Mean width and diameter of proportional sections of a symmetric convex body

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1 Introduction

Let $K$ be a symmetric convex body in $\mathbb{R}^n$. The purpose of this paper is to provide upper and lower bounds for the diameter of a random $[\lambda n]$-dimensional central section of $K$, where the proportion $\lambda \in (0, 1)$ is arbitrary but fixed. There are several aspects of our approach to this question that should be clarified right away:

1. We are interested in bounds expressed in terms of average parameters of the body $K$ which can be efficiently computed in a simple way, therefore being useful from the computational geometry point of view.

2. The dimension $n$ is understood to tend to infinity. Then, we say that our bounds hold for a random $[\lambda n]$-dimensional section of $K$ if they are satisfied by all $K \cap \xi$ where $\xi$ is in a subset of the appropriate Grassmannian with Haar probability measure greater than $1 - h(\lambda, n)$, and $h(\lambda, n) \to 0$ as $n$ tends to infinity.

3. We say that our estimates are tight for a class of bodies and a fixed proportion $\lambda$ if the ratio of our upper and lower bounds depends only on $\lambda$. It is clear that one cannot obtain tight bounds for the class of all symmetric convex bodies: it is not hard to describe almost degenerated bodies in $\mathbb{R}^n$ (for example, an ellipsoid with highly incomparable semiaxes) for which the diameter of $[\lambda n]$-sections does not concentrate around some value. So, it is an important question to see under what conditions on $K$ the estimates obtained by a method are tight.

We use the standard notation of the asymptotic theory of finite dimensional normed spaces (which can be found in [MS]). We consider a fixed Euclidean structure in $\mathbb{R}^n$ and write $|.|$ for the corresponding Euclidean norm. We denote the Euclidean unit ball and the unit sphere by $D_n$ and $S^{n-1}$ respectively, and we write $\sigma$ for the rotationally invariant probability measure on $S^{n-1}$.

If $W$ is a symmetric convex body in $\mathbb{R}^n$, then $W$ induces in a natural way a norm $\|.|_W$ on $\mathbb{R}^n$. As usual, the polar body $\{y \in \mathbb{R}^n : \max_{x \in W} |\langle y, x \rangle| \leq 1\}$ of $W$ is denoted by $W^\circ$. An important average parameter of $W$ is the average of the norm $\|.|_W$ on $S^{n-1}$, defined by

$$(1.1) \quad M(W) = \int_{S^{n-1}} \|\theta\|_W \sigma(d\theta).$$

With this notation, the quantity $M^*(W) := M(W^\circ)$ has a natural geometric meaning: it is half the mean width of $W$. 

1
Suppose that $K$ is a symmetric convex body in $\mathbb{R}^n$ such that $\frac{1}{2}D_n \subseteq K \subseteq aD_n$. Let $\lambda \in (0, 1)$ and set $k = [\lambda n]$. If $G_{n,k}$ is the Grassmannian of all $k$-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,k}$ and if $\xi \in G_{n,k}$, we have

\begin{equation}
M^*(K \cap \xi) = \int_{S(\xi)} \|\theta\|_{(K \cap \xi)^{\perp}} \sigma_\xi(d\theta) = \int_{S(\xi)} \max_{x \in K \cap \xi} |\langle x, \theta \rangle| \sigma_\xi(d\theta),
\end{equation}

and we can naturally define the function $S_K^* : (0, 1) \to (0, \infty)$ with

\begin{equation}
S_K^*(\lambda) = \int_{G_{n,k}} M^*(K \cap \xi) \nu_{n,k}(d\xi).
\end{equation}

In other words, $S_K^*(\lambda)$ gives the average mean width of the $[\lambda n]$-dimensional central sections of $K$. It is not hard to check that $S_K^*$ is increasing in $\lambda$. In particular, $S_K^*(\lambda) \leq M^*(K)$ for every $\lambda \in (0, 1)$. We view $S_K^*(\lambda)$ as an average parameter of the body $K$, although it is computationally more complex than the single quantity $M^*(K)$: the empirical distribution method (described in a similar setting in [BLM]) shows that given any $\delta$ and $\xi$ in $(0, 1)$, a random choice of $N = \lceil c\frac{\log(\frac{2}{\delta})}{\xi^2} \rceil + 1$ points $x_1, \ldots, x_N$ in $S^{n-1}$ satisfies

\begin{equation}
|M^*(K) - \frac{1}{N} \sum_{i=1}^{N} \|x_i\|_{K^*}| < \zeta M^*(K)
\end{equation}

with probability at least $1 - \delta$, where $c > 0$ is an absolute constant. Therefore, $M^*(K)$ can be efficiently “computed” with high probability to any given degree of accuracy. The computation of $S_K^*(\lambda)$ is more complicated and depends on whether the values of the function $M^*(K \cap \xi)$ on $G_{n,k}$ are concentrated around their mean value $S_K^*(\lambda)$. We shall deal with this question in Section 4.

The function $S_K^*$ is clearly related to our problem on the diameter of the sections of $K$: for every $\xi \in G_{n,k}$ we have $\text{diam}(K \cap \xi) \geq 2M^*(K \cap \xi)$. Therefore, if we define the average diameter function

\begin{equation}
D_K(\lambda) = \int_{G_{n,k}} \text{diam}(K \cap \xi) \nu_{n,k}(d\xi),
\end{equation}

we have the obvious lower bound $2S_K^*(\lambda) \leq D_K(\lambda)$ for every $\lambda \in (0, 1)$. In Section 3 we assume that our body $K$ satisfies a polynomial condition of the form $ab \leq n^t$ for some fixed $t > 0$ and show that an upper bound for $D_K(\lambda)$ in terms of $S_K^*$ is also possible:

**Theorem A.** Let $\frac{1}{b}D_n \subseteq K \subseteq aD_n$ and $ab \leq n^t$ for some $t > 0$. If $\lambda \in (0, 1)$, we have

\[2S_K^*(\lambda) \leq D_K(\lambda) \leq 5S_K^*(\lambda/\theta)/(1 - \theta)^{1/2},\]

for all $\theta \in (\lambda, 1)$ with $1 - \theta \geq c\lambda^{-1}t \log n/a$, where $c > 0$ is an absolute constant.
The point in the statement above is that $\theta$ can be chosen to be very close to 1, provided that the dimension $n$ is large and $t$ is fixed. The polynomial condition $ab \leq n^t$ is mild and, roughly speaking, prevents the body $K$ from being degenerated. For example, all the well–known natural representatives of any affine class of symmetric convex bodies satisfy a condition of this type with a small value of $t$: when the ellipsoid of maximal or minimal volume or the distance ellipsoid of $K$ is a ball we have $ab \leq \sqrt{n}$, when $K$ is in the isotropic position, in the $\ell$–position, or in $M$–position of any order $\alpha > \frac{1}{2}$ we also have $ab \leq n^t$ for a suitable $t > 0$ independent from $n$.

The double–sided estimate given by Theorem A determines the average diameter $D_K(\lambda)$ for “most” values of $\lambda \in (0,1)$. As a consequence of the polynomial condition $ab \leq n^t$, our function $S^*_K$ is forced to increase in a regular way on most of $(0,1)$ and this implies that the bounds of Theorem A are tight: one has the a–priori information that $S^*_K(\lambda) \simeq D_K(\lambda)$ for most values of $\lambda$ up to a constant depending only on $\lambda$ and $t$.

The duality relation

\begin{equation}
S^*_K(\lambda)S^*_{K^*}(\mu) \leq \frac{c}{1 - (\lambda + \mu)}
\end{equation}

holds true for every body $K$ satisfying a polynomial condition and every $\lambda, \mu \in (0,1)$ with $\lambda + \mu < 1$, provided that $n$ is large enough. The proof of this inequality is based on the second named author’s “distance lemma” (see Theorem 3.4).

In Section 4 we study the question of the diameter of a “random” $[\lambda n]$–section of $K$. Passing from the average diameter $D_K(\lambda)$ to the diameter of most sections requires some strong concentration of the function $M^*(K \cap \xi)$ on $G_{n,[\lambda n]}$ around its expectation $S^*_K(\lambda)$. We study the behavior of $M^*(K \cap \xi)$ and show that it satisfies a certain Lipschitz estimate. The resulting deviation inequality is relatively weak, however in the important case where $ab$ is roughly speaking $o(\sqrt{n})$ (or more generally $bM^*(K) = o(\sqrt{n})$) we prove an analogue of Theorem A for random sections (see Section 4 for variations of this result and more precise conditions on $ab$):

**Theorem B.** Let $K$ satisfy $D_n \subseteq K \subseteq \gamma(n)D_n$ with $\gamma(n) = o(n^{1/2})$. Let $\lambda \in (0,1)$ and $k = [\lambda n]$. Then, for every $\theta \in (\lambda, 1)$ we have

$$c_1 S^*_K(\lambda \theta) \leq \text{diam}(K \cap \xi) \leq c_2 S^*_K(\lambda/\theta)/(1 - \theta)^{1/2}$$

for most $\xi \in G_{n,k}$, provided that $n$ is large enough (depending on $\theta$).

Using this fact one can determine the diameter of a random $[\lambda n]$–section of a body $K$ with $ab = o(n^{1/2})$ for many values of $\lambda$. More precisely, we consider any $\lambda$–flag of subspaces $\mathbb{R}^n = E_0 \supset E_1 \supset \ldots \supset E_s$ with $\dim E_j = [\lambda n]$, $s = s(\lambda) \simeq \log[(1 - \lambda)n]/\log(1/\lambda)$ and prove that for most orthogonal transformations $T \in O(n)$ and most values of $j$, the diameter of $K \cap T(E_j)$ is determined by

$$\text{diam}(K \cap T(E_j)) \simeq S^*_K(\lambda^j)$$
up to a constant depending only on \( \lambda \). (Theorem 4.8). This is of interest being a statement for a generic body \( K \) whose minimal/maximal volume or distance ellipsoid is a Euclidean ball.

