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Why Did Liu Hui Fail to Derive the Volume of a Sphere?*

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Using the problem of deriving the volume of a sphere as its central focus, this paper tries to show the importance of different "heuristics" in Liu Hui and Zu Geng's ideas and theories of geometry. Rather than dismissing Liu's failure as due to inadequate time or effort, it argues that this failure was inherent in Liu's own heuristic, a powerful pattern of reasoning that enabled Liu to solve many geometrical problems, but also restrained him from finding the volume of a sphere. Zu Geng's heuristic, on the other hand, revealed its strength in problems concerning the sphere, although this does not imply that it could cover a wider range of geometrical problems than Liu's approach. Thus, directly beyond the problems concerned with the sphere, the central purpose of this paper is to use Liu's and Zu's heuristics (or patterns of geometrical reasoning) as guidelines in reconstructing and elucidating at least part of the historical structure of ancient Chinese geometry. From this perspective, light can also be thrown upon the geometrical reasonings of later figures such as Wang Xiaotong, Shen Kuo, Mei Wending, Jiao Xun, Li Huang, and Xu Youren. Thus, instead of using the classification scheme of 20th-century Western mathematics, the present approach preserves the historical context of these respective heuristics while organizing the different historical trends of Chinese geometrical reasoning into interconnected and consistent patterns.

劉徽為甚么沒求出球體積？本文從這個較少問的問題出發，首先排除了一般隨意的解釋，企圖指出：刘徽的幾何之「術」，雖然在其他領域達到了相當的成就，卻使得刘徽極難處理球類問題。祖暅日後能夠求出球体積，似也并非天才的偶然。本文點出：雖然刘、祖二人均对「刘祖原理」有認識，祖暅的幾何之「術」相當不同於刘徽之術，故二人從不同的角度去看刘祖原理，使用的方式也不盡相同。如果我們能充分區别開刘祖二人的幾何之術，則中国數學史中幾何学結構的演变就可更清楚。從這兩種不同的術出發，我们对刘祖以降的中国学者處理幾何問題，如王孝通、沈括、梅文鼎、焦循、李潢、徐有壬等，可以有更深一層的認識。進一步，如果我們不單純以西方當代數學為衡量準則，透過刘祖二術兩條線的歷史分合與演化，对中国幾何學史的發展也可以有新的視野與觀點。

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Das zentrale Anliegen dieses Aufsatzes ist die Ermittlung des Kugelvolumens. Er versucht, die Bedeutung verschiedener Heuristiken in Liu Huis and Zu Gengs Ideen und Theorien der Geometrie aufzuzeigen. Statt unpassende Zeit oder unzureichende Anstrengungen als Gründe zu verwenden, behauptet er, Lius Mißerfolg sei durch dessen eigene Heuristik bedingt. Diese Heuristik ist ein machtvolles Denkmuster, das Liu befähigt, viele geometrische Probleme zu lösen, aber auch daran hindert, das Kugelvolumen zu finden. Zu Gengs Heuristik auf der anderen Seite ist bei Aufgaben zur Kugel wirkungsvoll. Aber dies bedeutet nicht, daß sie einen größeren Bereich an geometrischen Problemen abdeckt als Lius Heuristik. Der Aufsatz geht daher über die Kugelprobleme hinaus. Seine grundlegende Vorgehensweise besteht darin, Lius und Zus verschiedene Heuristiken (oder Muster geometrischen Denkens) als Leitlinien zu verwenden, um wenigstens einen Teil der historischen Struktur der altchinesischen Geometrie zu rekonstruieren und zu unterscheiden. Aus dieser Sichtweise können auch die geometrischen Denkweisen späterer Autoren wie Wang Xiaotong, Shen Kuo, Mei Wending, Jiao Xun, Li Huang, Xu Youren und anderer chinesischer Mathematiker aufgehellt werden. Statt die westliche Mathematik des 20. Jahrhunderts als Klassifikationsschemata zu verwenden, baut diese Sichtweise die verschiedenen historischen Trends des chinesischen geometrischen Denkens eher innerhalb ihrer historischen Kontexte und Heuristiken zu untereinander verbundenen, konsistenten Mustern auf.

AMS 1991 subject classification: 01A25, 51-03, 00A30.

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Recently, a number of authors have discussed Liu Hui's and/or Zu Geng's general theory of geometrical volume, providing focused accounts of the "successes" of Liu Hui's derivation of the volume of a *yang-ma* (a rectangular or square pyramid with one side perpendicular to its base, see Glossary) and Zu's derivation of the volume of a sphere. Interesting as these derivations may be, little effort has been made to give a serious, historical account of Liu Hui's failure to derive the volume of a sphere, a failure from which Zu's subsequent efforts began. Regrets or apologies for his "near miss" have been the usual explanations for the failure of Liu Hui, one of the greatest of ancient Chinese mathematicians. Enthusiastic praises, on the other hand, have repeatedly been sung for the early discovery of the famous Cavalieri theorem (or the "Liu–Zu principle") by Liu and Zu, a discovery supposedly antedating that of Western mathematicians by more than one thousand years.

Such assessments raise several questions regarding the comparative perspectives of Chinese and Western histories of mathematics. What, we may ask, is the historical significance of this supported triumphant lead? Is it just a matter of chronological order? What, we may ask further, is the proper standard for comparing achievements in the history of science? Sometimes (in histories of mathematics) we simply regard the theorem *statement* itself and the temporal occasion of its enunciation as the standard for comparison. It seems to me, however, that whether a theorem is historically "important" or not depends on whether it generated fruitful subsequent developments after it was first discovered. Taking only the theorem statement itself as our standard automatically isolates the result from its historical context and significance. Thus, although Cavalieri's theorem was explicitly stated and proved in the 17th century, the idea was already implicitly assumed and used by some Greek mathematicians [1]. What, then, is the "exact" time of its discovery in the history of Western mathematics? If we choose the Greek period, then instead of a "triumphant lead," the discovery of the Liu–Zu principle (presented without proof) may actually "fall behind" the West. Depending on our "choice," we could assert that either East or West got the credit of "first discovery."

However, it seems to me that the important issue for historians is not to decide questions such as "who got it first," or related questions such as "why who got it first." In any event, we have spent more than enough time and energy on this kind of problematic. Rather, we should pay more attention to the historical significance and fruitfulness of a "discovery" *within* its own historical context. Perhaps we should pay more attention to the *research traditions* in which mathematical discoveries were embedded, and if we wish to compare histories of science, it would be far better to use the research tradition as the proper standard for comparison. Thus, in comparative perspective, our focus should be placed on two distinct historical contexts: one being the tradition of "indivisibles" [2] in 17th century Europe (with Kepler as an extremely interesting precursor), and the other centering on the applications of the Liu–Zu principle to general theories of volume as this tradition developed from medieval China down to Xu Youren (1800–1860). Measuring these two research traditions in terms of mathematical fruitfulness, it seems to me that Cavalieri's tradition led to mathematical developments far richer than those associated with the Liu–Zu principle. This kind of comparison, however, lies outside the scope of this paper, so let me therefore return to the nature and historical significance of Liu Hui's failure.

My aim in this paper is to show first that Liu's failure to derive the volume for a sphere was not a simple near miss nor a lapse that stemmed from Liu's inadequate effort. As a great mathematician, Liu Hui's failures, including his difficulties in deriving the surface of a *wan-tian* (the surface of a segment of the sphere), cannot be dismissed by such explanations. Rather, it is best explained by the particularity or limitations of the heuristic that governed Liu's geometrical reasoning: a heuristic I call "direct dissection and recombination" in Section one. Several hypothetical strategies designed according to Liu's heuristic for solving his special geometrical problem are therein described in order to show the difficulties in those strategies and hence the limitations of his heuristic. Moreover, the heuristic underlying Liu's geometrical reasoning also explains his successes in solving many other geometrical problems in the *Jiu zhang suansu*[a]. Hence we may achieve a unified explanation for both his successes and his failures by carefully evaluating the special pattern of reasoning he employed, a methodology adopted by other key figures in the subsequent history of Chinese geometry. This approach also naturally explains why Zu Geng, doing geometry with a somewhat different pattern of reasoning, succeeded where Liu Hui failed. In fact, Zu's new heuristic, referred to as "indirect construction" in Section three, can be used to solve other problems concerned with geometrical volumes. To make this point, I present a hypothetical new derivation of the volume of a *yang-ma* inspired by Zu's heuristic and quite different from Liu's construction.

