On the isotropic constant of non-symmetric convex bodies

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Abstract

We show that Bourgain's estimate $L_K \leq c\sqrt[4]{n}\log n$ for the isotropic constant holds true for non-symmetric convex bodies as well.

1 Introduction

Let K be a convex body in $\mathbb{R}^n$ with volume $|K| = 1$. Then, K is called *isotropic* if there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1}$$

for every $\theta \in S^{n-1}$. It is not hard to check (see [MP] for the origin symmetric case) that every convex body has an isotropic image under $GL(n)$. Moreover, this *isotropic position* is uniquely determined up to orthogonal transformations, hence the *isotropic constant* L_K is an invariant for the class $\{TK : T \in GL(n)\}$.

An important problem asks if there exists an absolute constant $C > 0$ such that $L_K \leq C$ for every isotropic convex body K with centroid at the origin o. This question has many equivalent reformulations: Let us mention the hyperplane problem which asks if every convex body of volume 1 has a hyperplane section through its centroid with "area" greater than an absolute constant.

Bourgain [B] has shown that $L_K \leq c\sqrt[4]{n}\log n$ for every origin symmetric isotropic convex body K in $\mathbb{R}^n$. This is the best known general estimate for the isotropic constant. Dar [D1] proved that $L_K \leq c'\sqrt{n}$ for every convex body with centroid at the origin. The purpose of this note is to extend Bourgain's estimate to non-symmetric isotropic bodies:

Theorem. *If K is an isotropic convex body in $\mathbb{R}^n$, then $L_K \leq c\sqrt[4]{n}\log n$.*

We shall actually follow Bourgain's argument, as presented in [D2]. We will make no assumption about the origin.

2 Proof of the Theorem

In what follows, K is an isotropic, not necessarily symmetric convex body in $\mathbb{R}^n$. The letters c, c', c_1, c_2 etc. will denote absolute positive constants.
Observe that (1) is equivalent to

$$\int_K \langle x, Tx\rangle dx = (\mathrm{tr}T)L_K^2 \tag{2}$$

for every $T \in L(\mathbb{R}^n)$. In particular, if $T \in SL(n)$ is symmetric and positive, the arithmetic-geometric means inequality gives $nL_K^2 \leq (\mathrm{tr}T)L_K^2$, which implies the following:

Lemma 1. *For every symmetric and positive $T \in SL(n)$ we have*

$$nL_K^2 \leq \int_K \langle x, Tx\rangle dx. \quad \square \tag{3}$$

Lemma 2. *For every $\theta \in S^{n-1}$,*

$$\int_K \exp\left(\frac{|\langle x,\theta\rangle|}{c_1 L_K}\right) dx \leq 2. \tag{4}$$

Proof: This is a consequence of Borell's lemma (see [MS], Appendix III): There exists $c_2 > 0$ such that

$$\left(\int_K |\langle x,\theta\rangle|^p dx\right)^{1/p} \leq c_2 p \int_K |\langle x,\theta\rangle| dx \tag{5}$$

for every $p \geq 1$ and $\theta \in S^{n-1}$. If K is isotropic, then $\int_K |\langle x,\theta\rangle| dx \leq L_K$ for every $\theta \in S^{n-1}$, and the Lemma follows from (5). $\square$

If V is a convex body in $\mathbb{R}^n$, the *mean width* $w(V)$ of V is the quantity

$$w(V) = \int_{S^{n-1}} \left\{\max_{z\in V}\langle z,\theta\rangle - \min_{z\in V}\langle z,\theta\rangle\right\} \sigma(d\theta),$$

where σ is the rotationally invariant probability measure on S^{n-1}. Well-known results from [L], [FT] and [P] show that for every symmetric convex body V in $\mathbb{R}^n$ there exists $T \in SL(n)$ for which

$$w(TV)w((TV)^\circ) \leq c_3 \log n, \tag{6}$$

where $(TV)^\circ$ is the polar body of TV. We will need the following extension to the non-symmetric case:

Lemma 3. *Let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. There exists a symmetric and positive $T \in SL(n)$ such that*

$$w(TK) \leq 2c_3\sqrt{n}\log n. \tag{7}$$

Proof: Consider the difference body $V = K - K$ of K. Then, we can find $T \in SL(n)$ such that $w(TV)w((TV)^\circ) \leq c_3 \log n$. Since mean width is invariant under orthogonal transformations, we may clearly assume that T is symmetric and positive. Now, if $\|\cdot\|$ is the norm induced to $\mathbb{R}^n$ by TV,

$$\begin{aligned} w((TV)^\circ) &= \int_{S^{n-1}} \|\theta\| \sigma(d\theta) \\ &\geq \left(\int_{S^{n-1}} \|\theta\|^{-n} \sigma(d\theta) \right)^{-1/n} = \left(\frac{|D_n|}{|TV|} \right)^{1/n}, \end{aligned}$$

where D_n is the Euclidean unit ball. Hence,

$$w(TV) \leq c_3 \left(\frac{|TV|}{|D_n|} \right)^{1/n} \log n \leq c_3 \sqrt{n} |TV|^{1/n} \log n. \tag{8}$$

