A Mathematical Theory of the Guillotine

PIERO VILLAGGIO

To Bernard Coleman for his 60th birthday

1. Prologue

In the year 1989, in a not yet well identified country of the world, a violent revolution exploded, unleashed by the excessive privileges of the upper class. The interim government, instituted by the democratic rebels, decided to reintroduce capital punishment by decapitation, in analogy to what was done in France two centuries before, and consequently the guillotine made its reappearance in history.

A man called *** was appointed superintendent of the executions and was charged with deciding the political and technical questions generated by the cruel decision of the government. Mr. *** was absolutely unable to carry out his task. He was an aristocrat masquerading as a populist, interested only in his own theoretical studies, and, above all, pathologically revolted by human blood.

Unfortunately he was not courageous enough to resign from his position for fear of raising suspicions about his past. He thus decided to exploit his knowledge of mechanics so as to render the guillotine more efficient and, perhaps, as painless as possible. In terms customary in his time, he solved an "optimization problem" in solid mechanics.

2. The Problem of Mr. ***

The traditional shape of the blade of the guillotine is trapezoidal (Figure 1) with the skew side sharpened in order to make penetration easier. The typical cross section of the blade, hatched in the figure, has a sharp lower edge where the first contact with the neck of the condemned occurs. But, while the lengths of the sides and thickness of the blade are prescribed, there is a certain freedom in modelling the profile which bounds the lower part of the cross section. Though in machines of the past it was an interval of a straight line, it is not necessarily true that this is the most favorable form of penetration. Granted that the thickness must be constant, the question arises whether a sharp or a blunt edge is more convenient.

Converted into more abstract terms, the problem faced by Mr. *** was that of optimizing the profile of the cutting edge of a rigid blade in order that, the load being equal, the penetration into a soft body would be highest, avoiding breaks.

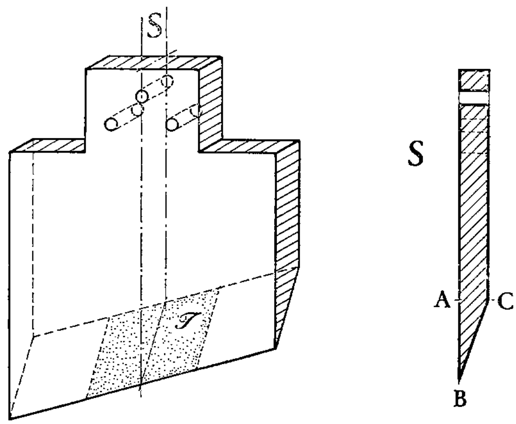

Figure 1

In practice the first contact does not occur along the entire lower side of the blade, but on a subinterval $\mathcal{T}$. If, however, the diameter of the body that must be cut is sufficiently large, this interval of first contact should be large enough that a substantial part of the edge enters into action at the same instant. On the other hand, if the blade is sufficiently thin, and the height of the sharpened part of the edge is also small compared with the thickness, the transverse penetration of the blade will be equal to its thickness just after the first instant of contact. These two considerations make plausible the assumption of regarding the state of deformation induced by the cross section of the blade into the flesh of a fat neck as that generated by the complete indentation of a rigid punch of variable profile on an elastic half-plane in a state of plane strain. The scheme of the position of the punch with respect to the half-plane after the first penetration is that indicated in Figure 2, where the length of AC, the thickness of the blade, and that of AB,

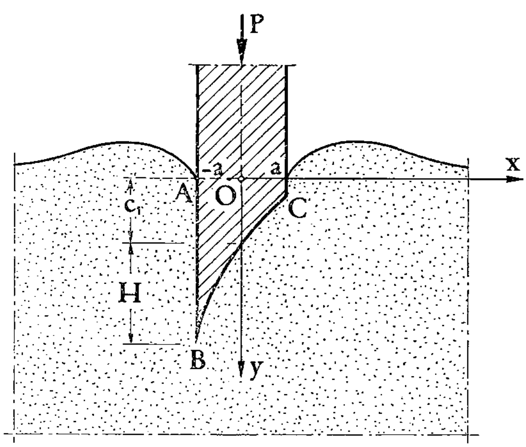

Figure 2

the height of the sharpened edge, are prescribed, while the shape of BC must be determined so as to improve the elastic penetration of the punch under constant load P.