In Section 5 we study the case of a body in \( M \)–position of order \( \alpha \) (an \( \alpha \)–regular body in the terminology of \([P1]\); see the beginning of Section 5 for the necessary definitions). In this case, we determine \( S_K^*(\lambda) \) up to a constant depending only on \( \lambda \) and \( \alpha \):

**Theorem C.** Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \), which is in \( M \)–position of order \( \alpha > 1/2 \). For every \( \lambda \in (0, 1) \),

\[
c_1 \lambda^\alpha \operatorname{v.rad}(K) \leq S_K^*(\lambda) \leq c_2 (1 - \lambda)^{-\alpha} \operatorname{v.rad}(K),
\]

where \( \operatorname{v.rad}(K) = (|K|/|D_n|)^{1/n} \) and \( c_1, c_2 > 0 \) are constants depending only on \( \alpha \).

Actually, the left hand side inequality holds true in the much stronger form

\[
M^*(K \cap \xi) \geq c_1 \lambda^\alpha \operatorname{v.rad}(K)
\]

for every \( \xi \in G_{n, \lfloor \lambda n \rfloor} \), while the right hand side inequality holds with \( M^*(K \cap \xi) \) replacing \( S_K^*(\lambda) \) for most \( \xi \in G_{n, \lfloor \lambda n \rfloor} \). One may view Theorem C as a proportional version of Urysohn’s inequality

\[
M^*(K) \geq \operatorname{v.rad}(K)
\]

for bodies in \( M \)–position of order \( \alpha \), which turns out to be an equivalence in this case: we have \( S_K^*(\lambda) \approx \operatorname{v.rad}(K) \) up to functions depending only on \( \lambda \) and \( \alpha \).

Using this information one determines \( \operatorname{diam}(K \cap \xi) \) for a random \( \xi \in G_{n,k} \) up to a function of \( \lambda \), for all \( \lambda \in (0, 1) \) (Theorem 5.3).

We close this paper with two upper bounds on the quantities \( S_K^*(\lambda) \) and \( M^*(K) \) in the case where \( K \) is in \( M \)–position of order \( \alpha \):

**Theorem D.** Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) in \( M \)–position of order \( \alpha > 1/2 \), and set \( \varepsilon = \alpha - 1/2 \).

(i) If \( 1 - \lambda \approx 1/\log n \), we have

\[
S_K^*(\lambda)S_{K^\circ}(\lambda) \leq c(\alpha) \log^{2\alpha} n.
\]

(ii) For the mean width of \( K \) we have the upper bound

\[
M^*(K) \leq c \varepsilon^{-5/4} n^{\varepsilon}.
\]

Both estimates in Theorem D should be compared to the following classical estimate of Pisier (see \([L]\), \([FT]\), and \([P2]\)): Every body \( K \) has a linear image \( K_1 \) such that \( M(K_1)M^*(K_1) \leq c \log n \). Part (i) of Theorem D demonstrates a regularity of the function \( F(\lambda) = S_K^*(\lambda)S_{K^\circ}(\lambda) \) in the \( M \)–position case: If \( K \) is in \( M \)–position of order \( \alpha \) (with \( \alpha \) even far from \( 1/2 \)), then the growth of \( F \) remains
logarithmic in \( n \) for \( \lambda \) even very close to 1. Pisier’s result implies that every body \( K \) has a linear image \( K_1 \) of volume \( |K_1| = |D_n| \) with mean width \( M^*(K_1) \leq c \log n \). Part (ii) of Theorem D shows that a body \( K \) in \( M^* \)-position of order \( \alpha \) sufficiently close to 1/2 has mean width logarithmic in \( n \). Simple examples show that for every value of \( \varepsilon = \alpha - \frac{1}{2} \), the \( n^\varepsilon \) bound for \( M^*(K) \) in Theorem D(ii) cannot be improved.

A different approach to the question of the diameter of random proportional sections was proposed in [GM1]. We briefly discuss this method in Section 2. Throughout the text, we compare the two methods whenever it is possible.

In the sequel, the letters \( c, c', c_1 \) etc. stand for absolute positive constants, not necessarily the same in every occurrence. The volume, the cardinality of a finite set and the Euclidean norm are all denoted by \( |.| \): this should cause no confusion.

## 2 The Low \( M^* \)-estimate and a first approach to the problem

This section is a survey of results from [GM1], [GM2] and describes the “\( M^*_K \)-approach” to the diameter problem.

A crucial inequality of the asymptotic theory of finite dimensional normed spaces is the second named author’s Low \( M^* \)-estimate which relates the diameter of proportional sections of a symmetric convex body \( W \) in \( \mathbb{R}^m \) to its mean width \( M^* (W) \). Roughly speaking, one has

\[
(2.1) \quad \text{diam}(W \cap \eta) \leq M^*(W)/h_1(\mu)
\]

for most \( \eta \in G_{m,[\mu m]} \), where \( h_1 \) is a function depending only on \( \mu \in (0,1) \). For proofs of (2.1) see [M1], [PT], [Go]: it is known that it holds true with \( h_1(\mu) = c(1 - \mu)^{1/2} \) and that this dependence on \( \mu \) is best possible. We shall make use of the precise probabilistic form of the Low \( M^* \)-estimate which can be found in [Go], [M4]:

### 2.1 Theorem (Low \( M^* \)-estimate)

If \( W \) is a symmetric convex body in \( \mathbb{R}^m \) and if \( \mu, \varepsilon \in (0,1) \), then we have

\[
(2.2) \quad \text{diam}(W \cap \eta) \leq \frac{2M^*(W)}{(1 - \varepsilon)^{1/\sqrt{1-\mu}}}
\]

for all \( \eta \) in a subset \( L_{m,k} \) of \( G_{m,k} \) of measure \( \nu_{m,k}(L_{m,k}) \geq 1 - c \exp(-c'\varepsilon^2(1-\mu)m) \), where \( k = [\mu m] \) and \( c, c' > 0 \) are absolute constants. \( \Box \)

Theorem 2.1 already shows that the diameter of a random section of proportional dimension is controlled by the mean width of the body. In [GM1] we exploit the idea of pushing the Low \( M^* \)-estimate to its limit in order to determine a reasonable “confidence interval” for the diameter of the \( [\lambda n] \)-sections of a body \( K \) in \( \mathbb{R}^n \) using average parameters of \( K \) with the same complexity as \( M^*(K) \).
To this end, we consider the function $M_K^* : \mathbb{R}^+ \to (0, 1]$ defined by

$$M_K^*(r) = M^*(K \cap rD_n)/r,$$

and as a simple consequence of Theorem 2.1 we see that if $r_1 > 0$ is the solution of the equation $M_K^*(r) = h_1(\lambda) = \frac{1}{2}(1 - \lambda)^{1/2}$ in $r$, then most $[\lambda n]$–sections of $K$ have diameter smaller than $2r_1$ (see [GM1], Theorem 2.1).

It turns out that this same function can provide a general lower bound for the diameter of the $[\lambda n]$–sections of $K$. The main new ingredient is a conditional Low $M$–estimate which is in a sense dual to Theorem 2.1:

2.2 Theorem (Conditional Low $M$–estimate) If $K$ is a symmetric convex body in $\mathbb{R}^n$ and if $\lambda \in (0, 1)$, then for the solution $r_2$ of the equation

$$M_K^*(r) = h_2(\lambda) := 1 - c \frac{\lambda}{1 - \lambda},$$

in $r$ we can find a subset $L_{n,k}$ of $G_{n,k}$ with $\nu_{n,k}(L_{n,k}) \geq 1 - c^k$, where $k = [\lambda n]$, such that

$$\text{diam}(K^o \cap \xi) \leq \frac{10}{r_2} C \lambda^{\frac{1}{1 - \lambda}}$$

for all $\xi \in L_{n,k}$, where $0 < c < 1$ and $C > 1$ are absolute constants, and $n$ is large enough. □

Theorem 2.2 shows that most $[\lambda n]$–projections of $K$ contain a Euclidean ball of radius proportional to $r_2$ up to a function depending only on $\lambda$. When $\lambda \in (\frac{1}{2}, 1)$, this fact combined with Borsuk’s antipodal theorem gives $r_2$ as a lower bound for the diameter of the $[\lambda n]$–sections of $K$. We thus have a double sided estimate of diam$(K \cap \xi)$ in terms of the function $M_K^*$:

2.3 Theorem ($M_K^*$ approach to the diameter problem) There exist two explicit functions $h_1, h_2 : (0, 1) \to (0, 1)$ such that for every $\lambda \in (\frac{1}{2}, 1)$ and every symmetric convex body $K$ in $\mathbb{R}^n$, solving the equations $M_K^*(r) = h_1(\lambda)$ and $M_K^*(r) = h_2(\lambda)$ in $r$ we find an upper estimate $r_1$ and a lower estimate $r_2$ for the diameter of a random $[\lambda n]$–section of $K$. □

The important point in Theorem 2.3 is that the functions $h_1$ and $h_2$ are universal and that the statement holds true for an arbitrary body $K$, the only restriction being that $n$ should be large enough depending on $\lambda$. Another advantage of Theorem 2.3 is that it makes use of the global (hence computationally simple) parameter $M^*$ of the body. On the other hand, being so general the estimates cannot be tight in full generality. Another disadvantage of the method in [GM1] is that the use of Borsuk’s theorem forces us to study only proportions $\lambda \in (\frac{1}{2}, 1)$. This first approach gives no information for small proportions.