Observe that $TV = T(K - K) = TK - TK$. From the Rogers-Shephard inequality [RS] we have $|TV| \leq \binom{2n}{n} |TK| \leq 4^n$. Hence,

$$w(TV) \leq 4c_3 \sqrt{n} \log n. \tag{9}$$

Finally,

$$\begin{aligned} w(TV) &= 2 \int_{S^{n-1}} \max_{z \in TK - TK} \langle z, \theta \rangle \sigma(d\theta) = 2 \int_{S^{n-1}} \left\{ \max_{z \in TK} \langle z, \theta \rangle - \min_{z \in TK} \langle z, \theta \rangle \right\} \sigma(d\theta) \\ &= 2w(TK). \end{aligned}$$

This shows that $w(TK) \leq 2c_3 \sqrt{n} \log n$. $\square$

The last ingredient of the proof is the Dudley-Fernique decomposition of a convex body A:

Lemma 4. *Let $A \subseteq RD_n$ be a convex body in $\mathbb{R}^n$, where $R > 0$. There exist finite sets $Z_j \subset \mathbb{R}^n$, $j \in \mathbb{N}$ with*

$$\log |Z_j| \leq c_5 n \left(\frac{2^j w(A)}{R} \right)^2,$$

which satisfy the following: For every $x \in A$ and every $m \in \mathbb{N}$ we can find $z_j \in Z_j \cap (3R/2^j) D_n$, $j = 1, \ldots, m$ and $w_m \in (R/2^m) D_n$ such that

$$x = z_1 + \ldots + z_m + w_m.$$

Proof: Recall that the covering number $N(A, tD_n)$ is the smallest integer N for which there exist N translates of tD_n whose union covers A. Using Sudakov's inequality [S] we see that

$$\log N(A, tD_n) \leq \log N(A - A, tD_n) \leq c_4 n \left(\frac{w(A - A)}{t} \right)^2 = 4c_4 n \left(\frac{w(A)}{t} \right)^2.$$

For every $j \in \mathbb{N}$ we find $N_j \subset \mathbb{R}^n$ with $|N_j| = N(A, (R/2^j)D_n)$ such that $A \subset \cup_{y \in N_j}(y + (R/2^j)D_n)$, and set $Z_j = N_j - N_{j-1}$, $j \geq 1$ (and $N_0 = \{o\}$). If $x \in A$ and $m \in \mathbb{N}$, for every $j \leq m$ there exists $y_j \in N_j$ such that $|x - y_j| \leq R/2^j$. We write

$$x = y_1 + (y_2 - y_1) + \ldots + (y_m - y_{m-1}) + (x - y_m),$$

and conclude the proof with $z_j = y_j - y_{j-1}$ and $w_m = x - y_m$. $\square$

Proof of the Theorem: Let K be an isotropic convex body. By Lemma 3, there exists a symmetric and positive $T \in SL(n)$ such that $w(TK) \leq 2c_3\sqrt{n}\log n$. Lemma 1 shows that

$$nL_K^2 \leq \int_K \langle x, Tx\rangle dx \leq \int_K \max_{z \in TK} |\langle z, x\rangle| dx. \tag{10}$$

Let $A = TK$ in Lemma 4, and consider the sets Z_j, $j \in \mathbb{N}$. Then, for every $x \in K$,

$$\begin{aligned}\max_{z \in TK} |\langle z, x\rangle| &\leq \sum_{j=1}^{m} \max_{z \in Z_j \cap (3R/2^j)D_n} |\langle z, x\rangle| + \max_{w \in (R/2^m)D_n} |\langle w, x\rangle| \\ &\leq \sum_{j=1}^{m} \frac{3R}{2^j} \max_{z \in Z_j \cap (3R/2^j)D_n} |\langle \overline{z}, x\rangle| + \frac{R}{2^m}|x|,\end{aligned}$$

where $\overline{z}$ is the unit vector parallel to z. Using the above and taking into account the fact that $\int_K |x| dx \leq \sqrt{n}L_K$, we see that

$$nL_K^2 \leq \sum_{j=1}^{m} \frac{3R}{2^j} \int_K \max_{z \in Z_j} |\langle \overline{z}, x\rangle| dx + \frac{R}{2^m}\sqrt{n}L_K. \tag{11}$$

Now, Lemma 2 shows that for every $t > 0$

$$\text{Prob}\left(x \in K : \max_{z \in Z_j} |\langle \overline{z}, x\rangle| \geq t\right) \leq 2|Z_j| \exp(-t/c_1 L_K),$$

and this implies that

$$\int_K \max_{z \in Z_j} |\langle \overline{z}, x\rangle| dx \leq c_6 L_K \log |Z_j| \leq c_7 n L_K \left(\frac{w(TK)2^j}{R}\right)^2.$$

Inserting this information into (11) we see that

$$nL_K^2 \leq c_8 L_K \left(n w^2(TK)\frac{2^m}{R} + \sqrt{n}\frac{R}{2^m}\right).$$

Choosing $m \in \mathbb{N}$ such that $R/2^m \simeq \sqrt[4]{n}\, w(TK)$, we get

$$nL_K^2 \leq c_9 n^{\frac{3}{4}} w(TK) L_K,$$

and the estimate $w(TK) \leq 2c_3\sqrt{n}\log n$ completes the proof. $\square$

Remark. If K is isotropic and has its centroid at the origin, then

$$L_K \simeq \int_K |\langle x, \theta\rangle| dx \simeq |K \cap \theta^{\perp}|^{-1}$$

for every $\theta \in S^{n-1}$ (see [F] for precise estimates). Therefore, in this case, the Theorem implies that all hyperplane sections of K through the origin have "area" greater than $1/c\sqrt[4]{n}\log n$, where $c > 0$ is an absolute constant.

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