Polynomial estimates towards a sharp Helly-type theorem for the diameter of convex sets

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Abstract

We discuss a problem posed by Bárány, Katchalski and Pach: if $\{P_i : i \in I\}$ is a family of closed convex sets in $\mathbb{R}^n$ such that $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$ then there exist $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that

$$\mathrm{diam}\,(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant C_n,$$

where $C_n \leqslant c\sqrt{n}$ for an absolute constant $c > 0$. We prove that this statement holds true with $C_n \leqslant cn^{11/2}$. All the previously known estimates for C_n were exponential or superexponential in the dimension n.

1 Introduction

In this note we provide a polynomial estimate for a question of Bárány, Katchalski and Pach on the quantitative version of Helly's theorem for the diameter of convex sets in Euclidean space. Helly's theorem states that, if $\mathcal{P} = \{P_i : i \in I\}$ is a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$ and if every $n+1$ or fewer members of $\mathcal{P}$ have non-empty intersection, then $\bigcap_{i\in I} P_i \neq \emptyset$. This classical result and its variants have found important applications in discrete and computational geometry (see e.g. [9], [10] and [1]).

Bárány, Katchalski and Pach obtained in [3] a quantitative version of Helly's theorem for the diameter:

Theorem 1.1. *Let $\{P_i : i \in I\}$ be a family of closed convex sets in $\mathbb{R}^n$ such that $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$. There exist $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that*

$$\mathrm{diam}\,(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant C_n, \tag{1.1}$$

where $C_n > 0$ is a constant depending only on the dimension.

The example of the cube $[-1,1]^n$ in $\mathbb{R}^n$, expressed as an intersection of exactly $2n$ closed half-spaces, shows that one cannot replace $2n$ by $2n-1$ in the statement above. The optimal growth of the constant C_n as a function of n is not completely understood. In [3] the authors established the bound $C_n \leqslant (cn)^{n/2}$ and conjectured that the bound should be polynomial in n; in fact they asked if $(cn)^{n/2}$ can be replaced by $c\sqrt{n}$.

In [8] we proved that there exists an absolute constant $\alpha > 2$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$, then there exist $s \leqslant \alpha n$ and $i_1, \dots, i_s \in I$ such that

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant cn^{3/2}, \tag{1.2}$$

where $c > 0$ is an absolute constant. Note that the estimate is polynomial in the dimension but the restriction $s \leqslant 2n$ is relaxed to $s \leqslant \alpha n$ for some absolute constant $\alpha > 2$. In this note we consider the original question of Bárány, Katchalski and Pach, and provide a polynomial estimate.

Theorem 1.2. *Let $\{P_i : i \in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$. We can find $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that*

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant cn^{11/2}, \tag{1.3}$$

where $c > 0$ is an absolute constant.

All the previously known estimates for the question were exponential or superexponential in the dimension. The main step for the proof of Theorem 1.2 is a Helly-type inclusion theorem.

Theorem 1.3. *Let $\{P_i : i \in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{int}\left(\bigcap_{i\in I} P_i\right) \neq \emptyset$. *For any $k > n$ there exist $z \in \mathbb{R}^n$, $s \leqslant k + n$ and $i_1, \ldots, i_s \in I$ such that*

$$z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_{k,n} n(n+2) \left(z + \bigcap_{i\in I} P_i \right), \tag{1.4}$$

where $\gamma_{k,n} = \left(\frac{\sqrt{k}+\sqrt{n}}{\sqrt{k}-\sqrt{n}}\right)^2$.

A main tool for the proof of Theorem 1.3 is the following theorem of Batson, Spielman and Srivastava [6]: If $v_1, \ldots, v_m \in S^{n-1}$ and $a_1, \ldots, a_m > 0$ satisfy "John's decomposition of the identity" $I_n = \sum_{j=1}^m a_j v_j \otimes v_j$, where $(v_j \otimes v_j)(y) = \langle v_j, y\rangle v_j$, then for every $d > 1$ there exists a subset $\sigma \subseteq \{1, \ldots, m\}$ with $|\sigma| \leqslant dn$ and $b_j > 0$, $j \in \sigma$, such that $I_n \preceq \sum_{j\in\sigma} b_j a_j v_j \otimes v_j \preceq \gamma_d^2 I_n$, where $\gamma_d := \frac{\sqrt{d}+1}{\sqrt{d}-1}$.