Nevertheless, there were also limitations inherent in Zu's heuristic, especially in those areas where Liu's was most fruitful. This theme is pursued in Section four, where I offer an interpretation of Wang Xiaotong's geometrical works in terms of the later developments in the tradition of Liu's heuristic and his important criticisms of Zu's geometrical works beyond those of the sphere. As to the famous Qing mathematician Mei Wending [b](1633–1721) of the 17th century and later figures, having been deeply influenced by the tradition of Liu's heuristic, their strengths lay in areas close to those in which Liu Hui excelled, whereas they were very weak in dealing with problems related to the sphere, its surfaces, etc. This pattern remained until Xu Youren [c] accidently rediscovered the geometrical heuristic of Zu Geng and its general significance in the 19th century. Thus the differences between Liu's and Zu's derivations of the volume of a sphere cannot be reduced to a contrast between their understandings of a "special vs. general" Liu–Zu principle, as some historians of Chinese geometry have argued. Rather, they resulted from a deeper difference in their general conceptions of the theory of volume. Indeed, the differences between Liu's and Zu's *theories of volume* actually help us to better understand, and to relate together, an important chapter in the history of Chinese geometry dealing not only with the properties of the sphere but also with many other geometrical problems. These differences, of course, also guided Chinese mathematicians in choosing how to use the Liu–Zu principle; hence the *different uses* of the Liu–Zu principle really depended upon the guiding heuristic and concrete historical situation. In short, directly beyond the problems concerned with the sphere, the central purpose of this paper is to employ these two heuristics as guidelines in reconstructing and elucidating at least part of the historical structure of ancient Chinese geometry.

I. LIU'S HEURISTIC AND THE NATURE OF HIS FAILURE

What is the basic "pattern of reasonings" or "heuristic" in Liu's theory of volume and even his theory of area? By the "heuristic" of Liu's theories, I mean the special pattern of operations he used in solving geometrical problems concerned with volumes and areas [3]. Historians of Chinese mathematics generally consider the principle of "*churu xiang bu*"[d] (the out–in complementary principle) as the basic intuitive method of ancient Chinese geometry—especially compared with the Western Euclidean deductive styles (see Wu [1978]). However, I prefer to think of Liu's specific method (especially for the determination of volumes) as a special heuristic of "direct dissection and recombination."

The usual conceptual operations that this heuristic comprises are these: first one directly dissects the solid in question into a (possibly infinite) number of smaller but more familiar solids; second one obtains its volume by summing up, or "recombining," the volumes of the smaller solids. How to cut the solid in question, and with it how to use some "proper" procedure to recombine the cut pieces, are the crucial elements of this heuristic. Following this approach, Liu solved many volume problems such as *fang-ting* (a square pyramid whose upper part has been cut off by a plane parallel to the base), *xien-chu* [g] (or drain, usually a tomb entrance tunnel sloping down into the ground), etc (see Kao [1984]), by employing small familiar solids (*yang-ma* and *bie-nao* [e] [a tetrahedron diagonally cut from a prism]) as "primary functional unit" [f]. Furthermore, Liu arrived at his celebrated solution for finding the volumes of a *yang-ma* and a *bie-nao* (see Wagner [1979]) by focusing on the "ratio of volumes" of the dissected *yang-ma* series and *bie-nao* series, which include smaller cubes, *qian-du* [h] (right-angled prism), *yang-ma*, and *bie-nao*. Liu's solution was certainly ingenious, but the general line of reasoning was still firmly guided by his own heuristic [4]. In a somewhat similar way, Liu employed the same heuristic to solve problems concerning areas, some related to the Pythagorean theorem.

In addition to the two aspects of Liu's heuristic just described, there is a *third* that we will encounter several times in this paper: if the original object is difficult to dissect, then one tries to dissect some new object which, geometrically speaking, *approaches* the original object in question. Liu Hui's own applications of the Liu–Zu principle were closely related to this aspect of his heuristic, and, in a sense, this idea is still very close to the original idea of "direct dissection and recombination." He solved, or at least attacked, important problems concerning the area of a circle or a portion of it (e.g., *hu tian* [a segment of a circle], *huan tian* [i] [annulus]) from this direction: i.e., using the "method of exhaustion" where a direct dissection of a figure increasingly approaches the original [5]. All of these notable achievements illustrate the actual *strength* of Liu's heuristic: Liu clearly showed how one can derive the volumes of many solids without resorting to any difficult or complex geometrical principles—all he required was the intuitive heuristic of dissection and recombination. In fact, he corrected many erroneous formulas concerning geometrical volumes in the *Jiu zhang*, replacing them with a series of more accurate ones (see Kao [1984]).

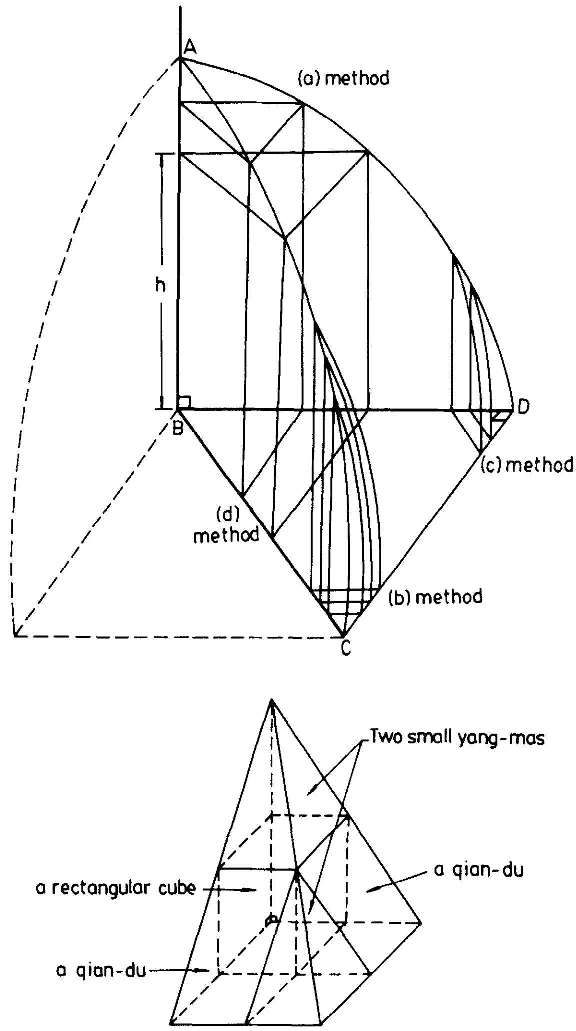
Notwithstanding all these successes, Liu's heuristic encountered difficulties when it came to finding the volume of a sphere. We shall postpone the discussion of the problem of *wan-tian* [j] (a section of the surface of a sphere) to the next section.

Before Liu, Zhang Heng [k] (78–139) had already studied the sphere. In considering it, Zhang assumed the ancient relation that the ratio of the volumes of a cylinder and the inscribed sphere was equal to the ratio of the areas of a square and the sectional circle of the sphere inscribed in this square. Liu must have examined Zhang's assumption and found his mistake. The supposed ratio of 4 : 3 (assuming that $\pi = 3$) is not that between the volumes of a cylinder and the inscribed sphere but rather the ratio between the volumes of a special solid (a *mou he fang gai* [l], abbreviated as **Mhfg** hereafter)—the intersection of *two* cylinders with orthogonal axes—and the inscribed sphere. This follows from the fact that any horizontal plane cuts the **Mhfg** in a square and the inscribed sphere in a circle. Since the ratio of the areas of these two sections is 4 : 3, by the Liu–Zu principle, the volumes of the **Mhfg** and the inscribed sphere are also in the same ratio. Thus, instead of using one cylinder to approach the sphere, Liu employed the intersection of two cylinders. This strategy is somewhat similar to Liu's method of approaching the area of a circle by polygons [6]. It conformed generally to Liu's geometrical heuristic because one tries to directly dissect the **Mhfg** which is viewed as *geometrically approaching* the sphere. In this way, Liu Hui attempted to find the volume of a sphere by first finding the volume of a **Mhfg** that circumscribes it. However, finding the volume of this **Mhfg** is clearly more difficult than finding the areas of polygons. It thus became a serious challenge for Liu's otherwise highly successful heuristic: could he employ "dissection and recombination" to determine the volume of a **Mhfg**?

The central importance of the sphere makes Liu's failure to derive its volume an issue that cannot simply be dismissed by explanations such as his "inadequate effort," offered by many historians, or "inadequate time" [m] (actually blamed by Zu Geng). We should not forget that Liu was engaging in a major effort to provide the *Jiu zhang suansu* with a theoretical commentary, and it was not uncommon that Liu, after pointing out an error in the *Jiu zhang*, admitted his own inability to find a satisfactory solution, as in his discussion of *wan-tian*. Liu Hui's failure seems to me, then, best explained by the particularity or even limitations of the heuristic of dissection and recombination. If Liu could have found the volume of a sphere (or a **Mhfg**) by operations similar to those in his celebrated solution of the volume of a *yang-ma*, there can be little doubt that he would have found the answer. The problem, however, is that it is extremely difficult to find the volume of a **Mhfg** (or, what amounts to the same, a one-eighth section of a **Mhfg**) by the heuristic of dissection and recombination. Let me explain why.

