty r^{p-1} f_x(r)\, dr \right)^{1/p}$$

is decreasing on $(0, \infty)$. A proof may be found in [9, Theorem 2.2.3]. Applying this fact one can check (see [9, Proposition 2.5.7]) that

$$\rho_{K_q(f)}(x) \leqslant \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} \rho_{K_p(f)}(x)$$

for all $0 < p \leqslant q$. □

We shall use the fact that

$$\begin{aligned} |K_n(f)| &= \frac{1}{n} \int_{S^{n-1}} \rho_{K_n(f)}(\xi)^n d\xi \\ &= \frac{1}{n} \int_{S^{n-1}} \frac{1}{f(0)} \int_0^\infty n r^{n-1} f(r\xi)\, dr\, d\xi = \frac{1}{f(0)} \int_{\mathbb{R}^n} f(x)\, dx. \end{aligned} \tag{2.6}$$

Similarly, for any $1 \leqslant k \leqslant n-1$ and $E \in G_{n,n-k}$ we see that

$$\begin{aligned} |K_{n-k}(f) \cap E| &= \frac{1}{n-k} \int_{S^{n-1} \cap E} \rho_{K_{n-k}(f)}(\xi)^{n-k} d\xi \\ &= \frac{1}{n-k} \int_{S^{n-1} \cap E} \frac{1}{f(0)} \int_0^\infty (n-k) r^{n-k-1} f(r\xi)\, dr\, d\xi \\ &= \frac{1}{f(0)} \int_E f(x)\, dx. \end{aligned} \tag{2.7}$$

We will also briefly consider s-concave measures. We say that a measure μ on $\mathbb{R}^n$ is s-concave for some $-\infty \leqslant s \leqslant 1/n$ if

$$\mu((1-\lambda)A + \lambda B) \geqslant ((1-\lambda)\mu^s(A) + \lambda \mu^s(B))^{1/s} \tag{2.8}$$

for any pair of compact sets A, B in $\mathbb{R}^n$ with $\mu(A)\mu(B) > 0$ and any $\lambda \in (0,1)$. We can also consider the limiting cases $s = 0$, where the right-hand side in (2.8) should be understood as $\mu(A)^{1-\lambda}\mu(B)^\lambda$, and

$s = -\infty$, where the right-hand side in (2.8) becomes $\min\{\mu(A), \mu(B)\}$. Note that 0-concave measures are the log-concave measures and that if μ is s-concave and $s' \leqslant s$ then μ is also s'-concave.

A function $f : \mathbb{R}^n \to [0, \infty)$ is called γ-concave for some $\gamma \in [-\infty, \infty]$ if

$$f((1-\lambda)x + \lambda y) \geqslant ((1-\lambda)f^\gamma(x) + \lambda f^\gamma(y))^{1/\gamma}$$

for all $x, y \in \mathbb{R}^n$ with $f(x)f(y) > 0$ and all $\lambda \in (0,1)$. One can also define the cases $\gamma = 0, +\infty$ appropriately. Borell [7] showed that if μ is a measure on $\mathbb{R}^n$ and the affine subspace F spanned by the support $\mathrm{supp}(\mu)$ of μ has dimension $\dim(F) = n$ then for every $-\infty \leqslant s < 1/n$ we have that μ is s-concave if and only if it has a non-negative density $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$ and f is γ-concave, where $\gamma = \frac{s}{1-sn} \in [-1/n, +\infty)$.

We shall extend some of our results to densities of s-concave measures with $s \in (-\infty, 0)$. Note that these classes of functions are strictly larger than the class of log-concave functions. Let μ be s-concave for some $s \in (-\infty, 0)$. Then, the density f of μ is $-\frac{1}{\alpha}$-concave, where $-\dfrac{1}{\alpha} = \dfrac{s}{1-sn}$, or equivalently,

$$\alpha = n - \frac{1}{s} > n.$$

Assume that $f(0) > 0$ and, for any $0 < p < \alpha$, define the star body $K_p(f)$ as in (2.2). It is proved in [18] that one has an analogue of Lemma 2.2: For any $0 < p \leqslant q < \alpha$,

$$K_q(f) \subseteq \frac{(qB(q, \alpha - q))^{1/q}}{(pB(p, \alpha - p))^{1/p}} K_p(f), \tag{2.9}$$

where $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function. Note that if $\alpha \geqslant q + 1$ then

$$c_1 \frac{q}{p} \leqslant \frac{(qB(q, \alpha - q))^{1/q}}{(pB(p, \alpha - p))^{1/p}} \leqslant c_2 \frac{q}{p}$$

where $c_1, c_2 > 0$ are absolute constants. This follows from [18, Lemma 11].

We refer to the classical monograph of Schneider [40] for the theory of convex bodies and to the books [1] and [9] for more information on asymptotic geometric analysis, isotropic convex bodies and log-concave probability measures. The study of classes of functions from a geometric point of view is a rapidly developing area of research; it is presented in [2, Chapter 9] where the reader may find a detailed exposition of the main ideas and several important functional inequalities that have been established.

3 Affine and dual affine quermassintegrals

The following inequality about sections of convex bodies was proved by Busemann and Straus [11], and independently by Grinberg [24]. If K is a convex body in $\mathbb{R}^n$ then, for any $1 \leqslant k \leqslant n-1$,

$$\int_{G_{n,k}} |K \cap E|^n d\nu_{n,k}(E) \leqslant \frac{\omega_k^n}{\omega_n^k} |K|^k. \tag{3.1}$$

Following Grinberg's argument one can check that this inequality is still true if we consider any bounded Borel set K in $\mathbb{R}^n$. This more general form appears in [20, Section 7]. Grinberg also observed that the integral in the left-hand side of (3.1) is invariant under volume preserving linear transformations of K.

Let $1 \leqslant k \leqslant n-1$. For every convex body K, or more generally for any bounded Borel set in $\mathbb{R}^n$, the

k-th dual affine quermassintegral of K is defined by

$$\Psi_k(K) := \left(\int_{G_{n,k}} |K\cap E^{\perp}|^n\, d\nu_{n,k}(E)\right)^{\frac{1}{kn}}.$$

Grinberg's inequality shows that if K is a bounded Borel set in $\mathbb{R}^n$ and B_K is the centered Euclidean ball with $|B_K| = |K|$ then

$$\Psi_k(K) \leqslant \Psi_k(B_K) = \left(\frac{\omega_{n-k}^n}{\omega_n^{n-k}}\right)^{\frac{1}{kn}} |K|^{\frac{n-k}{kn}} \leqslant \sqrt{e}|K|^{\frac{n-k}{kn}} \tag{3.2}$$

(the last inequality follows from the fact that $1 < \omega_{n-k}/\omega_n^{\frac{n-k}{n}} < e^{k/2}$; see [31, Lemma 2.1] for a proof). Assuming that K is a centered convex body in $\mathbb{R}^n$ there are two lower bounds on $\Psi_k(K)$, proved in [14]: one has that

$$\Psi_k(K) \geqslant \frac{c}{\psi_{n,k}}|K|^{\frac{n-k}{kn}}, \tag{3.3}$$

where $\psi_{n,k} := \min\left\{L_n, \left(\frac{n}{k}\log(en/k)\right)^{\frac{1}{2}}\right\}$. We consider the following natural generalization of Ψ_k for a non-negative bounded and integrable function $f : \mathbb{R}^n \to [0, +\infty)$:

$$\Psi_k(f) := \left(\int_{G_{n,k}} \|f\,|_{E^{\perp}}\|_1^n\, d\nu_{n,k}(E)\right)^{\frac{1}{kn}}.$$

Our aim is to give upper and lower bounds for $\Psi_k(f)$ in terms of $\|f\|_1$. A functional version of (3.1) is established in [15]: if $1 \leqslant m \leqslant n-1$ and f is a non-negative, bounded and integrable function on $\mathbb{R}^n$ then

$$\int_{G_{n,m}} \frac{\|f|_H\|_1^n}{\|f|_H\|_\infty^{n-m}}\, d\nu_{n,m}(H) \leqslant \frac{\omega_m^n}{\omega_n^m}\|f\|_1^m. \tag{3.4}$$

It follows that

$$\Psi_k(f) = \left(\int_{G_{n,n-k}} \|f\,|_H\|_1^n\, d\nu_{n,n-k}(H)\right)^{\frac{1}{kn}} \leqslant \|f\|_\infty^{1/n}\left(\int_{G_{n,n-k}} \frac{\|f|_H\|_1^n}{\|f|_H\|_\infty^k}\, d\nu_{n,n-k}(H)\right)^{\frac{1}{kn}}$$
$$\leqslant \|f\|_\infty^{1/n}\left(\frac{\omega_{n-k}^n}{\omega_n^{n-k}}\right)^{\frac{1}{kn}}\|f\|_1^{\frac{n-k}{kn}} = \sqrt{e}\|f\|_\infty^{1/n}\|f\|_1^{\frac{n-k}{kn}}.$$

In the next theorem we give an independent proof of the estimate $\Psi_k(f) \leqslant \sqrt{e}\|f\|_\infty^{1/n}\|f\|_1^{\frac{n-k}{kn}}$. Moreover, we show that if f is log-concave then a reverse inequality is also true.