It is clear that if we apply Theorem 1.3 with $k = n + 1$ then we obtain polynomial estimates (of order $O(n^4)$) for the diameter with $s \leqslant 2n + 1$. In order to reduce the number of the bodies P_{i_j} from $2n + 1$ to $2n$, and get the precise statement of Theorem 1.2, we use the idea of a lemma from [3] (see Lemma 2.3 in the next section).

In a different direction, Soberón proved in [15] that for any finite family of convex sets in $\mathbb{R}^n$ with the property that the intersection of every $2n$ of them has diameter at least 1, one can partition the family into a fixed number of subfamilies (depending only on n and $\varepsilon > 0$), each having an intersection with diameter at least $1 - \varepsilon$.

Closing this introductory section we mention that Bárány, Katchalski and Pach in [3] obtained also a quantitative Helly-type result for volume (see also [4]). They proved that if $\{P_i : i \in I\}$ is a family of closed convex sets in $\mathbb{R}^n$ such that $\left|\bigcap_{i\in I} P_i\right| > 0$ then we may find $s \leqslant 2n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant D_n \left|\bigcap_{i\in I} P_i\right|,$$

where $D_n > 0$ is a constant depending only on n. The bound in [3] was $O(n^{2n^2})$ and it was conjectured that one might actually have $D_n \leqslant n^{cn}$ for an absolute constant $c > 0$. Naszódi [13] has recently proved a volume version of Helly's theorem with $D_n \leqslant (cn)^{2n}$, where $c > 0$ is an absolute constant. In fact, a slight modification of Naszódi's argument leads to the exponent $\frac{3n}{2}$ instead of $2n$. In [7], relaxing the requirement that $s \leqslant 2n$ to the weaker one that $s = O(n)$, we showed that there exists an absolute constant $\alpha > 2$ with the following property: for every family $\{P_i : i \in I\}$ of closed convex sets in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume, there exist $s \leqslant \alpha n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (cn)^n\, |P|, \tag{1.5}$$

where $c > 0$ is an absolute constant.

Notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean inner product $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$ and the circumradius of K is the radius of the smallest ball which is centered at the origin and contains K:

$$R(K) = \max\{\|x\|_2 : x \in K\}.$$

Finally, given two symmetric positive definite matrices A and B we write $A \preceq B$ if $\langle Ax, x\rangle \leqslant \langle Bx, x\rangle$ for all $x \in \mathbb{R}^n$. We refer to the books [14] and [2] for basic facts from convex geometry.

2 Proof of the theorem

The proof of Theorem 1.3 is based on an extension to the non-symmetric case of the following fact, obtained by Gluskin-Litvak in [11] and Barvinok in [5]: If K is a symmetric convex body in $\mathbb{R}^n$ then for any $k > n$ there exist $N \leqslant k$ points $x_1, \dots, x_N \in K$ such that

$$\text{absconv}(\{x_1, \dots, x_N\}) \subseteq K \subseteq \gamma_{k,n}\sqrt{n}\,\text{absconv}(\{x_1, \dots, x_N\}). \tag{2.1}$$

We shall first prove the next proposition, which is a variant of a result from [8].

Proposition 2.1. *Let K be a convex body in $\mathbb{R}^n$, such that the ellipsoid of minimal volume containing K is the Euclidean unit ball B_2^n. For every $k > n$ there is a subset $X \subseteq K \cap S^{n-1}$ of cardinality* $\text{card}(X) \leqslant k + n$ *such that*

$$K \subseteq B_2^n \subseteq \left(\frac{\sqrt{k}+\sqrt{n}}{\sqrt{k}-\sqrt{n}}\right)^2 n(n+2)\,\text{conv}(X). \tag{2.2}$$

Proof. Since B_2^n is the minimal volume ellipsoid of K, by John's theorem [12] we may find $v_j \in K \cap S^{n-1}$ and $a_j > 0$, $j \in J$, such that

$$I_n = \sum_{j\in J} a_j v_j \otimes v_j \quad \text{and} \quad \sum_{j \in J} a_j v_j = 0. \tag{2.3}$$