Our method in this paper is based on the function $S_K^*$ which was defined in Section 1. It provides lower and upper bounds for the diameter of the sections of $K$ of any fixed proportion $\lambda \in (0, 1)$. We also show that the estimates obtained are tight for large classes of symmetric convex bodies and for most values of $\lambda$. Some of our results were announced in [GM2].
Let us close this section with an application of the $M^*_K$ approach: For every integer $t \geq 2$ we define the minimal circumradius of an intersection of $t$ rotations of a body $K$ by

$$r_t(K) = \min\{\rho > 0 : u_1(K) \cap \ldots \cap u_t(K) \subseteq \rho D_n \text{ for some } u_1, \ldots, u_t \in SO(n)\},$$

and the “upper radius” of a random $n/t$–dimensional central section of $K$ by

$$R_t(K) = \min\{R > 0 : \nu_{n,n/t}(\xi \in G_{n,n/t} : K \cap \xi \subseteq RD_\xi) \geq 1 - \frac{1}{t+1}\}.$$

It is proved in [M4] that

$$r_{2t}(K) \leq \sqrt{t} R_t(K)$$

for every $t \geq 2$ and every body $K$. In [GM2] we prove that the local parameter $R_t(K)$ and the global parameter $r_t(K)$ are closely related in the sense that an inverse inequality is possible in full generality:

**2.4 Theorem.** For every integer $t \geq 2$ and every symmetric convex body $K$ in $\mathbb{R}^n$, $n \geq n_0(t)$, we have

$$R_{f(t)}(K) \leq g(t) r_t(K),$$

where $g(t) = C^t$, $f(t) = \lceil g(t) \rceil$, and $C > 1$ is an absolute constant.

The proof of Theorem 2.4 is based on Theorems 2.1 and 2.2. The result is somehow unexpected for an arbitrary body $K$. The search for the best possible functions $f$ and $g$ in the statement above is likely to give more information and probably new ideas related to the $M^*_K$ approach.

**3 Average Mean Width and Diameter of Proportional Sections of a Symmetric Convex Body Satisfying Polynomial Bounds**

Let $K$ be a symmetric convex body in $\mathbb{R}^n$. Recall that the “average diameter” function $D_K : (0, 1) \to (0, \infty)$ is defined by

$$D_K(\lambda) = \int_{G_{n,k}} \text{diam}(K \cap \xi) \nu_{n,k}(d\xi),$$

where $k = \lfloor \lambda n \rfloor$. Since $2M^*(K \cap \xi) \leq \text{diam}(K \cap \xi)$ for every $\xi \in G_{n,k}$, we immediately compare $D_K(\lambda)$ with $S_K^*(\lambda)$:

$$2S_K^*(\lambda) \leq D_K(\lambda) \quad (3.1)$$

for every $\lambda \in (0, 1)$. Using Theorem 2.1 we shall give an upper bound for $D_K(\lambda)$ in terms of $S_K^*$. This is possible if we assume that $K$ satisfies a polynomial condition:
3.1 Theorem. Let $K$ be a symmetric convex body in $\mathbb{R}^n$, such that $\frac{1}{2}D_n \subseteq K \subseteq aD_n$ with $ab \leq n^t$. For every $\lambda \in (0, 1)$, we have

\begin{equation}
2S^*_K(\lambda) \leq D_K(\lambda) \leq 5 \inf \{ S^*_K(\lambda/\theta)/(1 - \theta)^{1/2} : \theta \in (\lambda, 1), 1 - \theta \geq c_t \lambda^{-1} \log n/n \},
\end{equation}

where $c_t > 0$ is an absolute constant.

Proof: Let $\theta \in (\lambda, 1)$ with $1 - \theta \geq c_t \lambda^{-1} \log n/n$, and fix a subspace $\eta$ with $\dim \eta = (\lambda/\theta)n$. There exists a subset $L_{(\lambda/\theta)n, \lambda n}(\eta)$ of $G_{(\lambda/\theta)n, \lambda n}(\eta)$ with measure $\nu(L_{(\lambda/\theta)n, \lambda n}(\eta)) \geq 1 - e^{(-c'(1 - \theta)n)}$ such that for every $\xi \in L_{(\lambda/\theta)n, \lambda n}(\eta)$

$$\text{diam}(K \cap \xi) \leq \frac{4M^*(K \cap \eta)}{\sqrt{1 - \theta}}.$$

Integrating over $G_{(\lambda/\theta)n, \lambda n}(\eta)$ we get:

\begin{equation}
\int_{G_{(\lambda/\theta)n, \lambda n}(\eta)} \text{diam}(K \cap \xi) \ \nu(d\xi) \leq \frac{4M^*(K \cap \eta)}{\sqrt{1 - \theta}} + abM^*(K \cap \eta) \exp(-t \log n),
\end{equation}

if $c_t > 0$ is chosen suitably large, where we made use of the fact that for every $\eta$ we have $\text{diam}(K \cap \xi) \leq 2a \leq 2abM^*(K \cap \eta)$. Since $ab \leq n^t$, it follows that

\begin{equation}
\int_{G_{(\lambda/\theta)n, \lambda n}(\eta)} \text{diam}(K \cap \xi) \ \nu(d\xi) \leq 5 \frac{M^*(K \cap \eta)}{\sqrt{1 - \theta}},
\end{equation}

Now, integrating (3.4) over $G_{n,(\lambda/\theta)n}$ and recalling (3.1) we obtain

$$2S^*_K(\lambda) \leq D_K(\lambda) \leq 5 \frac{S^*_K(\lambda/\theta)}{\sqrt{1 - \theta}},$$

and the proof is complete. \(\Box\)

3.2 Remark. If one has some information on the way $S^*_K(\lambda)$ increases as a function of $\lambda$, then Theorem 3.1 can be useful in order to determine the average diameter of the $\lambda n$-dimensional sections of $K$. It is however clear that the lower and upper bounds provided by Theorem 3.1 will be “close” only if $S^*_K(\lambda)$ increases in a regular way.

When $K$ satisfies a polynomial condition $ab \leq n^t$, then there will be many intervals of regularity for $S^*_K(\lambda)$. To make this more precise, let us fix some $\lambda \in (0, 1)$ and consider the finite sequence $k_j = \lfloor \lambda^j n \rfloor$, $j = 0, 1, \ldots, s(\lambda)$. The length $s(\lambda)$ of the sequence is the smallest positive integer $s$ for which $\lfloor \lambda^s n \rfloor = \lfloor \lambda^{s+1} n \rfloor$. It is easy to check that $s(\lambda) \approx \log((1 - \lambda)n)/\log(1/\lambda)$.

Since $S^*_K$ is increasing in $\lambda$, we have

$$M^*(K) = S^*_K(k_0) \geq S^*_K(k_1) \geq \ldots \geq S^*_K(k_{s(\lambda)}).$$
We set $d_j = k_{j-1}/k_j$, $j = 1, \ldots, s(\lambda)$. Given any small $\delta \in (0, 1)$ and any $\zeta > 1$, consider the set $J_\zeta = \{ j \leq s(\lambda) : d_j \geq \zeta \}$. Then, $|J_\zeta| \leq t \log n/\log \zeta \leq \delta s(\lambda)$ if $\zeta$ satisfies the condition $\log \zeta \geq c_1 \frac{s}{n} \log(b^{1/2}) \log(\frac{1}{\delta})$.

Choose any $j \in J_\zeta$. Then, Theorem 3.1 implies that

\begin{equation}
2S^*_{\lambda}(\lambda^j) \leq D_{\lambda}(\lambda^j) \leq 5(1 - \lambda)^{-1/2}S^*_{\lambda}(\lambda^{j-1}) \leq 5(1 - \lambda)^{-1/2}\zeta S^*_{\lambda}(\lambda^j).
\end{equation}

Thus, $D_{\lambda}(\lambda^j) \approx S^*_{\lambda}(\lambda^j)$ for all $j \leq s(\lambda)$ in a set of cardinality greater than $(1 - \delta)s(\lambda)$, up to a function depending only on $\lambda, t, \delta$. This observation has a meaning from the computational point of view, since $S^*_{\lambda}(\lambda)$ can be computed in contrast to $D_{\lambda}(\lambda)$. The degree of efficiency of this method clearly depends on the a-priori information one has for the concentration of the function $M^*(K \cap \xi), \xi \in G_{n,[\lambda n]}$ around $S^*_{\lambda}(\lambda)$ (see next section).

It is reasonable to expect that $S^*_{\lambda}$ increases faster as $\lambda \to 1^-$. If true, this would imply that $S^*_{\lambda}$ increases regularly on every interval $[0, \lambda_0)$, $\lambda_0 < 1$, when $K$ satisfies a polynomial condition and $n \geq n_0(\lambda_0, t)$. In particular, the bounds given by Theorem 3.1 would be tight for all “small” values of $\lambda$. Thus, we are lead to the following:

**Question:** Is it true that $S^*_{\lambda}$ is a “convex” function of $\lambda$ on $(0, 1)$?

It is also interesting to note some duality relations which are satisfied by $S^*_{\lambda}$: If $F : (0, 1] \times (0, 1] \to \mathbb{R}^+$ is defined by

$$F(\lambda, \mu) = S^*_{\lambda}(\lambda) S^*_{\mu}(\mu),$$

then one has upper bounds for $F(\lambda, \mu)$ which are independent of $K$ (assuming that $ab$ is polynomial in $n$), provided that $\lambda + \mu < 1$. We start with a simple lemma:

**3.3 Lemma.** Let $K$ be a symmetric convex body in $\mathbb{R}^n$ such that $\frac{1}{b}D_n \subseteq K \subseteq aD_n$, with $ab \leq n^t$. Then, if $\lambda, \varepsilon \in (0, 1)$, and if $r$ is the solution of the equation $M^*(K \cap rD_n) = (1 - \varepsilon)(1 - \lambda)^{1/2}r$, we have

$$S^*_{\lambda}(\lambda) \leq 2r,$$

provided that $n/\log n \geq c_1 t/\varepsilon^2(1 - \lambda)$, where $c_1 > 0$ is an absolute constant.