Of course such a model requires other tacit simplifications which must be kept in mind: the assumption that all the cross sections of the blade have the same initial penetration of magntiude H implies that the obliqueness of the lower side has been neglected; in addition, it is assumed that the contact is so smooth that the possible tangential stress due to friction can be disregarded; finally, the process is sufficiently slow to ignore dynamic effects. It must be also recalled that the hypothesis that H is small with respect to the thickness $2a$ is necessary to infer the strains small enough to be regarded as infinitesimal and consequently to determine them by the methods of plane linear elasticity. For, in particular, a flat punch (H = zero), the first solution of the corresponding elastic problem is due to SADOWSKI [1928]; and the same result was found through other techniques by MUSKHELISHVILI [1941] and by L. FÖPPL [1941].

The problem of modelling the shape of the edge is, however, different. In this case of force P acting on the punch, its thickness and height are prescribed, and the profile BC must be found which yields the largest penetration, that is the maximum displacement in the direction of the y-axis of a characteristic point of the blade taken to be the point with initial coordinates (0, 0). It may appear that the best shape is such that the profile BC has a very sharp vertex at B, but this solution is not physically acceptable for, if the vertex is too pointed, the excess of localized stress may cause the blade to break. The conclusion is that the solution must be a compromise between the need to sharpen the profile at B to favor penetration and the need to maintain a certain slope to avoid rupture.

The problem formulated in these terms is no longer a classical problem but one of shape optimization. Although these problems are very common in technical mechanics, their solution is difficult. That found here is only partial as it is assumed that the curve BC can be expressed by a polynomial, keeping in mind the obvious fact that, if the properties of regularity imposed on minimizing functions are removed, the minimum may be much lower or even not exist at all.

3. The Rectilinear Profile

Before adressing the case in which the profile is analytically represented by a polynomial it is expedient to consider a simpler example, like that in which BC is rectilinear and, therefore, the punch is triangular. The shape of the profile being prescribed, the question of optimizing it does not arise, since BC is simply the segment connecting these two points. Again, the condition that the angle with vertex at B is not too small imposes a limit on the magnitude of H.

With the x and y axes chosen as is shown in Figure 2, the Cartesian equation of BC is

$$y = f(x) = \frac{H}{2}\left(1 - \frac{x}{a}\right) + G, \qquad -a \leqq x \leqq a, \tag{3.1}$$

where G is a constant for the moment not determined. This curve also represents the normal displacement $v(x, 0)$ imparted to the boundary of the half-plane $y \geqq 0$ in the interval $-a \leqq x \leqq a$, the remaining part being free from stresses. Under the hypothesis that the half-plane is linearly elastic with Young's modulus E and Poisson's ratio σ, the problem was solved by FLORIN [1936]. The very simple solution is found by assuming the normal pressure $p(x)$, exerted upon the interval where penetration occurs, to have the form

$$p(x) = \frac{c_0 - c_1 \dfrac{x}{a}}{\sqrt{1 - \left(\dfrac{x}{a}\right)^2}}, \tag{3.2}$$

where c_0, c_1 are two indeterminate constants. This pressure is singular at the ends of the interval where the boundary conditions change type. The constants c_0, c_1 are determined by imposing the condition that the displacement $v(x, 0)$, or, better, its derivative $\dfrac{\partial v}{\partial x}(x, 0)$ is $f'(x)$:

$$\frac{\partial v}{\partial x}(x, 0) = -\frac{2(1 - \sigma^2)}{\pi E} \int_{-a}^{a} \frac{p(u)\, du}{x - u} = -\frac{H}{2a}. \tag{3.3}$$

In addition to this relation, it is known that the resultant of the pressure $p(x)$ is equivalent to the vertical load for unit of length P, that is

$$\int_{-a}^{a} p(u)\, du = P. \tag{3.4}$$

When the expression (3.2) for $p(u)$ is put into the above equations, with the abbreviation $\theta = \dfrac{\pi E}{2(1 - \sigma^2)}$ they become

$$-\frac{1}{\theta} \int_{-a}^{a} \frac{c_0 - c_1 \dfrac{u}{a}}{\sqrt{1 - \left(\dfrac{u}{a}\right)^2}\,(x - u)}\, du = -\frac{H}{2a}, \tag{3.5}$$

$$\int_{-a}^{a} \frac{c_0 - c_1 \dfrac{u}{a}}{\sqrt{1 - \left(\dfrac{u}{a}\right)^2}}\, du = P, \tag{3.6}$$

and these integrals can be evaluated in finite form by the change of variables $x = -a \cos \alpha$, $u = -a \cos \beta$ (*cf.* GRÖBNER & HOFREITER [1966, II 332]). After introducing the values of the integrals it is possible to derive c_0 and c_1:

$$c_0 = \frac{P}{\pi a}, \qquad c_1 = \frac{\theta H}{2\pi a}. \tag{3.7}$$

This is FLORIN's result, but the method used here is due to L. FÖPPL (*cf.* SZABÓ [1964, § 11]).