First, there is a basic difference between a *yang-ma* and a one-eighth section of a **Mhfg** (see Fig. 1(a)) with respect to Liu's heuristic. In the first series of dissections of a *yang-ma*, Liu utilized three cuts to dissect it into a cube, two *qian-du* and two smaller *yang-ma* (see [Wagner 1979] and see Fig. 1(b)). In this construction, the three perpendicular cutting planes meet in a point. For a one-eighth section of a **Mhfg**, on the other hand, there will be no such corresponding point so that one cannot dissect it into a number of smaller and familiar solids. It is very probable that Liu tried to find such a mathematical cut point but failed, since it would have been very natural for him to attempt to extend his successful heuristic to this new case.

Since this direct analogy does not work, could Liu Hiu still have dissected a **Mhfg** by a different method and then used the Liu–Zu principle directly? That is, could he have dissected a one-eighth section of a **Mhfg** into infinitely many parallel slices and then, by employing the Liu–Zu principle, added them together? This is a very vague suggestion, but it provides a good opportunity to observe the limitations of Liu's heuristic. There are basically four different ways this could be done. For simplicity, let us consider a one-sixteenth section of a **Mhfg** (see Fig. 1(a)) and try cutting figure ABCD: (a) into parallel triangles with respect to surface BCD, (b) into parallel surfaces with respect to surface ABD, (c) into parallel surfaces



Liu Hui's dissection of a big yang-ma

FIGURE 1

with respect to surface ABC, and finally (d) into parallel rectangles with sides parallel to lines CD and AB.

Using constructions (b) and (c), one obtains a series for a section of some arc-enclosed area. Both series are complicated, and, moreover, Liu Hui never even calculated the area of a single such arc-enclosed area (called a *hu-tian*[n] in *Jiu zhang*), which following his approach amounted to an infinite sum of triangular areas; Liu thought this "too complicated" to derive the formula. Thus, he might

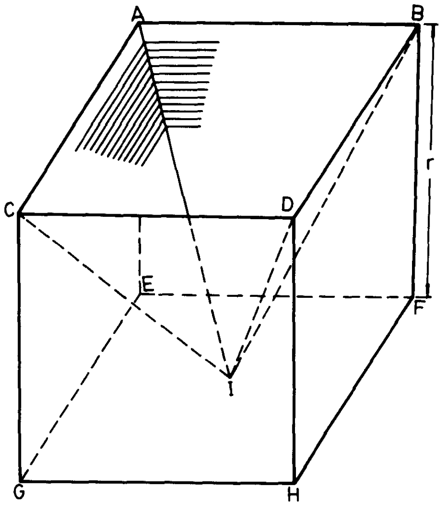


FIGURE 2

have thought of (b) and (c), but, in my judgment, he would have abandoned these constructions very quickly.

As to (a) and (d), here one obtains a series of right-angled triangles or a series of rectangles, so that these approaches appear more promising since it is easier to derive the areas of the basic constituents of both series. Let us consider (a) first. For any piece of a triangle with height h from the base triangle BCD, the area of that triangle is

$$\frac{1}{2}(r^2 - h^2) \tag{1}$$

where r is the radius of the sphere. Now, the entire volume obtained by summing this series of triangles with h ranging from 0 to r equals, according to the Liu–Zu principle, one half the volume of another "constructed" geometrical figure following the *suggestion* of formula 1: i.e., the volume of cube ABCDEFGH of length r with an inner *fang-zhui* [o](square pyramid) IABCD eliminated (see Fig. 2). We, as modern mathematicians, can therefore easily derive the entire volume of a **Mhfg**. This method, it seems to me, is even simpler than Zu's own derivation as recorded by Li Chunfeng [p] in the commentary to the *Jiu zhang*! The problem is why Liu Hui did not find this approach to deriving the volume of a **Mhfg**. Incidentally, we may also wonder why Zu, too, never hit upon this simpler method.

Here it should be noted that method (a) involves a special procedure for "constructing" an entirely different figure which has no geometrically intuitive relationship with the **Mhfg**. It therefore is *not* a straightforward application of Liu's own heuristic of "dissection and recombination." Within the context of Liu's heuristic, one could use the Liu–Zu principle *only* in two related situations: either for direct dissections of the originally given object or in connection with special objects, such as the **Mhfg**, which geometrically *approach* the original object in question. Clearly method (a) would involve a procedure of "indirect construction" much closer to Zu's heuristic. As I shall discuss later in section three, Zu's heuristic often relied on constructions indirectly suggested by an *algebraic* formula, such as formula 1 in the case of (a) [7].

Let us now turn to method (d). The area of the rectangle with height h from the base triangle BCD is

$$h \times (r^2 - h^2)^{1/2}. \tag{2}$$

Now the volume of a one-sixteenth section of a **Mhfg** equals the sum of these rectangles with h ranging from 0 to r. Here, however, it is not easy to indirectly construct, following the geometrical hint of formula 2, another solid whose cross-sections equal the areas of the rectangles calculated above. This requires, in modern terms, the integration of a trigonometrical function. Thus, construction (d) could not easily be employed at that time to derive the volume of a **Mhfg**; at least it would have been much more difficult than (a), and neither approach was available to Liu Hui.

Having considered these two basic approaches (i.e., taking Liu's proof for a *yang-ma* as an exemplar, or trying to employ the Liu–Zu principle), we must still admit that Liu might have considered yet other strategies for finding the volume of a **Mhfg** [8]. Nevertheless, the two approaches just discussed represent the basic ones easily accessible to him, and these not only reveal the peculiar limitations of Liu's heuristic of "direct dissection and recombination" but also indicate why he was unable to derive the volume of a **Mhfg**.

II. LIU'S RELATED FAILURE: THE SURFACE OF A *WAN-TIAN* [j]

In the first chapter *fan-tian* [q] of *Jiu zhang* one finds the difficult problem of finding the surface of a section of a sphere (at least this was the way that Liu conceived the problem). The original formula in the *Jiu zhang* is incorrect, as Liu Hui noted in his commentary. However, he was unable to derive the correct solution to this problem and admitted its difficulty. Could we ask again why Liu Hui was not able to solve this problem? Another occurrence of "inadequate" effort or time? Even Zu Geng said nothing about this problem, or about Liu's performance (at least there is no record of it if he did).

It seems to me that Liu's failure here, just as with the volume of a sphere, was not an isolated or insignificant one. On the contrary, Liu's failures for *wan-tian* and *wan* [r] (sphere) suggest the peculiarity and limitations of his approach to finding volumes and surfaces. Using Liu's heuristic of "dissection and recombination," it is difficult indeed to conceive a way of "dissecting" the surface of a section of a sphere [9]. Thus, it was by no means an accident that Xu Youren later derived the surface of a *wan-tian* primarily under the influence of Zu's heuristic and not Liu's.

In order to derive the surface area of a section of a sphere, Chinese mathematicians often used an important conception to relate the surface area and volume of

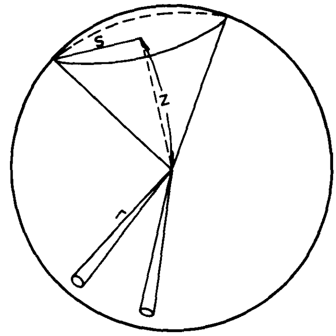


FIGURE 3

a solid (as found in Mei Wending, Xu, and *Shulijingyun* [s]). Consider, for example, the sphere. This conception suggests viewing the sphere as the sum of an infinite number of small cones having the radius of the sphere as their height. Since the volume of a cone is one-third the height multiplied by the base area of the cone, the total volume of these infinitely many small cones will be one-third the radius times the surface area of the sphere (see Fig. 3). This yields a definite relationship between the volume and the surface area of a sphere, and this relationship can be extended to obtain the volume of a "cone-like" dissection of the sphere (see Fig. 3) and its surface area. This conception appears to be a natural extension of Liu's heuristic of "dissection and recombination." Although the idea of "infinitely small cones" is involved, this would not have posed any difficulty for Liu, since similar ideas such as "infinitely small *hu*[t](arc)" were also involved in Liu's commentary on the area of a circle (*fang-tian* chapter, problem No. 32. [10]). However, Liu did not state or implicitly use the above conception in his *Jiu zhang* commentary (Mei Wending, under some partial influence of Liu's heuristic, did formulate this conception clearly later). Probably it would not have been of much use to him at that time for the following reasons: (i) the correct formulas for the volume or the surface area of a sphere were yet not known, and (ii) the volume or the surface area for a section of a sphere were not known either. Moreover, in order to derive either (i) or (ii) one usually resorted to Zu's heuristic, which was unavailable to Liu anyway.

III. ZU GENG'S HEURISTIC AND ITS DEVELOPMENTS

Although we are quite familiar with Zu's derivation of the volume of a **Mhfg** (see [Lam Lay-Yong & Shen Kangsheng 1985]), it seems to me that we still do not fully appreciate the differences between the heuristics of Liu and Zu. Zu's derivation was, in fact, based on patterns of reasoning quite different from Liu's. Instead of dissecting the **Mhfg** directly, Zu focused on an entirely new object, a cube *minus* the inscribed one-eighth portion of a **Mhfg**; then by the Liu–Zu principle, Zu demonstrated the equivalence of the volume of this object to that of a *constructed* pyramid. Similar to method (a) above, Zu conceived of this pyramid in connection with the algebraic formula for the cross-sectional areas of the gouged-out cube. From this he obtained the volume of the **Mhfg**. This strategy, which was qualitatively different from Liu's older heuristic, I prefer to call Zu's heuristic of "indirect construction with the Liu–Zu principle" (or more briefly "indirect construction").