**Proof:** Let $k = [\lambda n]$. It is clear that $r > 1/b$, hence for every $\xi \in G_{n,k}$ we have the obvious estimate $M^*(K \cap \xi) \leq a \leq abr \leq n^t r$. On the other hand, by the low $M^*$-estimate we know that $K \cap \xi \subseteq rD_\xi$ for all $\xi$ in a subset $L_{n,k}$ of $G_{n,k}$ with measure exceeding $1 - c \exp(-c\varepsilon^2(1 - \lambda)n)$. Therefore,

\begin{equation}
S^*_{\lambda}(\lambda) = \int_{G_{n,k}} M^*(K \cap \xi) d\nu_{n,k}(d\xi) \leq \nu_{n,k}(L_{n,k}^c)n^t r + \nu_{n,k}(L_{n,k}) r \\
\leq (c \exp(-c\varepsilon^2(1 - \lambda)n)n^t + 1)r \leq 2r,
\end{equation}

if $n$ is large enough. \(\square\)

We will also need the Distance Lemma from [M3]:

9
3.4 Lemma. Let $W$ be a symmetric convex body in $\mathbb{R}^n$ with $\rho D_n \subseteq W \subseteq r D_n$. Assume that $(M^*(W)/r)^2 + (M(W)\rho)^2 = s > 1$. Then,

\[(3.7) \quad \frac{r}{\rho} \leq \frac{1}{s-1}. \quad \Box \]

It is an obvious consequence of Hölder’s inequality that for every symmetric convex body $K$ in $\mathbb{R}^n$ the inequality

\[(3.8) \quad M(K)M^*(K) \geq 1 \]

holds true. Moreover, this inequality is in general far from being sharp: it holds as an equality if and only if $K$ is a multiple of the Euclidean ball. On the other hand, a well-known sequence of results of Figiel–Tomczak [FT], Lewis [L] and Pisier [P2] states that for every $K$ we can find a linear image $\overline{K}$ of $K$ for which

\[(3.9) \quad M(\overline{K})M^*(\overline{K}) \leq c \log n, \]

where $c > 0$ is an absolute constant.

It is not hard to check that $S^*_K(\lambda)S^*_K(\mu) \geq 1$ for every $\lambda, \mu \in (0, 1)$: If for example $\lambda \geq \mu$ we have

$S^*_K(\lambda)S^*_K(\mu) \geq S^*_K(\mu)S^*_K(\mu) \geq 1$

by the monotonicity of $S^*_K$ and Hölder’s inequality. Using Lemmas 3.3 and 3.4 one can see that for bodies satisfying a polynomial condition a weaker version of (3.9) is always true:

3.5 Theorem. Let $K$ be a symmetric convex body in $\mathbb{R}^n$ such that $\frac{1}{b} D_n \subseteq K \subseteq a D_n$, with $ab \leq n^t$. If $\lambda, \mu, \kappa \in (0, 1)$ and $\lambda + \mu = 1 - \kappa$, then

\[(3.10) \quad S^*_K(\lambda)S^*_K(\mu) \leq \frac{8}{\kappa}, \]

provided that $n$ is large enough (depending on $t$, $\lambda$ and $\mu$).

Proof: Let $n/\log n \geq 64 c_1 t/\kappa^2(1 - \lambda)$, where $c_1$ is the constant from Lemma 3.3. We apply Lemma 3.3 with $\varepsilon = \frac{1}{8} \kappa$: Find $r > 0$ such that $M^*(K \cap r D_n) = (1 - \kappa/8)(1 - \lambda)^{1/2}r$. Then, Lemma 3.3 shows that

\[(3.11) \quad S^*_K(\lambda) \leq 2r. \]

Next, find $\rho > 0$ such that $M^*(K^0 \cap \frac{1}{\rho} D_n) = (1 - \kappa/8)(1 - \mu)^{1/2} \frac{1}{\rho}$. Since $\frac{1}{a} D_n \subseteq K \subseteq b D_n$, Lemma 3.3 applies again to give

\[(3.12) \quad S^*_K(\mu) \leq \frac{2}{\rho}. \]
Without loss of generality we assume that \( \rho \leq r \). Let \( T = \co((K \cap rD_n) \cup \rho D_n) \). Then, \( \rho D_n \subseteq T \subseteq rD_n \), \( T \supseteq K \cap rD_n \), and \( T^o \supseteq K^o \cap \frac{1}{\rho} D_n \), therefore

\[
(3.13) \quad \left( \frac{M^*(T)}{r} \right)^2 + (M(T)\rho)^2 \geq \left( \frac{M^*(K \cap rD_n)}{r} \right)^2 + \left( \frac{M^*(K^o \cap \frac{1}{\rho} D_n)\rho}{r} \right)^2 = (1 - \kappa/8)^2(2 - \lambda - \mu)
\]

so that\( \rho D_n \subseteq T \subseteq rD_n \), and \( T \supseteq K \cap rD_n \), \( T \supseteq K^o \cap \frac{1}{\rho} D_n \), therefore\( (3.13) \quad (M(T)\rho)^2 \geq \frac{1}{8} \).

Since \( \rho D_n \subseteq T \subseteq rD_n \), the distance lemma implies that

\[
(3.14) \quad \frac{r}{\rho} \leq \frac{2}{\kappa}.
\]

Combining (3.11), (3.12) and (3.14), we obtain

\[
S_K^*(\lambda)S_K^*(\mu) \leq \frac{8}{\kappa}. \quad \square
\]

In Section 5 we shall see that in the case of a body in \( M \)–position of order \( \alpha \) one can avoid the restriction \( \lambda + \mu < 1 \). For \( \lambda \) and \( \mu \) both very close to 1, we have \( F(\lambda, \mu) \) bounded by a constant independent from \( n \).

### 4 On the Diameter of a Random Proportional Section

We proceed to see whether one can obtain more precise information about the diameter of a “random” \( [\lambda n] \)-dimensional section of \( K \). Here, we specify “random” as follows: for every \( \xi \) in a subset \( L_n, [\lambda n] \) of \( G_n, [\lambda n] \) with measure \( \nu_n, [\lambda n]((L_n, [\lambda n]) \geq 1 - h(\lambda, n) \), where \( h(\lambda, n) = o_n(1) \). This approaches 1 for every \( \lambda \) as the dimension \( n \) grows to infinity.

To this end, we first study the behavior of \( M^*(K \cap \xi) \) as a function of \( \xi \) on \( G_{n,k} \). We consider two distances \( \rho \) and \( d \) on \( G_{n,k} \), defined by

\[
(4.1) \quad \rho(\xi, \eta) = \min\{\left( \sum_{i=1}^{k} |e_i - f_i|^2 \right)^{1/2} : \{e_i\}_{i \leq k}, \{f_i\}_{i \leq k} \text{ are orthonormal bases of } \xi, \eta \}
\]

and

\[
(4.2) \quad d(\xi, \eta) = \max\{d_1(x, S(\eta)) : x \in S(\xi), \ d_1 \text{ the geodesic distance}\}.
\]

Our first Lemma compares \( \rho \) with \( d \). It is probably a well known fact but we did not find a convenient reference so we prove it below:

**4.1 Lemma.** Let \( \xi, \eta \in G_{n,k} \). Then,

\[
(4.3) \quad (2/\pi)d(\xi, \eta) \leq \rho(\xi, \eta) \leq (2k)^{1/2}d(\xi, \eta).
\]
Using Lemma 4.1, we can prove that
\[ \rho(4\lambda) \leq \lambda \text{ for every } \lambda > 0. \]

**Proof:** The left hand side inequality is clear: Let \( \{ e_i \} \) and \( \{ f_i \} \) be two orthonormal bases of \( \xi, \eta \) respectively. If \( x = \sum t_i e_i \in S(\xi) \), then
\[ |x - \sum t_i f_i| \leq (\sum |e_i - f_i|^2)^{1/2}, \]
therefore \( (2/\pi)d_1(x, S(\eta)) \leq (\sum |e_i - f_i|^2)^{1/2} \). It follows that \( (2/\pi)d(\xi, \eta) \leq \rho(\xi, \eta) \).

For the right hand side inequality, we use an inductive argument based on the following claim:

**Claim:** Let \( E, F \in G_{n,m}, m \geq 2 \), and \( x \in S(E) \) be such that \( |P_F(x)| \) is minimal. Then, for every \( x_1 \in E \cap x^\perp \) we have \( P_F(x_1) \perp P_F(x) \).

**Proof:** Suppose that \( x_1 \in S(E) \cap x^\perp \) with \( \beta = \langle P_F(x_1), P_F(x) \rangle \neq 0 \). Without loss of generality we may assume that \( \beta > 0 \). For every \( t > 0 \) we have
\[ |P_F(x - tx_1)| \geq |x - tx_1||P_F(x)|, \]
which implies that
\[ 2\beta \leq t(\|P_F(x_1)\|^2 - |P_F(x)|^2), \]
a contradiction if we let \( t \to 0^+ \).

We use the claim to choose orthonormal bases \( \{ e_i \} \) and \( \{ f_i \} \) of \( \xi, \eta \) as follows:
We choose \( e_1 \in S(\xi) \) such that \( P_0(e_1) \) is minimal. Observe that if \( P_0(e_1) = 0 \), then \( d(\xi, \eta) = \pi/2 \) and we have nothing to prove. If not, we set \( f_1 = P_0(e_1)/|P_0(e_1)| \).
If \( \{ e_i \}_{i \leq s} \) and \( \{ f_i \}_{i \leq s} \) have been chosen, we choose \( e_{s+1} \in S(\xi) \cap \langle e_i, i \leq s \rangle^\perp \) with \( |P_0(e_{s+1})| \) minimal. By the claim, \( P_0(e_{s+1}) \perp \langle f_i, i \leq s \rangle \), so we set \( f_{s+1} = P_0(e_{s+1})/|P_0(e_{s+1})| \).

With this construction,
\[ |e_i - f_i| = \sqrt{2}(1 - |P_0(e_i)|^2)^{1/2} \leq \sqrt{2}(1 - |P_0(e_1)|^2)^{1/2} \leq \sqrt{2}d(\xi, \eta). \]
It follows that \( \rho(\xi, \eta) \leq (2k)^{1/2}d(\xi, \eta) \). \( \Box \)

Using Lemma 4.1, we can prove that \( M^*(K \cap \xi) \) satisfies the following Lipschitz estimate:

**4.2 Lemma.** Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) with \( 1/2D_n \subseteq K \subseteq aD_n \), and fix \( \lambda \in (0, 1) \) and \( k = [\lambda n] \). Then,
\[ |M^*(K \cap \xi) - M^*(K \cap \eta)| \leq 6a^2bd(\xi, \eta) \]
for every \( \xi, \eta \in G_{n,k} \).