Once the local pressure is known, the partial resultant of this pressure starting from the end point $x = -a$, is given by

$$\mathscr{P}(x) = \int_{-a}^{x} p(u)\,du = \int_{-a}^{x} \frac{c_0 - c_1 \dfrac{u}{a}}{\sqrt{1 - \left(\dfrac{u}{a}\right)^2}}\,du$$

$$= \frac{P}{\pi}\left(\arcsin\frac{x}{a} + \frac{\pi}{2}\right) + \frac{\theta H}{2\pi}\sqrt{1 - \left(\frac{x}{a}\right)^2}, \tag{3.8}$$

which, of course, is P for $x = a$.

Still from the formula giving the pressure it is possible to obtain the displacement $v(x, 0)$ in the load region $-a \leqq x \leqq a$:

$$v(x, 0) = -\frac{1}{\theta}\int_{-a}^{a} p(u) \ln(x - u)\,du + \text{constant}; \tag{3.9}$$

since this displacement is defined to within a constant, it is customary to designate two points of the boundary $(\pm c, 0)$ with $c > a$ as fixed (*cf.* GIRKMANN [1963, § 32]). Thus $v(x, 0)$ assumes the form

$$v(x, 0) = -\frac{1}{\theta}\int_{-a}^{a} p(u) \ln\frac{|x - u|}{c}\,du. \tag{3.10}$$

The displacement of a characteristic point of the base, for example, the origin (0, 0) can be taken as a measure of the penetration of the blade under the load P. Then the formula (3.10) becomes

$$v_0 = v(0, 0) = -\frac{1}{\theta}\int_{-a}^{a} \frac{c_0 - c_1 \dfrac{u}{a}}{\sqrt{1 - \left(\dfrac{u}{a}\right)^2}}\left(\ln\frac{|u|}{a} - \ln\frac{c}{a}\right) du, \tag{3.11}$$

that is, by again using the results of GRÖBNER & HOFREITER [1966, II 324, 60]

$$v_0 = \frac{\pi a}{\theta} c_0 \ln\left(2\frac{c}{a}\right) = \frac{P}{\theta}\ln\left(2\frac{c}{a}\right). \tag{3.12}$$

Because v_0 is determined, also the constant G is known since $f(0)$ must be equal to v_0, and its value is $G = \dfrac{P}{\theta}\ln\left(2\dfrac{c}{a}\right) - \dfrac{H}{2}$.

At this point the problem would be solved if not for the fact that, if H is too large and the edge too sharp, the blade may break from excess pressure. In practice, since the pressure is infinite at B (as well at C), this condition must be rendered more precise: what is important to bound is the product

$$K = \lim_{x \to -a} p(x)\sqrt{1 + \frac{x}{a}}, \tag{3.13}$$

which is called the "stress concentration factor" and is a measure of how the pressure tends to infinity in the neighborhood of the vertex B. This quantity must obey a constraint of the type $K \leqq K_0$, this latter being a constant of the material composing the blade. Then, recalling (3.2), it is found that

$$K = \frac{1}{\sqrt{2}}(c_0 + c_1) = \frac{1}{\sqrt{2}}\left(\frac{P}{\pi a} + \frac{\theta H}{2\pi a}\right) \leqq K_0. \tag{3.14}$$

The concluding result is thus the following: at constant load P, the penetration v_0 is independent of H, but this cannot be too large in order to not violate (3.14) and the condition $f(a) = G \geqq 0$, ensuring complete penetration.

4. The Parabolic Profile

An alternative solution of the problem is that of adopting for BC, instead of a rectilinear profile, a parabolic one, described by the equation

$$y = f(x) = \frac{H}{4}\left(1 - \frac{x}{a}\right)^2 + G, \quad -a \leqq x \leqq a, \tag{4.1}$$

where G is a constant.

Also in this case an integral equation like (3.3) must be solved, but with the difference that $f'(x)$ is now $-\dfrac{H}{2}\left(1 - \dfrac{x}{a}\right)$. By taking the expression for the pressure to be

$$p(x) = \frac{c_0 - c_1 \dfrac{x}{a} + c_2\left(\dfrac{x}{a}\right)^2}{\sqrt{1 - \left(\dfrac{x}{a}\right)^2}}, \tag{4.2}$$

with c_0, c_1, c_2, being new constants, and placing it into (3.3) after modifying the term $f'(x)$, we obtain the new equation

$$-\frac{1}{\theta}\int_{-a}^{a} \frac{c_0 - c_1 \dfrac{u}{a} + c_2\left(\dfrac{u}{a}\right)^2}{\sqrt{1 - \left(\dfrac{u}{a}\right)^2}\,(x - u)}\,du = -\frac{H}{2a}\left(1 - \frac{x}{a}\right). \tag{4.3}$$