Theorem 3.1. *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a bounded integrable function. Then,*

$$\Psi_k(f) \leqslant \sqrt{e}\|f\|_\infty^{1/n}\|f\|_1^{\frac{n-k}{kn}}.$$

If f is also assumed to be log-concave and $f(0) > 0$ then

$$\Psi_k(f) \geqslant \frac{c}{\psi_{n,k}} f(0)^{1/n}\|f\|_1^{\frac{n-k}{kn}}$$

where $c>0$ is an absolute constant.

Proof. Applying (2.7) we write

$$\begin{aligned}\Psi_k(f) &= \left(\int_{G_{n,n-k}} (f(0)|K_{n-k}(f)\cap H|)^n \, d\nu_{n,n-k}(H)\right)^{\frac{1}{kn}} = f(0)^{\frac{1}{k}}\Psi_k(K_{n-k}(f)) \\ &\leqslant \sqrt{e} f(0)^{\frac{1}{k}} \, |K_{n-k}(f)|^{\frac{n-k}{kn}} = \sqrt{e} f(0)^{\frac{1}{k}} \left(\frac{K_{n-k}(f)|}{|K_n(f)|}\right)^{\frac{n-k}{kn}} |K_n(f)|^{\frac{n-k}{kn}} \\ &\leqslant \sqrt{e} f(0)^{\frac{1}{k}} \left(\frac{\|f\|_\infty}{f(0)}\right)^{\frac{n-k}{k}\left(\frac{1}{n-k}-\frac{1}{n}\right)} \left(\frac{\|f\|_1}{f(0)}\right)^{\frac{n-k}{kn}} = \sqrt{e}\|f\|_\infty^{\frac{1}{n}}\|f\|_1^{\frac{n-k}{kn}},\end{aligned}$$

using in the last steps (3.2) for the star body $K_{n-k}(f)$, Lemma 2.1 and (2.6).

For the lower bound, we apply (2.7) to write

$$\begin{aligned}\Psi_k(f) &= \left(\int_{G_{n,n-k}} (f(0)|K_{n-k}(f)\cap H|)^n \, d\nu_{n,n-k}(H)\right)^{\frac{1}{kn}} = f(0)^{\frac{1}{k}}\Psi_k(K_{n-k}(f)) \\ &\geqslant \frac{c}{\psi_{n,k}} f(0)^{\frac{1}{k}} \, |K_{n-k}(f)|^{\frac{n-k}{kn}} = \frac{c}{\psi_{n,k}} f(0)^{\frac{1}{k}} \left(\frac{K_{n-k}(f)|}{|K_n(f)|}\right)^{\frac{n-k}{kn}} |K_n(f)|^{\frac{n-k}{kn}} \\ &\geqslant \frac{c}{\psi_{n,k}} f(0)^{\frac{1}{k}} \frac{[(n-k)!]^{\frac{1}{k}}}{[n!]^{\frac{n-k}{kn}}} \left(\frac{\|f\|_1}{f(0)}\right)^{\frac{n-k}{kn}} = \frac{c}{\psi_{n,k}} \frac{[(n-k)!]^{\frac{1}{k}}}{[n!]^{\frac{n-k}{kn}}} f(0)^{\frac{1}{n}}\|f\|_1^{\frac{n-k}{kn}}.\end{aligned}$$

using in the last steps (3.3) for the convex body $K_{n-k}(f)$, Lemma 2.2 and (2.6). From Stirling's formula one may check that there exists a constant $c_1>0$ such that, for all $n\geqslant 2$ and $1\leqslant k\leqslant n-1$,

$$\frac{[(n-k)!]^{\frac{1}{k}}}{[n!]^{\frac{n-k}{kn}}} = \left(\frac{\Gamma(n-k+1)^{\frac{1}{n-k}}}{\Gamma(n+1)^{\frac{1}{n}}}\right)^{\frac{n-k}{k}} \geqslant c_1. \tag{3.5}$$

Therefore,

$$\Psi_k(f) \geqslant \frac{c'}{\psi_{n,k}} f(0)^{\frac{1}{n}}\|f\|_1^{\frac{n-k}{kn}},$$

with $c'=c_1c$. □

Note. For any $s\in(-\infty,0)$, using (2.9) we can modify the proof of Theorem 3.1 and extend it to the densities of s-concave measures.

Theorem 3.2. *Let $s\in(-\infty,0)$ and let $f:\mathbb{R}^n\to[0,\infty)$ be a non-negative integrable function which is the density of an s-concave measure μ. Then,*

$$\frac{c}{\delta_{n,k,s}^{\frac{n-k}{k}}\psi_{n,k}} f(0)^{\frac{1}{n}}\|f\|_1^{\frac{n-k}{kn}} \leqslant \Psi_k(f) \leqslant \sqrt{e}\|f\|_\infty^{1/n}\|f\|_1^{\frac{n-k}{kn}}.$$

where

$$\delta_{n,k,s} = \frac{(nB(n,-1/s))^{\frac{1}{n}}}{((n-k)B(n-k,k-1/s))^{\frac{1}{n-k}}}.$$

and $c>0$ is an absolute constant.

Sketch of the proof. Note that the upper bound holds for any bounded integrable function f. For the lower bound we note that

$$K_n(f) \subseteq \delta_{n,k,s} K_{n-k}(f)$$

by (2.9) and then, as in the proof of Theorem 3.1, we write

$$\begin{aligned}
\Psi_k(f) &= \left(\int_{G_{n,n-k}} (f(0)|K_{n-k}(f)\cap H|)^n \, d\nu_{n,n-k}(H) \right)^{\frac{1}{kn}} = f(0)^{\frac{1}{k}} \Psi_k(K_{n-k}(f)) \\
&\geqslant \frac{c}{\psi_{n,k}} f(0)^{\frac{1}{k}} \, |K_{n-k}(f)|^{\frac{n-k}{kn}} = \frac{c}{\psi_{n,k}} f(0)^{\frac{1}{k}} \left(\frac{K_{n-k}(f)|}{|K_n(f)|} \right)^{\frac{n-k}{kn}} |K_n(f)|^{\frac{n-k}{kn}} \\
&\geqslant \frac{c}{\psi_{n,k}} f(0)^{\frac{1}{k}} \frac{1}{\delta_{n,k,s}^{\frac{n-k}{k}}} \left(\frac{\|f\|_1}{f(0)} \right)^{\frac{n-k}{kn}} \geqslant \frac{c}{\delta_{n,k,s}^{\frac{n-k}{k}} \psi_{n,k}} f(0)^{\frac{1}{n}} \, \|f\|_1^{\frac{n-k}{kn}}.
\end{aligned}$$

□

Let $1 \leqslant k \leqslant n-1$. For every convex body K, or more generally for any bounded Borel set in $\mathbb{R}^n$, the k-th affine quermassintegral of K is defined by

$$\Phi_k(K) := \left(\int_{G_{n,k}} |P_E(K)|^{-n} \, d\nu_{n,k}(E) \right)^{-\frac{1}{kn}}.$$

Lutwak [35] conjectured that, among all convex bodies of volume 1, the affine quermassintegrals are minimized in the case of the Euclidean ball D_n of volume 1 and maximized in the case of the regular simplex S_n of volume 1 in $\mathbb{R}^n$:

$$\Phi_k(D_n) \leqslant \Phi_k(K) \leqslant \Phi_k(S_n), \tag{3.6}$$

for every convex body K of volume 1 in $\mathbb{R}^n$ and every $1 \leqslant k \leqslant n-1$. Note that (3.6) for $k=1$ it is equivalent to Blaschke-Santaló inequality and Mahler's conjecture, and that for $k = n-1$ it is equivalent to Zhang's inequality [42]. It is known that for every convex body K in $\mathbb{R}^n$,

$$c_1 \sqrt{n/k} \, |K|^{\frac{1}{n}} \leqslant \Phi_k(K) \leqslant c_2 \sqrt{n/k} \, \phi_{n,k} \, |K|^{\frac{1}{n}} \tag{3.7}$$

for some absolute constants $c_1, c_2 > 0$, where $\phi_{n,k} := \min\left\{ \log n, n/k \sqrt{\log(en/k)} \right\}$. The bounds on the right-hand side of (3.7) were proved in [14]. The second bound is better when k is proportional to n. The left-hand side inequality was proved in [39]. The recent work of E. Milman and Yehudayoff [36] establishes the sharp lower bound $\Phi_k(D_n) \leqslant \Phi_k(K)$ and verifies this part of Lutwak's conjecture, including a characterization of the equality cases, for all values of $k = 1, \ldots, n-1$: ellipsoids are the only local minimizers with respect to the Hausdorff metric.