**Proof:** Let \( \{ e_i \}_{i \leq k} \) and \( \{ f_i \}_{i \leq k} \) be two orthonormal bases of \( \xi \) and \( \eta \) respectively, such that \( \rho^2(\xi, \eta) = \sum_{i=1}^k |e_i - f_i|^2 \). Recall that
\[ M^*(K \cap \xi) \simeq \frac{1}{\sqrt{k}} \int_\Omega \left\| \sum_{i=1}^k g_i(\omega)e_i \right\|_{(K \cap \xi)}^2 d\omega = \frac{1}{\sqrt{k}} \int_\Omega \max_{x \in K \cap \xi} \left\| x, \sum_{i=1}^k g_i(\omega)e_i \right\| d\omega, \]

\[ 12 \]
where \( g_1, \ldots, g_k \) are independent standard Gaussian random variables on some probability space \( \Omega \) (with an analogous estimate holding for \( M^*(K \cap \eta) \) and the orthonormal basis \( \{f_i\}_{i \leq k} \)).

We define a function \( h \) on \( \mathbb{R}^n \) by

\[
(4.8) \quad h(x) = \max\{\langle z, x \rangle : z \in K \cap (\xi \cup \eta)\}.
\]

Then, we easily see that

\[
(4.9) \quad |M^*(K \cap \xi) - M^*(K \cap \eta)| \leq \int_{S(\xi)} [h(x) - \|x\|(K \cap \xi)] + \int_{S(\eta)} [h(x) - h(y)] + \int_{S(\eta)} [h(y) - \|y\|(K \cap \eta)].
\]

For the middle term note that, by Lemma 4.1,

\[
(4.10) \quad \left| \int_{S(\xi)} h(x) - \int_{S(\eta)} h(y) \right| \simeq \frac{1}{\sqrt{k}} \int_{\Omega} |h(\sum g_i e_i) - h(\sum g_i f_i)| d\omega \\
\leq \frac{a}{\sqrt{k}} \int_{\Omega} \|g_i(e_i - f_i)\| d\omega \\
\leq \frac{a \rho(\xi, \eta)}{\sqrt{k}} \leq \sqrt{2} a d(\xi, \eta).
\]

By symmetry, it remains to estimate the first term in (4.9). Given \( x \in S(\xi) \), suppose that \( h(x) = \langle z, x \rangle \) for some \( z \in K \cap \eta \) with \( \|z\| = 1 \) (if the max was attained for some \( z \in K \cap \xi \) we would simply have \( h(x) = \|x\|(K \cap \xi) = 0 \)). We can find \( x_0 \in \{z\}S(\xi) \) with \( |z - x_0| \leq |z| |d(\xi, \eta)\). If \( x_0 \in K \cap \xi \), then \( h(x) - \|x\|(K \cap \xi) \leq \langle z - x_0, x \rangle \leq |z - x_0| \leq a d(\xi, \eta) \). Assume that \( \|x_0\| > 1 \). Then, \( \|x_0\| \leq \|z\| + \|x_0 - z\| \leq 1 + abd(\xi, \eta) \), and we write

\[
(4.11) \quad h(x) - \|x\|(K \cap \xi) \leq \langle z - \frac{x_0}{\|x_0\|}, x \rangle + \langle (1 - \frac{1}{\|x_0\|})x_0, x \rangle \\
\leq a d(\xi, \eta) + \frac{abd(\xi, \eta)}{1 + abd(\xi, \eta)} |x_0| \\
\leq \left[ a + \frac{a^2 b}{1 + abd(\xi, \eta)} \right] d(\xi, \eta) \leq 2a^2 b d(\xi, \eta).
\]

Inserting this information into (4.9) we conclude the proof. \( \square \)

Lemma 4.2 and a well-known deviation inequality for a Lipschitz function on \( G_{n,k} \) (see [MS], Chapter 6 and Appendix V) give us the following estimate:
4.3 Lemma. Let $K$ be a symmetric convex body in $\mathbb{R}^n$ with $\frac{1}{2}D_n \subseteq K \subseteq aD_n$, and fix $\lambda \in (0, 1)$ and $k = \lfloor \lambda n \rfloor$. Then,

\[(4.12)\]

\[\nu_{n,k} \left( \{ \xi \in G_{n,k} : |M^*(K \cap \xi) - S^*_K(\lambda)| \geq \frac{1}{2}S^*_K(\lambda) \} \right) \leq \exp(-cn\|S^*_K(\lambda)\|^2),\]

where $c > 0$ is an absolute constant. $\square$

Lemma 4.3 provides a rather weak concentration of the values of the function $M^*(K \cap \xi)$ around its expectation $S^*_K(\lambda)$: when applied directly, it is practically useful only if $ab$ is considerably small. However, as a first step we can make use of this information in a quite interesting case: when $D_n \subseteq K \subseteq aD_n$ with $a = o(\sqrt{n})$.

In analogy to $M^*_K(r)$ we define the auxiliary function

\[(4.13)\]

\[S^*_K(r, \lambda) = \int_{G_{n,k}} M^*(K \cap rD_n \cap \xi) \nu_{n,k}(d\xi),\]

where $r > 0$, $\lambda \in (0, 1)$, and $k = \lfloor \lambda n \rfloor$. This is a function increasing in $r$ and $\lambda$. For fixed $\lambda \in (0, 1)$, the obvious inequality $\|\theta\|_{(K \cap rD_n)\cap(\xi)} \leq \|\theta\|_{(K \cap rD_n)\cap(\xi)}$ for $\theta \in S(\xi)$ shows that

\[(4.14)\]

\[S^*_K(r, \lambda) \leq M^*(K \cap rD_n) = rM^*_K(r)\]

for all $r > 0$. Furthermore, one has the following additional information:

4.4 Lemma. The functions $S^*_K(r, \lambda)$ and $rM^*_K(r) = M^*(K \cap rD_n)$ are concave in $r$.

Proof: We first show that $M^*(K \cap rD_n)$ is concave. Let $r_1, r_2 > 0$ and $0 < \beta < 1$. Given $\theta \in S^{n-1}$, there exist $x_i \in K \cap r_iD_n$, $i = 1, 2$, such that $\max_{x \in K \cap r_iD_n} \langle x, \theta \rangle = \langle x_i, \theta \rangle$. Then, $\beta x_1 + (1 - \beta)x_2 \in K \cap (\beta r_1 + (1 - \beta)r_2)D_n$, and

\[
\max_{x \in K \cap (\beta r_1 + (1 - \beta)r_2)D_n} \langle x, \theta \rangle = \beta \max_{x \in K \cap r_1D_n} \langle x, \theta \rangle + (1 - \beta) \max_{x \in K \cap r_2D_n} \langle x, \theta \rangle.
\]

Integrating over $S^{n-1}$ we get

\[
M^*(K \cap (\beta r_1 + (1 - \beta)r_2)D_n) \geq \beta M^*(K \cap r_1D_n) + (1 - \beta)M^*(K \cap r_2D_n).
\]

In exactly the same way we show that $M^*(K \cap rD_n \cap \xi)$ is concave in $r$ for every $\xi \in G_{n,k}$, and integrating on $G_{n,k}$ we see that $S^*_K(r, \lambda)$ is concave too. $\square$

Let $\{a_n\}$ be a sequence satisfying $a_n/\sqrt{n} \to 0$ as $n \to \infty$. Using Lemma 4.3 and the function $S^*_K(r, \lambda)$ we have the following information about the diameter of random proportional sections:

4.5 Proposition. Suppose that $D_n \subseteq K \subseteq a_nD_n$. Let $\lambda, \theta \in (0, 1)$ and $k = \lfloor \lambda n \rfloor$. We denote by $r^*_K(\lambda)$ the solution of the equation

\[(4.15)\]

\[S^*_K(r, \lambda) = (1 - \theta)^{1/2}r/3\]
in r. Then, if \( \lambda < \theta < 1 \) and \( n/a_n^2 \geq C(\lambda, \theta) \) we have

\[
(1 - \theta)^{1/2} r_K^\theta(\lambda)/3 \leq \text{diam}(K \cap \xi) \leq 2r_K^\theta(\lambda/\theta),
\]

for a random \( \xi \in G_{n,k} \).

Proof: Given \( \lambda \in (0,1) \), we find \( r_K^\theta(\lambda) \) solving (4.15) and then apply Lemma 4.3 to the body \( K \cap r_K^\theta(\lambda)D_n \); we can find a subset \( L_1 \) of \( G_{n,k} \) with measure

\[
\nu_{n,k}(L_1) \geq 1 - \exp(-c(1 - \theta)n/2) \geq 1 - h(\theta, n)
\]

where \( h(\theta, n) = a_n(1) \), such that for every \( \xi \in L_1 \),

\[
(1 - \theta) r_K^\theta\lambda/6 < M^*(K \cap r_K^\theta(\lambda)D_n \cap \xi) < (1 - \theta)^{1/2} r_K^\theta(\lambda)/2.
\]

The left hand side inequality clearly implies that for all \( \xi \in L_1 \) we have

\[
\text{diam}(K \cap \xi) \geq (1 - \theta)^{1/2} r_K^\theta(\lambda)/3.
\]

On the other hand, the right hand side of (4.17) shows that there exists a subset \( L_2 \) of \( G_{n,([\lambda/\theta])n} \) with measure \( \geq 1 - h(\theta, n)\), such that

\[
M^*(K \cap r_K^\theta(\lambda/\theta)D_n \cap \eta) < (1 - \theta)^{1/2} r_K^\theta(\lambda/\theta)/2
\]

for every \( \eta \in L_2 \), and the Low \( M^* \)-estimate implies that for most \( \xi \in G_{([\lambda/\theta])n,k}(\eta) \) we have

\[
\text{diam}(K \cap r_K^\theta(\lambda/\theta)D_n \cap \xi) \leq \frac{4 M^*(K \cap r_K^\theta(\lambda/\theta)D_n \cap \eta)}{\sqrt{1 - \theta}} < 2r_K^\theta(\lambda/\theta),
\]

which shows that

\[
\text{diam}(K \cap \xi) \leq 2r_K^\theta(\lambda/\theta),
\]

for all \( \xi \in L \subseteq G_{n,k} \) with \( \nu_{n,k}(L) \geq 1 - c_1 h(\theta, n) \). By (4.18) and (4.21), (4.16) holds true with probability greater than \( 1 - c_1 h(\theta, n) \). \( \Box \)

**4.6 Remark.** Let \( \gamma(\theta, n) = (c(1 - \theta)n/18 \log n)^{1/4} \), where \( c \) is the constant from Lemma 4.3. Assume that \( D_n \subseteq K \subseteq aD_n \), where \( a < \gamma(\theta, n) \). Then, a careful reading of the proof of Proposition 4.5 shows that it holds true with \( h(\theta, n) = n^{-2} \).

