Extremal problems and isotropic positions of convex bodies

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Abstract

Let $K$ be a convex body in $\mathbb{R}^n$ and let $W_i(K)$, $i = 1, \ldots, n-1$ be its quermassintegrals. We study minimization problems of the form $\min \{ W_i(TK) : T \in SL_n \}$ and show that bodies which appear as solutions of such problems satisfy isotropic conditions or even admit an isotropic characterization for appropriate measures. This shows that several well known positions of convex bodies which play an important role in the local theory may be described in terms of classical convexity as isotropic ones. We provide new applications of this point of view for the minimal mean width position.

1 Introduction

Given a convex body $K$ in $\mathbb{R}^n$ we consider the family $\{TK \mid T \in SL_n \}$ of its positions. One of the main problems in the asymptotic theory of finite dimensional normed spaces is introducing the right position of the unit ball $K_X$ of a space $X$. There exist many well-known positions which have been introduced and used for different purposes in this theory: John’s position, the $\ell$-position, $M$-positions are among them (see [MSch1], [Pi2] and [TJ] for a description and important applications). Because of the isomorphic nature of the results of the asymptotic theory, an isomorphic point of view dominates the study of these special positions as well. Even the definition of some of them (the $M$-position is such an example) is done in isomorphic form.

The purpose of this paper is to discuss the possibility of an isometric approach to these questions. The standard isotropic position of a convex body provides a good example for our point of view:

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Let $K$ be a convex body in $\mathbb{R}^n$ with centroid at the origin and volume equal to one. We say that $K$ is in isotropic position if

$$\int_K |x, \theta|^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$. It is not hard to see that every body $K$ of volume one has a position which is isotropic. Moreover, this position is uniquely determined up to an orthogonal transformation. Therefore, $L_K$ is an affine invariant which is called the isotropic constant of $K$.

The isotropic position is well studied and has several connections with classical convexity problems (see [MP1]). In particular, the question if $L_K \leq c$ for some absolute positive constant and every body $K$ is a major open problem. The starting point of our present discussion is the following remark:

**Fact I** A body $K$ is isotropic if and only if $\int_K |x|^2 dx \leq \int_K |x|^2 dx$ for every $T \in SL_n$, where $| \cdot |$ is the standard Euclidean norm.

The proof of the “if” part is given by a simple variational argument: If $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\varepsilon > 0$ is small enough, then $(I + \varepsilon T)/[\det(I + \varepsilon T)]^{1/n}$ is volume preserving, therefore

$$\int_K |x + \varepsilon T x|^2 dx \geq \frac{\det(I + \varepsilon T)^{2/n}}{\int_K |x|^2 dx}$$

Writting $|x + \varepsilon T x|^2 = |x|^2 + 2\varepsilon \langle x, Tx \rangle + O(\varepsilon^2)$ and $[\det(I + \varepsilon T)]^{2/n} = 1 + \frac{\text{tr}(T)\varepsilon}{n} + O(\varepsilon^2)$, and letting $\varepsilon \to 0^+$ we get

$$\int_K \langle x, Tx \rangle dx \geq \frac{\text{tr}(T)}{n} \int_K |x|^2 dx,$$

and replacing $T$ by $-T$ we see that there must be equality in (2) for every $T \in L(\mathbb{R}^n, \mathbb{R}^n)$. This in turn implies that $K$ is isotropic.

Starting with the functional $T \to f(TK) = \int_{TK} |x|^2 dx$ on $SL_n$ we saw that its minimum is achieved on some isotropic position (for the Lebesgue measure on $K$). In this paper we show that this is a general scheme which produces isometric descriptions for many classical positions of the theory.

As a second example, we mention the minimal surface area position: Let $K$ be a convex body, and write $\partial(K)$ for its surface area. We say that $K$ has minimal surface area if $\partial(K) \leq \partial(TK)$ for every $T \in SL_n$.

A characterization of the minimal surface area position was given by Petty ([Pe], see also [GP]):

**Fact II** A convex body $K$ has minimal surface area if and only if

$$\int_{S^{n-1}} \langle u, \theta \rangle^2 \sigma_K(du) = \frac{\partial(K)}{n}$$

for every $\theta \in S^{n-1}$, where $\sigma_K$ is the area measure of $K$. 

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Recall that the area measure \( \sigma_K \) of \( K \) is defined on \( S^{n-1} \) by
\[
\sigma_K(A) = \nu(\{x \in \text{bd}(K) : \text{the outer normal to } K \text{ at } x \text{ is in } A\}),
\]
where \( \nu \) is the \((n - 1)\)-dimensional surface measure on \( K \). The key point for the proof of the fact is the observation that
\[
\partial((T^{-1})^*K) = \int_{S^{n-1}} T x \sigma_K(\text{d}x).
\]

Then, we employ a variational argument identical to the one used for Fact 1. One can also check that the minimal surface position is unique up to orthogonal transformations (see [GP] for the details).

In view of the above result we give the following definition:

**Definition** A Borel measure \( \mu \) on \( S^{n-1} \) will be called isotropic if
\[
\int_{S^{n-1}} \langle u, \theta \rangle^2 \mu(\text{d}u) = \frac{\mu(S^{n-1})}{n}
\]
for every \( \theta \in S^{n-1} \).

In this terminology, a body \( K \) has minimal surface area if and only if its area measure is isotropic: The minimum of the functional \( T \to \partial(TK), T \in SL_n \) is again achieved on an isotropic position (for the appropriate measure on the sphere).

Surface area is one of the quermassintegrals \( W_i(K) \) of the body \( K \) (see Section 2 for notation and definitions). We consider the minimization problems
\[
\min \{ W_i(TK) \mid T \in SL_n \}, \quad i = 1, \ldots, n - 1.
\]
In every case, a necessary condition for the minimal position is that the corresponding mixed area measure \( S_{n-1}(K, \cdot) \) should be isotropic (see Section 4). In particular, in Section 3 we find a necessary and sufficient condition for the minimal mean width position: a body \( K \) has minimal mean width if and only if
\[
\int_{S^{n-1}} h_K(u) \langle u, \theta \rangle^2 \sigma(\text{d}u)
\]
does not depend on \( \theta \in S^{n-1} \), where \( h_K \) is the support function of \( K \) and \( \sigma \) is the rotationally invariant probability measure on the sphere. In the symmetric case, using a classical estimate of Pisier [Fil] (after work of Lewis [L] and Figiel and Tomczak-Jaegermann [FT]) we see that isotropicity of the measure \( h_K \sigma \text{d}x \) implies the inequality
\[
\int_{S^{n-1}} h_K(u) \sigma(\text{d}u) \leq c \log d(X_K, \ell_2^n) \left( \frac{|K|}{|D_n|} \right)^{1/n}.
\]

In Section 5 we see the maximal volume ellipsoid position (John’s position) as a solution of the problem
\[
\min \{ \| T : \ell_2^n \to X_K \| \mid T \in SL_n \}.
\]
Using the same general method we give a simple proof of John’s theorem in its full strength. In our present setting, John’s representation of the identity may be interpreted as an isotropic condition: a symmetric body $K$ is in John’s position if and only if there is an isotropic measure supported by its contact points with the inscribed ball.

Finally, in Section 6 we show that $M$-position may also be described in an isometric way. If $|K| = |D_n|$, we study the problem

\begin{equation}
\min \{ |T'K + D_n| : T \in SL_n \}
\end{equation}

and show that if $K$ is a solution, then $K + D_n$ must have minimal surface area. In view of Petty’s result, this opens the possibility of an isotropic $M$-position.

K. Ball [Ba1,2,3] realized that John’s representation of the identity could be combined with the Brascamp-Lieb inequality. This led him to sharp bounds for the volume ratio, the volume of the central sections of the cube, and an exact reverse isoperimetric inequality. The reverse Brascamp-Lieb inequality [Bar] has been recently applied for an estimate of the volume of the central sections of the difference body of a non-symmetric body [Ru]. Petty’s isotropic description of the minimal surface area position (combined with the Brascamp-Lieb inequality) leads to sharp inequalities for the volume of the projection body and its polar in terms of the minimal surface parameter [GP]. All these results show that the general isotropic point of view we propose in this paper might help towards a new understanding of several isomorphic results of the theory.

2 Definitions and preliminaries

We first recall some facts about mixed volumes and mixed area measures. For detailed proofs we refer the reader to [Sch].

2.1. Let $K_n$ denote the set of all non-empty, compact convex subsets of $\mathbb{R}^n$. We may view $K_n$ as a convex cone under Minkowski addition and multiplication by non-negative real numbers. Minkowski’s theorem (and the definition of the mixed volumes) asserts that if $K_1, \ldots, K_m \in K_n, m \in \mathbb{N}$, then the volume of $t_1 K_1 + \ldots + t_m K_m$ is a homogeneous polynomial of degree $n$ in $t_i > 0$. That is,

\begin{equation}
|t_1 K_1 + \ldots + t_m K_m| = \sum_{1 \leq i_1, \ldots, i_m \leq m} V(K_{i_1}, \ldots, K_{i_m}) t_{i_1} \ldots t_{i_m},
\end{equation}

where the coefficients $V(K_{i_1}, \ldots, K_{i_m})$ are chosen to be invariant under permutations of their arguments. The coefficient $V(K_1, \ldots, K_n)$ is called the mixed volume of $K_1, \ldots, K_n$.

2.2. Steiner’s formula may be seen as a special case of Minkowski’s theorem. The volume of $K + tD_n$, $t > 0$, can be expanded as a polynomial in $t$:

\begin{equation}
|K + tD_n| = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i,
\end{equation}
where $W_i(K) = V(K; n - i, D_n; i)$ is the $i$-th quermassintegral of $K$. Here and elsewhere we use the notation $L; j$ for $L, \ldots, L$ $j$-times. The quermassintegrals inherit properties of mixed volumes: they are monotone, continuous with respect to the Hausdorff metric, and homogeneous of degree $n - i$.

