A stability result for mean width of L_p-centroid bodies.

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Abstract

We give a different proof of a recent result of Klartag [12] concerning the concentration of the volume of a convex body within a thin Euclidean shell and proving a conjecture of Anttila, Ball and Perissinaki [1]. It is based on the study of the L_p-centroid bodies. We prove an almost isometric reverse Hölder inequality for their mean width and a refined form of a stability result.

1 Introduction

In this paper we study how the volume of a symmetric convex body concentrates within a very thin Euclidean shell. Let K be an isotropic convex body in $\mathbb{R}^n$ i.e. a symmetric convex body of volume 1 such that for some fixed $L_K > 0$,

$$\forall \theta \in S^{n-1}, \int_K \langle x, \theta\rangle^2 dx = L_K^2.$$

It is known that every symmetric convex body has an affine image which is isotropic. We denote by $|x|_2$ the Euclidean norm of $x \in \mathbb{R}^n$. In the paper [1], Anttila, Ball and Perissinaki asked if every isotropic convex body in $\mathbb{R}^n$ satisfy an ε-concentration hypothesis namely:
Concentration hypothesis. Does there exist ε_n such that $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$\left|\left\{x \in K, \left|\frac{|x|_2}{L_K\sqrt{n}} - 1\right| \geq \varepsilon_n\right\}\right| \leq \varepsilon_n?$$

We will prove the following

Theorem 1. *There exists c and c' such that for every isotropic convex body K in $\mathbb{R}^n$, and every $p \leq (\log n)^{1/3}$,*

$$1 \leq \left(\int_K |x|_2^p dx\right)^{1/p} \Big/ \left(\int_K |x|_2 dx\right) \leq 1 + c\,p/(\log n)^{1/3}.$$

In particular, for every $\varepsilon \in (0,1)$,

$$\left|\left\{x \in K, \left|\frac{|x|_2}{\sqrt{n}L_K} - 1\right| \geq \varepsilon\right\}\right| \leq 2e^{-\ c\sqrt{\varepsilon}(\log n)^{1/12}}. \tag{1}$$

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This implies that the concentration hypothesis holds with $\varepsilon_n = c(\log\log n)^2/(\log n)^{1/6}$. This result has been very recently obtained in full generality by Klartag [12], where he proved that (1) holds true with $2e^{-\varepsilon^2 \log n}$ for every isotropic convex body with center of mass at the origin. Our goal is to present a different approach via the notion of L_p-centroid bodies. To any star shape body with respect to the origin, $L \subset \mathbb{R}^n$, we associate its L_p-centroid body $Z_p(L)$ which is a symmetric convex body defined by its support function:

$$\forall y \in \mathbb{R}^n, h_{Z_p(L)}(y) = \left(\int_L |\langle x, y\rangle|^p dx\right)^{1/p}.$$

This body is homothetic to the L_p-centroid body defined by Lutwak and Zhang in [16] (see also [15]). For any symmetric convex body C, we define the p-th mean width as

$$W_p(C) = \left(\int_{S^{n-1}} h_C(\theta)^p d\sigma(\theta)\right)^{1/p}.$$

The main result of this paper compares the mean width of the L_p-centroid bodies of an isotropic convex body to the mean width of the L_p-centroid bodies of the Euclidean unit ball of volume 1.

Theorem 2. *There exists a constant c such that for any n, for every isotropic convex body K in $\mathbb{R}^n$, if $\widetilde{D}$ denotes the Euclidean unit ball in $\mathbb{R}^n$ of volume 1, for every $p \leq (\log n)^{1/3}$*

$$\frac{W_1(Z_p(K))}{W_1(Z_1(K))}\frac{W_1(Z_1(\widetilde{D}))}{W_1(Z_p(\widetilde{D}))} \leq 1 + cp/(\log n)^{1/3}.$$

Regarding K as a probability space, these techniques were used by the third named author [20] to prove that the L_q-norms of the Euclidean norm are almost constant for any $q \leq \sqrt{n}$, i.e. (see theorem 1.2 in [20])

$$\exists C \geq 1, \forall q \leq c\sqrt{n}, \left(\int_K |x|_2^q dx\right)^{1/q} \leq C\left(\int_K |x|_2^2 dx\right)^{1/2} = C\sqrt{n}\, L_K. \tag{2}$$

Theorem 1 is in fact an almost isometric version of this result (although it does not recover the full isomorphic one). It is also related to a weak form of Kannan, Lovász and Simonovits [11] conjecture about the Cheeger-type isoperimetric constant for convex bodies: does there exist $c > 0$ such that for any isotropic convex body K,

$$\sigma_K^2 := \frac{Var(|X|_2^2)}{nL_K^4} \leq c$$

where X is a random vector uniformly distributed on K? We refer to the paper of Bobkov [4] for more details between the full KLS-conjecture and this weaker form. Theorem 1 implies that $\lim_{n\to\infty} \sigma_K^2/n = 0$. Up to now, the only known upper bound was the trivial one, $\sigma_K \leq c\sqrt{n}$.