It is well-known that (2.3) implies that

$$\text{conv}\{v_1, \dots, v_m\} \supseteq \frac{1}{n} B_2^n. \tag{2.4}$$

Set $d = k/n > 1$ and $\gamma_{k,n} := \gamma_d = \left(\frac{\sqrt{k}+\sqrt{n}}{\sqrt{k}-\sqrt{n}}\right)^2$ and apply the theorem of Batson, Spielman and Srivastava to find a subset $\sigma \subseteq J$ with $|\sigma| \leqslant k$ and positive scalars b_j, $j \in \sigma$, such that $T := \sum_{j\in\sigma} b_j v_j \otimes v_j$ satisfies

$$I_n \preceq \sum_{j\in\sigma} b_j v_j \otimes v_j \preceq \gamma_{k,n} I_n.$$

Taking traces we see that

$$b := \sum_{j\in\sigma} b_j \leqslant \gamma_{k,n} n.$$

Note that the vector $w = -\frac{1}{bn}\sum_{j\in\sigma} b_j v_j$ has length $\|w\|_2 \leqslant \frac{1}{bn}\sum_{j\in\sigma} b_j = \frac{1}{n}$, and hence $w \in \text{conv}\{v_j, j \in J\}$ by (2.4). Therefore, we may find $\kappa \geqslant 1$ such that κw belongs to some facet of $\text{conv}\{v_j, j \in J\}$. Then, we apply Carathéodory's theorem to find $\tau \subseteq J$ with $|\tau| \leqslant n$ and $\rho_i > 0$, $i \in \tau$ such that

$$\kappa w = \sum_{i\in\tau} \rho_i v_i \quad \text{and} \quad \sum_{i\in\tau} \rho_i = 1. \tag{2.5}$$

We will show that

$$C := \text{conv}(\{v_j : j \in \sigma \cup \tau\}) \supseteq \frac{1}{\gamma_{k,n} n(n+2)} B_2^n. \tag{2.6}$$

Recall that the Minkowski functional of C, defined by $p_C(y) = \min\{t \geqslant 0 : y \in tC\}$, is subadditive and positively homogeneous. Given $x \in S^{n-1}$ we set $\delta = \min\{\langle x, v_j\rangle : j \in \sigma\}$ and observe that $|\delta| \leqslant 1$ and $\langle x, v_j\rangle - \delta \leqslant 2$ for all $j \in \sigma$. If $\delta < 0$, we write

$$\begin{aligned} p_C(T(x)) &\leqslant p_C\left(T(x) - \delta\sum_{j\in\sigma} b_j v_j\right) + p_C\left(\delta \sum_{j\in\sigma} b_j v_j\right) = p_C\left(\sum_{j\in\sigma} b_j(\langle x, v_j\rangle - \delta) v_j\right) + p_C\left(|\delta| bnw\right) \\ &\leqslant \sum_{j\in\sigma} b_j(\langle x, v_j\rangle - \delta) p_C(v_j) + |\delta| bn p_C(w). \end{aligned}$$

Since $p_C(v_j) \leqslant 1$ and $\langle x, v_j\rangle - \delta \leqslant 2$ for all $j \in \sigma$, we see that $\sum_{j\in\sigma} b_j(\langle x, v_j\rangle - \delta)p_C(v_j) \leqslant 2\sum_{j\in\sigma} b_j = 2b$. Since $w \in C$ we have $p_C(w) \leqslant 1$ and we also have $|\delta|bnp_K(w) \leqslant bn$. Therefore, if $\delta < 0$ then we finally get

$$p_C(T(x)) \leqslant 2b + bn = b(n+2) \leqslant \gamma_{k,n} n(n+2).$$

If $\delta \geqslant 0$ then $\langle x, v_j\rangle \geqslant 0$ for all $j \in \sigma$, therefore

$$p_C(T(x)) = p_C\left(\sum_{j\in\sigma} b_j\langle x, v_j\rangle v_j\right) \leqslant \sum_{j\in\sigma} b_j\langle x, v_j\rangle p_C(v_j) \leqslant \sum_{j\in\sigma} b_j \leqslant \gamma_{k,n} n. \tag{2.7}$$