Recall that given a non-negative measurable function $f : \mathbb{R}^n \to [0,\infty)$ and $E \in G_{n,k}$, the orthogonal projection of f onto E is the function $P_E f : E \to [0,\infty)$ defined by

$$(P_E f)(z) = \sup\{ f(y+z) : y \in E^{\perp} \}.$$

It is not hard to check that

$$R_t(P_E f) = P_E(R_t(f))$$

for every $t>0$, where, for a bounded non-negative integrable function g on $\mathbb{R}^n$, we set $R_t(g)=\{x:g(x)\geqslant t\}$. Note also that if $f=\mathbb{1}_K$ where K is a compact subset of $\mathbb{R}^n$, then $P_Ef=\mathbb{1}_{P_E(K)}$.

Assume that for every $t>0$ the set $R_t(f)$ is compact. A definition of the k-th affine quermassintegral of f was proposed in [16]: for every $1\leqslant k\leqslant n-1$ let

$$\Phi_k'(f):=\int_0^\infty \Phi_k(R_t(f)))\,dt=\int_0^\infty\left(\int_{G_{n,k}}|P_E(R_t(f))|^{-n}d\nu_{n,k}(E)\right)^{-\frac{1}{kn}}dt.$$

One can check that if $f_r(x):=f(x/r)$ for $r>0$ and $(f\circ T)(x)=f(T^{-1}(x))$ for $T\in GL(n)$ then

$$R_t(f\circ T)=T(R_t(f))\quad\text{and}\quad R_t(f_r)=rR_t(f)$$

for any $t>0$. From the 1-homogeneity and the affine invariance of the usual affine quermassintegrals, we check that

$$\Phi_k'(f_r)=r\Phi_k'(f)\quad\text{and}\quad\Phi_k'(f\circ T)=\Phi_k'(f)\tag{3.8}$$

for any $1\leqslant k\leqslant n-1$ and any affine volume preserving transformation T and $r>0$. It was proved in [16, Theorem 5.5] that $\Phi_k'(f)\geqslant\Phi_k'(f^*)$, where f^* is the symmetric decreasing rearrangement of f.

We consider the following variant of $\Phi_k'(f)$:

$$\Phi_k(f):=\left(\int_0^\infty(\Phi_k(R_t(f)))^n\,dt\right)^{1/n}.$$

Note that (3.8) continues to hold with $\Phi_k'(f)$ replaced by $\Phi_k(f)$ and that

$$\begin{aligned}\Phi_k(\mathbb{1}_K)&=\left(\int_0^\infty(\Phi_k(R_t(\mathbb{1}_K)))^n\,dt\right)^{1/n}\\&=\left(\int_0^\infty\left(\int_{G_{n,k}}|P_E(R_t(\mathbb{1}_K))|^{-n}d\nu_{n,k}(E)\right)^{-\frac{1}{k}}dt\right)^{1/n}\\&=\left(\int_0^1\left(\int_{G_{n,k}}|P_E(K)|^{-n}d\nu_{n,k}(E)\right)^{-\frac{1}{k}}dt\right)^{1/n}=\Phi_k(K)\end{aligned}$$

for every convex body K in $\mathbb{R}^n$. Moreover, if f is a geometric log-concave integrable function then $R_t(f)=\varnothing$ for all $t>1$, and hence

$$\Phi_k'(f)=\int_0^1\Phi_k(R_t(f)))\,dt\leqslant\left(\int_0^1(\Phi_k(R_t(f)))^n\,dt\right)^{1/n}=\Phi_k(f)$$

by Hölder's inequality. We shall prove a two-sided estimate for $\Phi_k(f)$.

Theorem 3.3. *Let $f:\mathbb{R}^n\to[0,+\infty)$ be a bounded integrable function such that $R_t(f)$ is a compact convex set for all $t>0$. Then,*

$$c_1\sqrt{n/k}\,\|f\|_1^{\frac{1}{n}}\leqslant\Phi_k(f)\leqslant c_2\sqrt{n/k}\,\psi_{n,k}\|f\|_1^{\frac{1}{n}}$$

for every $1\leqslant k\leqslant n-1$, where $c_1,c_2>0$ are absolute constants.

Proof. Note that

$$\Phi_k(f) = \left(\int_0^\infty \left(\int_{G_{n,k}} |P_E(R_t(f))|^{-n} d\nu_{n,k}(E)\right)^{-1/k} dt\right)^{1/n}.$$

From (3.7) we know that

$$c_1\sqrt{n/k}|R_t(f)| \leqslant \left(\int_{G_{n,k}} |P_E(R_t(f))|^{-n} d\nu_{n,k}(E)\right)^{-1/k} \leqslant c_2\sqrt{n/k}\psi_{n,k}|R_t(f)|$$

for any function f such that $R_t(f)$ is a compact convex set for all $t > 0$. Since

$$\left(\int_0^\infty |R_t(f)|\, dt\right)^{1/n} = \left(\int_0^\infty |\{f \geqslant t\}|\, dt\right)^{1/n} = \|f\|_1^{\frac{1}{n}}$$

we immediately obtain the result. □

Note. For any log-concave integrable function $f : \mathbb{R}^n \to [0, +\infty)$ it is clear that $R_t(f) = \{f \geqslant t\}$ is compact and convex for all $t > 0$ (we easily see that $R_t(f)$ is bounded using the fact that there exist constants $A, B > 0$ such that $f(x) \leqslant Ae^{-B|x|}$ for all $x \in \mathbb{R}^n$; see [9, Lemma 2.2.1]). Therefore, Theorem 3.3 holds true for all $f \in \mathcal{F}(\mathbb{R}^n)$.

4 Sections versus projections

In this section we present a proof of the following more general version of Theorem 1.3.

Theorem 4.1. *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a geometric log-concave integrable function and $g : \mathbb{R}^n \to [0, +\infty)$ be a bounded integrable function with $\|g\|_\infty = 1$. Assume that for some $1 \leqslant k \leqslant n-1$ we have that*

$$\|P_E f\|_1 \leqslant \|g\big|_E\|_1 \qquad \text{for all } E \in G_{n,n-k}.$$

Then,

$$\|f\|_1 \leqslant \frac{n!}{[(n-k)!]^{\frac{n}{n-k}}}\|g\|_1.$$

The proof makes use of the notion of mixed integrals of functions, introduced in [38] (see also [5]). First, recall the definition of mixed volumes. By a classical theorem of Minkowski, if $K_1, \dots, K_m$ are non-empty, compact convex subsets of $\mathbb{R}^n$, then the volume of $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is a homogeneous polynomial of degree n in $\lambda_i > 0$. One can write

$$|\lambda_1 K_1 + \cdots + \lambda_m K_m| = \sum_{1 \leqslant i_1, \dots, i_n \leqslant m} V(K_{i_1}, \dots, K_{i_n})\lambda_{i_1} \cdots \lambda_{i_n},$$

where the coefficients $V(K_{i_1}, \dots, K_{i_n})$ are invariant under permutations of their arguments. The coefficient $V(K_{i_1}, \dots, K_{i_n})$ is the mixed volume of $K_{i_1}, \dots, K_{i_n}$. In particular, if K and T are two convex bodies in $\mathbb{R}^n$ then the function $|K + \lambda T|$ is a polynomial in $\lambda \in [0, \infty)$:

$$|K + \lambda T| = \sum_{k=0}^n \binom{n}{k} V_{n-k}(K, T)\ \lambda^k,$$

where $V_{n-k}(K,T)=V((K,n-k),(T,k))$ is the k-th mixed volume of K and T (we use the notation (T,k) for the k-tuple $(T,\dots,T)$). If $T=B_2^n$ then we set $W_k(K):=V_{n-k}(K,B_2^n)=V((K,n-k),(B_2^n,k))$; this is the k-th quermassintegral of K. Kubota's integral formula expresses the quermassintegral $W_k(K)$ as an average of the volumes of $(n-k)$-dimensional projections of K:

$$W_k(K)=\frac{\omega_n}{\omega_{n-k}}\int_{G_{n,n-k}}|P_E(K)|d\nu_{n,n-k}(E). \tag{4.1}$$

Aleksandrov's inequalities (see [40]) imply that if we set

$$Q_k(K)=\left(\frac{W_{n-k}(K)}{\omega_n}\right)^{\frac{1}{k}}=\left(\frac{1}{\omega_k}\int_{G_{n,k}}|P_E(K)|\,d\nu_{n,k}(E)\right)^{\frac{1}{k}}, \tag{4.2}$$

then $k\mapsto Q_k(K)$ is decreasing.