On the way, we will need a new type of stability result for the L_p-centroid bodies. Let K and L be symmetric convex bodies of volume 1 in $\mathbb{R}^d$, if $Z_p(L)$ is close to $Z_p(K)$ for the geometric distance, what can we say about the geometric distance between K and L? This type of question has been studied by Bourgain and Lindenstrauss [6] in the case of projection bodies i.e. $p = 1$. We will prove a more precise result when one of the bodies is the Euclidean unit ball D. The geometric distance between two symmetric convex bodies K and L is defined by

$$d(K, L) = \inf\{ab \mid a, b > 0 \text{ and } 1/aK \subset L \subset bK\}.$$

Theorem 3. *There exists $c > 0$ such that for every integer d greater than 3 and any odd integer $p \le d$, we have the following property:*
if K is a symmetric convex body in $\mathbb{R}^d$ such that for some $\alpha > 1$ and $\varepsilon \in (0, (c\,\alpha)^{-2d^3})$

$$d(K, D) \le \alpha \quad \text{and} \quad d(Z_p(\widetilde{D}), Z_p(\widetilde{K})) \le 1 + \varepsilon$$

where $\widetilde{K} = |K|^{-1/d}K$ and $\widetilde{D} = |D|^{-1/d}D$ then

$$d(K, D) \le 1 + h(\varepsilon) \quad \text{and} \quad (1 - h(\varepsilon))Z_p(\widetilde{D}) \subset Z_p(\widetilde{K}) \subset (1 + h(\varepsilon))Z_p(\widetilde{D})$$

where $h(\varepsilon) = (c\,\alpha)^{d+p+1}\varepsilon^{1/d^2}$.

It was proved in [1] that the concentration hypothesis implies some type of central limit theorem. The conjecture about a central limit theorem for convex sets stated by Anttila, Ball, Perissinaki [1] and Brehm, Voigt [7] has been recently proved by Klartag [12] and we refer to that paper for more precise references on this subject.

The paper is organized as follows. In Section 2, we shall explain how we reduce the study of concentration of the volume of an isotropic convex body to the study of its L_p-centroid bodies. We shall prove the main Theorem 2 in Section 3. The proof of Theorem 3 is given in Section 4 and uses standard tools coming from the theory of spherical harmonics.

Notations. Throughout this paper, D will be the Euclidean ball in $\mathbb{R}^n$ and S^{n-1} the unit sphere. The volume is denoted by $|\cdot|$. We write ω_n for the volume of D and σ for the rotationally invariant probability measure on S^{n-1}. By $\widetilde{L}$ we denote the convex body that is homothetic to $L \subset \mathbb{R}^n$ and has volume 1, that is $\widetilde{L} = |L|^{-1/n}L$ and $R(L)$ will be the circumradius of L i.e. the smallest real number such that $L \subset R(L)D$. The letter c will allways be used as being a universal constant and it can change from line to line.

Acknowledgement. The authors would like to thank K. Ball for several useful discussions.

2 Reduction to L_p centroid bodies.

For any isotropic convex body K, we define $I_p(K) = \left(\int_K |x|_2^p dx\right)^{1/p}$. It is easy to check that there exists a constant $c_{n,p}$ such that for every $\theta \in S^{n-1}$

$$c_{n,p}^p \int_{S^{n-1}} |\langle \theta, x\rangle|^p d\sigma(\theta) = |x|_2^p, \quad \text{i.e.} \quad c_{n,p} = \left(\frac{\sqrt{\pi}\ \Gamma(\frac{p+n}{2})}{\Gamma(\frac{p+1}{2})\ \Gamma(\frac{n}{2})} \right)^{1/p}.$$

Note that $c_{n,p}$ is of the same magnitude than $\sqrt{(n+p)/p}$. By the Fubini theorem and the definition of $W_p(Z_p(K))$, $I_p(K) = c_{n,p}\ W_p(Z_p(K))$. We first need some precise computations in the case of the Euclidean ball of volume 1.

Lemma 1. *Let D be the Euclidean unit ball in $\mathbb{R}^n$, then for any $p \le n$,*

$$I_p(\widetilde{D})/I_1(\widetilde{D}) \le 1 + cp/n^2. \tag{3}$$

Let k be an integer and $p \le k \le n$ and denote by $\widetilde{D_F}$ the Euclidean unit ball of volume 1 in any k-dimensional subspace F of $\mathbb{R}^n$ then

$$\left(W_1(Z_1(\widetilde{D}))/W_1(Z_p(\widetilde{D}))\right)\ \left(W_1(Z_p(\widetilde{D_F}))/W_1(Z_1(\widetilde{D_F}))\right) \le 1 + c\ p/k.$$

Proof. For any $1 \le p \le n$, we have

$$\frac{c_{n,p}}{c_{n,1}} \frac{W_p(Z_p(\widetilde{D}))}{W_1(Z_1(\widetilde{D}))} = \left(\int_{\widetilde{D}} |x|_2^p dx \right)^{1/p} \Big/ \int_{\widetilde{D}} |x|_2 dx = (1+1/n)(1+p/n)^{-1/p} \le 1 + cp/n^2.$$