It is then not hard to compare the solution \( r_K^\theta(\lambda) \) of the equation \( S_K^r(r, \lambda) = (1 - \theta)^{1/2} r/3 \) with the function \( S_K^* \) itself. We clearly have \( S_K^r(r, \lambda) \leq S_K^*(\lambda) \) for every \( r > 0 \), and hence

\[
r_K^\theta(\lambda) \leq 3S_K^*(\lambda)/(1 - \theta)^{1/2}
\]

for all \( \lambda, \theta \in (0,1) \). On the other hand, assuming that \( D_n \subseteq K \subseteq \gamma(\theta, n)D_n \), by Proposition 4.5 for every \( \lambda < \theta < 1 \) we can find \( L \subseteq G_{n,k} \) with \( \nu(L) \leq cn^{-2} \) such
that $\text{diam}(K \cap \xi) \leq 2r^\theta_K(\lambda/\theta)$ for all $\xi \in L$. Since $r^\theta_K(\lambda/\theta) \geq 1$, a simple estimate gives

$$(4.23) \quad S^*_K(\lambda) \leq \frac{1}{2} \int_{G_{n,k}} \text{diam}(K \cap \xi)d\xi \leq r^\theta_K(\lambda/\theta)[1 + c_\gamma(\theta, n)n^{-2}] \leq c'r^\theta_K(\lambda/\theta)$$

where $c' > 0$ is an absolute constant. Therefore, we obtain an analogue of Proposition 4.5 in which the process of “solving the equation in $r^\lambda$” is avoided:

**4.7 Theorem.** Let $\lambda, \theta \in (0, 1)$ with $\lambda < \theta$, and $k = [\lambda n]$. For every symmetric convex body $K$ in $\mathbb{R}^n$, $n \geq n_0(\theta)$, satisfying $D_n \subseteq K \subseteq \gamma(\theta, n)D_n$, there exists a subset $L_{n,k}(\theta)$ of $G_{n,k}$ with measure greater than $1 - cn^{-2}$, such that

$$(4.24) \quad c_1S^*_K(\lambda \theta) \leq \text{diam}(K \cap \xi) \leq c_2S^*_K(\lambda \theta)/(1 - \theta)^{1/2}$$

for all $\xi \in L_{n,k}(\theta)$, where $c_1, c_2 > 0$ are absolute constants. \(\square\)

It is clear that the new point here lies in the left hand side inequality. The right hand side of (4.24) is a consequence of the Low $M^*$–estimate (under milder assumptions on $K$: see section 3).

Using Proposition 4.5 we proceed to obtain a-priori information on the diameter of a random $[\lambda n]$-section of $K$ for “most” $\lambda \in (0, 1)$. To this end, for every $\lambda \in (0, 1)$ we define a $\lambda$-flag of subspaces of $\mathbb{R}^n$ to be a finite sequence of subspaces

$$\mathbb{R}^n = E_0 \supset E_1 \supset \ldots \supset E_{s(\lambda)},$$

of dimension $\dim(E_j) = k_j = [\lambda^j n]$, $j = 0, 1, \ldots, s(\lambda)$. The length of the $\lambda$-flag is the smallest integer $s$ for which $k_s = k_{s+1}$. As in Remark 3.2, one easily checks that $s(\lambda) \geq \log((1 - \lambda)n)/\log(1/\lambda)$.

Let $r_j$, $j = 0, 1, \ldots, s(\lambda)$, be the solution of the equation

$$(4.25) \quad S^*_K(r, k_j) = (1 - \lambda)^{1/2}r/3$$

in $r$. In the notation of Proposition 4.5 we have $r_j = r^\lambda_K(\lambda^j)$. We shall show that for most $j \leq s(\lambda)$ the diameter of a random $k_j$-section of $K$ with $D_n \subseteq K \subseteq \gamma(\lambda, n)D_n$ is equal to $r_j$ up to a function depending only on $\lambda$:

**4.8 Theorem.** Let $\lambda \in (0, 1)$ and $K$ be a symmetric convex body in $\mathbb{R}^n$ with $D_n \subseteq K \subseteq \gamma(\lambda, n)D_n$. Let $\{E_j\}_{j \leq s(\lambda)}$ be any $\lambda$-flag of subspaces of $\mathbb{R}^n$. For every $\beta \in (0, 1)$ we can find a set of indices $J \subseteq \{0, 1, \ldots, s(\lambda)\}$ with $|J| \geq (1 - \beta)s(\lambda)$ and a subset $L$ of the orthogonal group $O(n)$ with Haar measure $\nu(L) \geq 1 - \frac{1}{n}$, such that

$$(4.26) \quad c_1r_j \leq \text{diam}(K \cap T(E_j)) \leq c_2g(\lambda, \beta)r_j,$$

for every $T \in L$ and every $j \in J$, where $c_1, c_2 > 0$ are absolute constants and $g(\lambda, \beta)$ is an explicit function depending only on $\lambda$ and $\beta$.\[16\]
Proof: We first observe that the sequence \( r_j, j = 0, 1, \ldots, s(\lambda) \) is decreasing: we write \( f_j(r) \) for the function \( S^*_K(r, \lambda') \). Then \( r_j \) is the solution of the equation 
\[
  f_j(r) = (1 - \lambda)^{1/2} / 3 r.
\]
Since \( \lambda^{j-1} > \lambda' \), we have \( f_{j-1} \geq f_j \) on \((0, \infty)\) for every \( j = 1, \ldots, s(\lambda) \). Also, by Lemma 2.1 each \( f_j \) is a concave increasing function of \( r \). It is then clear that the unique points \( r_j \) where \( f_j \) intersect the line \( y = (1 - \lambda)^{1/2} r/3 \) satisfy the inequality \( r_{j-1} \geq r_j \) for all \( j \leq s(\lambda) \).

We denote by \( d_j \) the ratio \( r_{j-1}/r_j, j \leq s(\lambda) \). Since \( D_n \subseteq K \subseteq \gamma(\lambda, n) D_n \), we have
\[
  (4.27) \quad d_1 \ldots d_{s(\lambda)} = r_0 / r_{s(\lambda)} \leq \gamma(\lambda, n).
\]
It follows that for every \( \zeta > 1 \), if \( J_\zeta = \{ j \leq s(\lambda) : d_j \geq \zeta \} \) we must have \( |J_\zeta| \leq \log n / \log \zeta \). This means that \( |J_\zeta| \leq \beta(\lambda) \) provided that \( \log \zeta \geq \frac{\zeta}{\pi} \log(\frac{\zeta}{\gamma(1 - \lambda, 1)^2}) \log(1/\lambda) \).

Consider any \( j \in J_\zeta \). By Proposition 4.5, for all \( T \) in a subset \( L_j \) of \( O(n) \) with measure greater than \( 1 - c_1 n^{-2} \) we have
\[
  (4.28) \quad (1 - \lambda)^{1/2} r_j / 3 \leq \text{diam}(K \cap T(E_j)) \leq 2r_{j-1} \leq 2\zeta r_j.
\]
Set \( L = \bigcap_{j \in J_\zeta} L_j \). Then, \( \nu(L) \geq 1 - n^{-1} \) if \( n \) is large enough, and for every \( j \in J_\zeta \) and every \( T \in L \) we have 
\[
  \text{diam}(K \cap T(E_j)) \approx r_j \text{ up to } \zeta/(1 - \lambda)^{1/2}.
\]
Recall that \( |J_\zeta| \geq (1 - \beta)s(\lambda) \) if \( \zeta \geq g(\lambda, \beta) = \frac{\zeta}{\pi} \log(\frac{\zeta}{\gamma(1 - \lambda, 1)^2}) \log(1/\lambda) \), and the proof is complete. \( \square \)

4.9 Remark. In view of (4.22), (4.23) and (4.25), one can replace \( r_j \) by \( S^*_K(\lambda) \) in Theorem 4.8. An argument similar to the one in the proof of Theorem 4.8 shows that this is true for most \( j \in J \).

4.10 Remark. Suppose that the maximal/minimal volume ellipsoid or the distance ellipsoid of \( K \) is a Euclidean ball. Without loss of generality we may assume that \( D_n \subseteq K \subseteq aD_n \) with \( a \leq \sqrt{n} \). If \( \gamma(\theta, n) \leq a \leq \sqrt{n} \), we may apply the results of this section to the body \( K_1 = K \cap \gamma(\theta, n) D_n \). Since \( K_1 \subseteq K \subseteq c(\theta) \sqrt{\log n} K_1 \), all statements will hold true up to a \( \sqrt{\log n} \)-factor for the body \( K \) as well.

Let us also note that an additional application of the Low \( M^* \)-estimate shows that the results of this section hold for every symmetric convex body \( K \) with \( \frac{1}{2} D_n \subseteq K \subseteq aD_n \) and \( bM^*(K) = o(\sqrt{n}) \). It would be interesting to know if the condition can be replaced by the weaker \( M(K) M^*(K) = o(\sqrt{n}) \).