In more general terms, Zu's heuristic of "indirect construction" consists of the following elements: (i) in order to find the volume of the object in question, one focuses on some new object (usually obtained by subtracting the original object from a larger one that contains it), and/or one *constructs* some new objects that often bear no geometrically intuitive relationship to the original; (ii) the strategy for constructing new objects was often suggested by an algebraic formula used in conjunction with the Liu–Zu principle; (iii) the volume of the original object is obtained, via the Liu–Zu principle, from the volume of the constructed new object which is usually much easier to calculate.

Historians have sometimes sought to explain Liu's failure and Zu's success as due to the latter's use of a "generalized" Liu–Zu principle; this presumes that Liu's conception of the Liu–Zu principle was much narrower than Zu's [Wagner 1978, 62; Mei 1984, 117; Kao 1984, 56–59]. I disagree. Liu and Zu's theories were actually two very different kinds of exploration, one employing the idea of direct dissection and recombination, the other indirect construction. Although both used the Liu–Zu principle, the fundamental differences between their heuristics directed their respective *uses* of that principle. To invoke a somewhat *teleological* explanation [11] in terms of "special vs. general" applications of a single principle in order to explain these differences seems to me missing the point. Had Liu changed his otherwise extremely successful heuristic to Zu's, he would have had no trouble deriving the volume of a **Mhfg** as we have seen before. Since Liu's own heuristic was very successful, he had sufficient reason to *see* and *use* the Liu–Zu principle from his own perspective and not Zu's.

Nor was Zu's heuristic of "indirect construction" trivial or arbitrary. Zu's approach represents a real historical *break* [12] with Liu's older method of "dissection and recombination" within the grand tradition of "*churu xiang bu*." This break has its own epistemological roots in the sense that Zu's heuristic could have been used to change a portion of the derivation structure in the landscape of ancient Chinese geometry. This further reveals the problem of treating the differences between Liu and Zu in terms of "special vs. general" forms of the Liu–Zu principle. If that were the case, Zu's efforts should only have served to "expand" Liu's territory of successful derivations, but not to have "changed" (or even "subtracted") from them. Let me illustrate several points along this important line.

First, Zu's heuristic can easily be applied to other fundamental geometrical problems, such as a new derivation of the volume of a *yang-ma* (to be discussed

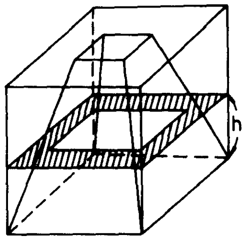


FIGURE 4

in the next section). It may conceivably be extended to other solids such as *fang-ting* [g], etc., although this is a matter still in need of further study (see Fig. 4 for an indication of the problematic). The relevance of this point is that Zu's treatment of *fang-ting* was criticized later by Wang Xiaotong [u]. It is at this point that we may begin to realize the limitations and particularity of Zu's heuristic and the strength of Liu's.

Second, to find the volume of a sphere, one can use Zu's heuristic without resorting to a "strange" object, the **Mhfg**, as historically demonstrated by Xu Youren much later. The fact that Zu did use a **Mhfg** may indicate how Zu's new heuristic emerged from a historically continuous context, a context that Xu did not share. Moreover, Xu extended the area of application of Zu's heuristic to the unsolved problem of finding the surface area of a *wan-tian* in *Jiu zhang*. With patterns of reasoning different from those employed in Greek geometry or 17th century Western mathematics (e.g., Kepler's *Nova stereometria*) [13], Xu also solved the problems of finding the surface area of a sphere and the volume of a section of the sphere. Xu's expansion of Zu's exemplar cannot simply be understood by saying that Xu grasped the "general" spirit of the Liu–Zu principle but rather that Xu understood Zu's heuristic as embodied in Zu's original derivation. Historically speaking, Liu and Zu's concrete derivations of volumes served as two different kinds of heuristic exemplars for Chinese mathematicians to follow. To interpret the history of Chinese theories of volume in terms of two distinct heuristics (with their different uses of the Liu–Zu principle) can help us to understand better, and link together, many geometrical derivations in history: not only those for the sphere, but also others as well.

IV. A NEW DERIVATION OF *YANG-MA* AND SOME FURTHER HISTORICAL NOTES CONCERNING ZU'S HEURISTIC

Let me give one example illustrating how Zu's heuristic could have led to derivations of volumes quite different from Liu's. Consider Liu's celebrated example of the *yang-ma*, which, following Zu, we inscribe in a rectangular solid with sides a, b, and c (see Fig. 5).

One cuts *yang-ma* AEFGH with a plane parallel to the base EFGH at height h (YH), so that this plane also cuts the rectangular solid. Now consider the area

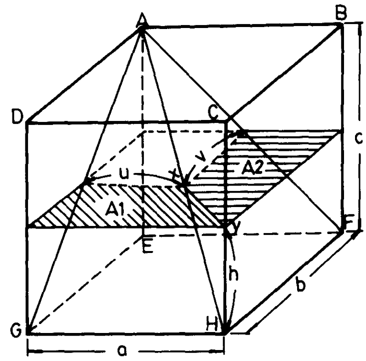


FIGURE 5

enclosed between the rectangular solid and the pyramid. Note here that this approach is analogous to what Zu had focused on: i.e., the area enclosed between the rectangular solid and one-eighth of a **Mhfg**. This enclosed area (in Fig. 5) can be divided by line XY into two smaller areas, A1 and A2. We assert that the areas of A1 and A2 are equal for any height h ranging from 0 to c. The proof of this assertion is simple. The area of A1 is equal to $\frac{1}{2}(a + u)(hb/c)$; whereas A2 equals $\frac{1}{2}(b + v)(ha/c)$. But since $v = ub/a$, area A1 equals A2. According to the Liu–Zu principle, we can now assert that the volumes of the two pyramids (ACBHF with base CBHF, and ADCGH with base DCGH) are equal. Since the rectangular solid can be divided into three pyramids, AEFGH, ACBHF, ADCGH, we can repeat the same proof procedure with respect to pyramid ACBHF and prove the equivalence of the volumes of pyramids AEFGH and ADCGH. Thus, the volume of pyramid AEFGH must be one-third of the volume of the rectangular solid ABCDEFGH, and the proof is established.

Several points can be stated here as consequences of the proof just established. First, this proof is analogous to, and actually modeled upon, Zu's original proof concerning the volume of a **Mhfg**, and is, therefore, established by the heuristic of "indirect construction." Instead of directly dissecting the *yang-ma* in question, one focuses upon two different objects which, geometrically speaking, are not obviously related to the first *yang-ma*. Furthermore, although we did not construct any new object, we employed the Liu–Zu principle only after a nontrivial algebraic calculation. The possibility of such a proof shows that Zu's proof of a **Mhfg** should not be viewed as a singular or isolated case. The heuristic embedded in Zu's proof is by no means insignificant, and it can actually be extended to other cases *just as* Liu Hui took his heuristic of "dissection and recombination" as a general line of approach capable of solving many geometrical problems. Some historical records strongly suggest that this was what Zu had actually done in his classic, but now lost, *Zhui Shu* [v]. As is well known, Wang Xiaotong, in his Preface to *Xugu suanjing* [w], criticized *Zhui Shu*:

祖暅之綴術，時人稱之精妙
曾不覺方邑進行之術，全錯不通
芻甍方亭之問，於理未盡，臣今更作新術

[Zu Geng's *Zhui Shu* was praised by his contempories as brilliant and marvelous. They had never, however, been aware that Zu's operative method concerning problems of "square city" (*Fang-yi*: probably concerning complicated areas and Pythagorean problems) was totally wrong. And his discussions of problems concerning "fodder loft" (*chu-meng*) and "square frustum" (*fang-ting*) were narrowly conceived without reaching the depth of reason. Your humble servant now wishes to make a new method . . . .]

As I briefly indicated in the previous section, we have reason to believe that Zu would have had difficulties in deriving the volume of a *fang-ting* if he had tried, and this may partly explain Wong's criticisms of Zu [14] (also see Fig. 4). In more general terms, concerning the *Zhui Shu*, Li Chunfeng stated briefly in *Shui Shu* "*Lu Li Zhi*" [x] that

又設開差冪、開差立，兼以正圓參之
指要精密，算氏之最也

[[*Zhui shu*] constructed operative methods of length-finding from subtractions of areas and volumes, aided further by the operations concerning the sphere. Its ideas are excellent and subtle, and could be considered as the best of all mathematicians.]