In [38] the definition of mixed volumes was extended to the setting of functions. Given a bounded integrable function $f:\mathbb{R}^n\to[0,\infty)$ recall that $R_t(f)=\{f\geqslant t\}$ for every $t>0$. If $f_1,\dots,f_n:\mathbb{R}^n\to[0,\infty)$ are bounded integrable functions such that $R_t(f_j)$ is bounded and convex for all $1\leqslant j\leqslant n$ and all $t>0$, their mixed integral is defined by

$$V(f_1,f_2,\dots,f_n):=\int_0^\infty V(R_t(f_1),R_t(f_2),\dots,R_t(f_n))\,dt,$$

where we agree that $V(R_t(f_1),R_t(f_2),\dots,R_t(f_n))=0$ if $R_t(f_j)=\varnothing$ for some $1\leqslant j\leqslant n$. Note that

$$V(f,\dots,f)=\int_0^\infty|\{f\geqslant t\}|\,dt=\|f\|_1.$$

The k-th quermassintegral of f is defined as

$$W_k(f)=V((f,n-k),(\mathbb{1}_{B_2^n},k)).$$

V. Milman and Rotem proved in [38] that if $u(x)=e^{-|x|}$ then for every log-concave function f with $f(0)=\|f\|_\infty=1$ the function $k\mapsto(W_k(f)/W_k(u))^{\frac{1}{n-k}}$, $0\leqslant k\leqslant n-1$ is increasing. In particular,

$$\left(\frac{\|f\|_1}{\|u\|_1}\right)^{\frac{n-k}{n}}\leqslant\frac{W_k(f)}{W_k(u)} \tag{4.3}$$

for all $1\leqslant k\leqslant n-1$.

It is not hard to compute $\|u\|_1$ and, more generally, $W_k(u)$. Note that, by the definition,

$$\begin{aligned}W_k(u)=V((u,n-k),(\mathbb{1}_{B_2^n},k))&=\int_0^\infty V(R_t(u),\dots,R_t(u),R_t(\mathbb{1}_{B_2^n}),\dots,R_t(\mathbb{1}_{B_2^n})\,dt\\&=\int_0^1 V(R_t(u),\dots,R_t(u),B_2^n,\dots,B_2^n)\,dt=\int_0^\infty e^{-s}V(R_{e^{-s}}(u),\dots,R_{e^{-s}}(u),B_2^n,\dots,B_2^n)\,ds\\&=\int_0^\infty e^{-s}V(sB_2^n,\dots,sB_2^n,B_2^n,\dots,B_2^n)\,ds=\omega_n\int_0^\infty s^{n-k}e^{-s}ds=(n-k)!\omega_n,\end{aligned}$$

and similarly

$$\|u\|_1=V(u,\dots,u)=n!\omega_n.$$

Therefore, (4.3) takes the form

$$\|f\|_1^{\frac{n-k}{n}} \leqslant \frac{(n!)^{\frac{n-k}{n}}}{(n-k)!\,\omega_n^{\frac{k}{n}}} W_k(f). \tag{4.4}$$

Proof of Theorem 4.1. Using the assumption and Kubota's formula we write

$$\begin{aligned}
W_k(f) &= \int_0^{\infty} V((R_t(f), n-k), (R_t(B_2^n), k))\, dt \\
&= \frac{\omega_n}{\omega_{n-k}} \int_0^{\infty} \int_{G_{n,n-k}} |P_E(R_t(f))| d\nu_{n,n-k}(E)\, dt \\
&= \frac{\omega_n}{\omega_{n-k}} \int_{G_{n,n-k}} \left(\int_0^{\infty} |R_t(P_E f))|\, dt \right) d\nu_{n,n-k}(E) \\
&= \frac{\omega_n}{\omega_{n-k}} \int_{G_{n,n-k}} \left(\int_0^{\infty} |\{x : P_E f(x) \geqslant t\}|\, dt \right) d\nu_{n,n-k}(E) \\
&= \frac{\omega_n}{\omega_{n-k}} \int_{G_{n,n-k}} \|P_E(f)\|_1\, d\nu_{n,n-k}(E) \\
&\leqslant \frac{\omega_n}{\omega_{n-k}} \int_{G_{n,n-k}} \|g\big|_E\|_1\, d\nu_{n,n-k}(E).
\end{aligned}$$

From the proof of Theorem 3.1 we see that

$$\begin{aligned}
\int_{G_{n,n-k}} \|g\big|_E\|_1\, d\nu_{n,n-k}(E) &\leqslant \left(\int_{G_{n,n-k}} \|g\big|_E\|_1^n\, d\nu_{n,n-k}(E) \right)^{1/n} \\
&\leqslant \frac{\omega_{n-k}}{\omega_n^{\frac{n-k}{n}}} \|g\|_\infty^{k/n} \|g\|_1^{\frac{n-k}{n}} = \frac{\omega_{n-k}}{\omega_n^{\frac{n-k}{n}}} \|g\|_1^{\frac{n-k}{n}}.
\end{aligned} \tag{4.5}$$

Combining the above with (4.4) we get

$$\|f\|_1^{\frac{n-k}{n}} \leqslant \frac{(n!)^{\frac{n-k}{n}}}{(n-k)!\,\omega_n^{\frac{k}{n}}} W_k(f) \leqslant \frac{(n!)^{\frac{n-k}{n}}}{(n-k)!\,\omega_n^{\frac{k}{n}}} \frac{\omega_n}{\omega_{n-k}} \frac{\omega_{n-k}}{\omega_n^{\frac{n-k}{n}}} \|g\|_1^{\frac{n-k}{n}} = \frac{(n!)^{\frac{n-k}{n}}}{(n-k)!} \|g\|_1^{\frac{n-k}{n}}.$$

This proves the theorem. $\square$

5 Sections of log-concave functions

The classical Busemann-Petty problem [10] asks if for any pair of origin-symmetric convex bodies K, T in $\mathbb{R}^n$ that satisfy

$$|K \cap \xi^{\perp}| \leqslant |T \cap \xi^{\perp}| \tag{5.1}$$

for all $\xi \in S^{n-1}$ it follows that $|K| \leqslant |T|$. It is known that the answer is affirmative if $n \leqslant 4$, and it is negative if $n \geqslant 5$ (the history of the problem is presented in the books of Koldobsky [29] and Gardner [19]). The isomorphic Busemann-Petty problem, posed in [37], asks whether the inequalities (5.1) imply $|K| \leqslant C\, |T|$, where C is an absolute constant.

The lower dimensional Busemann-Petty problem is the following question: Let $1 \leqslant k \leqslant n-1$ and let

$\beta_{n,k}$ be the smallest constant $\beta > 0$ with the following property: For every pair of centered convex bodies K and T in $\mathbb{R}^n$ that satisfy

$$|K \cap F| \leqslant |T \cap F| \tag{5.2}$$

for all $F \in G_{n,n-k}$, one has

$$|K|^{\frac{n-k}{n}} \leqslant \beta^k \, |T|^{\frac{n-k}{n}}. \tag{5.3}$$

Then, one may ask if it is true that there exists an absolute constant $C > 0$ such that $\beta_{n,k} \leqslant C$ for all n and k. The following bounds are known for the constants $\beta_{n,k}$. For every $1 \leqslant k \leqslant n-1$ we have $\beta_{n,k} \leqslant c_1 L_n$ where $c_1 > 0$ is an absolute constant and we also have the bound $\beta_{n,k} \leqslant c_2\sqrt{n/k}\,(\log(en/k))^{\frac{3}{2}}$ where $c_2 > 0$ is an absolute constant (which is stronger for codimensions k that are proportional to n). In fact, the last upper bound follows from the inequality $\beta_{n,k} \leqslant d_{\rm ovr}(K, \mathcal{B}P_k^n)$, where $d_{\rm ovr}(K, \mathcal{B}P_k^n)$ is the outer volume ratio distance from K to the class $\mathcal{B}P_k^n$ of k-intersection bodies in $\mathbb{R}^n$, and the estimate $d_{\rm ovr}(K, \mathcal{B}P_k^n) \leqslant c\sqrt{n/k}\ln^{3/2}(en/k)$ obtained in [33].