5 The case of a body in \( M \)-Position of order \( \alpha \)

If \( A, B \) are symmetric convex bodies in \( \mathbb{R}^n \) we define as usual the covering number \( N(A, B) \) of \( A \) by \( B \) to be the smallest integer \( N \) for which we can find \( y_i \in \mathbb{R}^n, i = 1, \ldots, N \) such that \( A \subseteq \bigcup_{i \leq N} (y_i + B) \). It is known that given any symmetric convex body \( K \) in \( \mathbb{R}^n \) and any \( \alpha > \frac{1}{2} \), there exists a linear image \( K' \) of \( K \) satisfying the following two conditions:

(i) The volume radius of \( K \) is 1: \( |K| = |D_n| \).
(ii) \( \max\{N(K,tD_n), N(D_n,tK), N(K^o,tD_n), N(D_n,tK^o)\} \leq \exp(c(\alpha) \frac{n}{\alpha^2}) \), for every \( t \geq 1 \), where \( c(\alpha) > 0 \) is a constant depending only on \( \alpha \): \( c(\alpha) = O(1/(\alpha - \frac{1}{2})^{1/2}) \) as \( \alpha \to \frac{1}{2} \).

Condition (i) is just a normalization. We could have omitted it and replaced \( D_n \) by \( sD_n \), where \( |K| = |sD_n| \), in (ii). The fact that a body \( K \) which satisfies (i) and (ii) exists in every affine class for every \( \alpha > \frac{1}{2} \) is an improvement of Pisier (see e.g. [P1], Chapter 7) on previous work of Milman related to the inverse Brunn–Minkowski inequality [M2], where (ii) had been established for \( \alpha = 1 \).

In this section we assume that \( K \) is a symmetric convex body satisfying (i) and (ii), and say that \( K \) is in \( M \)-position of order \( \alpha \) ((\( \alpha \)-regular in the terminology of [P1]). One of the consequences of (i) and (ii) is the inverse Brunn-Minkowski inequality [M2] which will be used below in the following precise form: If \( u_1, \ldots, u_s \) are orthogonal transformations of \( \mathbb{R}^n \), then \( u_i(K) \) is in \( M \)-position of order \( \alpha \) for every \( i \leq s \), and

\[
| \frac{1}{s} \sum_{i=1}^{s} u_i(K) \mid^\frac{1}{n} \leq c'(\alpha)s^\alpha |K|^{\frac{1}{n}}. \tag{5.1}
\]

The constant \( c'(\alpha) \) in (5.1) is related to \( c(\alpha) \) in (ii) as follows: \( c'(\alpha) \leq \exp(2c(\alpha)) \) [P1]. Observe also that, if \( r > 0 \) then \( K \cap rD_n \) and \( \text{co}(K \cup rD_n) \) (normalized so that their volume will be \( |D_n| \)) are also in \( M \)-position of order \( \alpha \), with \( c(\alpha) \) replaced by \( c'c(\alpha) \), where \( c' \) is an absolute constant.

We shall prove that in this case \( S_K^*(\lambda) \) is determined by the volume radius of \( K \) up to explicit functions depending only on \( \lambda \) and \( \alpha \):

**5.1 Theorem.** Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) which is in \( M \)-position of order \( \alpha > \frac{1}{2} \). For every \( \lambda \in (0,1) \) we have

\[
c_1(\alpha)\lambda^\alpha \left( \frac{|K|}{|D_n|} \right)^\frac{1}{n} \leq S_K^*(\lambda) \leq \frac{c_2(\alpha)}{(1-\lambda)^\alpha} \left( \frac{|K|}{|D_n|} \right)^\frac{1}{n}, \tag{5.2}
\]

where \( c_1(\alpha), c_2(\alpha) \) are constants depending only on \( \alpha \).

The lower bound may be viewed as a proportional version of Urysohn’s inequality. Together with the upper bound it shows that if we accept a small loss of dimension in computing the mean width of the body, then in the case of a body in \( M \)-position of order \( \alpha \) we have an equivalence

\[
S_K^*(\lambda) \simeq v.\text{rad}(K)
\]

up to a function depending only on \( \lambda \) and \( \alpha \). For the proof of the lower bound we shall need the following geometric Lemma which is based on measure concentration arguments:

**5.2 Lemma.** Let \( \gamma \geq 1, p > 0, 0 < \lambda < 1, \) and \( W \) be a symmetric convex body in \( \mathbb{R}^m \) such that

\[
N(W,tD_m) \leq \exp(\gamma \frac{m}{\lambda mp})
\]

for every \( t \geq 1 \), where \( c(\alpha) > 0 \) is a constant depending only on \( \alpha \): \( c(\alpha) = O(1/(\alpha - \frac{1}{2})^{1/2}) \) as \( \alpha \to \frac{1}{2} \).

Condition (i) is just a normalization. We could have omitted it and replaced \( D_n \) by \( sD_n \), where \( |K| = |sD_n| \), in (ii). The fact that a body \( K \) which satisfies (i) and (ii) exists in every affine class for every \( \alpha > \frac{1}{2} \) is an improvement of Pisier (see e.g. [P1], Chapter 7) on previous work of Milman related to the inverse Brunn–Minkowski inequality [M2], where (ii) had been established for \( \alpha = 1 \).

In this section we assume that \( K \) is a symmetric convex body satisfying (i) and (ii), and say that \( K \) is in \( M \)-position of order \( \alpha \) ((\( \alpha \)-regular in the terminology of [P1]). One of the consequences of (i) and (ii) is the inverse Brunn-Minkowski inequality [M2] which will be used below in the following precise form: If \( u_1, \ldots, u_s \) are orthogonal transformations of \( \mathbb{R}^n \), then \( u_i(K) \) is in \( M \)-position of order \( \alpha \) for every \( i \leq s \), and

\[
| \frac{1}{s} \sum_{i=1}^{s} u_i(K) \mid^\frac{1}{n} \leq c'(\alpha)s^\alpha |K|^{\frac{1}{n}}. \tag{5.1}
\]

The constant \( c'(\alpha) \) in (5.1) is related to \( c(\alpha) \) in (ii) as follows: \( c'(\alpha) \leq \exp(2c(\alpha)) \) [P1]. Observe also that, if \( r > 0 \) then \( K \cap rD_n \) and \( \text{co}(K \cup rD_n) \) (normalized so that their volume will be \( |D_n| \)) are also in \( M \)-position of order \( \alpha \), with \( c(\alpha) \) replaced by \( c'c(\alpha) \), where \( c' \) is an absolute constant.

We shall prove that in this case \( S_K^*(\lambda) \) is determined by the volume radius of \( K \) up to explicit functions depending only on \( \lambda \) and \( \alpha \):

**5.1 Theorem.** Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) which is in \( M \)-position of order \( \alpha > \frac{1}{2} \). For every \( \lambda \in (0,1) \) we have

\[
c_1(\alpha)\lambda^\alpha \left( \frac{|K|}{|D_n|} \right)^\frac{1}{n} \leq S_K^*(\lambda) \leq \frac{c_2(\alpha)}{(1-\lambda)^\alpha} \left( \frac{|K|}{|D_n|} \right)^\frac{1}{n}, \tag{5.2}
\]

where \( c_1(\alpha), c_2(\alpha) \) are constants depending only on \( \alpha \).

The lower bound may be viewed as a proportional version of Urysohn’s inequality. Together with the upper bound it shows that if we accept a small loss of dimension in computing the mean width of the body, then in the case of a body in \( M \)-position of order \( \alpha \) we have an equivalence

\[
S_K^*(\lambda) \simeq v.\text{rad}(K)
\]

up to a function depending only on \( \lambda \) and \( \alpha \). For the proof of the lower bound we shall need the following geometric Lemma which is based on measure concentration arguments:

**5.2 Lemma.** Let \( \gamma \geq 1, p > 0, 0 < \lambda < 1, \) and \( W \) be a symmetric convex body in \( \mathbb{R}^m \) such that

\[
N(W,tD_m) \leq \exp(\gamma \frac{m}{\lambda mp})
\]
for every \( t \geq 1 \). Then, there exists a subspace \( \eta \in G_{m,[m/2]} \) such that

\[
W \cap \eta \subseteq c_1 \gamma^{1/p} \lambda^{-1/p} D_{\eta},
\]

where \( c > 0 \) is an absolute constant.

**Proof:** Let \( t \geq 1 \). We can find \( N \leq \exp(\gamma m/\lambda p) \) and \( x_i \in \mathbb{R}^m, i = 1, \ldots, N \), such that \( W \subseteq \bigcup_{i \leq N} (x_i + t D_m) \). Consider the sphere \( RS^{m-1} \), where \( R > 0 \) is a constant to be chosen.

Let \( \sigma_R \) denote the normalized rotationally invariant measure on \( RS^{m-1} \). It is easy to see that for every \( i \leq N \) the intersection \( A_i = (x_i + 2t D_m) \cap RS^{m-1} \) has measure

\[
\sigma_R(A_i) \leq \sigma(B(2t/R)),
\]

where \( B(\varepsilon) \) denotes a cap of angular radius \( \varepsilon > 0 \) in \( S^{m-1} \). We estimate \( \sigma(B(2t/R)) \) in a standard way:

\[
\sigma(B(2t/R)) = \int_0^{\sin^{-1}(2t/R)} \sin^{m-2} s ds \leq \left( \frac{c_1 t}{R} \right)^{m-1},
\]

for some absolute constant \( c_1 > 0 \). This implies that if we set \( A = \bigcup_{i \leq N} A_i \), then

\[
\sigma_R(A \cap RS^{m-1}) \leq \exp(\gamma m/\lambda p) \left( \frac{c_1 t}{R} \right)^{m-1}.
\]

Assuming that \( R \) is chosen large enough, this is exponentially small in \( m \). More precisely, since the cardinality of a \( t \)-net in \( RS^{[m/2]} \) is bounded by \( (2R/t)^{m/2} \), a standard argument (see [MS], Chapter 4) shows that if

\[
\exp(\gamma m/\lambda p) \left( \frac{2R}{t} \right)^{1/2} \left( \frac{c_1 t}{R} \right)^{m-1} < 1,
\]

then we can find a subspace \( \eta \in G_{m,[m/2]} \) and a \( t \)-net \( C(\eta) \) for \( \eta \cap RS^{m-1} \) such that \( A \cap C(\eta) = \emptyset \). Analyzing the condition on \( R \), we see that it is enough to choose

\[
R = c_2 t \exp(3\gamma m/\lambda p),
\]

for some constant \( c_2 > c_1 \). We can now easily show that with this choice of \( R \) we have \( W \cap \eta \subseteq RD_{\eta} \). Suppose not. Then, we can find \( x \in RS^{m-1} \) which is also in \( W \cap \eta \). It follows that \( |x - x_i| \leq t \) for some \( i \leq N \), and \( |x - y| \leq t \) for some \( y \in C(\eta) \). But then, \( |y - x_i| \leq 2t \), which means that \( A \cap C(\eta) \neq \emptyset \), a contradiction.