Li Di [y] and others [15] suggested that

差冪

and

差立

referred to, or could be translated as, "the subtraction of areas" and "the subtraction of volumes" respectively. Thus, they suggested that Zu's *Zhui Shu* was closely related to Zu's studies of the sphere, since the subtraction of volumes was also involved in Zu's derivation of the volume of a sphere, and since

兼以正圓參之

("[it was] aided further by the operations concerning the sphere"). From the perspective of this paper, the method of "subtraction of areas and volumes" is very congenial to Zu's heuristic of "indirect construction" but not to Liu's. In Zu's geometrical reasoning, the focus was usually on a foreign object which is obtained by subtracting the original object from a larger one that circumscribes it. It is interesting to note then that Li Di's interpretation of these passages in a sense confirms my position concerning the unique heuristic of Zu's theory of volume. Although doubts [16] remain concerning how exactly to interpret that passage in *Shui Shu*, there are further reasons to support Li Di's interpretation. In Wang's Preface to *Xugu*, his primary concern was clearly geometrical matters. Having noticed the "limit" of the *Jiu zhang's "Shang Gong"* [z] (a chapter dealing primarily with volumes), Wang proposed a "new method" to solve some special "inclined *xian chu*." Thus, his criticism of Zu's *Zhui Shu* is most likely related to Zu's geometrical theory but not to some new methods concerned with extracting roots of equations as some historians have believed. Moreover, Wang's own heuristic for solving the volume of the inclined *xian chu* [aa], according to one historian's reconstruction [17], is very close to Liu Hui's own heuristic. From Liu's perspective, it is indeed very difficult to derive the volume of "*xian chu*" or that of "*fang ting*" along Zu's line of reasoning, hence Wang's criticism of Zu. It is at this point that we can clearly see the limitation of Zu's heuristic, no matter how he used the Liu–Zu principle. This again shows that the difference between the heuristics of Liu and Zu is more important in the history of Chinese geometry than the distinction between the "special vs. general" applications of a single principle.

Consequently, we may say that the "problem domain" [18] most suitable to Zu's heuristic at best only partially overlaps with that of Liu's. There are, in fact, some volume problems which are very suitable to Liu's heuristic but *not* to Zu's. Problems such as finding the volumes of *fang-ting, xian-chu, chu-tong* [ab] (frustum of a prism) are naturally too complicated for Zu's heuristic, but these problems can be solved by Liu's heuristic in a straightforward manner. Liu's major achievements and his modifications of the original formula in *Jiu zhang* [Kao 1984] fall precisely within this specific problem domain. Because the problem domain most suitable to Liu's heuristic assumed a central position in the history of Chinese geometry and its "practical" applications, this pattern of reasoning more or less dominated the later development of Chinese geometry (perhaps from Wang Xiaotong to Mei Wending and even later). Wang Xiaotong made a successful extension of Liu's heuristic to a new set of problems, the inclined *xian chu*, question 3 of the *Xugu Suanjing*, being perhaps the most famous one. Later, Shen Kuo [ac] also made a successful extension of Liu's heuristic to yet another set of problems concerning sums of series. As we know, Shen's "*Xi Ji shu*" [ad] (a method of finding the volume of an object that contains slits and holes) is best explicated in terms of Liu Hui's geometrical heuristic but not some algebraic methods which are quite unrelated to the mathematical traditions in which Shen Kuo lived [19].

Although Zu Geng had already opened the way to new approaches in ancient

Chinese theories of volume, due to the dominant position of Liu Hui, these only came to flower with Xu Youren in a much later age.

V. THE REVIVAL OF ZU'S HEURISTIC IN XU YOUREN'S *JIE QIU JIE YI*

In the time of Mei Wending [b], Chinese mathematicians already had some familarity with Western sources and translations of geometry. Although they still knew some of the ancient Chinese geometry, many preferred to do geometry along Western lines. Mei Wending, for example, though he established his general theory of volume with fundamental notions similar to Liu Hui's *qian-du* and *bie-nao* (in Mei's *Qian-du celiang* [*The Measurement with qian-du*] [ae] and *Jihe bu bian* [*Supplement to geometry*] [af]), adopted a general strategy and heuristic for his theory of volume that appear to have been strongly influenced by the West [20]. In developing properties of the sphere in *Fang yuan mi ji* [ag] (*On areas and volumes of cubes and spheres*), Mei discussed problems and properties quite foreign to the problem domain of ancient Chinese geometry, and he never mentioned the proofs of Liu and Zu related to geometrical problems in the *Jiu zhang*.

Years later, in a very peculiar historical context, it was Xu Youren [c] who "rediscovered" Zu's heuristic and explicitly applied it to some new problems. By this time, most Chinese mathematicians already had a fair amount of Western geometry at their disposal, especially after the publication of the encyclopedic *Shuli jing yun* [s] (*The essence of mathematics*). At the very beginning of his essay *Jie qiu jieyi* [z] (*Analyzing the dissections of a sphere*), Xu stated some propositions concerning surfaces of a sphere from the so-called "*Jihe yuanben*" [ai] (may mean *The origin and source book of geometry*, actually books two to four of *Shuli jingyun* [21]). He then complained that the text failed to offer arguments showing why these propositions were correct, nor did the books by Mei Wending that explained them. As a result of this situation, Xu adopted an alternative approach. Searching through the materials of ancient Chinese geometry, he finally hit upon Zu's derivation of the volume of a sphere.

幾何原本謂...而不直抉其所以然，遍檢梅氏
諸書，亦未能明釋之也，蓄疑於心久矣．
近讀李淳風九章注，乃得其解，因釋之，以告同志．...
甚矣，索解人之難也，今釋幾何原本，
而淳風之注，因是以明．

[*Jihe yuanben* stated [propositions concerning surfaces of a sphere] . . . without offering reasons why those propositions must be so. I had surveyed through the books by Mei, but they threw no light upon those propositions. Doubts persisted in my mind for a long time. They were not solved until recently reading Li Chunfeng's commentary on *Jiu zhang*. I elucidated them subsequently in order to inform my friends. . . . They are so difficult that few people can understand them! I now elucidate *Jihe yuanben* with the following explanation, and Li Chunfeng's commentary becomes clear as well.]

Before I comment upon Xu's novel extension (or "application") of Zu's heuristic, some further discussion is needed concerning the nature of Xu's criticism of *The origin*. In Propositions 8 through 11 in Chapter ten of *The origin* (Book three of *Shuli jingjun*), we find "proofs" of the propositions related to and stated by Xu Youren. The "proof" of basic proposition 8 ("if the radius of a circle is equal to that of a sphere, then the area of that circle is one fourth of that of the sphere") (pp. 130–131) was based upon a faulty geometrical construction and circular reasoning. The authors of *The origin* here tried constructing a small cone, the base circle of which has a diameter equal to the diameter of the sphere in question *and* also equal to the base radius of a big cone whose volume is equal to the volume of the same sphere in question. Essentially, this construction (p. 131) begs the whole question. After Proposition 8, we find Propositions 10 and 11 dealing with what Xu mentioned at the beginning of *Jie qiu*. Although proposition 11 (pp. 133-5) is followed by a very tedious "proof"—based on Proposition 8—about the surface area of a section of a sphere, the error in the "proof" of Proposition 8 falsified the entire project.

Elsewhere in Book 26 ("on the curved volumes") of *Shuli jing yun*, one finds problems and solutions concerning "*Jie qiu*" [aj] (dissections of a sphere); their derivations also resort to Propositions 10 and 11 discussed above. We may also note that the terminology, phrasing, and framework used in *Jie qiu jieyi* are very similar to that of book 26, e.g., the term "*Jie qiu*." Book 26 thus seems to be an important reference point for Xu Youren. Moreover, in Book 26 we find a "new way" to find the surface area of a section of a sphere, and this new method was neither mentioned nor proved in book three. A very bad "explanation" of it, perhaps intended as a "brief proof," was given however in the same Book 26 (pp. 1111–1112). But this very "explanation" (again using circular reasoning) seems to come from Mei Wending's *Fang yuan mi ji* (pp. 14–15). This interesting relationship may partly explain why Xu, after being dissatisfied with Books three and twenty-six of *Jingyun*, searched through the books of Mei for further arguments. Incidentally, this new way of finding surface areas actually can be found in Archimedes (Propositions 35–44, Book I of *On the sphere and cylinder*); but neither Mei nor *Shuli jing yun* replicated the constructions in his proofs.

Let me now comment upon Xu's proofs concerning the following propositions: (i) the volume of a sphere is two-thirds that of its circumscribed cylinder, and (ii) the volume of a spherical sector is two-thirds that of the cylinder of the same height and with the same radius as the sphere, and the surface of that sector is the same as the lateral surface of that cylinder (see [Lam and Shen 1985], or [Shen 1982]).