An extension of the Busemann-Petty problem to arbitrary measures in place of volume was considered in [43]. Let K, T be origin-symmetric convex bodies in $\mathbb{R}^n$, and let f be a locally integrable non-negative function on $\mathbb{R}^n$. Suppose that for every $\xi \in S^{n-1}$

$$\int_{K\cap\xi^\perp} f(x)dx \leqslant \int_{T\cap\xi^\perp} f(x)dx, \tag{5.4}$$

where integration is with respect to Lebesgue measure on $\xi^\perp$. Does it necessarily follow that

$$\int_K f(x)dx \leqslant s_n \int_T f(x)dx \tag{5.5}$$

where the constant s_n does not depend on f, K, T? It was proved in [43] that, for any strictly positive function f, if one asks for such an inequality with constant 1 then the solution is the same as in the case of volume (where $f \equiv 1$): affirmative if $n \leqslant 4$ and negative if $n \geqslant 5$. However, it was proved in [32] that the answer to the isomorphic question is affirmative, namely $s_n \leqslant \sqrt{n}$. It is not known whether the $\sqrt{n}$ estimate is optimal.

A lower dimensional version of inequality (5.5) was also proved in [30]. If K is a star body in $\mathbb{R}^n$, f is a continuous non-negative function on K, and $1 \leqslant k \leqslant n-1$, then

$$\int_K f(x)\,dx \ \leqslant C^k \ (d_{\rm ovr}(K, \mathcal{BP}_k^n))^k \ |K|^{k/n} \ \max_{E\in G_{n,n-k}} \int_{K\cap E} f(x)\,dx, \tag{5.6}$$

where C is an absolute constant. This inequality is an immediate consequence of the following more general result that was proved in [22].

Theorem 5.1. *Let K and T be star bodies in $\mathbb{R}^n$, let $0 < k < n$ be an integer, and let f, g be non-negative continuous functions on K and T, respectively, so that $\|g\|_\infty = g(0) = 1$. Then,*

$$\frac{\int_K f(x), dx}{\left(\int_T g(x)\,dx\right)^{\frac{n-k}{n}} |K|^{\frac{k}{n}}} \leqslant \frac{n}{n-k}\left(d_{\rm ovr}(K, \mathcal{BP}_k^n)\right)^k \max_{E\in G_{n,n-k}} \frac{\int_{K\cap E} f(x)\,dx}{\int_{T\cap E} g(x)\,dx}. \tag{5.7}$$

We shall obtain an analogue of Theorem 5.1, with improved constants, under the assumption that f is log-concave.

Theorem 5.2. *Let $f,g:\mathbb{R}^n\to[0,\infty)$ be non-negative integrable functions such that f is log-concave with $f(0)>0$ and $\|g\|_\infty=g(0)=1$. If K is a convex body in $\mathbb{R}^n$ and T is a compact subset of $\mathbb{R}^n$ with $0\in K\cap T$, then*

$$\left(\frac{f(0)^{-1}\int_K f(x)\,dx}{\int_T g(x)\,dx}\right)^{\frac{n-k}{n}}\leqslant (c\beta_{n,k})^k\max_{E\in G_{n,n-k}}\frac{f(0)^{-1}\int_{K\cap E}f(x)\,dx}{\int_{T\cap E}g(x)\,dx},\tag{5.8}$$

where $c>0$ is an absolute constant.

Proof. The function $f_K=f\cdot\mathbb{1}_K$ is log-concave, with $f_K(0)=f(0)$. From (2.6) and (2.7) we immediately get

$$f(0)|K_n(f_K)|=\int_K f(x)\,dx\tag{5.9}$$

and, for any $1\leqslant k\leqslant n-1$ and $E\in G_{n,n-k}$,

$$f(0)|K_{n-k}(f_K)\cap E|=\int_{K\cap E}f(x)\,dx.\tag{5.10}$$

Similarly, for the function $g_T=g\cdot\mathbb{1}_T$, using also the assumption that $g_T(0)=1$, we check that

$$|K_n(g_T)|=\int_T g(x)\,dx\quad\text{and}\quad|K_{n-k}(g_T)\cap E|=\int_{T\cap E}g(x)\,dx\tag{5.11}$$

for every $E\in G_{n,n-k}$. Now, let

$$\tau^{n-k}:=f(0)\quad\text{and}\quad\varrho^{n-k}:=\max_{E\in G_{n,n-k}}\frac{\int_{K\cap E}f(x)\,dx}{\int_{T\cap E}g(x)\,dx}.$$

From (5.10) and (5.11) we see that

$$|\tau K_{n-k}(f_K)\cap E|\leqslant|\varrho K_{n-k}(g_T)\cap E|$$

for all $E\in G_{n,n-k}$. Therefore,

$$\tau^{n-k}|K_{n-k}(f_K)|^{\frac{n-k}{n}}=|\tau K_{n-k}(f_K)|^{\frac{n-k}{n}}\leqslant\beta_{n,k}^k\,|\varrho K_{n-k}(g_T)|^{\frac{n-k}{n}}=\beta_{n,k}^k\,\varrho^{n-k}|K_{n-k}(g_T)|^{\frac{n-k}{n}}.$$

This shows that

$$|K_{n-k}(f_K)|\leqslant\tau^{-n}\beta_{n,k}^{\frac{kn}{n-k}}\,\varrho^n|K_{n-k}(g_T)|.$$

We apply Lemma 2.2 to write

$$\frac{[(n-k)!]^{\frac{1}{n-k}}}{[n!]^{\frac{1}{n}}}K_n(f_K)\subseteq K_{n-k}(f_K)\tag{5.12}$$

and Lemma 2.1 together with the assumption that $g(0)=\|g\|_\infty=1$ to write

$$K_{n-k}(g_T)\subseteq\left(\frac{\|g\|_\infty}{g(0)}\right)^{\frac{1}{n-k}-\frac{1}{n}}K_n(g_T)=K_n(g_T).\tag{5.13}$$

Taking into account (2.6) and (5.11) we get

$$\begin{aligned}\int_K f(x)\,dx = f(0)\,|K_n(f_K)| &\leqslant \frac{n!}{[(n-k)!]^{\frac{n}{n-k}}}\,\tau^{n-k}\,|K_{n-k}(f_K)| \\ &\leqslant \frac{n!}{[(n-k)!]^{\frac{n}{n-k}}}\,\tau^{n-k}\tau^{-n}\beta_{n,k}^{\frac{kn}{n-k}}\,\varrho^n|K_{n-k}(g_T)| \\ &\leqslant \frac{n!}{[(n-k)!]^{\frac{n}{n-k}}}\,\tau^{-k}\beta_{n,k}^{\frac{kn}{n-k}}\,\varrho^n|K_n(g_T)| \\ &= \frac{n!}{[(n-k)!]^{\frac{n}{n-k}}}\,\tau^{-k}\beta_{n,k}^{\frac{kn}{n-k}}\,\varrho^n\int_T g(x)\,dx \\ &= \frac{n!}{[(n-k)!]^{\frac{n}{n-k}}}\,f(0)^{-\frac{k}{n-k}}\beta_{n,k}^{\frac{kn}{n-k}}\,\varrho^n\int_T g(x)\,dx.\end{aligned}$$

Using Stirling's formula we see that

$$\frac{n!}{[(n-k)!]^{\frac{n}{n-k}}} \leqslant c^{\frac{kn}{n-k}},$$

and combining the above we get

$$\left(\frac{\int_K f(x)\,dx}{\int_T g(x)\,dx}\right)^{\frac{n-k}{n}} \leqslant f(0)^{-\frac{k}{n}}(c\beta_{n,k})^k\varrho^{n-k} = f(0)^{-\frac{k}{n}}(c\beta_{n,k})^k \max_{E\in G_{n,n-k}}\frac{\int_{K\cap E} f(x)\,dx}{\int_{T\cap E} g(x)\,dx}$$

as claimed. □

Let $f\in\mathcal{F}_0(\mathbb{R}^n)$ (more generally, assume that f is log-concave with $f(0)=1$). Given a measurable function $g:\mathbb{R}^n\to[0,\infty)$ with $\|g\|_\infty=g(0)=1$ and a pair (K,T) where K is a convex body, T is compact and $0\in K\cap T$, we may apply Theorem 5.2 to get

$$\frac{\int_K f(x)\,dx}{\left(\int_T g(x)\,dx\right)^{\frac{n-k}{n}}} \leqslant (c\beta_{n,k})^k\left(\int_K f(x)\,dx\right)^{\frac{k}{n}} \max_{E\in G_{n,n-k}}\frac{\int_{K\cap E} f(x)\,dx}{\int_{T\cap E} g(x)\,dx}.$$

Moreover, if we assume that $f\in\mathcal{F}_0(\mathbb{R}^n)$ then

$$\int_K f(x)\,dx \leqslant |K|.$$

The same inequality holds true, by Jensen's inequality, if f is log-concave and centered with $f(0)=1$. Combining the above we get an analogue of Theorem 5.2 involving the volume of K.