Since for any $p \ge 1$, $W_1(Z_p(\widetilde{D})) = W_p(Z_p(\widetilde{D}))$ and $x\Gamma(x) = \Gamma(x+1)$, we get

$$\frac{W_1(Z_1(\widetilde{D}))\ W_1(Z_p(\widetilde{D_F}))}{W_1(Z_p(\widetilde{D}))\ W_1(Z_1(\widetilde{D_F}))} = \left(\frac{\Gamma\left(1+\frac{n+p}{2}\right)\Gamma\left(1+\frac{k}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(1+\frac{k+p}{2}\right)} \right)^{1/p} \frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(1+\frac{k+1}{2}\right)}{\Gamma\left(1+\frac{n+1}{2}\right)\Gamma\left(1+\frac{k}{2}\right)}.$$

Easy computations involving the Γ function give the stated estimate when $p \le k$. □

For any fixed symmetric convex body L, Litvak, Milman and Schechtman [14] studied the behavior of $W_p(L)$ as a function of p.

Lemma [14] *Let L be a symmetric convex body of $\mathbb{R}^n$ then for any $p \le c_1 n \left(W_1(L)/R(L)\right)^2$,*

$$|W_p(L) - W_1(L)| \le \|h_L(u) - W_1(L)\|_p \le c_2 \sqrt{\frac{p}{n}} R(L) \tag{4}$$

where c_1 and c_2 are universal constants.
The next lemma was essentially proved in [20].

Lemma 2. *There exists $c > 0$ such that for every isotropic convex body $K \subset \mathbb{R}^n$, for every $1 \le p \le c\sqrt{n}$,*

$$R(Z_p(K)) \le c\sqrt{p}\, W_1(Z_p(K)). \tag{5}$$

Proof. We briefly indicate a proof. In isotropic position, $R(Z_p(K)) \le cpR(Z_2(K)) = cpL_K$. Corollary 3.11 in [20] means that if $p \le c\sqrt{n}$, $W_p(Z_p(K))$ is similar up to universal constants to $W_1(Z_p(K))$. Observe that $W_p(Z_p(K) \ge c\sqrt{p/n} I_p(K) \ge c\sqrt{p} L_K$ and $\sqrt{p}\, W_1(Z_p(K)) \ge cpL_K \ge cR(Z_p(K))$. □

Proof of Theorem 1. We write

$$\frac{I_p(K) I_1(\widetilde{D})}{I_1(K) I_p(\widetilde{D})} = \frac{W_p(Z_p(K))}{W_1(Z_p(K))} \left(\frac{W_1(Z_p(K))}{W_1(Z_1(K))} \frac{W_1(Z_1(\widetilde{D}))}{W_1(Z_p(\widetilde{D}))} \right). \tag{6}$$

From (4) and (5), we get $1 \le W_p(Z_p(K))/W_1(Z_p(K)) \le 1 + c\ \frac{p}{\sqrt{n}}$ if $p \le c\sqrt{n}$. Hence Theorem 1 is proved using (3), (6) and Theorem 2. In particular,

$$\int_K \left(\frac{|x|_2^2}{nL_K^2} - 1 \right)^2 dx = \frac{I_4^4(K)}{I_2^4(K)} - 1 \le c/(\log n)^{1/3}. \tag{7}$$

The function $f(x) = \left(\frac{|x|_2^2}{nL_K^2} - 1 \right)$ is a polynomial of degree 2 and we can use the results of Bobkov [3] about L_r-norms of polynomials. Indeed, theorem 1 of [3] states that there exists a universal constant $c > 0$ such that $\int_K e^{\hat{f}(x)/c \int_K \hat{f}(x)dx} dx \le 2$ where $\hat{f} = |f|^{1/2}$. For every $\varepsilon \in (0,1)$, since $\int_K \hat{f}(x)dx \le \left(\int_K f^2(x)dx \right)^{1/4}$, we get by (7) and by the Chebychev inequality

$$\left| \left\{ x \in K, \left| \frac{|x|_2}{\sqrt{n} L_K} - 1 \right| \ge \varepsilon \right\} \right| \le \left| \left\{ x \in K, \left| \frac{|x|_2^2}{nL_K^2} - 1 \right| \ge \varepsilon \right\} \right| \le 2e^{-\ c\sqrt{\varepsilon}\ (\log n)^{1/12}}.$$

□

3 Proof of Theorem 2

We now introduce some notations and recall some well known facts from local theory of Banach spaces. For a given subspace $F \subset \mathbb{R}^n$, denote by E the orthogonal subspace to F and for every $\phi \in S_F$, the Euclidean sphere in F, we define $E(\phi)$ to be $\{x \in \mathrm{span}\{E, \phi\}, \langle x, \phi\rangle \geq 0\}$. For any $q \geq 0$, define the star body B_q by its radial function

$$\forall \phi \in S_F,\ r_{B_q}(\phi) = \left(\int_{K\cap E(\phi)} |\langle x, \phi\rangle|^q dx\right)^{1/(q+1)}.$$

A theorem of Ball [2] asserts that when K is a symmetric convex body in $\mathbb{R}^n$, this radial function defines a symmetric convex body in F. These balls are related to the L_p-centroid bodies by the following proposition (see proposition 4.3 in [20]).