In any case,

$$p_{T^{-1}(C)}(x) \leqslant \gamma_{k,n} n(n+2) p_{B_2^n}(x) \tag{2.8}$$

for all $x \in S^{n-1}$. Since $I_n \preceq T$, we also have $B_2^n \subseteq T(B_2^n)$, and hence

$$K \subseteq B_2^n \subseteq T(B_2^n) \subseteq \gamma_{k,n} n(n+2)C. \tag{2.9}$$

Since $\mathrm{card}(\sigma \cup \tau) \leqslant k + n$, the proof is complete. □

Theorem 2.2. *Let $\{P_i : i \in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{int}\left(\bigcap_{i\in I} P_i\right) \neq \emptyset$. For any $k > n$ there exist $z \in \mathbb{R}^n$, $s \leqslant k + n$ and $i_1, \dots i_s \in I$ such that*

$$z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_{k,n} n(n+2)\left(z + \bigcap_{i\in I} P_i\right), \tag{2.10}$$

In particular, assuming that $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$ we get that for every $k > n$ there exist $s \leqslant k + n$ and $i_1, \dots i_s \in I$ such that

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant \gamma_{k,n} n(n+2). \tag{2.11}$$

Therefore, if we choose $k = n + 1$, we get that there exist $s \leqslant 2n + 1$ and $i_1, \dots i_s \in I$ such that

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant 16n(n+2)(n+1)^2. \tag{2.12}$$

Proof. Let $P = \bigcap_{i\in I} P_i$. We may assume that $0 \in \mathrm{int}(P)$ and that the minimal volume ellipsoid of the polar body

$$P^\circ = \mathrm{conv}\left(\bigcup_{i\in I} P_i^\circ\right) \tag{2.13}$$

of P is the Euclidean unit ball. Using Proposition 2.1 for $K = P^\circ$ we may find $X = \{v_1, \dots, v_s\} \subset P^\circ \cap S^{n-1}$ with $\mathrm{card}(X) = s \leqslant k + n$ such that

$$P^\circ \subseteq \gamma_{k,n} n(n+2)\mathrm{conv}(\{v_1, \dots, v_s\}). \tag{2.14}$$

Since $v_1, \dots, v_s$ are contact points of P° with B_2^n, we can easily check that we actually have $v_j \in \bigcup_{i\in I} P_i^\circ$ for all $j = 1, \dots, s$. In other words, we may find $i_1, \dots, i_s \in I$ such that $v_j \in P_{i_j}^\circ$, $j = 1, \dots, s$. Then, (2.14) implies that

$$P^\circ \subseteq \gamma_{k,n} n(n+2)\mathrm{conv}(P_{i_1}^\circ \cup \cdots \cup P_{i_s}^\circ), \tag{2.15}$$

and passing to the polar bodies, we get

$$P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_{k,n} n(n+2)P \tag{2.16}$$

as claimed. Since $\gamma_{n+1,n} = (\sqrt{n+1} + \sqrt{n})^4 \leqslant 16(n+1)^2$, the proof is complete. □

For the final step of the proof of Theorem 1.2 we use the idea of a lemma from [3] which will allow us to further reduce the number of the bodies P_{i_j} from $2n+1$ to $2n$. We include a sketch of its proof for the reader's convenience.

Lemma 2.3. *Let $P_1,\dots,P_{2n+1}$ be convex bodies in $\mathbb{R}^n$ such that $0\in P_1\cap\cdots\cap P_{2n+1}$. If the circumradius of $P_1\cap\cdots\cap P_{2n+1}$ is equal to R then we can find $1\leqslant j\leqslant 2n+1$ such that the circumradius of $\bigcap_{i=1,i\neq j}^{2n+1}P_i$ is at most R/t_n, where $t_n=\sin((2n^{3/2})^{-1})\geqslant\frac{1}{\pi n^{3/2}}$.*