Corollary 5.3. *Let $f\in\mathcal{F}_0(\mathbb{R}^n)$ and let g be a non-negative measurable function such that $\|g\|_\infty=g(0)=1$. If K is a convex body in $\mathbb{R}^n$ and T is a compact subset of $\mathbb{R}^n$ with $0\in K\cap T$, then*

$$\frac{\int_K f(x)\,dx}{\left(\int_T g(x)\,dx\right)^{\frac{n-k}{n}}} \leqslant (c\beta_{n,k})^k|K|^{\frac{k}{n}} \max_{E\in G_{n,n-k}}\frac{\int_{K\cap E} f(x)\,dx}{\int_{T\cap E} g(x)\,dx}, \tag{5.14}$$

where $c>0$ is an absolute constant.

Theorem 5.2 has a number of consequences if we make a specific choice of g and/or T. First, we can obtain a comparison theorem for the Radon transform. If, in addition to the conditions of Corollary 5.3, we assume that

$$\int_{K\cap E} f(x)\,dx \leqslant \int_{T\cap E} g(x)\,dx$$

for all $E \in G_{n,n-k}$, then we get

$$\int_K f(x)\,dx \leqslant (c\beta_{n,k})^k |K|^{\frac{k}{n}} \left(\int_T g(x)\,dx\right)^{\frac{n-k}{n}}.$$

This is a sharpening (in the log-concave setting) of a result established in [34].

From Corollary 5.3 we also obtain the next lower estimate for the sup-norm of the Radon transform.

Theorem 5.4. *Let $f \in \mathcal{F}_0(\mathbb{R}^n)$. Then, for every $1 \leqslant k \leqslant n-1$ we have that*

$$\int_K f(x)\,dx \leqslant (c\beta_{n,k})^k |K|^{\frac{k}{n}} \max_{E\in G_{n,n-k}} \int_{K\cap E} f(x)\,dx$$

where $c > 0$ is an absolute constant.

Proof. The theorem follows directly from Corollary 5.3 if we choose $T = B_2^n$ and $g \equiv 1$. We get

$$\int_K f(x)\,dx \leqslant \frac{\omega_n^{\frac{n-k}{n}}}{\omega_{n-k}} (c\beta_{n,k})^k |K|^{\frac{k}{n}} \max_{E\in G_{n,n-k}} \int_{K\cap E} f(x)\,dx$$

and recall that $\omega_n^{\frac{n-k}{n}}/\omega_{n-k} \leqslant 1$ (see [31, Lemma 2.1]). $\square$

Theorem 5.4 is a sharpening (in the log-concave setting) of Koldobsky's slicing inequality for arbitrary functions from [30]. The terminology comes from the slicing problem of Bourgain asking if there exists an absolute constant $C_1 > 0$ such that for every $n \geqslant 1$ and every centered convex body K in $\mathbb{R}^n$ one has

$$|K|^{\frac{n-1}{n}} \leqslant C_1 \max_{\xi\in S^{n-1}} |K \cap \xi^\perp|. \tag{5.15}$$

It is well-known that this problem is equivalent to the question if $L_n \leqslant C$. Consider the best constant $\alpha_{n,k} > 0$ with the following property: For every centered convex body K in $\mathbb{R}^n$ one has

$$|K| \leqslant \alpha_{n,k}^k\, |K|^{\frac{k}{n}} \max_{E\in G_{n,n-k}} |K \cap E|. \tag{5.16}$$

The lower dimensional slicing problem asks if there exists an absolute constant $\alpha > 0$ such that $\alpha_{n,k} \leqslant \alpha$ for all n and k. Theorem 5.4 states that a more general inequality than (5.16) holds for geometric log-concave functions, with a constant $\alpha_{n,k} = O(L_n)$.

Another application of Corollary 5.3 is a mean value inequality for the Radon transform. Given $f \in \mathcal{F}_0(\mathbb{R}^n)$, if we choose $K = T$ and $g \equiv 1$ we get

$$\frac{1}{|K|}\int_K f(x)\,dx \leqslant (c\beta_{n,k})^k \max_{E\in G_{n,n-k}} \frac{1}{|K\cap E|}\int_{K\cap E} f(x)\,dx.$$

Finally, choosing $f \equiv 1$ and $g \equiv 1$ in Theorem 5.2 we see that if K is a convex body and T is a compact subset of $\mathbb{R}^n$ with $0 \in K \cap T$ then,

$$\left(\frac{|K|}{|T|}\right)^{\frac{n-k}{n}} \leqslant (c\beta_{n,k})^k \max_{E\in G_{n,n-k}} \frac{|K\cap E|}{|T\cap E|}.$$

for every $1 \leqslant k \leqslant n-1$.

Note. For any $s \in (-\infty, 0)$, using (2.9) we can modify the proof of Theorem 5.2 and extend it (together with all its consequences) to the densities of s-concave measures. An extension of Corollary 5.3 in the same spirit, but with a less direct proof, was obtained in [41].

Theorem 5.5. *Let $s \in (-\infty, 0)$ and let $f, g : \mathbb{R}^n \to [0, \infty)$ be non-negative integrable functions such that f is the density of an s-concave measure μ, $f(0) > 0$ and $\|g\|_\infty = g(0) = 1$. If K is a convex body in $\mathbb{R}^n$ and T is a compact subset of $\mathbb{R}^n$ with $0 \in K \cap T$, then*

$$\left(\frac{f(0)^{-1}\int_K f(x)\,dx}{\int_T g(x)\,dx}\right)^{\frac{n-k}{n}} \leqslant \delta_{n,k,s}^{n-k}\beta_{n,k}^k \max_{E \in G_{n,n-k}} \frac{f(0)^{-1}\int_{K\cap E} f(x)\,dx}{\int_{T\cap E} g(x)\,dx}, \tag{5.17}$$

where

$$\delta_{n,k,s} = \frac{(nB(n,-1/s))^{\frac{1}{n}}}{((n-k)B(n-k,k-1/s))^{\frac{1}{n-k}}}.$$

Sketch of the proof. We consider the functions $f_K = f \cdot \mathbb{1}_K$ and $g_T = g \cdot \mathbb{1}_T$. Note that f_K is $(-1/\alpha)$-concave, where $\alpha = n - \frac{1}{s} > n$. From (2.9) we get

$$K_n(f_K) \subseteq \delta_{n,k,s} K_{n-k}(f_K), \tag{5.18}$$

and Lemma 2.1 together with the assumption that $g(0) = \|g\|_\infty = 1$ gives

$$K_{n-k}(g_T) \subseteq \left(\frac{\|g\|_\infty}{g(0)}\right)^{\frac{1}{n-k}-\frac{1}{n}} K_n(g_T) = K_n(g_T). \tag{5.19}$$

Let

$$\varrho^{n-k} := \max_{E \in G_{n,n-k}} \frac{\int_{K\cap E} f(x)\,dx}{\int_{T\cap E} g(x)\,dx}.$$

Taking into account (2.6) and (5.11) we get

$$\begin{aligned}\int_K f(x)\,dx = f(0)\,|K_n(f_K)| &\leqslant \delta_{n,k,s}^n f(0)\,|K_{n-k}(f_K)| \leqslant f(0)^{-\frac{k}{n-k}}\,\delta_{n,k,s}^n \beta_{n,k}^{\frac{kn}{n-k}}\,\varrho^n |K_{n-k}(g_T)| \\ &\leqslant f(0)^{-\frac{k}{n-k}}\,\delta_{n,k,s}^n \varrho^n |K_n(g_T)| = f(0)^{-\frac{k}{n-k}}\,\delta_{n,k,s}^n \beta_{n,k}^{\frac{kn}{n-k}}\,\varrho^n \int_T g(x)\,dx.\end{aligned}$$