Proposition [20] *Let K be a symmetric convex body in $\mathbb{R}^n$ and let $1 \leq k \leq n-1$. For every subspace F of $\mathbb{R}^n$ of dimension k and every $q \geq 1$, we have*

$$P_F(Z_q(K)) = (k+q)^{1/q}\, Z_q(B_{k+q-1}) = (k+q)^{1/q}\, |B_{k+q-1}|^{1/k+1/q} Z_q(\widetilde{B_{k+q-1}}). \tag{8}$$

Moreover, an application of a result of Borell [5] gives comparison between these norms.

Lemma [5] *For f being a* log*-concave non-increasing function on* $[0, +\infty)$*, define*

$$F : t \mapsto \frac{1}{\Gamma(t)}\int_0^{+\infty} x^{t-1} f(x)dx,\ \ G : t \mapsto t\int_0^{+\infty} x^{t-1} f(x)dx$$

then F is log*-concave and G is* log*-convex on* $(0, +\infty)$.

Proposition 3. *Let K be a symmetric convex body in $\mathbb{R}^n$, let F be a k-dimensional subspace of $\mathbb{R}^n$, and for any $t \geq 1$, define the symmetric convex body B_{t-1} in F as before. For every $\phi \in S_F$ and every $1 \leq s \leq t \leq u$, we have*

$$\|\phi\|_{B_{t-1}}^t \leq \frac{\Gamma(s)^{(1-\lambda)}\Gamma(u)^\lambda}{\Gamma(t)} \|\phi\|_{B_{s-1}}^{(1-\lambda)s} \|\phi\|_{B_{u-1}}^{u\lambda} \ \text{ and }\ \|\phi\|_{B_{t-1}}^t \geq \frac{t}{s^{(1-\lambda)}u^\lambda} \|\phi\|_{B_{s-1}}^{(1-\lambda)s} \|\phi\|_{B_{u-1}}^{u\lambda}$$

where $t = (1-\lambda)s + \lambda u$.

Proof. Let $f_\phi(y) = |K \cap (E + y\phi)\}|$ for $y \in \mathbb{R}_+$ then by the Brunn-Minkowski inequality, f_ϕ is a log-concave function and non-increasing. By Fubini, for every $\phi \in S_F$,

$$\|\phi\|_{B_{t-1}}^{-t} = \int_0^{+\infty} y^{t-1} f_\phi(y)dy = t^{-1}G(t) = \Gamma(t)F(t)$$

and the conclusion follows easily by the above lemma. □

We will also use a refinement of Dvoretzky's theorem proved by Milman [17] (see also [18]).

Theorem [17] *There exist constants c_1, c_2 such that for any n, any $\varepsilon > 0$ and any symmetric convex body $L \subset \mathbb{R}^n$, if $k \leq c_1 \left(\varepsilon^2/\log(1/\varepsilon)\right) n \left(W_1(L)/R(L)\right)^2$, the set of subspaces $F \in \mathcal{G}_{n,k}$ such that*

$$(1-\varepsilon)W_1(L)D_F \subset P_F L \subset (1+\varepsilon)W_1(L)D_F$$

(where D_F is the Euclidean unit ball of F) has Haar measure greater than $1 - e^{-c_2 k}$.

It was proved by Gordon [9] that we may take ε^2 instead of $\varepsilon^2/\log(1/\varepsilon)$.

Proof of Theorem 2. Let K be an isotropic convex body in $\mathbb{R}^n$. Hence from (5), for every $1 \le q \le c\sqrt{n}$, $R(Z_q(K)) \le c\sqrt{q}\, W_1(Z_q(K))$. Without loss of generality, we can assume that p is an odd integer. Let k and $\varepsilon \in (0, 1/3)$ (to be chosen later) be such that $k^2 \le c\varepsilon^2 n$ and $k \ge p$. Since Dvoretzky's theorem holds with high probability, we can choose a subspace F of $\mathbb{R}^n$ of dimension k such that five conditions hold simultaneously: for every $q \in \{1, p, k, 2k - p, 2k\}$,

$$(1-\varepsilon)\frac{W_1(Z_q(K))}{W_1(Z_q(\widetilde{D_F}))} Z_q(\widetilde{D_F}) \subset P_F Z_q(K) \subset (1+\varepsilon)\frac{W_1(Z_q(K))}{W_1(Z_q(\widetilde{D_F}))} Z_q(\widetilde{D_F}).$$