2.3. The mixed area measures were introduced by Alexandrov [All2] and may be viewed as a local generalization of the mixed volumes. For any $(n - 1)$-tuple $\mathcal{C} = K_1, \ldots, K_{n-1} \in \mathcal{K}_n$, the Riesz representation theorem guarantees the existence of a Borel measure $S(\mathcal{C}, \cdot)$ on the unit sphere $S^{n-1}$ such that

$$V(L, K_1, \ldots, K_{n-1}) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS(\mathcal{C}, u)$$

for every $L \in \mathcal{K}_n$, where $h_L$ is the support function of $L$. The local analogue of Minkowski’s theorem is

$$S_{n-1}(\sum_{i=1}^{m} t_i K_i, \omega) = \sum_{1 \leq i_1, \ldots, i_m \leq m} S(K_{i_1}, \ldots, K_{i_m}, \omega)t_{i_1} \ldots t_{i_m},$$

for all Borel $\omega \subseteq S^{n-1}$, $t_i > 0, K_i \in \mathcal{K}_n$, $m \in \mathbb{N}$ (see below for the definition of $S_{n-1}$).

The $j$-th area measure of $K$ is defined by $S_j(K, \cdot) = S(K; j, D_n; n - j - 1, \cdot)$, $j = 0, 1, \ldots, n - 1$. It follows that the quermassintegrals of $K$ can be represented by

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS_{n-i-1}(K, u), \quad i = 0, 1, \ldots, n - 1$$

or, alternatively,

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} dS_{n-i}(K, u), \quad i = 1, \ldots, n.$$

2.4. Let $K_i \in \mathcal{K}_n$ and assume for simplicity that $h_{K_i}$ is twice continuously differentiable. Then, the mixed area measure of $K_1, \ldots, K_{n-1}$ has a continuous density $s(K_1, \ldots, K_{n-1}, \cdot)$ with respect to the Lebesgue measure on $S^{n-1}$, the mixed discriminant of the second differentials of $h_{K_i}$. We write $s_j(K, u)$ for $s(K; j, D_n; n - j - 1, u)$. It follows that

$$\int_{S^{n-1}} h_{K_1}(u)s(K_2, K_3, \ldots, K_n, u)du = \int_{S^{n-1}} h_{K_2}(u)s(K_1, K_3, \ldots, K_n, u)du.$$

In particular, for $i = 1, \ldots, n - 1$ we have

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} s_{n-i}(K, u)du = \frac{1}{n} \int_{S^{n-1}} h_K(u)s_{n-i-1}(K, u)du.$$
2.5. Let \( f \) be a real function on \( \mathbb{R}^n \setminus \{o\} \). We write \( \hat{f} \) for the restriction of \( f \) to \( S^{n-1} \). If \( F \) is defined on \( S^{n-1} \), the radial extension \( f \) of \( F \) to \( \mathbb{R}^n \setminus \{o\} \) is given by \( f(x) = F(x/|x|) \). If \( F \) is a twice differentiable function on \( S^{n-1} \), we define

\[
\Delta_o F = (\hat{\Delta} f) \quad \text{and} \quad \nabla_o F = (\hat{\nabla} f),
\]

where \( f \) is the radial extension of \( F \). The operator \( \Delta_o \) is usually called the Laplace-Beltrami operator, while \( \nabla_o \) is referred to as the gradient. As a consequence of Green’s formula we have

\[
\int_{S^{n-1}} F \Delta_o G = \int_{S^{n-1}} G \Delta_o F = -\int_{S^{n-1}} (\nabla_o F) \cdot (\nabla_o G).
\]

For more details we refer the reader to [Gr].

2.6. If \( K \) is an origin symmetric convex body in \( \mathbb{R}^n \), then \( K \) induces a norm \( \| \cdot \|_K \) on \( \mathbb{R}^n \) in a natural way. We shall write \( X_K \) for the normed space with unit ball \( K \), and \( K_X \) for the unit ball of \( X \). The polar body of \( K \) is defined by

\[
\|x\|_K^* = \max_{y \in K} |\langle x, y \rangle| = h_K(x),
\]

and will be denoted by \( K^c \).

We consider the average

\[
M(K) = \int_{S^{n-1}} \|x\|_K \sigma(dx)
\]

of the norm \( \| \cdot \|_K \) on \( S^{n-1} \), and define \( M^*(K) = M(K^c) \).

If \( K \) and \( L \) are bodies in \( \mathbb{R}^n \), their multiplicative distance \( d(K, L) \) is defined by

\[
d(K, L) = \inf \{ab : a, b > 0, K \subseteq bL, L \subseteq aK\}.
\]

The Banach-Mazur distance between \( X_K \) and \( X_L \) is

\[
d(X_K, X_L) = \inf \{d(K, TL) : T \in GL_n\}.
\]

Whenever we write \( (1/a)|x| \leq \|x\|_K \leq b|x| \), we assume that \( a, b \) are the smallest positive numbers for which this inequality holds true for every \( x \in \mathbb{R}^n \). In particular, we then have \( d(K, D_n) = ab \).

Finally, we denote by \( G_{n,k} \) the Grassmannian of all \( k \)-dimensional subspaces of \( \mathbb{R}^n \), equipped with the Haar probability measure \( \nu_{n,k} \). We write \( |K| \) for the volume of \( K \), and \( \omega_n \) for the volume of the Euclidean unit ball. The letters \( c, c', C \) etc. are reserved for absolute positive constants.

3 Minimal mean width

Let \( K \) be a convex body in \( \mathbb{R}^n \) (without loss of generality we may assume that \( o \in \text{int}K \)). The mean width \( w(K) \) of \( K \) is the quantity

\[
w(K) = 2 \int_{S^{n-1}} h_K(u)\sigma(du).
\]
This is equal to $2M^*(K)$ in the symmetric case. From 2.3 we see that

$$W_{n-1}(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) du = \omega_n \int_{S^{n-1}} h_K(u) \sigma(du),$$

hence,

$$w(K) = \frac{2W_{n-1}(K)}{\omega_n}.$$

We say that $K$ has minimal mean width if $w(K) \leq w(TK)$ for every $T \in SL_n$. This notion was heavily used in the literature under a different name: $K$ has minimal mean width if and only if the $L$-ellipsoid of $K^0$ is a multiple of $D_n$ [FT]. Our purpose is to find necessary and sufficient conditions for a body $K$ to have minimal mean width. We assume for simplicity that $h_K$ is twice continuously differentiable (we then say that $K$ is smooth enough).

**Theorem 3.1** A smooth enough convex body $K$ in $\mathbb{R}^n$ has minimal mean width if and only if

$$2 \int_{S^{n-1}} \langle \nabla h_K(u), Tu \rangle \sigma(du) = \frac{\text{tr} T}{n} w(K)$$

for every $T \in L(\mathbb{R}^n, \mathbb{R}^n)$. Moreover, this minimal mean width position is unique up to an orthogonal transformation.

**Proof:** Assume first that $K$ has minimal mean width. Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then $(I + \varepsilon T)^* [\det(I + \varepsilon T)]^{1/n}$ is volume preserving, and this means that

$$\int_{S^{n-1}} h_K(u + \varepsilon Tu) \sigma(du) \geq [\det(I + \varepsilon T)]^{1/n} \int_{S^{n-1}} h_K(u) \sigma(du).$$

Since $h_K(u + \varepsilon Tu) = h_K(u) + \varepsilon \langle \nabla h_K(u), Tu \rangle + O(\varepsilon^2)$ and $[\det(I + \varepsilon T)]^{1/n} = 1 + \varepsilon \frac{\text{tr} T}{n} + O(\varepsilon^2)$, letting $\varepsilon \to 0^+$ we obtain

$$2 \int_{S^{n-1}} \langle \nabla h_K(u), Tu \rangle \sigma(du) \geq \frac{\text{tr} T}{n} w(K).$$

Replacing $T$ by $-T$ in (6) we see that there must be equality in (4) for every $T \in L(\mathbb{R}^n, \mathbb{R}^n)$.

Conversely, assume that (4) is satisfied and let $T \in SL_n$. Up to an orthogonal transformation we may assume that $T^*$ is symmetric positive-definite. Then,

$$w(TK) = 2 \int_{S^{n-1}} h_{TK}(u) \sigma(du) = 2 \int_{S^{n-1}} h_K(T^*u) \sigma(du).$$

It is a known fact that $\nabla h_K(u)$ is the unique point on the boundary of $K$ at which $u$ is the outer normal to $K$ (see [Sch], pp.40). In particular, $\nabla h_K(u) \in K$, which implies

$$\langle \nabla h_K(u), z \rangle \leq h_K(z)$$
for every \( z \in \mathbb{R}^n \). Therefore, by (7), (8) and (4) we get

\[
(9) \quad w(TK) \geq 2 \int_{S^{n-1}} \langle \nabla h_K(u), T^* u \rangle \sigma(du) = \frac{\text{tr}T^*}{n} w(K) \geq w(K).
\]

This shows that \( K \) has minimal mean width. Moreover, we can have equality in (9) only if \( T \) is the identity. This proves uniqueness of the minimal mean width position up to \( U \in O(n) \). \( \square \)

Consider the measure \( \nu_K \) on \( S^{n-1} \) with density \( h_K \) with respect to \( \sigma \). We shall prove that a smooth enough convex body \( K \) has minimal mean width if and only if \( \nu_K \) is isotropic.

**Lemma 3.2** Let \( K \) be a smooth enough convex body in \( \mathbb{R}^n \). We define

\[
(10) \quad I_K(\theta) = \int_{S^{n-1}} \langle \nabla h_K(u), \theta \rangle \langle u, \theta \rangle \sigma(du), \quad \theta \in S^{n-1}.
\]

Then,

\[
(11) \quad \frac{w(K)}{2} + I_K(\theta) = (n+1) \int_{S^{n-1}} h_K(u) \langle u, \theta \rangle^2 \sigma(du)
\]

for every \( \theta \in S^{n-1} \).