It would seem to me inaccurate to describe Xu's extensions of Zu's heuristic simply as an "application of Cavalieri's principle," since such a characterization would miss the important distinction between Liu and Zu's heuristics. What Xu had learned from Li Chunfeng's [p] commentary on the *Jiu zhang* was not Liu's method of "dissection and recombination" but rather Zu's "indirect construction." Without using Liu Hui's **Mhfg**, Xu's new method for finding the volume of a sphere represents an extension by "analogy" following the exemplar of Zu's proof [22]. Zu's method had considered the horizontal square area in a cube *minus* the area of a cross-section of an inscribed **Mhfg** with the same height as the basic unit (area) for the Liu–Zu principle. Xu now took the horizontal circular area of a cylinder *minus* the cross-sectional area of an inscribed hemisphere with the same height. Xu thus explained the basic idea of his proof that "whereas Chunfeng used the square, I used the circle here; the underlying ideas are not different "[ak] (*Jie qiu* p. 1). Again, guided by the algebraic formula obtained from subtracting the areas, Xu indirectly constructed a cone with a base area equal to that of the cylinder and a height equal to its radius. Thus the volume of this cone is, by the Liu–Zu principle, equal to the volume of a cylinder minus an inscribed hemisphere. Then by subtracting volumes (cylinder minus an inscribed cone), Xu obtained the volume of a sphere. Having derived the volume of a sphere, Xu then calculated the surface of that sphere under the *assumption* that the volume of a sphere equals one-third the product of the surface area and the radius.

It is interesting to note that although Xu criticized *The origin* and Mei's books, he did not question this assumption throughout the *Jie qiu jieyi*. Xu later used a similar assumption to derive the surface area of a cone-like dissection of a sphere from its known volume (see Fig. 3). This assumption, based on the idea that a sphere can be dissected into infinitely many small cones, was discussed earlier in section two where I raised the possibility that it may have stemmed from Liu's heuristic of "dissection and recombination." Although this assumption (or proposition) figured prominently in Book two of *Shuli jing yun* and might have come from Kepler's *Nova stereometria*, certainly Mei Wending relied heavily upon this idea in many of his geometrical works. It is beyond the scope of this paper, however, to give a detailed discussion of the origin of this idea in Mei's works, but it is possible that Mei hit upon this idea from Liu's original heuristic of "dissection and recombination" when he seriously started dealing with the trigonometric problems in *Qian-du celiang* [23].

From this and other evidence, Mei Wending may be seen at least as a partial follower of Liu Hui's heuristic of "dissection and recombination." However, Mei seems not to have appreciated or noticed Zu's alternative heuristic in his proof of the volume of a sphere. Usually Mei merely assumed the formula of the volume and surface of a sphere; probably he thought these were simply basics of Western geometrical knowledge. Mei's works therefore disappointed Xu Youren as Xu pondered a slightly different set of geometrical problems in *Jie qiu jieyi*, and this situation caused Xu to recover a different heuristic from the sources of *Jiu zhang*.

Having discussed the relationship between the surface of a sphere and the lateral surface of its circumscribed cylinder, Xu began pondering the problem of finding the volume and surface of a segment of a sphere (i.e., the ancient problem of *wantian*). Guided by Zu's heuristic, Xu passed from the spherical sector in question and focused upon the volume obtained by subtracting the inscribed spherical sector from a cylinder with height z (see Fig. 6). Any horizontal plane of height h cuts

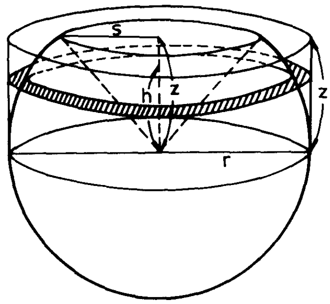


FIGURE 6

this subtracted volume in a ring with area

$$\pi\,(\mathrm{sq}(\mathrm{h})). \tag{3}$$

Using formula 3, Xu could easily *construct* a cone with both base radius and height equal to z. Xu called this the "exterior cone," and the inverse small cone with base radius s and height z the "interior cone." The volume of the spherical sector in question therefore equals the volume of the above cylinder minus the exterior and interior cones. By means of typical operations of "indirect construction," Xu finally derived the volume and surface of a spherical sector. This proof seems to me again a beautiful "extension" analogous to Zu's original exemplary proof in his commentary on the *Jiu zhang*.

Considering the nature of Xu's two proofs, and contrasting the approaches and proof techniques of Mei Wending with Xu Youren, we are therefore much more confident concerning the reality of two different heuristics employed in the *Jiu zhang* commentaries discussed in previous sections. In general, roughly two lines of geometrical development, originating from the different heuristics of Liu Hui and Zu Geng, can accordingly be traced through some intermediary developments down to the achievements of Mei Wending and Xu Youren in the Qing period. A genealogy of volume derivation in the history of Chinese geometry can be drawn to show the distinct developments within these two lines. Of course, this does not mean that Xu Youren did not understand Liu Hui's heuristic; a mathematician like Xu can be said to have mastered both kinds of heuristic. It is much more difficult, however, to say that Xu had learned a "general" Liu–Zu principle whereas Mei Wending had not. As the general theme of this essay has indicated, the differences between the heuristics of Liu and Zu help us to *understand* better, and to *relate* together, an important portion of the history of Chinese geometry. These differences also guided Chinese mathematicians in deciding how to use the Liu–Zu principle; thus the different uses made of the Liu–Zu principle really depended upon the guiding heuristic and the concrete historical situation.

VI. SOME FURTHER HISTORICAL NOTES CONCERNING THE INTERPRETATIONS OF THE DIFFERENCE BETWEEN LIU AND ZU

Mei Wending only read the first five chapters of *Jiu zhang*, which includes Zu's proof on the volume of a sphere, from the South Sung block-printed edition in the home of Huang Yu Ji [bq]; however, Mei probably did not pay sufficient attention to Zu's proof. Instead of saying that Mei could not understand the "general" properties of the Liu–Zu principle, we may interpret this oversight as a symptom of the dominance of Liu's heuristic in Chinese geometry, a paradigm under which Mei did much of his mathematical research. Furthermore, it is difficult to explain why Mei's understanding of the Liu–Zu principle remained confined exactly to the so-called "special" level of Liu's, unless we regard Mei's orientation as more or less fixed by Liu's paradigm, so that he was prevented from appreciating Zu's quite different heuristic (perhaps even "invisible" to Mei?).

After the various new editions of *Jiu zhang* in the generation of Dai Zhen [br] (see [Kao 1989]), Chinese mathematicians would appear to have had much better opportunity to appreciate Zu's proof, and they also would seem to have been much better equipped to deal with the set of problems that confronted Xu Youren many years later. Interestingly enough, this was not the case. Take the example of Jiao Xun [bs], working at the very end of the 18th century. In his otherwise very interesting piece, *Jia-Jian-Cheng-Chu Shi* [bt] (roughly, "A mathematical explication in terms of plus, minus, multiple, and divide"), Jiao made no explicit notes concerning the nature of Zu's proof, a proof which presumably could have served as an important example for Jiao's abstract and theoretical generalizations [Wu 1986] of the heuristic of *Jiu zhang* [24]. As an important mathematician of his time, Jiao's theoretical generalizations of the volume heuristic in *Jiu zhang* were conducted along the lines of Liu Hui and Mei Wending [25], especially those concerning problems related to the sphere. Again it is difficult to explain this by saying that even Jiao Xun's understanding of the Liu–Zu principle was somehow restricted to the "specific" level of Liu and Mei, unless we can interpret this as a further sign of the actual strength and dominance of Liu's heuristic. It may well be the case that Jiao's confinement within Liu's heuristic was in general quite comfortable, and permitted further interesting developments along Liu's line of reasoning such as Jiao's abstract algebraic reasonings following *Jiu zhang*'s heuristic and his further elaborations on Liu Hui and Wang Xiaotong's works on volume problems [26].

Consider, as another example, the case of Li Huang [bu] in the early 19th century. In his influential *(Jiu zhang) Xicao Tushuo* [bv] (roughly "A detailed calculation and pictorial explanation of *Jiu zhang*"), Li could not but touch upon Zu's proof, and somehow expressed a sense of "joy and surprising rediscovery." He paid much attention to Zu's proof of the volume of a sphere, supplying pictures and discussions that may even surpass many modern discussions of the same subject [27]. Nevertheless, Li treated Zu's proof as an interesting but "isolated" case, but did not reveal its general significance as a possible alternative heuristic to Liu's. A case in point is Li Huang's discussion of the problem of finding the surface of a segment of a sphere (*wan-tian*). Except for a focused look at Liu's discussions, Li did not at all address the set of problems confronted by Xu Youren years later. He faithfully presented Liu's own reasonings on *wan-tian*, but Li only added Mei Wending's (Western) way of calculating the surface area of a section of a sphere without any further explanation [28]. Should Li's understanding of the

Liu–Zu principle, after his full explication of Zu's proof, be considered "special" or "general"? This teleological interpretation of the historical understanding of Liu–Zu principle in terms of mechanical, ordered "stages" again seems to reveal serious problems. What we should do, it seems to me, is to interpret these situations in terms of the actual developmental strength and dominance of Liu Hui's heuristic, which prevailed in the face of Zu's very insightful alternative until the special historical context confronted by Xu Youren led him to break with it.