Indeed, observe that $\forall q \in \{1, p, k, 2k-p, 2k\}$, $k \le c\varepsilon^2 n/q \le c_1 \varepsilon^2 n (W_1(Z_q(K))/R(Z_q(K)))^2$. From (8), these inclusions mean that for every $q \in \{1, p, k, , 2k - p, 2k\}$,

$$(1-\varepsilon)\gamma_q Z_q(\widetilde{D_F}) \subset Z_q(\widetilde{B_{k+q-1}}) \subset (1+\varepsilon)\gamma_q Z_q(\widetilde{D_F}) \tag{9}$$

where

$$\gamma_q = \frac{W_1(Z_q(K))}{(k+q)^{1/q} |B_{k+q-1}|^{1/k+1/q} W_1(Z_q(\widetilde{D_F}))}. \tag{10}$$

The first step is to prove the following

Claim: there is a universal constant c such that, for $q \in \{1, p\}$, $d(B_{k+q-1}, D_F) \le c$.

Indeed, since B_{k+q-1} is a symmetric convex body in a k-dimensional space, it is well known that there exists a universal constant c such that $c\widetilde{B_{k+q-1}} \subset Z_q(\widetilde{B_{k+q-1}}) \subset \widetilde{B_{k+q-1}}$ for $q \ge k$ (see for example lemma 4.1 in [19] or lemma 3.1.1 in [8]). For $q \in \{k, 2k - p, 2k\}$, we deduce from (9) that $d(B_{k+q-1}, D_F) \le c$ where c is a universal constant. Now, for $q \in \{1, p\}$, Proposition 3 with $s = k + q, t = 2k, u = 3k - q$ (i.e. $t = (1 - \lambda)s + \lambda u$ with $\lambda = 1/2$) gives

$$\|\phi\|_{B_{2k-1}}^{2k} \le \frac{\Gamma(k+q)^{1/2}\,\Gamma(3k-q)^{1/2}}{\Gamma(2k)} \|\phi\|_{B_{k+q-1}}^{(k+q)/2} \|\phi\|_{B_{3k-q-1}}^{(3k-q)/2},$$

$$\|\phi\|_{B_{2k-1}}^{2k} \ge \frac{2k}{(k+q)^{1/2}\,(3k-q)^{1/2}} \|\phi\|_{B_{k+q-1}}^{(k+q)/2} \|\phi\|_{B_{3k-q-1}}^{(3k-q)/2}$$

for every $\phi \in S_F$. Since $q \le p \le k$, it is easy to conclude the proof of the claim.

In the second step, we apply Theorem 3. Indeed, for $q \in \{1, p\}$, we get from (9) that $d(Z_q(\widetilde{B_{k+q-1}}), Z_q(\widetilde{D_F})) \le (1+\varepsilon)/(1-\varepsilon) \le 1 + 3\varepsilon$ and we have seen that $d(B_{k+q-1}, D_F) \le c$ therefore, Theorem 3 (since q is a non even number) states that there exists a universal constant c such that

$$1 - h_k(\varepsilon) \le \gamma_q \le 1 + h_k(\varepsilon) \tag{11}$$

and for every $\theta, \theta_0 \in S_F$,

$$(1 + h_k(\varepsilon))^{-1} \|\theta_0\|_{B_{k+q-1}} \le \|\theta\|_{B_{k+q-1}} \le (1 + h_k(\varepsilon)) \|\theta_0\|_{B_{k+q-1}} \tag{12}$$

where $h_k(\varepsilon) = c^{2k}(3\varepsilon)^{1/k^2}$. We want that this last quantity goes to 0 when k goes to infinity hence we choose $\varepsilon = (2c)^{-2k^3}$ in such a way that $h_k(\varepsilon) \le e^{-k}$. In order to use Dvoretzky's theorem, k has been chosen such that $k^2 = c\varepsilon^2 n$ which means that $k \ge c'(\log n)^{1/3}$. By (10) and (11),

$$\frac{W_1(Z_p(K))}{W_1(Z_1(K))} \frac{W_1(Z_1(\widetilde{D_F}))}{W_1(Z_p(\widetilde{D_F}))} \le \frac{(1+e^{-k})(k+p)^{1/p} |B_{k+p-1}|^{1/k+1/p}}{(1-e^{-k})(k+1)|B_k|^{1/k}}. \tag{13}$$

To conclude, it is left to observe that $|K| = 1$ can be written as

$$1 = |K| = k\omega_k \int_{S_F} \int_{K \cap E(\theta)} |\langle x, \theta\rangle|^{k-1} dx\, d\sigma_F(\theta) = k\omega_k \int_{S_F} \|\theta\|_{B_{k-1}}^{-k} d\sigma_F(\phi)$$

so that there exists a $\theta_0 \in S_F$ such that $1 = k\omega_k \|\theta_0\|_{B_{k-1}}^{-k}$. Using relation (12),

$$\begin{aligned} \frac{(k+p)^{1/p}|B_{k+p-1}|^{1/k+1/p}}{(k+1)|B_k|^{1+1/k}} &= \frac{(k+p)^{1/p}\left(\omega_k \int_{S_F} \|\theta\|_{B_{k+p-1}}^{-k} d\sigma_F(\theta)\right)^{1/k+1/p}}{(k+1)\left(\omega_k \int_{S_F} \|\theta\|_{B_k}^{-k} d\sigma_F(\theta)\right)^{1+1/k}} \\ &\leq \frac{\left(1+e^{-k}\right)^{k+2+k/p} (k+p)^{1/p} \|\theta_0\|_{B_k}^{k+1}}{(k+1)\ \omega_k^{1-1/p} \ \|\theta_0\|_{B_{k+p-1}}^{1+k/p}}. \end{aligned}$$