**Proof:** Let \( \theta \in S^{n-1} \), and consider the function \( f(x) = \langle x, \theta \rangle^2 / 2 \). A direct computation shows that

\[
(12) \quad (\nabla \hat{f})(u) = \langle u, \theta \rangle - \langle u, \theta \rangle^2 u
\]

and

\[
(13) \quad (\Delta \hat{f})(u) = 1 - n \langle u, \theta \rangle^2.
\]

Since \( h_K \) is positively homogeneous of degree 1, we have \( (\nabla \hat{h_K})(u) = \nabla h_K(u) - h_K(u)u \) and \( h_K(u) = \langle \nabla h_K(u), u \rangle, \ u \in S^{n-1} \). Taking into account (12) we obtain

\[
(14) \quad \langle (\nabla \hat{f})(u), (\nabla \hat{h_K})(u) \rangle = \langle \nabla h_K(u), \theta \rangle \langle u, \theta \rangle - h_K(u) \langle u, \theta \rangle^2.
\]

Integrating on the sphere and using Green’s formula (see 2.5), we have

\[
(15) \quad I_K(\theta) - \int_{S^{n-1}} h_K(u) \langle u, \theta \rangle^2 \sigma(du) = - \int_{S^{n-1}} h_K(u) (\Delta \hat{f})(u) \sigma(du),
\]

which is equal to

\[
- \frac{w(K)}{2} + n \int_{S^{n-1}} h_K(u) \langle u, \theta \rangle^2 \sigma(du)
\]

by (13). This proves (11). \( \square \)

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Theorem 3.3 A smooth enough convex body $K$ has minimal mean width if and only if
\begin{equation}
\int_{S^{n-1}} h_K(u) \langle u, \theta \rangle^2 \sigma(du) = \frac{w(K)}{2n}
\end{equation}
for every $\theta \in S^{n-1}$ (equivalently, if $\nu_K$ is isotropic).

Proof: It is not hard to check that (4) is true for every $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ if and only if
\begin{equation}
I_K(\theta) = \frac{w(K)}{2n}
\end{equation}
for every $\theta \in S^{n-1}$. The result now follows from Theorem 3.1 and Lemma 3.2. \qed

Remark. The smoothness assumption in Theorem 3.3 is not really needed. Assume for example that $K$ is any convex body for which $\nu_K$ is isotropic. Given $\varepsilon > 0$, we may approximate $K$ by a smooth body $K_\varepsilon$, so that $I_{K_\varepsilon}(\theta)$ is up to $\varepsilon$ constant on $S^{n-1}$. If $T_{\varepsilon}(K_\varepsilon)$ has minimal mean width for some symmetric and positive $T_{\varepsilon} \in SL_n$, we easily check from (9) that $tr T \leq (1 + O(\varepsilon))n$, and the stability of the arithmetic-geometric means inequality implies that $T_{\varepsilon}$ is close to the identity. Passing to the limit as $\varepsilon \to 0^+$ and taking into account the fact that $T_{\varepsilon}(K_\varepsilon)$ has minimal mean width, we see that $K$ has the same property. The other direction can be treated in a similar way.

The fact that (4) and (16) are linear in $K$ has the following immediate consequence:

Corollary 3.4 Let $K_1$ and $K_2$ be smooth enough convex bodies in $\mathbb{R}^n$.

(i) If $K_1$ and $K_2$ have minimal mean width, then their Minkowski sum $K_1 + K_2$ has also minimal mean width.

(ii) If $K_1 + K_2$ has minimal mean width, then $K_2$ has also minimal mean width.

Proof: Obvious from Theorem 3.1 or 3.3, since $h_{K_1 + K_2} = h_{K_1} + h_{K_2}$ and $w(K_1 + K_2) = w(K_1) + w(K_2)$. \qed

In the symmetric case, it is a well-known fact [L], [FT], [Pil] that if $W$ has minimal mean width then $M(W)M^*(W) \leq c \log d(\mathcal{X}_W, \ell_2^n)$. As an application of this estimate and of Corollary 3.4 we obtain:

Theorem 3.5 Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$ and assume that its unit ball $K$ has the property $M(K) \leq M(TK)$ for every $T \in SL_n$. Then, for every $\lambda \in (0, 1)$, there exists a $[(1 - \lambda)n]$-dimensional section $K \cap E$ of $K$ such that
\begin{equation}
d(K \cap E, D_n \cap E) \leq c \frac{b}{M \sqrt{\lambda}} \log \left( \frac{2b}{M \sqrt{\lambda}} \right),
\end{equation}
where $c > 0$ is an absolute constant.
Proof: Without loss of generality we may assume that \( M(K) = 1 \) and \( \lambda < 1/2 \).
Let \( t_0 \) be the smallest integer \( t \) for which \( \log^{(t)}(bM^*(K)) \leq 2 \) (where \( \log^{(t)} \) denotes the \( t \)-th iterated logarithm). The Low \( M^* \)-estimate [M1, PT, Go] implies that, for some absolute constant \( \delta > 0 \),

\[
\|x\| \geq \frac{\delta \sqrt{\lambda/2^{t_0}}}{M^*(K)} |x|
\]

for all \( x \in E_0 \) or \( x \in E_0^\perp \), where \( E_0 \) is in a subset \( L_0 \) of \( G_{n_0,|\{1-2^{-t_0}\lambda|n|} \) of measure greater than \( p(\lambda, n, t_0) = 1 - c_1 \exp(-c_2 2^{-t_0} \lambda n) \), and \( c_1, c_2 > 0 \) are absolute constants.

Consider the orthogonal transformation \( U = U(E_0) = P E_0 - P E_0^\perp \), \( E_0 \in L_0 \). Then,

\[
\frac{\|x\| + \|Ux\|}{2} \geq \frac{\delta \sqrt{\lambda/2^{t_0}}}{\sqrt{2} M^*(K)} |x|
\]

for all \( x \in \mathbb{R}^n \). Define a new body \( K_1 = K_1(E_0) \) by \( K_1^o = K_1^{u+U^*K^o} \). Then, by Corollary 3.4, \( K_1^o \) has minimal mean width equal to \( M(K_1) = 1 \). It follows that

\[
M^*(K_1) = M(K_1) M^*(K_1) \leq c \log \left( \frac{\sqrt{2^t M^*(K_1)}}{\delta \sqrt{\lambda}} \right).
\]

Observe that \( \|x\|_{K_1} = \|x\|_{K_1(E_0)} \) on \( E_0 \), for every \( E_0 \in L_0 \).

We now iterate this step: assume that \( L_i \subset G_{n_i,|\{1-2^{-t_{i+1}}\lambda|n|} \), \( E_i \in L_i \), and \( K_{i+1}(E_0, \ldots, E_i), i = 0, \ldots, s - 1 \) have been defined and satisfy the following

(i) \( (K_{i+1})^o \) has minimal mean width, and \( M(K_{i+1}) = 1 \).
(ii) \( M^*(K_{i+1}) \leq c \log(\sqrt{2^{t_{i+1} - 1} M^*(K_i)} b/\delta \sqrt{\lambda}) \).
(iii) \( \|x\|_{K_{i+1}} = \|x\|_{K_i} = \ldots = \|x\| \), for all \( x \in F_i = E_0 \cap \ldots \cap E_i \).

We apply the Low \( M^* \)-estimate to \( K_s \), and find \( L_s \subset G_{n_s,|\{1-2^{-t_{s+1}}\lambda|n|} \) with measure \( p(\lambda, n, t_0 - s) \) such that

\[
b|x| \geq \|x\|_{K_s} \geq \frac{\delta \sqrt{\lambda/2^{t_0 - s}}}{M^*(K_s)} |x|
\]

on \( E_s \) and on \( E_s^\perp \), for every \( E_s \in L_s \). If \( E_s \in L_s \), we define \( K_{s+1} \) by \( K_{s+1}^o = K_s^{u+U^*(E_s)K_s^o} \). Then,

\[
b|x| \geq \|x\|_{K_{s+1}} \geq \frac{\delta \sqrt{\lambda/2^{t_0 - s}}}{\sqrt{2} M^*(K_s)} |x|
\]

on \( \mathbb{R}^n \), and

\[
\|x\|_{K_{s+1}} = \|x\|_{K_s} = \ldots = \|x\|
\]
for every $x \in F_s = E_0 \cap \ldots \cap E_s$. This means that

\begin{equation}
(25) \quad d(K \cap F_s, D_n \cap F_s) \leq \delta \sqrt{2^{n-s+1}/\lambda bM^*(K_s)}.
\end{equation}

We stop the procedure when $s = t_0$. Note that if $(E_0, E_1, \ldots, E_{t_0})$ is a sequence as above, we have $\dim F_{t_0} \geq (1 - \lambda)n$. Also, since each $(K_s)^{c}$ has minimal mean width, exactly as in (21) we get

\begin{equation}
(26) \quad M^*(K_{s+1}) \leq c \log \left( \frac{\sqrt{2^{n-s+1}M^*(K_s)b}}{\delta \sqrt{\lambda}} \right),
\end{equation}

and this implies that $M^*(K_{t_0}) \leq C \log \left( \frac{b}{\sqrt{\lambda}} \right)$. By (25), we have

\begin{equation*}
 d(K \cap F_{t_0}, D_n \cap F_{t_0}) \leq \frac{b}{\sqrt{\lambda}} \log \left( \frac{b}{\lambda} \right). \quad \square
\end{equation*}