A SHORT GLOSSARY OF SOME TECHNICAL TERMS:

BIE-NAO: tetrahedron diagonally cut from prism.
CHU-MENG: fodder loft.
CHU-TONG: fodder boy, frustum of prism, used in fort earthwork, bank, etc.
FANG-TIAN: square.
FANG-TING: frustum or square pavilion, a square pyramid whose upper part has been cut off by plane parallel to base.
FANG-ZHUI: square awl, or square pyramid.
HU-TIAN: segment of a circle, arc field.
QIAN-DU: right-angled prism.
WAN-TIAN: the surface of a segment of a sphere.
XIAN-CHU: drain, usually tomb entrance tunnel sloping down into the ground.
YANG-MA: a rectanglar (or square) pyramid with one side perpendicular to the base. For further explanations of these fundamental Chinese geometrical terms, please consult [Wagner 1978; 1979; Li Yan 1987; or Martzloff 1987].
HISTORICAL BREAK: Despite its superficial similarity, a subsequent scientific theory may have a break with its historical antecedent by forming different fundamental principles, or different heuristics and problem domains. It need not be a fundamental revolution, preceeded by a grand crisis, as perhaps envisaged by Kuhn, but more often than not, the antecedent theory has troubles with its problem domain which in turn provoke the emergence of a new theory.
EXEMPLAR or PARADIGM: A standard or even outstanding solution of a typical problem offered by a theory such that it serves as an exemplar for the followers of that theory to model upon, to draw inspiration from, as they are exploring new problems in its problem domain.
TELEOLOGICAL EXPLANATION: To interpret the historical evolution of science or mathematics as an ordered and progressively oriented process toward some predestined truths (usually the received scientific knowledge of the 20th century). Later stages of the evolution are usually conceived as more advanced than the earlier stages since they are more similar to 20th century science, and hence more rational, objective, universal, and less "sensational."
FRUITFULNESS of a scientific tradition: the ability of a scientific tradition with a set of powerful heuristics to solve more difficult problems of a field than its competitors, and to explore and cultivate new problem domains for the "consumptive" development of that tradition, without the necessary assumption that the scientific tradition is evolving toward the final goal of 20th century science and mathematics.

GLOSSARY OF CHINESE EXPRESSIONS

[a] 九章算術
[b] 梅文鼎
[c] 徐有壬
[d] 出入相補
[e] 鼈臑
[f] 功實之主
[g] 方亭、羨除
[h] 塹堵
[i] 弧田，環田
[j] 宛田
[k] 張衡
[l] 牟合方蓋
[m] 未暇校新
[n] 弧田
[o] 方錐
[p] 李淳風
[q] 方田
[r] 丸
[s] 數理精蘊
[t] 觚
[u] 王孝通
[v] 綴術
[w] 緝古算經
[x] 隋書律曆志
[y] 李迪
[z] 商功
[aa] 羨除
[ab] 芻童
[ac] 沈括
[ad] 隙積術
[ae] 塹堵測量
[af] 幾何補篇
[ag] 方圓冪積
[ah] 截球解義
[ai] 幾何原本
[aj] 截球
[ak] 蓋淳風用方，今用圓，其理則無二也
[al] 白尚恕
[am] 胡道靜
[an] 夢溪筆談校證
[ao] 郭書春
[ap] 科技史文集
[aq] 藍麗蓉、沈康身
[ar] 李繼閔
[as] 科學普及
[at] 梅榮照
[au] 康熙
[av] 商務
[aw] 梅氏叢書輯要
[ax] 藝文印書館
[ay] 錢寶琮
[az] 算經十書
[ba] 中華書局
[bb] 知不足齋
[bc] 務民義齋算學
[bd] 吳文俊
[be] 中国古代科技成就
[bf] 中国青年出版社
[bg] 割之又割，以至於不可割，則與圓合體，而無所失
[bh] 立三角法
[bi] 李迪
[bj] 遼寧人民
[bk] 山東教育
[bl] 李儼
[bm] 許蒓舫
[bn] 徐光啓
[bo] 棊驗法
[bp] 中国數學簡史
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[bs] 焦循
[bt] 加減乘除釋
[bu] 李潢
[bv] 九章算術細草圖說
[bw] 吳裕賓
[bx] 里堂學算記
[by] 焦氏叢書
[bz] 李兆華

NOTES

1. Democritus and Archimedes are perhaps the outstanding examples for their explorations and uses of "indivisibles" in the history of Greek mathematics (see [Heath 1981 I, 180; II, 19–20]). Furthermore, the reason that Greek mathematics contains few explicit references to "indivisibles" is not that the

Greeks did not anticipate Cavalieri's principle, but rather because of the paradoxes posed by Zeno of Elea. The Greeks, however, still used this principle very often as a fruitful heuristic, as can clearly be seen from Archimedes' *The Method of Mechanical Theorems* (see [Dijksterhuis 1987, 148, 313–321]). See also [Knorr 1986, 265–267] and [Martzloff 1987, 276–279]. As has been pointed out by many, the construction and volume calculation of a **Mhfg** also appear in Archimedes' *The Method* and the preface to his letter to Eratosthenes. Archimedes considered the volume of a **Mhfg** special since it is equal to "solid figures bounded by planes," which is not the case for other complicated curved figures.

2. See [Boyer 1949, Chap. 4; Baron 1969, Chap. 4; Anderson 1985].

3. I have borrowed the notion of "heuristic" from [Lakatos 1970].

4. In [Wagner 1979], except for the problem of infinitesimals, it is not clear whether Wagner would consider Liu's solution of a *yang-ma* as something decidedly different from Liu's "general method" [see p. 168 & pp. 182–183]. On the other hand, it seems to me that Liu's method in his derivation of a *yang-ma* is, in spirit, congenial to his "general method" except that he pushes it to its extreme. It order to find the volume of a *yang-ma*, Liu Hui first dissected a *qian-du* (see Glossary) into a *yang-ma* and a *bie-nao*, then he dissected the latter two figures into two series of smaller figures, and found the ratio of the volumes of the two series to be 2 : 1.

5. Sometimes Liu Hui's method for finding areas relating to circles is called Chinese style "method of exhaustion." Like the method of exhaustion, which does not involve a direct "dissection," it uses a dissection of figures (polygons) that increasingly approach a given figure (e.g., a circle). Of course it also involves auxiliary operations concerning square roots, the Pythagorean theorem, etc.

6. Liu might have thought of taking infinitely many intersecting cylinders to approach the volume of a sphere, but that again presupposes finding a way to calculate the volume of a **Mhfg** first. Here, Liu did not even have the algorithm to calculate the simple intersected volume; thus, he was in a worse situation than that in the calculation of "*hu-tian*" [n]. One thing more to note is that Liu's strategy here certainly does not belong to Zu's heuristic of "indirect construction" to be discussed later.

7. In [Bai 1981, 154], he disputed Zu Geng's blame of "inadequate time" on Liu; rather, Bai believed, it was Liu's "modesty" which prevented Liu from saying more about the volume of a sphere. Bai stated further that Zu's proof is not elegant enough, and he proposed a simpler one. Bai's proposed proof is identical to the method (a) discussed here. However, he did not recognize the fundamental difference in the lines of reasoning between Liu and Zu. Bai's explanation of Liu's failure in terms of Liu's "modesty" seems to me a bad explanation as well. On the other hand, as pointed out by one reviewer of this paper, it might be "modesty" on Zu's part to explain Liu's failure in terms of "inadequate time."

8. Liu might have obtained the volume of a sphere through the use of Archimedes' method to get the surface of a sphere first, and then, using ideas similar to Kepler's to derive the volume of a sphere (see Chaps. 1 and 4 of Baron [1969]; for more details, see [Dijksterhuis 1987], Archimedes *On the sphere and cylinder*, Bk.I, Propositions 21–34.). It is true that Archimedes' method is not an easy one, but basically that method can be understood in terms of Liu's heuristic of "dissection and recombination," or the "method of exhaustion" Chinese style. The fact that Liu Hui did not think of Archimedes' proof may indicate Liu's *real* failure within the context of Liu's own specific heuristic.

9. Of course, we might think again of Archimedes' method and its further application (see [Dijksterhuis 1987], Archimedes's *On the sphere and cylinder*, Bk.I, Propositions 35–44.). But as I discussed in the previous note, that method is not an easy one. Curiously enough, although Liu had thought of the "Liu–Zu principle," which is a geometrical principle stating an interesting relationship between areas and volumes, it seems that he did not think of some possible "analogous" principle stating the relationship between lengths of lines (possibly curved) and areas (or surfaces). Is it possible that Liu would have thought of cutting the *wan-tian* into infinitely many "circles" parallel to the base circle and equating the surface of a *wan-tian* to the "sum" of the length of these infinitely many circles?