The condition that the resultant pressure is still P is

$$\int_{-a}^{a} \frac{c_0 - c_1 \dfrac{u}{a} + c_2\left(\dfrac{u}{a}\right)^2}{\sqrt{1 - \left(\dfrac{u}{a}\right)^2}}\,du = P. \tag{4.4}$$

By again using the tables of GRÖBNER & HOFREITER the two foregoing integrals can be evaluated explicitly so as to arrive at the two equations

$$c_1 - c_2\frac{x}{a} = \frac{\theta H}{2\pi a}\left(1 - \frac{x}{a}\right), \qquad c_0 + \frac{c_2}{2} = \frac{P}{\pi a},$$

and hence

$$c_0 = \frac{P}{\pi a} - \frac{\theta H}{4\pi a}, \qquad c_1 = \frac{\theta H}{2\pi a}, \qquad c_2 = \frac{\theta H}{2\pi a}. \tag{4.5}$$

The constants c_0, c_1, c_2 being known, penetration is now given by the expression

$$v_0 = v(0,0) = -\frac{1}{\theta}\int_{-a}^{a} \frac{c_0 - c_1\frac{u}{a} + c_2\left(\frac{u}{a}\right)^2}{\sqrt{1-\left(\frac{u}{a}\right)^2}}\left(\ln\frac{|u|}{a} - \ln\frac{c}{a}\right) du,$$

the calculation of which furnishes

$$v_0 = \frac{\pi a}{\theta} c_0 \ln\left(2\frac{c}{a}\right) - \frac{\pi a}{\theta} c_2\left(\tfrac{1}{4} - \tfrac{1}{2}\ln\left(2\frac{c}{a}\right)\right), \tag{4.4}$$

or, after introducing the values of c_0 and c_2,

$$v_0 = \frac{P}{\theta}\ln\left(2\frac{c}{a}\right) + \frac{H}{8}, \tag{4.5}$$

and a comparison with the formula (3.12) shows that penetration is increased by the amount $\frac{H}{8}$ for a parabolic profile. From (4.5) the derivation of the constant $G = \frac{P}{\theta}\ln\left(2\frac{c}{a}\right) - \frac{H}{8}$ is also immediate, and G must be non-negative in order to guarantee complete penetration.

The factor K, which is still defined by (3.13), becomes

$$K = \frac{1}{\sqrt{2}}(c_0 + c_1 + c_2) = \frac{1}{\sqrt{2}}\left(\frac{P}{\pi a} + 3\frac{\theta H}{4\pi a}\right), \tag{4.6}$$

and, to prevent possible ruptures, K must not be larger than K_0.

Since K is now higher than in the case of a rectilinear profile, the conclusion is that a parabolic profile improves the elastic penetration but worsens the specific pressure at the vertex.

5. The Optimal Profile

As the two simple cases of rectilinear and parabolic profile are, in some sense, in conflict with each other, the first more effective for local pressure concentration at the vertex, the second for elastic penetration, it is natural to ask whether it is possible to design a mixed profile, effecting a compromise between the two counterpoised requirements.

For this purpose it is convenient to start with a profile having the equation

$$y = f(x) = a_1\left(1 - \frac{x}{a}\right) + a_2\left(1 - \frac{x}{a}\right)^2 + G, \tag{5.1}$$

where, this time, a_1 and a_2 are constants to be determined and G is the constant introduced before. However, the coefficient of the equation (5.1) must obey the condition that the height remains H, that is

$$f(-a) - f(a) = 2a_1 + 4a_2 = H; \tag{5.2}$$

and that the graph of BC nowhere crosses the x-axis within the interval $-a \leqq x \leqq a$. A simple, though more restrictive, condition ensuring the latter constraint is that

$$f(a) \geqq 0, \quad f'(x) \leqq 0,$$

or, in terms of the coefficients a_1, a_2, G,

$$G \geqq 0, \quad a_1 \geqq 0, \quad a_1 + 2a_2 \geqq 0. \tag{5.3}$$

The pressure $p(x)$ can still be assumed to have the form (4.2), determining it in terms of the three constants c_0, c_1, c_2, which must satisfy an equation like (4.3) but with the right-hand side modified: $f'(x) = -\frac{a_1}{a} - 2\frac{a_2}{a}\left(1 - \frac{x}{a}\right)$. The condition on the resultant remains (4.4). Calculation of the integrals yields the relations

$$c_1 - c_2\frac{x}{a} = \frac{\theta}{\pi a}\left[a_1 + 2a_2\left(1 - \frac{x}{a}\right)\right], \quad c_0 + \frac{c_2}{2} = \frac{P}{\pi a},$$

and hence

$$c_0 = \frac{P}{\pi a} - \frac{\theta a_2}{\pi a}, \quad c_1 = \frac{\theta}{\pi a}(a_1 + 2a_2), \quad c_2 = +\frac{\theta}{\pi a}2a_2.$$