Theorem 3.5 should be compared to an analogous result for the $M$-position: In [MSch2] it is proved that if $K$ is in $M$-position of order $\alpha > 1/2$, and if there exist $t$ orthogonal transformations $U_1, \ldots, U_t$ such that $\frac{1}{t} \sum_{i=1}^{t} U_i K^c$ is $c$-equivalent to a ball, then for every $\lambda \in (0, 1)$ there exists a subspace $F \in G_{n,(1-\lambda)n}$ such that $d(K \cap F, D_n \cap F) \leq C(t, \lambda, c)$. We can now show that the same is true for the minimal mean width position:

**Corollary 3.6** Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$ and assume that its unit ball $K$ has the property $M(K) \leq M(TK)$ for every $T \in SL_n$. Assume further that for some $t$ orthogonal transformations $U_1, \ldots, U_t$ and for some $0 < r, C < \infty$,

\begin{equation}
(27) \quad r |x| \leq \frac{1}{t} \sum_{i=1}^{t} ||U_i x|| \leq C r |x|
\end{equation}

for all $x \in \mathbb{R}^n$. Then, for every $\lambda \in (0, 1)$, there exists a $\lfloor (1 - \lambda)n \rfloor$-dimensional section $K \cap E$ of $K$ such that

\begin{equation}
(28) \quad d(K \cap E, D_n \cap E) \leq C \frac{r}{\sqrt{\lambda}} \log \left( \frac{2C \sqrt{t}}{\sqrt{\lambda}} \right).
\end{equation}

**Proof:** Lemma 2.1 from [MSch2] and (27) imply that

\begin{equation}
(29) \quad b(K) = \max_{x \in S^{n-1}} ||x|| \leq C r \sqrt{t}.
\end{equation}

Since $M(K) \geq r$, we have

\begin{equation}
(30) \quad \frac{b(K)}{M(K)} \leq C \sqrt{t}.
\end{equation}

The result is now a consequence of Theorem 3.5. \quad \square
An inspection of the argument we used for Theorem 3.5 shows that the statement holds true for a random \([(1-\lambda)n]\)-dimensional section of \(K\). This allows a “global” reformulation of Corollary 3.6:

**Corollary 3.7** With the same hypotheses as in Corollary 3.6, there exists one orthogonal transformation \(U\) such that, for some \(r' > 0\),

\[
  r'|x| \leq \|x\| + \|Ux\| \leq r'C\sqrt{t}\log(2C\sqrt{t})|x|
\]

for all \(x \in \mathbb{R}^n\).

The example of \(X = \ell_1^{n/10} \oplus \ell_\infty^{n/10}\) from [MSch2] shows that such a statement cannot hold in general.

Let \(t(K)\) be the smallest integer \(t\) for which there exist orthogonal transformations \(U_1, \ldots, U_t\) such that

\[
  \frac{M(K)}{2}|x| \leq \frac{1}{t} \sum_{i=1}^t \|U_i x\| \leq 2M(K)|x|
\]

for all \(x \in \mathbb{R}^n\). In [MSch2] it is shown that \(t(K) \simeq (b/M(K))^2\). We will prove below an “isomorphic” version of this fact for bodies in \(\ell\)-position. We fix \(s \in \{2, \ldots, t(K)\}\) and ask how close to Euclidean can a norm \(\|x\|_s = \frac{1}{s} \sum_{i=1}^s \|U_i x\|\), \(U_i \in O(n)\) be. More precisely, let \(g_K(s)\) be the smallest \(A > 0\) for which there exist \(r > 0\), \(m \leq s\), and \(U_1, \ldots, U_m \in O(n)\) satisfying

\[
r|x| \leq \frac{1}{m} \sum_{i=1}^m \|U_i x\| \leq rA|x|, \quad x \in \mathbb{R}^n.
\]

From Lemma 2.1 in [MSch2] (see also the proof of Corollary 3.6), we must have \(b(K) \leq rA\sqrt{m} \leq M(K)A\sqrt{s}\). This shows that

\[
g_K(s) \geq c\sqrt{t(K)/s}.
\]

We shall show that if \(K\) has minimal mean width (if \(K\) has minimal \(M\)), then this estimate is sharp.

**Theorem 3.8** Let \(\|\cdot\|\) be a norm on \(\mathbb{R}^n\) such that its unit ball \(K\) satisfies \(M(K) \leq M(TK), T \in SL_n\). Then,

\[
c_1 \sqrt{\frac{t(K)}{s}} \leq g_K(s) \leq c_2 \sqrt{\frac{t(K)}{s}} \log \left(\frac{2t(K)}{s}\right),
\]

where \(c_1, c_2 > 0\) are absolute constants.

**Proof:** Let \(s \in \{2, \ldots, t(K)\}\), and set \(b = b(K), M = M(K)\). Following the proof of Theorem 2 in [BLM], one can check that there exist \(s_1 = \lfloor s/2 \rfloor\) and \(U_1, \ldots, U_{s_1} \in O(n)\) such that

\[
  \|x\|_{s_1} := \frac{1}{s_1} \sum_{i=1}^{s_1} \|U_i x\| \leq c' \frac{b}{\sqrt{s_1}} |x| \leq c' \frac{b}{\sqrt{s}} |x|
\]

\(\Box\)
for all $x \in \mathbb{R}^n$. Let $K_1$ be the unit ball of $\|\cdot\|_s$, and set $b_1 = b(K_1)$, $M_1 = M(K_1)$. Since $M_1 = M$, (34) implies that

$$t(K_1) \leq c^s t(K)/s.$$  (35)

Observe that $K_1$ has minimal $M$, therefore we can apply Corollary 3.7 with $C = 4$ and $t = t(K_1)$ to find $r > 0$ and $V \in O(n)$ such that

$$r|x| \leq \|x\|_{s_1} + \|Vx\|_{s_1} \leq c^m r \sqrt{t(K_1)} \log(2t(K_1))|x|$$

for all $x \in \mathbb{R}^n$. Setting $U_{s_1+i} = U_iV$, $i = 1, \ldots, s_1$, and taking into account (35) we conclude the proof. \hspace*{1cm} \Box

Remark. Let $k(K)$ be the largest integer $k$ for which a random $k$-dimensional central section of $K$ is $4$-equivalent to Euclidean. In [Msch2] it is proved that $\frac{1}{\log n} \leq t(K)k(K) \leq Cn$, where $C > 0$ is an absolute constant. Having this duality in mind, one may view Theorem 3.8 as a global analogue (for bodies with minimal $M$) of the isomorphic version of Dvoretzky’s theorem proved in [Msch3] (see also [GGM]): There exists a constant $c > 0$ such that, for every $k \geq c \log n$ every $n$-dimensional space $K$ has a $k$-dimensional subspace $F$ with $d(F, \ell_2^k) \leq c \sqrt{k/\log(n/k)}$.

Let us also mention the following common property of the $M$-position and the minimal mean width position: If both $a/M^*$ and $b/M$ are bounded by some constant $C$, then the space is $f(C)$-isomorphic to $\ell_2^s$. This is proved in [MSch2] for the $M$-position, and follows from Pisier’s inequality

$$MM^* \leq c \log(ab) \leq c \log(C^2 MM^*)$$  (37)

for the minimal mean width position. The space $X = \ell_1^{n/2} \oplus \ell_\infty^{n/2}$ shows that the position of the unit ball is crucial for this statement as well.

We close this section with a variation of the minimal mean width position. Consider a symmetric convex body $K$ in $\mathbb{R}^n$, and the problem of minimizing $M(TK)M^*(TK)$ over all $T \in SL_n$. Repeating the procedure of Theorems 3.1 and 3.3 we obtain the following condition for the minimum position:

**Theorem 3.9** Let $K$ be a symmetric convex body in $\mathbb{R}^n$, and assume that $M(K)M^*(K) \leq M(TK)M^*(TK)$ for every $T \in SL_n$. Then,

$$\frac{1}{M} \int_{S^{n-1}} \|u\|_{K} \langle u, \theta \rangle^2 \sigma(du) = \frac{1}{M^*} \int_{S^{n-1}} \|u\|_{K^*} \langle u, \theta \rangle^2 \sigma(du),$$

for every $\theta \in S^{n-1}$.

**Proof:** Without loss of generality we may assume that $K$ is smooth enough. Let $R \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\varepsilon > 0$ be small enough, and write $T^{-1} = I + \varepsilon R$. Then, $T^* = (I + \varepsilon R^*)^{-1} = I + \sum_{k=1}^{\infty} (-1)^k \varepsilon^k (R^k)^*$, and our assumption about $K$ takes the form

$$M(K)M^*(K) \leq \int_{S^{n-1}} \|u + \varepsilon Ru\|_K \sigma(du) \int_{S^{n-1}} \|u - \varepsilon Ru\|_{K^*} \sigma(du) + O(\varepsilon^2),$$

13
which implies
\[ M M' \leq \left( M + \varepsilon \int_{S^{n-1}} \langle \nabla h_K(u), Ru \rangle \right) \left( M' - \varepsilon \int_{S^{n-1}} \langle \nabla h_K(u), R^* u \rangle \right) + O(\varepsilon^2). \]

Letting \( \varepsilon \to 0^+ \) and replacing \( R \) by \( -R \), we have
\[ \frac{1}{M} \int_{S^{n-1}} \langle \nabla h_K^*(u), Ru \rangle \sigma(du) = \frac{1}{M'} \int_{S^{n-1}} \langle \nabla h_K(u), R^* u \rangle \sigma(du) \]
for every \( R \in GL_n \). Using (40) with \( R_0(x) = \langle x, \theta \rangle \theta, \theta \in S^{n-1} \), we get
\[ \frac{1}{M} \int_{S^{n-1}} \langle \nabla h_K^*(u), \theta \rangle \langle u, \theta \rangle \sigma(du) = \frac{1}{M'} \int_{S^{n-1}} \langle \nabla h_K(u), \theta \rangle \langle u, \theta \rangle \sigma(du) \]
for every \( \theta \in S^{n-1} \). Taking into account Lemma 3.2, we conclude the proof. □

We do not know if (38) implies the minimality condition of Theorem 3.9 (nevertheless, we find (38) quite appealing, since it demonstrates once again the deep relation between a body and its polar.