Proposition 3 with $s = k, t = k+1, u = k+p$ (i.e. $t = (1-\lambda)s + \lambda u$ with $\lambda = 1/p$) gives

$$\|\theta_0\|_{B_k}^{k+1} \leq \frac{\Gamma(k)^{1-1/p}\,\Gamma(k+p)^{1/p}}{\Gamma(k+1)} \|\theta_0\|_{B_{k-1}}^{k(1-1/p)} \|\theta_0\|_{B_{k+p-1}}^{1+k/p}.$$

Since $\|\theta_0\|_{B_{k-1}}^{k} = k\omega_k$ and $p \leq k$, easy computations involving the Γ function gives

$$\begin{aligned} \frac{(k+p)^{1/p}|B_{k+p-1}|^{1/k+1/p}}{(k+1)|B_k|^{1+1/k}} &\leq \left(1+e^{-k}\right)^{2k} \frac{(1+p/k)^{1/p}}{(1+1/k)} \frac{\Gamma(k)^{1-1/p}\,\Gamma(k+p)^{1/p}}{\Gamma(k+1)} \\ &= \left(1+e^{-k}\right)^{2k} \frac{1}{k+1} \left(\frac{\Gamma(k+p+1)}{\Gamma(k+1)}\right)^{1/p} \leq 1 + cp/k. \end{aligned}$$

Combining this last inequality with (13) and with Lemma 1, we conclude that if $p \leq k$

$$\frac{W_1(Z_p(K))}{W_1(Z_1(K))} \frac{W_1(Z_1(\widetilde{D}))}{W_1(Z_p(\widetilde{D}))} \leq 1 + cp/(\log n)^{1/3}$$

for a universal constant c. □

4 Stability result for L_p-centroid bodies

In Theorem 3, the equality case (i.e. $\varepsilon = 0$) may be treated via the use of the Funck-Hecke theorem. This is why we will follow an approach using the decomposition in spherical harmonics and we refer to the chapter 3 of the book of Groemer [10] for more detailed explanation. This technique was also used by Bourgain and Lindenstrauss [6].

Let p be an odd integer with $p \leq d$, we consider the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) = |t|^p$ and we define the operator J_ϕ on $L_2(S^{d-1})$ by

$$J_\phi(F)(u) = \int_{S^{d-1}} \phi(\langle u, v\rangle) F(v) d\sigma(v)$$

for any $u \in S^{d-1}$. By the Funck-Hecke theorem, for every harmonic polynomial H homogeneous of degree l on the sphere S^{d-1} we have $\langle J_\phi(F), H\rangle = \alpha_{d,l}(\phi)\langle F, H\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L_2(S^{d-1})$ and

$$\alpha_{d,l}(\phi) = \frac{(-1)^l \pi^{(d-1)/2}}{2^{l-1}\Gamma(l+\frac{d-1}{2})} \int_{-1}^{1} \phi(t) \frac{d^l}{dt^l}(1-t^2)^{(l+\frac{d-3}{2})} dt.$$

These coefficients are known, see [21] or Lemma 1 in [13]. Hence, for any odd values of l, $\alpha_{d,l}(\phi) = 0$ and for any even values of l,

$$\alpha_{d,l}(\phi) = \frac{\pi^{d/2-1}\Gamma(p+1)\sin(\pi(l-p)/2)\Gamma((l-p)/2)}{2^{p-1}\Gamma((l+d+p)/2)}.$$

Standard computations involving the Γ function give a universal constant c such that for any even integer l,

$$\frac{1}{\alpha_{d,l}(\phi)^{1/(p+d/2)}} \leq c \max(d, l). \tag{14}$$

For a continuous function $F : S^{d-1} \to \mathbb{R}$ such that $F^\vee : \mathbb{R}^d \to \mathbb{R}$ defined by $F^\vee(x) = F(x/|x|_2)$ is differentiable on $\mathbb{R}^d \setminus \{0\}$, we set for any $u \in S^{d-1}$, $\nabla_0 F(u) = \nabla F^\vee(u)$. The next proposition is a standard trick using spherical harmonics [10].

