Intrinsic volumes and linear contractions

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Abstract

It is shown that intrinsic volumes of a convex body decrease under linear contractions.

Let $C \subset \mathbb{R}^N$ be a convex body and B_2^N the Euclidean ball in $\mathbb{R}^N$. The Steiner formula expresses the volume of the Minkowski sum $C + \varepsilon B_2^N$ in terms of the intrinsic volumes $V_0, V_1, \ldots, V_N$ of C:

$$\mathrm{vol}_N\left(C + \varepsilon B_2^N\right) = \sum_{n=0}^{N} \omega_n V_{N-n}(C)\varepsilon^n.$$

Here $\mathrm{vol}_N(\cdot)$ denotes N-dimensional Lebesgue measure and $\omega_n = \mathrm{vol}_n(B_2^n)$. Of particular interest are V_1, V_{N-1} and V_N, which are multiples of the mean-width, surface area and volume, respectively. We refer the reader to [5] for background on intrinsic volumes. In addition to their role in convex geometry, intrinsic volumes also appear in connection with Gaussian processes; see, e.g., [9], [10] and the references therein.

The purpose of this note is to prove the following.

Proposition 1.1. *Let $C \subset \mathbb{R}^N$ be a convex body and let S be a linear contraction, i.e., $\|Sx\|_2 \leqslant \|x\|_2$ for each $x \in \mathbb{R}^N$. Then for $n = 1, \ldots, N$,*

$$V_n(SC) \leqslant V_n(C).$$

The case of V_1 and arbitrary contractions (not necessarily linear) is well-studied [6, Theorem 2 in §5], [1, Theorem 1]; see also [2, pg 177]. Of course for V_N one has $V_N(SC) = |\det(S)|\,\mathrm{vol}_N(C)$. For other intrinsic volumes, we were unable to find Proposition 1.1 in the literature but noticed that it follows from some results in [4] and thought it was worthwhile to show the details.

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Particularly useful for our purpose is the Gaussian representation of intrinsic volumes, as in [10]; see also [8]. If $\Gamma_{N,n} = [\gamma_{ij}]$ is an $n \times N$ matrix with independent N(0,1) Gaussian entries, then the n-th intrinsic volume of $C \subset \mathbb{R}^N$ is given by

$$V_n(C) = \frac{(2\pi)^{n/2}}{\omega_n n!} \mathbb{E}\, \mathrm{vol}_n \left(\Gamma_{N,n} C \right).$$

As in [4], we say that a function $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ satisfies **Groemer's Convexity Condition**, or simply (**GCC**), if for every $z \in \mathbb{R}^n$ and for every $y_1 \ldots, y_N \in z^\perp$ the function $F_Y : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$F_Y(t) = F(y_1 + t_1 z, \ldots, y_N + t_N z)$$

is even and convex. The latter definition was motivated by isoperimetric-type problems for random convex sets in [3]. In particular, by adapting [3, Lemma 3], it was shown in [4, Proposition 4.1] that for a convex body $C \subset \mathbb{R}^N$, the function $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ defined by

$$F(x_1, \ldots, x_N) = \mathrm{vol}_n \left([x_1 \ldots x_N] C \right), \tag{1}$$

where $[x_1 \ldots x_N]$ denotes the $n \times N$ matrix with columns $x_1, \ldots, x_N$, viewed as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$, satisfies (**GCC**). The latter property fits well with symmetrization techniques and can be used in various isoperimetric-type problems for the volume of random (and non-random) sets [4, Theorem 1.1].

For our present purpose, we require less than the (**GCC**) condition. In fact, we will use only the following consequence.

Lemma 1.2. *If $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ satisfies (**GCC**) then for any $x_1, \ldots, x_N \in \mathbb{R}^n$ and any $1 \leqslant j \leqslant N$, the function*

$$\mathbb{R} \ni s \mapsto F(x_1, \ldots, s x_j, \ldots, x_N) \tag{2}$$

is convex.

The lemma is immediate since the restriction of a convex function to a line is itself convex.

Additionally, we will make use of the following elementary lemma (the proof is given in [4, Lemma 3.7]).

Lemma 1.3. *Let $\rho : \mathbb{R}^n \to \mathbb{R}^+$ be a function such that*

$$\mathbb{R} \ni s \mapsto \rho(sx)$$

is convex for each $x \in \mathbb{R}^n$. If X is a symmetric random vector with values in $\mathbb{R}^n$, then

$$\mathbb{R}^+ \ni s \mapsto \mathbb{E}\rho(sX)$$

is an increasing function.

Here and elsewhere, we use the term "increasing" in the non-strict sense.

Proof of Proposition 1.1. As noted above, the function $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ defined according to (1) satisfies (**GCC**). Let $g_1, \ldots, g_N$ denote the columns of the Gaussian random matrix $\Gamma_{N,n}$. If $g_1, \ldots, g_N$ are fixed, then

$$\mathbb{R} \ni s \mapsto F(g_1, \ldots, g_{j-1}, sg_j, g_{j+1}, \ldots, g_N)$$

is convex by Lemma 1.2. Letting $\mathbb{E}_j$ denote expectation with respect g_j and applying Lemma 1.3, we have that

$$\mathbb{R}^+ \ni s \mapsto \mathbb{E}_j F(g_1, \ldots, g_{j-1}, sg_j, g_{j+1}, \ldots, g_N)$$

is an increasing function.

Suppose first that S is represented by the $N \times N$ diagonal matrix $S = \mathrm{diag}(1, \ldots, 1, s_j, 1, \ldots, 1)$ where $s_j \in [0, 1]$ is in the j^{th}-column. Then

$$\mathbb{E}_j F(g_1, \ldots, g_{j-1}, s_j g_j, g_{j+1}, \ldots, g_N) \leqslant \mathbb{E}_j F(g_1, \ldots, g_{j-1}, g_j, g_{j+1}, \ldots, g_N)$$

and hence

$$\begin{aligned} (2\pi)^{-n/2}\omega_n n! V_n(SC) &= \mathbb{E}\,\mathrm{vol}_n\left(\Gamma_{N,n} SC\right) \\ &= \mathbb{E} F(g_1, \ldots, g_{j-1}, s_j g_j, g_{j+1}, \ldots, g_N) \\ &\leqslant \mathbb{E} F(g_1, \ldots, g_N) \\ &= (2\pi)^{-n/2}\omega_n n! V_n(C). \end{aligned}$$

In the general case, using singular value decomposition, one writes $S = UDV^T$, where D is the diagonal matrix $\mathrm{diag}(s_1, \ldots, s_N)$, and U and V are orthogonal. Since S is a contraction, its singular values satisfy $0 \leqslant s_i \leqslant 1$ for $i = 1, \ldots, N$. To conclude, we use the fact that intrinsic volumes are invariant under orthogonal transformations and apply the latter argument iteratively.

□

Remark 1.4. The latter proof uses ideas from [7, Lemma 2.7].

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