4 Quermassintegrals and volume preserving transformations

We say that a convex body \( K \) minimizes \( W_i \) if \( W_i(K) \leq W_i(TK) \) for every volume preserving linear transformation \( T \). Since \( nW_1(K) = \partial(K) \), a body \( K \) minimizes \( W_1 \) if and only if it has minimal surface area. Also, since \( 2W_{n-1}(K) = \omega_n w(K) \), a body \( K \) minimizes \( W_{n-1} \) if and only if it has minimal mean width.

Our purpose is to find necessary and sufficient conditions for a convex body \( K \) to minimize \( W_i, i = 1, \ldots, n - 1 \). We first show that such a body is a solution of a much more general problem:

Proposition 4.1 Let \( i = 1, \ldots, n - 1 \), and assume that the convex body \( K \) minimizes \( W_i \). Then,

\[ V(T_1 K, \ldots, T_{n-i} K, D_n; i) \geq W_i(K) \]

for any \( T_1, \ldots, T_{n-i} \in SL_n \).

Proof: We have \( W_i(T_j K) \geq W_i(K), j = 1, \ldots, n - i \). As a consequence of the Alexandrov-Fenchel inequality we see that

\[ V(T_1 K, \ldots, T_{n-i} K, D_n; i) \geq W_i(T_1 K)^{\frac{1}{n-i}} \ldots W_i(T_{n-i} K)^{\frac{i}{n-i}}, \]

and this proves our claim. □
The arguments we used for the surface area and the mean width apply to every quermassintegral and provide necessary conditions for the minimal position:

**Proposition 4.2** Assume that $K$ is smooth enough and minimizes $W_i$. Then,

$$
\int_{S_{n-i}} \langle \nabla h_K(u), Ru \rangle dS_{n-i-1}(K, u) = [\text{tr} R] W_i(K)
$$

for any $R \in L(\mathbb{R}^n, \mathbb{R}^n)$.

**Proof:** Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then, $(I + \varepsilon T)/[\det(I + \varepsilon T)]^{1/n}$ is volume preserving. Therefore,

$$
[\det(I + \varepsilon T)]^{\frac{1}{n}} W_i(K) \leq V((I + \varepsilon T)K; n - i, D_n; i).
$$

Since $(I + \varepsilon T)K \subseteq K + \varepsilon TK$, using the monotonicity of the mixed volumes we get

$$
[\det(I + \varepsilon T)]^{\frac{1}{n}} W_i(K) \leq V(K + \varepsilon TK; n - i, D_n; i).
$$

We have $[\det(I + \varepsilon T)]^{\frac{1}{n}} = 1 + \varepsilon \frac{1}{n} \text{tr} T + O(\varepsilon^2)$, and linearity of the mixed volumes with respect to its arguments shows that $V(K + \varepsilon TK; n - i, D_n; i) = W_i(K) + (n - i)\varepsilon V(TK, K; n - i - 1, D_n; i) + O(\varepsilon^2)$. Letting $\varepsilon \to 0^+$ we see that

$$
\frac{\text{tr} T}{n} W_i(K) \leq V(TK, K; n - i - 1, D_n; i) = \frac{1}{n} \int_{S_{n-i}} h_{TK}(u) dS_{n-i-1}(K, u).
$$

Now, let $R \in L(\mathbb{R}^n, \mathbb{R}^n)$ and set $T^* = I + \varepsilon R$ where $\varepsilon > 0$. Since $h_{TK}(u) = h_K(T^* u) = h_K(u + \varepsilon Ru)$, we get

$$
W_i(K) + \varepsilon \frac{\text{tr} R}{n} W_i(K) \leq \frac{1}{n} \int_{S_{n-i}} h_{K}(u + \varepsilon Ru) dS_{n-i-1}(K, u).
$$

But, $h_K(u + \varepsilon Ru) = h_K(u) + \varepsilon \langle \nabla h_K(u), Ru \rangle + O(\varepsilon^2)$, so letting $\varepsilon \to 0^+$ and using (2.5), we have

$$
\frac{\text{tr} R}{n} W_i(K) \leq \frac{1}{n} \int_{S_{n-i}} \langle \nabla h_K(u), Ru \rangle dS_{n-i-1}(K, u).
$$

Replacing $R$ by $-R$ we get the reverse inequality, therefore

$$
[\text{tr} R] W_i(K) = \int_{S_{n-i}} \langle \nabla h_K(u), Ru \rangle dS_{n-i-1}(K, u)
$$

for every $R \in L(\mathbb{R}^n, \mathbb{R}^n)$.

**Proposition 4.3** Let $i = 1, \ldots, n - 1$. If a convex body $K$ in $\mathbb{R}^n$ minimizes $W_i$, then $S_{n-i}(K, \cdot)$ is isotropic.

**Proof:** Assume that $K$ minimizes $W_i$. For every $U \in SL_n$ we have

$$W_i(UK) = V(K; n - i, U^{-1} D_n; i) \geq W_i(K)$$

for every $R \in L(\mathbb{R}^n, \mathbb{R}^n)$. □
Let \( T \in L(\mathbb{R}^n, \mathbb{R}^n) \) and \( \varepsilon > 0 \) be small enough. Then, \( U^{-1} = (I + \varepsilon T)/[\det(I + \varepsilon T)]^{1/n} \) is volume preserving, therefore

\[
(11) \quad V(K; n - i, D_n + \varepsilon TD_n; i) \geq [\det(I + \varepsilon T)]^{i/n} W_i(K).
\]

Observe that the right hand side is \( W_i(K) + \frac{\varepsilon \text{tr} T}{n} W_i(K) + O(\varepsilon^2) \), while the left hand side is \( W_i(K) + \frac{\varepsilon}{n} V(K; n - i, D_n; i - 1, TD_n) + O(\varepsilon^2) \). Letting \( \varepsilon \to 0^+ \) and taking into account (2.3) we get

\[
(12) \quad \frac{1}{n} \int_{S^{n-1}} h_{TD_n}(u)dS_{n-i}(K, u) \geq \frac{\text{tr} T}{n} W_i(K)
\]

for every \( T \in L(\mathbb{R}^n, \mathbb{R}^n) \). Let \( R \in L(\mathbb{R}^n, \mathbb{R}^n) \) and set \( T^* = I + \varepsilon R \). We have \( h_{TD_n}(u) = |T^*u| = |u + \varepsilon Ru| = 1 + \varepsilon \langle u, Ru \rangle + O(\varepsilon^2) \), so (12) becomes

\[
(13) \quad \int_{S^{n-1}} \{1 + \varepsilon \langle u, Ru \rangle + O(\varepsilon^2) \}dS_{n-i}(K, u) \geq n W_i(K) + \varepsilon \text{tr} R W_i(K).
\]

Letting \( \varepsilon \to 0^+ \), using (2.6) and replacing \( R \) by \(-R\) we conclude that

\[
(14) \quad \int_{S^{n-1}} \langle u, Ru \rangle dS_{n-i}(K, u) = \text{tr} R W_i(K)
\]

for every \( R \in L(\mathbb{R}^n, \mathbb{R}^n) \). This shows that \( S_{n-i}(K, \cdot) \) is isotropic. \( \square \)

In order to proceed we need to introduce some terminology and notation. If \( A \) is a selfadjoint linear transformation of \( \mathbb{R}^n \), we denote by \( s_j(A) \) the \( j \)-th elementary symmetric function \( s_j(\lambda_1, \ldots, \lambda_n) \) of the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( A \):

\[
(15) \quad s_j(A) = \sum_{1 \leq k_1 < \cdots < k_j \leq n} \lambda_{k_1} \cdots \lambda_{k_j}.
\]

The \( j \)-th Newton operator of \( A \) is defined by

\[
(16) \quad T_j(A) = s_j(A)I - s_{j-1}(A)A + \cdots + (-1)^jA^j.
\]

We set \( s_0(A) = 1 \) and \( T_0(A) = I \). We also agree that \( T_j(A) = 0 \) if \( j < 0 \).

Some known properties of \( s_j(A) \) and \( T_j(A) \) are listed in the Proposition below (see e.g. Reilly [Re]):

**Proposition 4.4** Let \( A \in L(\mathbb{R}^n, \mathbb{R}^n) \) be selfadjoint, and assume that it has matrix \((a_{kl})\) with respect to some basis of \( \mathbb{R}^n \). Then, \( T_j(A) \) is selfadjoint and

(i) \( s_j(A) = \frac{1}{j!} \sum a_{k_1 \cdots k_j}a_{k_1 \cdots a_{k_j}l_j} \)

(ii) \( T_j(A)_{kl} = \frac{1}{j!} \sum a_{k_1 \cdots k_j}a_{k_1 \cdots a_{k_j}l_j} \)

(iii) \( \text{tr} (T_j(A) \circ A) = (j + 1)s_{j+1}(A) \)

(iv) \( T_j(A) = s_j(A)I - T_{j-1}(A) \circ A \)

(v) \( \text{tr}(T_j(A)) = (n - j)s_j(A) \).
Here, we denote by $\delta_{i_1 \ldots i_j}^{k_1 \ldots k_j}, 1 \leq j \leq n$, the Kronecker symbol which has the value +1 (respectively, −1) if $k_1, \ldots, k_j$ are distinct and $(l_1, \ldots, l_j)$ is an even (respectively, odd) permutation of $(k_1, \ldots, k_j)$. If not, then the symbol takes the value 0. □

We will also use the following consequence of Green’s formula (see [Fi]):

**Proposition 4.5** Let $f : \mathbb{R}^n \setminus \{o\} \to \mathbb{R}$ and $F : \mathbb{R}^n \setminus \{o\} \to \mathbb{R}^n$ be homogeneous functions of degree $p$ and $q$ respectively. Assume that $\nabla f$ and div $F$ are continuous. Then,

$$
(17) \quad \int_{S^{n-1}} f(u) \text{div} F(u) \sigma(du) = (p + q + n - 1) \int_{S^{n-1}} (f(u)F(u), u)\sigma(du)
$$

$$
- \int_{S^{n-1}} (\nabla f(u), F(u))\sigma(du). \quad \Box
$$

Note that Lemma 3.2 is a special case of Proposition 4.5: choose $f(x) = h_K(x)$ and $F(x) = \langle x, \theta \rangle \theta$.