Proposition 4. *There exists a universal constant c such that for any continuous even function $F : S^{d-1} \to \mathbb{R}$ such that $\nabla_0 F$ exists,*

$$\|F\|_2 \leq c \|J_\phi(F)\|_2^{2/(d+2p+2)} \left(\|\nabla_0 F\|_2^2 + d^2 \|F\|_2^2\right)^{\frac{1}{2}(1-2/(d+2p+2))}.$$

Proof. Let $F \sim \sum Q_l(F)$ be the decomposition in spherical harmonics of F (with $Q_l(F)$ spherical harmonics of degree l) then by Corollary 3.2.12 in [10]

$$\|\nabla_0 F\|_2^2 = \sum_{l \geq 0} l(l+d-2)\|Q_l(F)\|_2^2.$$

For any odd l, $\alpha_{d,l}(\phi) = 0$ and since F is even, $Q_l(F) = 0$. Hence from the Parseval equality

$$\|F\|_2^2 = \sum_{l \text{ even}} \|Q_l(F)\|_2^2 = \sum_{l \text{ even}} \left(\alpha_{d,l}(\phi)\|Q_l(F)\|_2\right)^\beta \|Q_l(F)\|_2^{2-\beta} \alpha_{d,l}(\phi)^{-\beta}$$

where $\beta \in (0, 2)$ is chosen such that $2\beta/(2-\beta) = 2/(p+d/2)$. By the Hölder inequality,

$$\|F\|_2^2 \leq \left(\sum_{l \text{ even}} \alpha_{d,l}(\phi)^2 \|Q_l(F)\|_2^2\right)^{\beta/2} \left(\sum_{l \text{ even}} \|Q_l(F)\|_2^2 \alpha_{d,l}(\phi)^{-2\beta/(2-\beta)}\right)^{1-\beta/2}.$$

By the Funck-Hecke theorem, $\|J_\phi(F)\|_2^2 = \sum_{l \text{ even}} \alpha_{d,l}(\phi)^2 \|Q_l(F)\|_2^2$ and by the inequality (14),

$$\begin{aligned} \sum_{l \text{ even}} \|Q_l(F)\|_2^2 \alpha_{d,l}(\phi)^{-2/(p+d/2)} &\leq c^2 \sum_{l \text{ even}} \max(d^2, l^2)\|Q_l(F)\|_2^2 \\ \leq c^2 \left(d^2 \sum_{l \text{ even}, l \leq d} \|Q_l(F)\|_2^2 + \sum_{l \text{ even}, l \geq d} l^2 \|Q_l(F)\|_2^2\right) &\leq c^2 \left(d^2 \|F\|_2^2 + \|\nabla_0(F)\|_2^2\right). \end{aligned}$$

This proves that $\|F\|_2 \leq c\|J_\phi(F)\|_2^{2/(d+2p+2)} \left(\|\nabla_0 F\|_2^2 + d^2\|F\|_2^2\right)^{\frac{1}{2}(1-2/(d+2p+2))}$. □

We will also need the following simple lemma.

Lemma 5. *Let $F : S^{d-1} \to \mathbb{R}$ be a Lipschitz function and let $M = \max(\|F\|_2, \|F\|_{\text{Lip}})$ then*

$$\|F\|_\infty \leq 5 M^{(d-1)/(d+1)} \|F\|_2^{2/(d+1)}.$$

Proof. Let $u \in S^{d-1}$ such that $|F(u)| = \|F\|_\infty$ and let $C(u, R)$ be the spherical cap of radius R centered at u. For any $\delta \geq 1$, define $A_\delta = \{v \in S^{d-1}, |F(v)| \leq \delta \|F\|_2\}$ then by the Chebychev inequality, $\sigma(A_\delta) \geq 1 - 1/\delta^2$. For any $R \in (0, 2)$, it is well known that $\sigma(C(u, R)) \geq \frac{1}{2}\left(\frac{R}{2}\right)^{d-1}$. If R is chosen such that $\frac{1}{2}\left(\frac{R}{2}\right)^{d-1} = \frac{1}{\delta^2}$ then $A_\delta \cap C(u, R) \neq \emptyset$. In that case, take $v \in A_\delta \cap C(u, R)$ then

$$|F(u)| \leq |F(u) - F(v)| + |F(v)| \leq \|F\|_{\text{Lip}} |u - v|_2 + \delta \|F\|_2 \leq RM + \delta \|F\|_2.$$

Since $R = 2(2/\delta^2)^{1/(d-1)}$, we get the estimate taking $\delta = (M/\|F\|_2)^{(d-1)/(d+1)} \geq 1$. $\square$

Proof of Theorem 3. Using the support functions, $d(Z_p(\widetilde{K}), Z_p(\widetilde{D})) \leq 1+\varepsilon$ implies that there exists $\gamma > 0$ such that

$$\gamma h_{Z_p(\widetilde{D})} \leq h_{Z_p(\widetilde{K})} \leq (1+\varepsilon)\gamma h_{Z_p(\widetilde{D})}. \tag{15}$$