Proof. If C is a spherical cap such that $\mathrm{dist}(0,\mathrm{conv}(C))=t$ then we can write it as a geodesic ball $C=B(v,\pi/2-\delta)$ (for some $v\in S^{n-1}$) where $t=\sin\delta$. Then,

$$\sigma(C)=\sigma(B(v,\delta))=\frac{1}{2I_{n-1}}\int_0^{\frac{\pi}{2}-\delta}(\sin\theta)^{n-1}d\theta,$$

where σ is the standard rotationally invariant probability measure on the sphere and $I_k=\int_0^{\pi/2}(\cos\theta)^k d\theta$ (see e.g. [2, Chapter 3]). Therefore, we will have $\sigma(C)>\frac{n}{2n+1}$ if

$$\frac{1}{I_{n-1}}\int_0^{\frac{\pi}{2}-\delta}(\sin\theta)^{n-1}d\theta=\frac{1}{I_{n-1}}\int_\delta^{\pi/2}(\cos u)^{n-1}du>\frac{2n}{2n+1},$$

or equivalently

$$\frac{1}{I_{n-1}}\int_0^{\delta}(\cos u)^{n-1}du<\frac{1}{2n+1}.$$

It is known $\sqrt{k}I_k\geqslant 1$ for all $k\geqslant 1$ and we trivially have $\cos u\leqslant 1$ for all $u\in[0,\delta]$. If we choose $\delta_n=\frac{1}{2n^{3/2}}$ then we get

$$\int_0^{\delta_n}(\cos u)^{n-1}du\leqslant\delta_n=\frac{1}{2n^{3/2}}<\frac{1}{(2n+1)\sqrt{n-1}}\leqslant I_{n-1}\cdot\frac{1}{2n+1}.$$

Therefore, if $\mathrm{dist}(0,\mathrm{conv}(C))=t_n=\sin\delta_n$ we have that

$$\sigma(C)>\frac{n}{2n+1}.$$

We assume that for any $1\leqslant j\leqslant 2n+1$ the circumradius of $\bigcap_{i=1,i\neq j}^{2n+1}P_i$ is greater than 1 and we will show that the circumradius of $P_1\cap\cdots\cap P_{2n+1}$ is greater than t_n. We can choose $y_j\in\bigcap_{i=1,i\neq j}^{2n+1}P_i$ with $\|y_j\|_2=1$ and then we consider the spherical cap C_j with center y_j and $\mathrm{dist}(0,\mathrm{conv}(C_j))=t_n$. We claim that there exists $v\in S^{n-1}$ which belongs to at least $n+1$ of the C_j's; otherwise, each point of S^{n-1} would be covered by at most n of the C_j's and this would imply that

$$n\geqslant\sum_{j=1}^{2n+1}\sigma(C_j)>(2n+1)\cdot\frac{n}{2n+1}=n,$$

a contradiction. Now, consider the spherical cap $C(v)$ with center v and $\mathrm{dist}(0,\mathrm{conv}(C(v)))=t_n$. We have at least $n+1$ of the y_j's in $C(v)$, and we may assume that $y_1,\dots,y_{n+1}\in C(v)$. Each line segment $[0,y_j]$, $j\leqslant n+1$, meets the bounding hyperplane H of $C(v)$ at some point $w_j\in\bigcap_{i=1,i\neq j}^{2n+1}P_i$. Applying Radon's theorem for the points $w_1,\dots,w_{n+1}$ in H, we find a point $u\in\bigcap_{j=1}^{n+1}\left(\bigcap_{i=1,i\neq j}^{2n+1}P_i\right)=P_1\cap\cdots\cap P_{2n+1}$. Since $u\in H$, we have $\|u\|_2\geqslant t_n$. $\square$

Now, let $\{P_i:i\in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I}P_i\right)=1$. We may assume that $0\in\bigcap_{i\in I}P_i$. First we apply Theorem 2.2 to find $s\leqslant 2n+1$ and $i_1,\dots i_s\in I$ such that

$$\mathrm{diam}(P_{i_1}\cap\cdots\cap P_{i_s})\leqslant c_1n^4, \tag{2.17}$$

where $c_1 > 0$ is an absolute constant. If $s \leqslant 2n$ then there is nothing to do, otherwise $s = 2n+1$ and then we apply Lemma 2.3 and keep $2n$ of the P_{i_j}'s so that the diameter of their intersection is bounded by

$$c_1 n^4 \cdot \pi n^{3/2} \leqslant c_2 n^{11/2}, \tag{2.18}$$

where $c_2 > 0$ is an absolute constant. This completes the proof of Theorem 1.2.

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