Let $K$ be a convex body in $\mathbb{R}^n$, and assume that $h_K$ is a $C^2$-function. For every $x \in \mathbb{R}^n \setminus \{o\}$ the Hessian $H_x := (\nabla^2 h_K)_x$ of $h_K$ defines a selfadjoint linear transformation of $\mathbb{R}^n$. If $u \in S^{n-1}$, then $s_j(H_u) = s_j(K, u)$ (for simplicity we will write $s_j(u)$). In this context, one has the following additional properties of the Newton operator $T_j(H_u)$ (see [BH]):

**Proposition 4.6** Assume that $h_K$ has continuous partial derivatives of order three in $\mathbb{R}^n \setminus \{o\}$. Then,

(i) $(j + 1)s_{j+1}(x) = \text{div} [(T_j(H_x)) (\nabla ^2 h_K(x))], \quad j = 0, \ldots, n - 2.

(ii) $H_x (x) = 0$, $(T_j(H_x))(x) = s_j(x)x.

Combining the above results we obtain the following:

**Theorem 4.7** Let $K$ be a convex body in $\mathbb{R}^n$, whose support function $h_K$ is $C^3$. Then, for every $j = 0, 1, \ldots, n - 2$ and any $\theta \in S^{n-1}$, we have

$$
(18) \quad \int_{S^{n-1}} [(n + 1 - j)h_K(u)s_j(u) - (j + 1)s_{j+1}(u)]\langle u, \theta \rangle^2 \sigma(du)
$$

$$
= 2 \int_{S^{n-1}} \langle T_j(H_u) (\nabla ^2 h_K(u)), \theta \rangle \langle u, \theta \rangle \sigma(du).
$$

**Proof:** Let $f(x) = \langle x, \theta \rangle^2$. By Proposition 4.6(i),

$$
(19) \quad \int_{S^{n-1}} (j + 1)s_{j+1}(u)\langle u, \theta \rangle^2 \sigma(du) = \int_{S^{n-1}} f(u) \text{div} [(T_j(H_u)) (\nabla ^2 h_K(u))]\sigma(du).
$$

Since $f$ and $T_j$ are homogeneous of degree 2 and $-j$ respectively, Proposition 4.5 shows that this last integral is equal to

$$
(20) \quad (n + 1 - j) \int_{S^{n-1}} \langle T_j(\nabla h_K(u)), u \rangle \langle u, \theta \rangle^2 \sigma(du) - 2 \int_{S^{n-1}} \langle T_j(\nabla h_K(u)), \theta \rangle \langle u, \theta \rangle \sigma(du).
$$
To complete the proof, observe that since $T_j$ is selfadjoint by Proposition 4.6(ii) we have

$$
\langle T_j(\nabla h_K(u)), u \rangle = \langle \nabla h_K(u), T_j(u) \rangle = s_j(u)\langle \nabla h_K(u), u \rangle = s_j(u)h_K(u). \quad \Box
$$

Note first that Theorem 3.3 is a consequence of Theorem 4.7: When $j=0$, (18) takes the form

$$
(n+1) \int_{S^{n-1}} h_K(u)\langle u, \theta \rangle^2 \sigma(du)
$$

$$
= \int_{S^{n-1}} \langle u, \theta \rangle^2 dS_1(K,u) + 2 \int_{S^{n-1}} \langle \nabla h_K(u), \theta \rangle \langle u, \theta \rangle \sigma(du).
$$

By Theorem 3.1 and Proposition 4.3, the last two integrals are independent of $\theta \in S^{n-1}$, hence $\nu_K = h_Kd\sigma$ is isotropic.

We now consider the case $j=1$, which corresponds to the quermassintegral $W_{n-2}$:

**Theorem 4.8** Let $K$ be a convex body in $\mathbb{R}^n$, whose support function $h_K$ is $C^1$. If $K$ minimizes $W_{n-2}$, then the measures $s_2(u)\sigma(du)$ and $[h_K(u)s_1(u) + |\nabla h_K(u)|^2] \sigma(du)$ are isotropic.

**Proof:** We have $T_1(\mathcal{H}_u)(\nabla h_K(u)) = s_1(u)\nabla h_K(u) - \mathcal{H}_u(\nabla h_K(u))$. Then, Theorem 4.7 implies that for every $\theta \in S^{n-1}$,

$$
n \int_{S^{n-1}} h_K(u)s_1(u)\langle u, \theta \rangle^2 \sigma(du) + 2 \int_{S^{n-1}} \langle \mathcal{H}_u(\nabla h_K(u)), \theta \rangle \langle u, \theta \rangle \sigma(du)
$$

$$
= 2 \int_{S^{n-1}} \langle u, \theta \rangle^2 dS_2(K,u) + 2 \int_{S^{n-1}} \langle \nabla h_K(u), \theta \rangle \langle u, \theta \rangle dS_1(K,u).
$$

Assume that $K$ minimizes $W_{n-2}$. By Propositions 4.2 and 4.3, the expression on the right hand side of (22) does not depend on $\theta$. On the other hand, it is easy to check that

$$
2\mathcal{H}_u(\nabla h_K(u)) = \nabla (|\nabla h_K(u)|^2).
$$

Applying Proposition 4.5 with $F(x) = \langle x, \theta \rangle$ and $f(x) = |\nabla h_K(x)|^2$, we get

$$
\int_{S^{n-1}} |\nabla h_K(u)|^2 \sigma(du) = n \int_{S^{n-1}} |\nabla h_K(u)|^2 \langle u, \theta \rangle^2 \sigma(du)
$$

$$
-2 \int_{S^{n-1}} \langle \mathcal{H}_u(\nabla h_K(u)), \theta \rangle \langle u, \theta \rangle \sigma(du).
$$

Inserting this into (22) we see that

$$
\int_{S^{n-1}} [s_1(u)h_K(u) + |\nabla h_K(u)|^2] \langle u, \theta \rangle^2 \sigma(du)
$$

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does not depend on \( \theta \). This completes the proof. \( \square \)

Using the same tools one can obtain analogous necessary isotropic conditions for the position which minimizes each quermassintegral. It is an interesting question to determine a set of necessary and sufficient isotropic conditions for the position minimizing \( W_i, i = 2, \ldots, n - 2 \).

5 John’s theorem

A classical result of F. John [Jo] states that \( d(X, \ell_2^n) \leq \sqrt{n} \) for every \( n \)-dimensional normed space \( X \), where \( \ell_2^n \) is Euclidean space, and \( d \) stands for the Banach-Mazur distance. One comes up with this estimate while studying the following extremal problem:

Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \). Maximize \( \|\det T\| \) over all \( T : \ell_2^n \to X = X_K \) with \( \|T\| = 1 \).

If \( T_0 \) is a solution of this problem, then \( T_0D_n \) is the ellipsoid of maximal volume which is inscribed in \( K \). One can easily establish existence and uniqueness of such an ellipsoid. In the spirit of our discussion, we may equivalently formulate the problem as follows:

Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \). Minimize \( \|T : \ell_2^n \to X_K\| \) over all volume preserving transformations \( T \).

We shall see that our standard variational argument provides all the available information about this “maximal volume ellipsoid position”. In particular, one may naturally interpret the well-known John’s representation of the identity as an isotropic condition.

To this end, assume that the identity map \( I \) is a solution of the problem, and normalize so that

\[
\|I : \ell_2^n \to X_K\| = 1 = \min\{\|T : \ell_2^n \to X_K\| : \|\det T\| = 1\}.
\]

This means that the Euclidean unit ball \( D_n \) is the maximal volume ellipsoid of \( K \). Our first result provides a necessary “trace condition” on \( K \):

**Theorem 5.1** Let \( K \) be a smooth enough symmetric convex body in \( \mathbb{R}^n \) and assume that \( D_n \) is the maximal volume ellipsoid of \( K \). Then, for every \( T \in L(\mathbb{R}^n, \mathbb{R}^n) \) we can find a contact point \( x \) of \( K \) and \( D_n \) such that

\[
\langle x, Tx \rangle \geq \frac{\text{tr}T}{n}.
\]

**Proof:** Let \( S \in L(\mathbb{R}^n, \mathbb{R}^n) \). We shall first show that there exists a contact point \( x \) of \( K \) and \( D_n \) such that

\[
\|Sx\|_K \geq \frac{\text{tr}S}{n}.
\]
Let $\varepsilon > 0$ be small enough. From (1) we have

$$
(4) \quad \|I + \varepsilon S : \ell_2^n \to X_K\| \geq \left[\det (I + \varepsilon S) \right]^{1/n} = 1 + \varepsilon \frac{\text{tr} S}{n} + O(\varepsilon^2).
$$

Choose any $x_\varepsilon \in S^{n-1}$ such that $\|x_\varepsilon + \varepsilon S x_\varepsilon\|_K = \|I + \varepsilon S\|$. Since $D_n \subseteq K$, we have $\|x_\varepsilon\|_K \leq 1$. Therefore, combining (4) with the triangle inequality for $\| \cdot \|_K$ we see that

$$
(5) \quad \|S x_\varepsilon\|_K \geq \frac{\text{tr} S}{n} + O(\varepsilon).
$$

By compactness, we may find $x \in S^{n-1}$ and a sequence $\varepsilon_m \to 0$ such that $x_{\varepsilon_m} \to x$. By (5) we obviously have $\|S x\|_K \geq \frac{\text{tr} S}{n}$. On the other hand,

$$
(6) \quad \|x\|_K = \lim_{m \to \infty} \|x_{\varepsilon_m} + \varepsilon_m S x_{\varepsilon_m}\|_K = \lim_{m \to \infty} \|I + \varepsilon_m S\| = \|I\| = 1.
$$

This shows that $x$ is a contact point of $K$ and $D_n$, which proves (3).