For any symmetric convex body $L \subset \mathbb{R}^d$, by integration in polar coordinates,

$$h_{Z_p(L)}(u)^p = \int_L |\langle x, u\rangle|^p dx = \frac{d\omega_d}{d+p}\int_{S^{d-1}} |\langle v, u\rangle|^p \frac{1}{\|v\|_L^{d+p}} d\sigma(v)$$

hence applying it for $L = \widetilde{K}$ and $L = \widetilde{D}$, we get for any $u \in S^{d-1}$,

$$\left| \int_{S^{d-1}} |\langle v, u\rangle|^p \left(\frac{1}{\|v\|^{d+p}} - \frac{\gamma^p}{\omega_d^{1+p/d}} \right) d\sigma(v) \right| \leq ((1+\varepsilon)^p - 1)\frac{\gamma^p}{\omega_d^{1+p/d}} \int_{S^{d-1}} |\langle v, u\rangle|^p d\sigma(v)$$

where $\|\cdot\|$ is the norm with unit ball $\widetilde{K}$. For every $u \in S^{d-1}$, let $F(u) = \frac{\omega_d^{1+p/d}}{\gamma^p \|u\|^{p+d}} - 1$. Since $\forall u \in S^{d-1}$, $\int_{S^{d-1}} |\langle v, u\rangle|^p d\sigma(v) \leq 1$, we get

$$\|J_\phi(F)\|_2 \leq \|J_\phi(F)\|_\infty \leq ((1+\varepsilon)^p - 1). \tag{16}$$

Since $d(K, D) \leq \alpha$, there exits $a, b > 1$ such that $1/a\widetilde{D} \subset \widetilde{K} \subset b\widetilde{D}$ and $ab = \alpha$.

$$\forall y \in S^{d-1}, \gamma^p(1+\varepsilon)^p h^p_{Z_p(\widetilde{D})}(y) \geq h^p_{Z_p(\widetilde{K})}(y) \geq \int_{\widetilde{D}/a} |\langle x, y\rangle|^p dx = h^p_{Z_p(\widetilde{D})}(y)/a^{d+p}$$

therefore $1/\gamma^p \leq a^{d+p}(1+\varepsilon)^p$. For any $x \in \mathbb{R}^d$, $\omega_d^{1/d} b^{-1}|x|_2 \leq \|x\| \leq a\omega_d^{1/d}|x|_2$ and for $u \in S^{d-1}$, $\nabla F^\vee(u) = \frac{\omega_d^{1+p/d}}{\gamma^p} \frac{(p+d)}{\|u\|^{p+d}} \left(u - \frac{\nabla\|\cdot\|(u)}{\|u\|} \right)$, therefore

$$\|\nabla_0 F\|_2 \leq \|\nabla_0 F\|_\infty \leq \frac{(p+d)b^{p+d}}{\gamma^p}(1+ab) \leq 4\ d\ \alpha^{d+p+1}(1+\varepsilon)^p. \tag{17}$$

We also have $\|F\|_2 \leq \|F\|_\infty \leq 1 + b^{p+d}/\gamma^p \leq 2\alpha^{p+d}(1+\varepsilon)^p$. Using Proposition 4 with (16) and (17), we get

$$\|F\|_2 \leq c((1+\varepsilon)^p - 1)^{2/(d+2p+2)}(6\ d\ \alpha^{d+p+1}(1+\varepsilon)^p)^{1-2/(d+2p+2)} \leq c\varepsilon^{2/(d+2p+2)}(4\alpha)^{d+p+1}.$$

Moreover, for any $u, v \in S^{d-1}$, $F(u) - F(v) = \omega_d^{1+p/d}/\gamma^p \left(1/\|u\|^{p+d} - 1/\|v\|^{p+d}\right)$ and

$$|F(u) - F(v)| \leq \frac{\omega_d^{1+p/d}}{\gamma^p}\|u - v\| \sum_{i=0}^{d+p-1} \|u\|^{-(d+p-i)}\|v\|^{-(i+1)} \leq 2\ d\ \alpha^{d+p+1}(1+\varepsilon)^p|u - v|_2.$$

Therefore $\max(\|F\|_2, \|F\|_{\mathrm{Lip}}) \leq (4\alpha)^{d+p+1}$ and by Lemma 5,

$$\|F\|_\infty \leq c(4\alpha)^{d+p+1}\varepsilon^{4/(d+1)(d+2p+2)} \leq c(4\alpha)^{d+p+1}\varepsilon^{1/d^2} := f(\varepsilon).$$

Recalling the definition of F, $F(u) = -1 + \omega_d^{1+p/d}/\gamma^p\|u\|^{p+d}$, $\forall u \in S^{d-1}$, we have proved

$$(1 - f(\varepsilon))^{1/d+p}\gamma^{p/d+p}\widetilde{D} \subset \widetilde{K} \subset (1 + f(\varepsilon))^{1/d+p}\gamma^{p/d+p}\widetilde{D}. \tag{18}$$

Since $|\widetilde{K}| = |\widetilde{D}| = 1$, $(1 + f(\varepsilon))^{-1} \leq \gamma^p \leq (1 - f(\varepsilon))^{-1}$ and choosing $\varepsilon \leq (c\alpha)^{-2d^3}$, (15) and (18) prove the assertions of Theorem 3. $\square$

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