The elastic penetration is still given by (4.4), but, with the new values for the constants it becomes

$$v_0 = \frac{P}{\theta}\ln\left(2\frac{c}{a}\right) + \frac{a_2}{2}, \tag{5.4}$$

and the goemetrical condition $v_0 = f(0)$, gives the relation determining G:

$$v_0 = a_1 + a_2 + G.$$

Lastly, (4.6) gives the factor of pressure concentration, while the condition of safety against breaking assumes the form

$$K = \frac{1}{\sqrt{2}}(c_0 + c_1 + c_2) = \frac{1}{\sqrt{2}}\left(\frac{P}{\pi a} + \frac{\theta}{\pi a}a_1 + 3\frac{\theta}{\pi a}a_2\right) \leqq K_0. \tag{5.5}$$

The optimum profile will then be that which maximizes v_0, regarded as a function of a_1, a_2, G, with the condition (5.2) and the constraints (5.3), (5.5). Since

(5.2) permits the elimination of a_2, the other constraints reduce to:

$$a_1 \leqq \tfrac{4}{3}\frac{P}{\theta}\ln\left(2\frac{c}{a}\right)-\frac{H}{6}, \quad a_1 \geqq 0, \quad a_1 \geqq \tfrac{3}{2}H+2\frac{P}{\theta}-2\sqrt{2}\frac{\pi a}{\theta}K_0, \tag{5.6}$$

and v_0 can be written

$$v_0=\frac{P}{\theta}\ln\left(2\frac{c}{a}\right)+\frac{H}{8}-\frac{a_1}{4}. \tag{5.7}$$

The discussion of the problem is simple. It is clear that the constraints are compatible only if

$$\tfrac{4}{3}\frac{P}{\theta}\ln\left(2\frac{c}{a}\right)-\frac{H}{6}\geqq \max\left\{0, \tfrac{3}{2}H+2\frac{P}{\theta}-2\sqrt{2}\frac{\pi a}{\theta}K_0\right\}. \tag{5.8}$$

Provided that this inequality is satisfied, two cases are possible: if the maximum of the right-hand side is 0, then the solution is $a_1=0$ and $a_2=\frac{H}{4}$, which corresponds to the simple parabola; if, alternatively, the second term on the right-hand side of (5.8) is the maximum, then the solution is $a_1=\tfrac{3}{2}H+2\frac{P}{\theta}-2\sqrt{2}\frac{\pi a}{\theta}K_0$, $a_2=\sqrt{2}\frac{\pi a}{\theta}K_0-\frac{P}{\theta}-\frac{H}{2}$, and the profile is mixed. In either case it is convenient to thin the profile at the vertex.

6. Epilogue

The reprisals of 1989 never took place because every citizen could be proven to be a revolutionary. Mr. *** received solemn awards and his theory was applied to grind knives for slicing bread and salami. Only the pigs protested.

References

[1928] SADOWSKI, M. A.: „Zweidimensionale Probleme der Elastizitätstheorie." Zeit. Angew. Math. Mech. V. 8, pp. 107–121.

[1941] MUSKHELISHVILI, N. I.: "The fundamental boundary problems of the theory of elasticity for the half-plane." Soobshceniya A. N. Gruz. S.S.E. V. III, n. 10, pp. 873–880.

[1941] FÖPPL, L.: „Elastische Beanspruchung des Erdbodens unter Fundamenten." Forschg. Ing. Wes. V. 12, p. 31.

[1948] FLORIN, V. A.: *Analysis of Foundation of Hydraulic Structures*. Stroiizdat: Moscow.

[1963] GIRKMANN, K.: *Flächentragwerke*. 6th Ed. Wien: Springer.

[1964] SZABÓ, I.: *Höhere Technische Mechanik*. Berlin/Göttingen/Heidelberg: Springer.

[1965] GRÖBNER, W., & HOFREITHER, N.: *Integraltafel*. Wien/New York: Springer.

Istituto di Scienza delle Costruzioni
Università di Pisa

(*Received June 27, 1989*)