Now, let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and write $S = I + \varepsilon T$, $\varepsilon > 0$. We can find $x_\varepsilon$ such that $\|x_\varepsilon\|_K = \|x_\varepsilon\|_K = 1$ and

$$
(7) \quad \|x_\varepsilon + \varepsilon T x_\varepsilon\|_K \geq \frac{\text{tr}(I + \varepsilon T)}{n} = 1 + \varepsilon \frac{\text{tr} T}{n}.
$$

We write $\|x_\varepsilon + \varepsilon T x_\varepsilon\|_K = 1 + \varepsilon \langle \nabla \|x_\varepsilon\|_K, T x_\varepsilon \rangle + O(\varepsilon^2)$, and from (7) we get

$$
\langle \nabla \|x_\varepsilon\|_K, T x_\varepsilon \rangle \geq \frac{\text{tr} T}{n} + O(\varepsilon).
$$

Choosing again $\varepsilon_m \to 0^+$ such that $x_{\varepsilon_m} \to x \in S^{n-1}$, we see that $x$ is a contact point of $K$ and $D_n$ which satisfies

$$
(8) \quad \langle \nabla \|x\|_K, T x \rangle \geq \frac{\text{tr} T}{n}.
$$

Moreover, since $\nabla \|x\|_K$ is the point on the boundary of $K$ at which the outer unit normal is parallel to $x$ and $x$ is a contact point of $K$ and $D_n$, we must have $\nabla \|x\|_K = x$. This proves the theorem. \qed

From Theorem 5.1 we can easily recover all the well-known properties of the maximal volume ellipsoid:

**Theorem 5.2** Let $D_n$ be the maximal volume ellipsoid of $K$. Then, $K \subset \sqrt{n}D_n$.

**Proof:** Let $x \in \mathbb{R}^n$ and consider the map $Ty = \langle y, x \rangle x$. By Theorem 5.1, we can find a contact point $z$ of $K$ and $D_n$ such that

$$
(9) \quad \langle z, Tz \rangle \geq \frac{\text{tr} T}{n} = \frac{\|x\|^2}{n}.
$$

But,

$$
(10) \quad \langle z, Tz \rangle = \langle z, x \rangle^2 \leq \|z\|_K^2 \|x\|^2_\mathcal{K} = \|x\|^2_\mathcal{K}.
$$

Therefore, $\|x\| \leq \sqrt{n}\|x\|_K$. This is equivalent to the assertion of the theorem. \qed

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Theorem 5.2 provides the estimate $d(X, \ell^2_2) \leq \sqrt{n}$ for the Banach-Mazur distance from an arbitrary $n$-dimensional normed space to $\ell^2_2$. From Theorem 5.1 we can also deduce the Dvoretzky-Rogers lemma:

**Theorem 5.3** Let $D_n$ be the maximal volume ellipsoid of $K$. There exist pairwise orthogonal vectors $y_1, \ldots, y_n$ in $\mathbb{R}^n$ such that

$$\left(\frac{n - i + 1}{n}\right)^{1/2} \leq \|y_i\|_{\mathcal{K}} \leq |y_i| = 1, \quad i = 1, \ldots, n.$$

**Proof:** We define the $y_i$'s inductively. The first vector $y_1$ can be any of the contact points of $K$ and $D_n$. Assume that $y_1, \ldots, y_{i-1}$ have been defined. Let $F_i = \text{span}\{y_1, \ldots, y_{i-1}\}$. Then, $\text{tr}(P_{F_i^\perp}) = n - i + 1$, and by Theorem 5.1 there exists a contact point $x_i$ such that

$$|P_{F_i^\perp} x_i|^2 = \langle x_i, P_{F_i^\perp} x_i \rangle \geq \frac{n - i + 1}{n}. \quad (11)$$

It follows that $\|P_{F_i^\perp} x_i\| \leq |P_{F_i^\perp} x_i| \leq \sqrt{(i-1)/n}$. We set $y_i = P_{F_i^\perp} x_i / |P_{F_i^\perp} x_i|$. Then,

$$1 = |y_i| \geq \|y_i\|_{\mathcal{K}} \geq \langle x_i, y_i \rangle = |P_{F_i^\perp} x_i| \geq \left(\frac{n - i + 1}{n}\right)^{1/2}. \quad \square \quad (12)$$

Note that the argument shows that for every $k$-dimensional subspace $F$ there exists a contact point $x$ of $K$ and $D_n$ such that $|P_F x|^2 = \langle x, P_F x \rangle \geq k/n$.

Finally, a separation argument and Theorem 5.1 give us John's representation of the identity.

**Theorem 5.4** Let $D_n$ be the maximal volume ellipsoid of $K$. There exist contact points $x_1, \ldots, x_m$ of $K$ and $D_n$ and positive real numbers $\lambda_1, \ldots, \lambda_m$ such that

$$I = \sum_{i=1}^m \lambda_i x_i \otimes x_i.$$

**Proof:** Consider the convex hull $C$ of all operators $x \otimes x$, where $x$ is a contact point of $K$ and $D_n$. One can easily see that the assertion of the theorem is equivalent to $I/n \in C$. If this is not true, there exists $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\langle T, I/n \rangle > \langle x \otimes x, T \rangle \quad (13)$$

for every contact point $x$. But, $\langle T, I/n \rangle = \text{tr}T/n$ and $\langle x \otimes x, T \rangle = \langle x, Tx \rangle$. Therefore, (13) would contradict Theorem 5.1. \quad \square

Theorem 5.4 implies that

$$\sum_{i=1}^m \lambda_i \langle x_i, \theta \rangle^2 = 1 \quad (14)$$
for every $\theta \in S^{n-1}$. In our terminology, the measure $\mu$ on $S^{n-1}$ that gives mass $\lambda_i$ to the point $x_i$, $i = 1, \ldots, m$, is isotropic. In this sense, John’s position is an isotropic position. Conversely, following [Ba4] we have:

**Proposition 5.5** Let $K$ be a symmetric convex body in $\mathbb{R}^n$ which contains the Euclidean unit ball $D_n$. Assume that there exists an isotropic Borel measure $\mu$ on $S^{n-1}$ which is supported by the contact points of $K$ and $D_n$. Then, $D_n$ is the maximal volume ellipsoid of $K$.

**Proof:** Let $\|\mu\| = \mu(S^{n-1})$ and $A \subset S^{n-1}$ be the support of $\mu$. Define

\begin{equation}
L = \{ y \in \mathbb{R}^n : |\langle x, y \rangle| \leq 1, x \in A \}.
\end{equation}

Since $K \subseteq L$, it clearly suffices to prove that $D_n$ is the maximal volume ellipsoid of $L$. Let

\begin{equation}
E = \{ y \in \mathbb{R}^n : \sum_{j=1}^{n} \alpha_j^{-2} \langle y, v_j \rangle^2 \leq 1 \},
\end{equation}

where $\{v_j\}$ is an orthonormal basis of $\mathbb{R}^n$ and $\alpha_j > 0$. Assume that $E \subseteq L$. For every $x \in A$ we have

\begin{equation}
y(x) = \left( \sum_{j=1}^{n} \alpha_j^2 \langle x, v_j \rangle^2 \right)^{-1/2} \sum_{j=1}^{n} \alpha_j \langle x, v_j \rangle v_j \in E \subseteq L,
\end{equation}

hence, $|\langle x, y(x) \rangle| \leq 1$ gives

\begin{equation}
\sum_{j=1}^{n} \alpha_j^2 \langle x, v_j \rangle^2 \leq 1, \quad x \in A.
\end{equation}

Our hypotheses imply that

\begin{equation}
\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \alpha_j \|\mu\| \int_{S^{n-1}} \langle x, v_j \rangle^2 \mu(dx)
\end{equation}

\begin{equation}
= \frac{n}{\|\mu\|} \int_{S^{n-1}} \sum_{j=1}^{n} \alpha_j \langle x, v_j \rangle^2 \mu(dx).
\end{equation}

Using (18) and the Cauchy-Schwarz inequality we see that

\begin{equation}
\sum_{j=1}^{n} \alpha_j \langle x, v_j \rangle^2 \leq \left( \sum_{j=1}^{n} \alpha_j^2 \langle x, v_j \rangle^2 \right)^{1/2} \left( \sum_{j=1}^{n} \langle x, v_j \rangle^2 \right)^{1/2} \leq 1
\end{equation}

for every $x \in A$. Then, (19) becomes

\begin{equation}
\sum_{j=1}^{n} \alpha_j \leq n.
\end{equation}

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By the arithmetic-geometric means inequality we get $\prod \alpha_j \leq 1$. That is $|E| \leq |D_n|$. Moreover, we can have equality only if all $\alpha_j$’s are equal to 1, which shows that $D_n$ is the unique maximal volume ellipsoid of $L$. □

Theorem 5.4 and Proposition 5.5 provide the following characterization of John’s position:

“Let $K$ be a symmetric convex body in $\mathbb{R}^n$ which contains the Euclidean unit ball $D_n$. Then, $D_n$ is the maximal volume ellipsoid of $K$ if and only if there exists an isotropic measure $\mu$ supported by the contact points of $K$ and $D_n$.”