It follows that

$$\left(\frac{\int_K f(x)\,dx}{\int_T g(x)\,dx}\right)^{\frac{n-k}{n}} \leqslant f(0)^{-\frac{k}{n}}\delta_{n,k,s}^{\frac{n-k}{n}}\beta_{n,k}^k \varrho^{n-k} = f(0)^{-\frac{k}{n}}\delta_{n,k,s}^{n-k}\beta_{n,k}^k \max_{E\in G_{n,n-k}} \frac{\int_{K\cap E} f(x)\,dx}{\int_{T\cap E} g(x)\,dx}$$

as claimed. $\square$

Remark 5.6. We close this section by mentioning the next result from [12] which provides a different general estimate for the Busemann-Petty problem in the case where $f = g$ is an even log-concave density: If K is a symmetric convex body in $\mathbb{R}^n$ and T is a compact subset of $\mathbb{R}^n$ such that

$$\int_{K\cap E} f(x)\,dx \leqslant \int_{T\cap E} f(x)\,dx \tag{5.20}$$

for all $E \in G_{n,n-k}$, then

$$\int_K f(x)\,dx \leqslant \big(ckL_{n-k}\big)^k \int_T f(x)\,dx, \tag{5.21}$$

where $c > 0$ is an absolute constant.

6 Projections of log-concave functions

In this section we obtain a functional inequality related to the classical Shephard problem. The original question is the following: Let K and T be two centrally symmetric convex bodies in $\mathbb{R}^n$ such that $|P_{\xi^\perp}(K)| < |P_{\xi^\perp}(T)|$ for every $\xi \in S^{n-1}$. Does it follow that $|K| < |T|$? Although the answer is affirmative if $n = 2$ (simply because the assumptions imply that $K \subset T$), it is negative in higher dimensions as shown, independently, by Petty who gave an explicit counterexample in $\mathbb{R}^3$, and by Schneider for all $n \geqslant 3$. It is natural to ask for the order of growth (as $n \to \infty$) of the best constant S_n for which the assumptions of Shephard's problem imply that $|K| \leqslant S_n|L|$. Recall that the projection body ΠK of a convex body K is the centrally symmetric convex body whose support function is defined by

$$h_{\Pi K}(\xi) = |P_{\xi^\perp}(K)|, \qquad \xi \in S^{n-1}.$$

An argument which is based on the identity

$$V(K, \ldots, K, \Pi T) = V(T, \ldots, T, \Pi K)$$

as well as on John's theorem and the fact that ellipsoids are projection bodies, shows that if K and T are two centrally symmetric convex bodies in $\mathbb{R}^n$ such that $|P_{\xi^\perp}(K)| < |P_{\xi^\perp}(T)|$ for every $\xi \in S^{n-1}$ then

$$|K|^{\frac{n-1}{n}} \leqslant \sqrt{n}|T|^{\frac{n-1}{n}}. \tag{6.1}$$

This shows that $S_n \leqslant c_1\sqrt{n}$ for some absolute constant $c_1 > 0$. In fact, a result of K. Ball [4] shows that, conversely, there exists an absolute constant $c_2 > 0$ such that $S_n \geqslant c_2\sqrt{n}$. We refer to [2, Section 4.6.2] for a concise, but more detailed, exposition of all the above and references.

In view of the above, one may consider the next lower dimensional Shephard problem. Let $1 \leqslant k \leqslant n-1$ and let $S_{n,k}$ be the smallest constant $S > 0$ with the following property: For every pair of convex bodies K and T in $\mathbb{R}^n$ that satisfy $|P_E(K)| \leqslant |P_E(T)|$ for all $E \in G_{n,n-k}$, one has that $|K|^{\frac{n-k}{n}} \leqslant S^k\,|T|^{\frac{n-k}{n}}$. Goodey and Zhang [23] proved that $S_{n,k} > 1$ if $n-k > 1$. General estimates for $S_{n,k}$ were obtained in [21]: If K and T are two convex bodies in $\mathbb{R}^n$ such that $|P_E(K)| \leqslant |P_E(T)|$ for every $E \in G_{n,n-k}$, then

$$|K|^{\frac{1}{n}} \leqslant c_1\tilde{S}_{n,k}\,|T|^{\frac{1}{n}},$$

where $c_1 > 0$ is an absolute constant, and

$$\tilde{S}_{n,k} = \min\left\{\sqrt{\tfrac{n}{n-k}}\ln\left(\tfrac{en}{n-k}\right), \ln n\right\}.$$

It follows that $S_{n,k} \leqslant (c_1\tilde{S}_{n,k})^{\frac{n-k}{k}}$, and in particular that $S_{n,k} \leqslant C^{\frac{n-k}{k}}$ if $\frac{k}{n-k}$ is bounded.

The next lemma provides a third estimate for $S_{n,k}$, which is reasonably good if k is small.

Lemma 6.1. *For every $1 \leqslant k \leqslant n-1$ we have that $S_{n,k} \leqslant \sqrt{n}$.*

Proof. We use inductively the following claim. For any $1 \leqslant s \leqslant n-1$ let a_{n-s} be the best positive constant with the property that $|P_E(K)| \leqslant \alpha_{n-s}|P_E(T)| = |P_E(\alpha_{n-s}^{\frac{1}{n-s}}T)|$ for all $E \in G_{n,n-s}$. Then,

$$a_{n-s+1} \leqslant \alpha_{n-s}^{\frac{n-s+1}{n-s}}(n-s+1)^{\frac{n-s+1}{2(n-s)}}. \tag{6.2}$$

To see this, consider any $F \in G_{n,n-s+1}$. Note that if $E \in G_{n,n-s}$ and $F \subset E$ then $P_E(P_F(K)) = P_E(K)$ and $P_E(P_F(T)) = P_E(T)$. Since the assumption is true for all 1-codimensional subspaces of F, from (6.1) we see that

$$|P_F(K)|^{\frac{n-s}{n-s+1}} \leqslant (n-s+1)^{\frac{1}{2}}|P_F(\alpha_{n-s}^{\frac{1}{n-s}}T)|^{\frac{n-s}{n-s+1}} = (n-s+1)^{\frac{1}{2}}\alpha_{n-s}|P_F(T)|^{\frac{n-s}{n-s+1}},$$

therefore $|P_F(K)| \leqslant \alpha_{n-s}^{\frac{n-s+1}{n-s}}(n-s+1)^{\frac{n-s+1}{2(n-s)}}|P_F(T)|$ for all $F \in G_{n,n-s+1}$. Assume that $a_{n-k} = 1$, i.e. $|P_E(K)| \leqslant |P_E(T)|$ for all $E \in G_{n,n-k}$. From (6.2) we see that $a_{n-k+s} \leqslant \prod_{j=1}^{s}(n-k+j)^{\frac{n-k+s}{2(n-k+j-1)}}$, and in particular,

$$a_n \leqslant \prod_{j=1}^{k}(n-k+j)^{\frac{n}{2(n-k+j-1)}} \leqslant n^{\frac{n}{2}\sum_{j=1}^{k}\frac{1}{n-k+j-1}} \leqslant n^{\frac{nk}{2(n-k)}}.$$

This means that $|K| \leqslant n^{\frac{nk}{2(n-k)}}|T|$, therefore

$$|K|^{\frac{n-k}{n}} \leqslant n^{\frac{k}{2}}|T|^{\frac{n-k}{n}},$$

which shows that $S_{n,k} \leqslant \sqrt{n}$. □

Note. It is not hard to check that if $k \ll \frac{n}{\log n}$ then $\sqrt{n} \ll \tilde{S}_{n,k}^{\frac{n-k}{k}}$, and hence, in order to estimate $S_{n,k}$ in this range it is preferable to use the upper bound of Lemma 6.1.

Our goal is to obtain an analogue of the above estimates for $S_{n,k}$ in the setting of geometric log-concave integrable functions. Our substitute for the assumption that all k-codimensional projections of K gave smaller volume than the corresponding projections of T is the following: we consider $f, g \in \mathcal{F}_0(\mathbb{R}^n)$ such that, for some $1 \leqslant k \leqslant n-1$ and for all $E \in G_{n,n-k}$ and $t > 0$ we have that

$$|R_t(P_Ef)| \leqslant |R_t(P_Eg)|. \tag{6.3}$$

Recall that (6.3) can be equivalently written as $|P_E(R_t(f))| \leqslant |P_E(R_t(g))|$. Moreover, if $f = \mathbb{1}_K$ and $g = \mathbb{1}_T$ are the indicator functions of two convex bodies in $\mathbb{R}^n$ then for every $0 \leqslant t \leqslant 1$ we have that $R_t(P_E\mathbb{1}_K) = P_E(K)$ and $R_t(P_E\mathbb{1}_T) = P_E(K)$, and hence (6.3) is equivalent to the assumption $|P_E(K)| \leqslant |P_E(T)|$ in Shephard's problem.