10. Liu's idea is roughly this: "we take the enclosed polygon and cut it again and again until it cannot be cut further; then the polygon is "embodied" in the circle, and nothing is lost" [bg].

11. In Kao [1984, 56–59], he admirably pointed out the important difference concerning the "uses" of horizontal cross-sections of a volume as used by Liu and others before him. But this does not mean that he is justified in arranging a single teleological line of "three stages" from the development of the pre-Liu "sensational" stage, via Liu's "rational but particular" stage, to Zu's final "rational and universal" stage.

12. I resist seeing the difference between Liu and Zu's heuristics as some kind of "revolution." But it is not entirely unreasonable to consider this perhaps "middle range" break as some kind of small mathematical revolution in Kuhn's [1970] sense, since in the postcript to *Structure* [1970], Kuhn indicated that an occasion of "scientific revolution" does not have to be on such a grand scale as the Copernican "revolution." It seems to me more natural to see this difference as a genuine historical *break*, and to call it such has the advantage of emphasizing the historical significance of this difference.

13. Kepler's *Nova stereometria* is a fascinating source of modern Western "intuitive" reasonings on geometrical problems very similar to the Chinese tradition of geometry, especially the well known general "out–in complementary principle." However, Kepler's "heuristic" seems to lie between the Chinese "dissection and recombination" and "indirect construction." That may simply mean that Kepler is not in the Chinese geometrical tradition. Kepler had a beautiful series of "indirect constructions" concerning the volumes of a series of solids: apple, apple ring, spherical ring, lemon, sphere. His indirect constructions are similar to method (a) in Section I, but more complicated. See Sect. 4.1 of [Baron 1969], especially pp. 112–114. Although Western mathematicians could be said to have grasped the so-called "general" Liu–Zu principle, their heuristics and uses of that principle are quite different from Zu's.

14. It seems to me that it is very difficult to use the Liu–Zu principle, along with Zu's heuristic, to derive the volume of a *fang-ting*. The best one can do is to equate "the volume of a cube minus its inscribed *fang-ting*" with "the volume of a special *yang-ma* with conic section curves sides minus a regular *yang-ma*." But since the properties of conic section curves were not accessible to Zu, this became a difficult problem to Zu's heuristic. Probably this is one of the reason for Wang Xiaotong's criticism of Zu.

15. See Li Di [bi] [1984, 119–120, 131]. Also see *Zhongguo Shuxue Jianshi* [bp] [1986, 177–178]. Incidentally, the latter book is much richer in content than Li Di's book.

16. Basically following Qian Baocong's line of interpretation, Mei [1984, 117–121] offered a modified interpretation; he did not think that this passage indicated anything closely related to Zu's theory of volumes or his famous derivation of the volume of a sphere. Rather, Mei believed that the passage really refers to some special methods of extracting square or cube roots of a polynomial expression or equation, with "negative" coefficients involved. Mei thought that it was this negative coefficient that marked the historical importance of Zu's *Zhui Shu*. It is possible, however, that Li Di's interpretation of Zu's ideas could be related to some new techniques concerning extracting roots with negative coefficients. Incidentally, Professor Mei Rongzhao has remarked that Liu's geometrical theory always used "addition" but almost never used "subtraction," whereas Zu's derivation of the volume of a sphere is a clear example of "subtraction." Also consider the discussions of Jiao Xun in Section VI.

17. See [*Zhongguo Shuxue Jianshi* 1986, 198–205]. We may note here that, according to this reconstruction of Question 3 of *Xugu*, it is indeed not possible to have equations with negative coefficients; this is just what Mei [1984] has asserted. Due to difficulties of communication between China and Taiwan, I was not able to see Shen Kangshen's 1964 paper on this same subject.

18. This notion refers to the set of problems that a theory or research programme, using its heuristics, takes as its primary target of attack. Different theories using different heuristics usually do not have the identical problem domain. Each theory usually would take its own problem domain as the "most important" area in the discipline, and deemphasize the importance of other theory's problem domain. See [Fu 1986] for further elaborations in the case of 17th century Western optics. As to the strategies and "logic" of competition between rival Chinese theories, I have discussed these in my study of Chinese ancient mathematical cosmology. See [Fu 1988].

19. How Shen Kuo derived his famous Xi Ji formula (in *Meng Qi Bi Tan* [an] "Brush Talks," Book 18, entry 301) is an old problem in the history of Chinese mathematics. I do not think that Li Yan [bl] or Xu Chunfang [bm] have the right solutions; see Hu Daojing [am] [1956, 579–584] (same page references in Hu's 1986 new ed.). Li Jimin [ar] [1974] and the reconstruction in [*Zhongguo Shuxue Jianshi* 1986, 263–267] (by Li Zhaohua [bz]) have offered a much better solution which has the merit of firmly putting Xi Ji Shu back into its proper historical context, i.e., the tradition of *Jiu zhang* "Shang Gong" [z] and Liu Hui. According to their reconstruction, it is very clear that "Xi Ji Shu" was strongly influenced by Liu's heuristic of dissection and recombination. Shen's basic strategy is to "see" the problem of finding sums of series as a special problem for finding geometrical volumes. He then derives the volume of a special figure by direct dissection and recombination.

20. Of course, it is beyond the scope of this paper to discuss Mei's position and his strategies of how to "integrate" Chinese and Western traditions of mathematics.

21. "*Jihe yuanben*" may refer to Xu Quangqi's [bn] translation of first six chapters of C. Clavius' *Euclidis Elementorum libri XV* (1574). However, there is no discussion of volumes or surfaces in Xu's translation, although Clavius added many more examples in his version of Euclid's famous work.

22. Many examples discussed in this paper, either historical or hypothetical, are clearly "modeled" on original exemplars like Liu Hui's proof of *yang-ma*, or Zu Geng's proof of **Mhfg**. These examples are perhaps some of the clearest examples of T. Kuhn's [1970] idea of scientific developments in "normal science" which model upon the original "exemplars"—a notion that Kuhn especially stressed in his postscript of [1970].

23. In [1982, 129–131], Shen believed that the idea just discussed here is the crucial assumption of Xu. I disagree. This idea was actually common knowledge from Mei to Xu. The crucial idea of Xu's proofs actually comes from Zu's proof, hence the importance of Zu's heuristic and its difference from Liu's heuristic. We know also that in Mei's *Qian-du celiang* [ae] he explicitly stated that the inspiration and "ancient method" from the *Jiu zhang* was a major source of his own method. Thus as Mei pondered the problems of spherical trigonometry, the ideas of direct dissection of a sphere and the ancient dissection of a *qian-du* by Liu Hui became related to each other. This can be clearly seen from the section of "*li san jiao fa*" [bh] (the method of standing triangle, i.e., a tetrahedron) in *Qian-du celiang*. Whereas Liu Hui took "*yang-ma*" and "*bie-nao*" as the basic conceptual units of his theory of volume, Mei Wending took "*bie-nao*" (tetrahedron in general) (p. 10). As to Mei's theory of volume, see also Martzloff [1981, 284–290].

24. See Jiao's ideas concerning the purposes of his work: ". . . whereas *Jiu zhang* cannot exhaust the uses of addition, subtraction, multiplication, and division, these four principles can exhaust the content and limit of *Jiu zhang*. . . ." See Chap. 1, p. 2 of Jiao's preface. Jiao considered the theories of volume in *Jiu zhang* as results originating from the multiplication principle (Chap. 8, p. 18); I wonder how he conceived the nature of Zu's proof of the volume of a sphere from the perspective of "multiplication"!? . . .

25. See *Jia-Jian-Cheng-Chu shi*, Chap. 1, pp. 4–5, Chap. 3, pp. 24–38, and Chap. 4, pp. 7–14. For his vision of the "architecture" and "ordering" of geometrical problems, especially related to the circle and sphere, see Chap. 3, pp. 37–38. His vision seems to follow naturally from the lines of Liu Hui and Mei Wending, and did not pay sufficient attention to the particularity and limitations of Liu's heuristics and Mei's geometrical conceptions. Wu [1986, 127] did not notice Jiao's problems at this point.

26. For examples on the more general specifications of Liu's heuristics concerning volumes, see *Jia-Jian-Cheng-Chu shi*, Chap. 3, p. 26, and Chap. 4, pp. 11–12; on the further elaborations of Wang Xiaotong's problematics, see Chap. 3, pp. 27–37.

27. See Li Huang's *Xicao Tushuo*, Chap. 4, pp. 46–55. Li used more space in *Xicao* to discuss Zu's proof than any other geometrical problem: Zu's proof took 10 pages, Liu's discussions of *xian-chu* 8 pages, and Liu's celebrated proof of *yang-ma* a mere 4 pages.

28. See *Xi Chao Tu Shuo*, Chap. 1, pp. 44–45. Li somehow tried to indicate a possible "continuity" between Liu's discussion and Mei's formula. We have discussed this formula in the previous section

(a "new way" of finding surface area); it originated from Archimedes, appeared in Mei's work *Fang yuan mi ji*, and Book 26 of *Shuli Jingyun*.

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