Let us discuss one more problem of the same nature: Let $K$ be a symmetric convex body in $\mathbb{R}^n$ and $\| \cdot \|$ be the corresponding norm. Assume that $(1/\alpha)||x|| \leq \|x|| \leq \beta||x||$ for every $x \in \mathbb{R}^n$. It is clear that $M(K)1(K) \geq 1$, and we are interested in

$$\min \{ M(TK) | T \in GL_n, a(TK) = 1 \}. $$

The condition $a(TK) = 1$ means that $TK \subseteq D_n$ but there exist contact points of $TK$ and $D_n$. We then have the following condition for the minimum position:

**Theorem 5.6** Let $K$ be a symmetric convex body in $\mathbb{R}^n$ satisfying $a(K) = 1$ and $M(K) \leq M(TK)$ for every $T \in GL_n$ with $a(TK) = 1$. Then, for every $\theta \in S^{n-1}$ we can find contact points $x_1, x_2$ of $K$ and $D_n$ such that

$$1 + \langle x_1, \theta \rangle^2 \leq \frac{n + 1}{M} \int_{S^{n-1}} ||u||_K (u, \theta)^2 \sigma(du) \leq 1 + \langle x_2, \theta \rangle^2. $$

**Proof:** Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then $T_1 := (\min_{S^{n-1}} ||x + \varepsilon Tx||)(I + \varepsilon T)^{-1}$ satisfies $a(T_1 K) = 1$. Therefore,

$$\int_{S^{n-1}} ||u + \varepsilon Tu|| \sigma(du) \geq M(K) \min_{x \in S^{n-1}} ||x + \varepsilon Tx||. $$

If we write $||u + \varepsilon Tu|| = ||u|| + \varepsilon \langle \nabla h_{K^*}(u), Tu \rangle + O(\varepsilon^2)$, we see that

$$\int_{S^{n-1}} \langle \nabla h_{K^*}(u), Tu \rangle \sigma(du) + O(\varepsilon) \geq M(K) \min_{x \in S^{n-1}} ||x + \varepsilon Tx|| - 1. $$

Let $x_c$ be a point on $S^{n-1}$ at which the minimum is attained. If $x$ is a contact point of $K$ and $D_n$, we must have $1 + \varepsilon ||T|| \geq ||x + \varepsilon Tx|| \geq ||x_c + \varepsilon Tx_c|| \geq ||x_c|| - \varepsilon ||T||$, where $||T|| := ||T : l_2^n \rightarrow X_K||$. It follows that

$$1 \leq ||x_c|| \leq 1 + 2\varepsilon ||T||. $$

Since $x_c \in S^{n-1}$ and $|| \cdot || \geq | \cdot |$, (25) takes the form

$$\int_{S^{n-1}} \langle \nabla h_{K^*}(u), Tu \rangle \sigma(du) + O(\varepsilon) \geq M(K) \frac{||x_c + \varepsilon Tx_c|| - 1}{\varepsilon} $$

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Now, we can find a sequence \( \varepsilon_m \to 0 \) and a point \( x \in S^{n-1} \) such that \( x_{\varepsilon_m} \to x \). Letting \( m \to \infty \) in (27), we obtain

\[
\int_{S^{n-1}} \langle \nabla h_{K^*}(u), Tu \rangle \sigma(du) \geq M(K) \langle x, Tx \rangle.
\]

Also, \( x \in S^{n-1} \) and using (26) we see that \( \|x\| = \lim_{m} \|x_{\varepsilon_m}\| = 1 \). That is, \( x \) is a contact point of \( K \) and \( D_n \). Replacing \( T \) by \( -T \) we find another contact point \( x' \) of \( K \) and \( D_n \) such that

\[
\int_{S^{n-1}} \langle \nabla h_{K^*}(u), Tu \rangle \sigma(du) \leq M(K) \langle x', Tx' \rangle.
\]

Choosing \( T_\theta(x) = \langle x, \theta \rangle \theta, \theta \in S^{n-1} \), and applying Lemma 3.2, we obtain (23). \( \square \)

The condition of the Theorem shows in a sense that the minimum position of the problem is rich in contact points with the circumscribed ball. The dual problem of maximizing \( M \) under the condition \( b = 1 \) has exactly the same answer.

6 Minimal surface area and \( M \)-position

If \( K \) and \( L \) are convex bodies in \( \mathbb{R}^n \), we write \( N(K, L) \) for the covering number of \( K \) by \( L \) (that is, the minimum number of translates of \( L \) whose union covers \( K \)). If \( |K| = |D_n| \), we say that \( K \) is in \( M \)-position (with parameter \( \delta > 0 \)) if

\[
N(K, D_n) \leq \exp(\delta n).
\]

One can then prove (see [MP2] for the non-symmetric case) that

\[
N(K, D_n) \cdot N(D_n, K) \cdot N(K^\circ, D_n) \cdot N(D_n, K^\circ) \leq \exp(\delta_1 n),
\]

where \( \delta_1 = c\delta \), and \( c > 0 \) is an absolute constant. Moreover, condition (1) is equivalent to

\[
|K + D_n|^{1/n} \leq c|D_n|^{1/n}.
\]

This isomorphically defined position is the best representative of the affine class of a body in volume computations: this is mainly due to the fact that reverse Brunn-Minkowski inequalities hold for bodies in \( M \)-position [M2].

We define a function \( f : [0, +\infty) \to \mathbb{R} \) by

\[
f(t) = \min\{|TK + tD_n| \mid T \in SL_n \}.
\]

For every \( t > 0 \) there exists a volume preserving \( T_t \) such that \( |T_tK + tD_n| = f(t) \). It is clear that \( UT_t \) has the same property for every \( U \in O(n) \). By (3) we see
that $T_i K$ is in $M$-position. This suggests that $M$-position can be described as the
solution of a minimum problem similar to the ones we discussed in the previous
sections.

We start with the following observation:

**Lemma 6.1** Let $K$ be a convex body in $\mathbb{R}^n$. Then,

\begin{equation}
|K + tA_1 D_n + sA_2 D_n| \geq \min \{ |K + (t + s)A_1 D_n|, |K + (t + s)A_2 D_n| \}
\end{equation}

for every $A_1, A_2 \in GL_n$ and $t, s > 0$.

**Proof:** It is an immediate consequence of the Brunn-Minkowski inequality, since

\begin{equation}
K + tA_1 D_n + sA_2 D_n \supseteq \frac{t}{t + s} (K + (t + s)A_1 D_n) + \frac{s}{t + s} (K + (t + s)A_2 D_n) \quad \square
\end{equation}

**Theorem 6.2** Let $K$ be a convex body in $\mathbb{R}^n$. Assume that

\begin{equation}
|K + tD_n| = f(t)
\end{equation}

for some $t > 0$. Then, $K + tD_n$ has minimal surface area.

**Proof:** Let $T \in SL_n$. From Steiner’s formula we see that

\begin{equation}
|T(K + (t - \varepsilon)D_n) + \varepsilon D_n| - |T(K + (t - \varepsilon)D_n)|
= n\varepsilon W_1(T(K + (t - \varepsilon)D_n)) + o(\varepsilon^2).
\end{equation}

By the continuity of $W_1$ with respect to the Hausdorff metric,

\begin{align}
\partial(T(K + tD_n)) &= nW_1(T(K + tD_n)) = n \lim_{\varepsilon \to 0^+} W_1(T(K + (t - \varepsilon)D_n)) \\
&= \lim_{\varepsilon \to 0^+} \frac{|T(K + (t - \varepsilon)D_n) + \varepsilon D_n| - |T(K + (t - \varepsilon)D_n)|}{\varepsilon} \\
&= \lim_{\varepsilon \to 0^+} \frac{|K + (t - \varepsilon)D_n + \varepsilon T^{-1}D_n| - |K + (t - \varepsilon)D_n|}{\varepsilon}
\end{align}

Since $|K + tD_n| = f(t)$, Lemma 6.1 implies that $|K + (t - \varepsilon)D_n + \varepsilon T^{-1}D_n| \geq |K + tD_n|$. Hence,

\begin{equation}
\partial(T(K + tD_n)) \geq \lim_{\varepsilon \to 0^+} \frac{|K + tD_n| - |K + (t - \varepsilon)D_n|}{\varepsilon} = \partial(K + tD_n).
\end{equation}

This shows that $K + tD_n$ has minimal surface area. \quad \square

**Remark.** It is not hard to show that

\begin{equation}
f'(t) = \partial(T_i K + tD_n)
\end{equation}
for every $t > 0$. It follows that for every $t > s > 0$ we have

$$
\int_s^t \partial (T_w K + x D_n) dx \geq \int_s^t \partial (T_0 K + x D_n) dx,
$$

with equality if $s = 0$.

In the planar case, a convex body $K$ has minimal perimeter (surface area) if and only if it has minimal mean width. Since $|T_t K + t D_n| = f(t)$, Theorem 6.2 shows that $T_t K + t D_n$ has minimal mean width and, using Corollary 3.4(ii) we see that $T_t K$ has minimal mean width. Moreover, $T_t$ is constant up to an orthogonal transformation. That is, the solution of Problem (4) is the minimal mean width position, independently of $t > 0$.

**Corollary 6.3** A convex body $K$ in $\mathbb{R}^2$ satisfies $|K + t D_n| \leq |TK + t D_n|$ for every $T \in SL_n$ and every $t > 0$ if and only if it has minimal mean width. \(\square\)

It would be interesting to see if the minimal surface area position is an $M$-position in higher dimensions. This would provide an isometric description of the $M$-position. Observe that, by Theorem 6.2, the limit of $T_t K$ as $t \to 0^+$ is the minimal surface position and, by Steiner’s formula, the limit of $T_t K$ as $t \to +\infty$ is the minimal mean width position.

### References


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