Theorem 6.2. *Let $f, g \in \mathcal{F}_0(\mathbb{R}^n)$ and $1 \leqslant k \leqslant n-1$. Assume that $|R_t(P_Ef)| \leqslant |R_t(P_Eg)|$ for all $E \in G_{n,n-k}$ and $0 \leqslant t \leqslant 1$. Then,*

$$\|f\|_1^{\frac{n-k}{n}} \leqslant S_{n,k}^k\|g\|_1^{\frac{n-k}{n}}.$$

Proof. The assumption $|R_t(P_Ef)| \leqslant |R_t(P_Eg)|$ for all $E \in G_{n,n-k}$ implies that $|R_t(f)| \leqslant S_{n,k}^{\frac{kn}{n-k}}|R_t(g)|$ for all $0 \leqslant t \leqslant 1$ by the definition of $S_{n,k}$. Then,

$$\|f\|_1 = \int_0^1 |R_t(f)|\,dt \leqslant S_{n,k}^{\frac{kn}{n-k}}\int_0^1 |R_t(g)|\,dt = S_{n,k}^{\frac{kn}{n-k}}\|g\|_1$$

and the result follows. □

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References

[1] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Vol. I*, Mathematical Surveys and Monographs, 202. American Mathematical Society, Providence, RI, 2015. xx+451 pp.

[2] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Vol. II*, Mathematical Surveys and Monographs., 262. American Mathematical Society, Providence, RI, 2021. xxxvii+645 pp.

[3] K. M. Ball, *Logarithmically Concave Functions and Sections of Convex Sets in* $\mathbb{R}^n$. Studia Math. 88 (1988), 69–84.

[4] K. M. Ball, *Shadows of convex bodies*, Trans. Amer. Math. Soc. 327 (1991), 891-901.

[5] S. G. Bobkov, A. Colesanti and I. Fragalá, *Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities*, Manuscripta Mathematica, 143, 1-2, (2014),131–169.

[6] C. Borell, *Convex measures on locally convex spaces*, Ark. Mat. 12 (1974), 239–252.

[7] C. Borell, *Convex set functions in d-space*, Period. Math. Hungar. 6 (1975), 111–136.

[8] J. Bourgain, *On the distribution of polynomials on high dimensional convex sets*, Geometric aspects of functional analysis (1989-90), 127–137, Lecture Notes in Math., 1469, Springer, Berlin, 1991.

[9] S. Brazitikos, A. Giannopoulos, P. Valettas and B-H. Vritsiou, *Geometry of isotropic convex bodies*, Mathematical Surveys and Monographs, 196. American Mathematical Society, Providence, RI, 2014. xx+594 pp.

[10] H. Busemann, C. M. Petty, *Problems on convex bodies*, Math. Scand. 4 (1956), 88–94.

[11] H. Busemann and E. G. Straus, *Area and Normality*, Pacific J. Math. 10 (1960), 35–72.

[12] G. Chasapis, A. Giannopoulos and D. Liakopoulos, *Estimates for measures of lower dimensional sections of convex bodies*, Adv. Math. 306 (2017), 880–904.

[13] Y. Chen, *An Almost Constant Lower Bound of the Isoperimetric Coefficient in the KLS Conjecture*, Geom. Funct. Anal. 31 (2021), no. 1, 34–61.

[14] N. Dafnis and G. Paouris, *Estimates for the affine and dual affine quermassintegrals of convex bodies*, Illinois Journal of Mathematics 56 (2012), 1005-1021.

[15] S. Dann, G. Paouris and P. Pivovarov, *Bounding marginal densities via affine isoperimetry*, Proc. Lond. Math. Soc. (3) 113 (2016), no. 2, 140–162.

[16] S. Dann, G. Paouris and P. Pivovarov, *Affine isoperimetric inequalities on flag manifolds*, Canadian Journal of Mathematics (to appear).

[17] M. Fradelizi, *Sections of convex bodies through their centroid*, Arch. Math. (Basel) 69 (1997), no. 6, 515–522.

[18] M. Fradelizi, O. Guédon and A. Pajor, *Thin-shell concentration for convex measures*, Studia Math. 223 (2014), no. 2, 123–148.

[19] R. J. Gardner, *Geometric tomography*, Second edition. Encyclopedia of Mathematics and its Applications, 58. Cambridge University Press, New York, 2006. xxii+492 pp.

[20] R. J. Gardner, *The dual Brunn-Minkowski theory for bounded Borel sets: dual affine quermassintegrals and inequalities*, Adv. Math. 216 (2007), 358–386.

[21] A. Giannopoulos and A. Koldobsky, *Variants of the Busemann-Petty problem and of the Shephard problem,* Int. Math. Res. Not. (IMRN), 2017, no. 3, 921–943.

[22] A. Giannopoulos, A. Koldobsky and A. Zvavitch, *Inequalities for the Radon transform on convex sets*, Int. Math. Res. Not. (IMRN), 2022, no. 18, 13984–14007.

[23] P. Goodey and G. Zhang, *Inequalities between projection functions of convex bodies*, Amer. J. Math. 120 (1998), 345–367.

[24] E. L. Grinberg, *Isoperimetric inequalities and identities for k-dimensional cross-sections of convex bodies*, Math. Ann. 291 (1991), 75–86.

[25] A. Jambulapati, Y. T. Lee ans S. Vempala, *A slightly improved bound for the KLS constant*, arxiv:2208.11644v2.

[26] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. 16 (2006), 1274–1290.

[27] B. Klartag, *Logarithmic bounds for isoperimetry and slices of convex sets*, Ars Inven. Anal. (2023), Paper No. 4, 17 pp.

[28] B. Klartag and J. Lehec, *Bourgain's slicing problem and KLS isoperimetry up to polylog*, Geom. Funct. Anal. 32 (2022), no. 5, 1134–1159.

[29] A. Koldobsky, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs, 116. American Mathematical Society, Providence, RI, 2005. vi+170 pp.

[30] A. Koldobsky, *Slicing inequalities for measures on convex bodies*, Adv. Math. 283 (2015), 473–488.

[31] A. Koldobsky and M. Lifshits, *Average volume of sections of star bodies*, Geometric aspects of functional analysis, 119–146, Lecture Notes in Math., 1745, Springer, Berlin, 2000.

[32] A. Koldobsky and A. Zvavitch, *An isomorphic version of the Busemann-Petty problem for arbitrary measures*, Geom. Dedicata 174 (2015), 261–277.

[33] A. Koldobsky, G. Paouris and M. Zymonopoulou, *Isomorphic properties of intersection bodies,* J. Funct. Anal. 261 (2011), 2697–2716.

[34] A. Koldobsky, G. Paouris and A. Zvavitch, *Measure comparison and distance inequalities for convex bodies*, Indiana Univ. Math. J. 71 (2022), no. 1, 391–407.

[35] E. Lutwak, *Inequalities for Hadwiger's harmonic Quermassintegrals*, Math. Ann. 280 (1988), 165-175.

[36] E. Milman and A. Yehudayoff, *Sharp Isoperimetric Inequalities for Affine Quermassintegrals*, Preprint.

[37] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geometric aspects of functional analysis (1987-88), 64–104, Lecture Notes in Math., 1376, Springer, Berlin, 1989.

[38] V. D. Milman and L. Rotem, *Mixed integrals and related inequalities*, J. Funct. Analysis, 264 (2013), no. 2, 570–604.

[39] G. Paouris and P. Pivovarov, *Small-ball probabilities for the volume of random convex sets*, Discrete Comput. Geom. 49 (2013), no. 3, 601–646.

[40] R. Schneider, *Convex bodies: The Brunn-Minkowski theory*, Second expanded edition. Encyclopedia of Mathematics and its Applications, 151. Cambridge University Press, Cambridge, 2014. xxii+736 pp.

[41] D. Wu, *The isomorphic Busemann-Petty problem for s-concave measures*, Geom. Dedicata 204 (2020), 131–148.

[42] G. Zhang, *Restricted chord projection and affine inequalities*, Geom. Dedicata 39 (1991), 213-222.

[43] A. Zvavitch, *The Busemann-Petty problem for arbitrary measures*, Math. Ann. 331, no. 4 (2005), 867–887.

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