We choose \( t = (\gamma/\lambda)^{1/p} \geq 1 \). Then, \( R = c_1^{1/p} \lambda^{-1/p} \) and the proof is complete. \( \square \)

We can now pass to the proof of the Theorem:
Lower bound: Let $\lambda \in (0, 1)$, $k = [\lambda n]$, and consider any $\xi \in G_{n,k}$. The projection $P_\xi(K^n)$ of $K^n$ onto $\xi$ satisfies

\begin{equation}
N(P_\xi(K^n), tD_\xi) \leq N(K^n, tD_\xi) \leq \exp(c(\alpha) \frac{k}{\lambda^p}),
\end{equation}

for every $t \geq 1$. We may clearly assume that $c(\alpha) \geq 1$. We apply Lemma 5.2 with $W = P_\xi(K^n)$, $m = k$, $\gamma = c(\alpha)$, and $p = 1/\alpha$: There exists $\eta \in G_{k,[\lambda/2]}(\xi)$ for which

\begin{equation}
P_\xi(K^n) \cap \eta \subseteq c[c(\alpha)]'^\alpha \lambda^{-\alpha} D_\eta := [c_1(\alpha)]^{-1} \lambda^{-\alpha} D_\eta.
\end{equation}

Taking polars in $\eta$ we see that $P_\eta(K \cap \xi) \geq c_1(\alpha) \lambda^\alpha D_\eta$. Recall that for every symmetric convex body $W$ in $\mathbb{R}^m$ and every $\eta \in G_{m,s}$ the inequality $M(W \cap \eta) \leq \sqrt{m/s} M(W)$ holds, so we get

\begin{equation}
M^*(K \cap \xi) = M((K \cap \xi)^o) \geq \frac{1}{\sqrt{2}} M((K \cap \xi)^o \cap \eta) = \frac{1}{\sqrt{2}} M^*(P_\eta(K \cap \xi)) \geq c_1(\alpha) \lambda^\alpha.
\end{equation}

It is then obvious that $S^*_K(\lambda) \geq c_1(\alpha) \lambda^\alpha$. □

[It is interesting to note that the lower bound (5.8) holds true for every subspace $\xi \in G_{n,k}$. Observe also that $c_1(\alpha) \geq c/(\alpha - 1/2)^{\alpha/2}$].

Upper bound: Let $\lambda \in (0, 1)$ and $k = [\lambda n]$. Find $r > 0$ such that

\begin{equation}
M^*(K \cap rD_n) = \frac{1}{2} (1 - \lambda)^{1/2} r.
\end{equation}

By the Low $M^*$-estimate there exists a subset $L_{n,k}$ of $G_{n,k}$ with measure $\nu_{n,k}(L_{n,k}) \geq 1 - c \exp(-c'(1 - \lambda)n)$, such that

\begin{equation}
M^*(K \cap \xi) \leq \frac{1}{2} \text{diam}(K \cap \xi) \leq r
\end{equation}

for every $\xi \in G_{n,k}$. On the other hand (see [BLM]), we can find $s \leq \frac{e}{\lambda}$ and orthogonal transformations $u_1, \ldots, u_s$, satisfying

\begin{equation}
\frac{1}{4} (1 - \lambda)^{1/2} r D_n \leq \frac{1}{s} \sum_{i=1}^{s} u_i(K \cap rD_n) \subseteq (1 - \lambda)^{1/2} r D_n.
\end{equation}

Set $K_1 = \frac{1}{s} \sum u_i(K \cap rD_n)$. Then, for every $\xi \in G_{n,k}$ we have

\[K_1 \cap \xi \geq \frac{1}{s} \sum_{i=1}^{s} [u_i(K \cap rD_n) \cap \xi],\]

which, together with (5.11), implies that

\[(1 - \lambda)^{1/2} r \geq M^*(K_1 \cap \xi) \geq \frac{1}{s} \sum_{i=1}^{s} M^*[u_i(K \cap rD_n) \cap \xi],\]
and an integration over $G_{n,k}$ shows that

\[(5.12) \quad S^*_{K \cap rD_n}(\lambda) \leq (1 - \lambda)^{1/2}r.\]

We give an upper bound for $r$ using the inverse Brunn-Minkowski inequality: $K \cap rD_n$ is $\alpha$-regular, therefore by (5.1) and (5.11) we obtain

\[(5.13) \quad \frac{1}{4}(1 - \lambda)^{1/2}r \leq \left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}} \leq c' \alpha s^\alpha \left(\frac{|K \cap rD_n|}{|D_n|}\right)^{\frac{1}{n}} \leq \frac{c_2'^c c^\alpha}{(1 - \lambda)^\alpha} \left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}.\]

Finally, we can compare $S^*_{K \cap rD_n}(\lambda)$ with $S^*_K(\lambda)$: Observe first that there is a constant $c_3(\alpha)$ such that $K \subseteq c_3(\alpha)n^\alpha D_n$ and $K^\circ \subseteq c_3(\alpha)n^\alpha D_n$. This follows immediately from the bounds (ii) of the covering numbers of $K$ and $K^\circ$ by large balls. Choosing $t = c_2' c(\alpha)/n^\alpha$ for some absolute constant $c_2$, we can make both $N(K, tD_n)$ and $N(K^\circ, tD_n)$ smaller than 2. Using this information and the fact that the set $L_{n,k}$ has almost full measure, we easily check that for $W = K$ or $K \cap rD_n$,

\[
\int_{L_{n,k}} M^*(W \cap \xi) \nu_{n,k}(d\xi) \simeq \int_{G_{n,k}} M^*(W \cap \xi) \nu_{n,k}(d\xi),
\]

up to absolute constants. But, $M^*(K \cap \xi) = M^*(K \cap rD_n \cap \xi)$ for every $\xi \in L_{n,k}$, which implies that

\[(5.14) \quad S^*_K(\lambda) \leq cS^*_{K \cap rD_n}(\lambda).
\]

Combining (5.12), (5.13) and (5.14), we get

\[
S^*_K(\lambda) \leq c_2(\alpha)(1 - \lambda)^{-\alpha} \left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}.
\]

This completes the proof of the theorem. Observe that $c_2(\alpha) \leq c_4 c'(\alpha) \leq c_4 \exp(2c(\alpha))$ for some absolute constant $c_4 > 0$. \qed

A careful reading of the proof above shows that the diameter of “most” sections $K \cap \xi$, $\xi \in G_{n,\lambda n}$, is determined up to constants depending only on $\lambda$.

5.3 Theorem. Let $K$ be a symmetric convex body in $\mathbb{R}^n$ which is in $M$-position of order $\alpha$. Then, for every $\lambda \in (0, 1)$ and for most $\xi \in G_{n,\lambda n}$ we have

\[
c_1(\alpha) \lambda^\alpha \left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}} \leq \text{diam}(K \cap \xi) \leq \frac{c_2(\alpha)}{(1 - \lambda)^{\alpha + \frac{1}{2}}} \left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}.
\]

Proof: Since $\text{diam}(K \cap \xi) \geq 2M^*(K \cap \xi)$, the lower estimate holds true for every $\xi \in G_{n,\lambda n}$ by (5.8). According to the proof of the upper estimate in Theorem 5.1, if $r$ is the solution of the equation $M^*(K \cap rD_n) = (1 - \lambda)^{1/2}r/2$, then

\[
\text{diam}(K \cap \xi) \leq 2r \leq \frac{c_2(\alpha)}{(1 - \lambda)^{\alpha + \frac{1}{2}}} \left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}},
\]

21
for all \( \xi \in L_{n,\lambda n} \subseteq G_{n,\lambda n} \), where \( \nu_{n,\lambda n}(L_{n,\lambda n}) \geq 1 - c \exp(-c'(1 - \lambda)n) \) (see (5.10) and (5.13)). □

5.4 Remark. The discussion above also shows that the \( M^*_K \) approach is equivalent to the \( S^*_K \) approach in the \( M \)-position: If \( r = r(\lambda) \) is the solution of the equation \( M^*_K(r) = \frac{1}{2}\sqrt{1 - \lambda} \), then

\[
\begin{align*}
  r & \approx S^*_K(\lambda) \approx \text{diam}(K \cap \xi)
\end{align*}
\]

for all \( \lambda \in (0, 1) \) and for most \( \xi \in G_{n,|\lambda n|} \), up to functions depending only on \( \lambda \) and \( \alpha \). This is clear from the lower bound in Theorem 5.1 and the inequalities (5.12), (5.13) and (5.14).

It should also be noted that for an \( \alpha \)-regular body \( K \) in \( \mathbb{R}^n \), as a consequence of the upper estimate in Theorem 5.1 and of the Blaschke – Santaló inequality, we have the following analogue of the duality relation given by Theorem 3.4:

5.5 Corollary. Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \), which is in \( M \)-position of order \( \alpha > 1/2 \). For every \( \lambda, \mu \in (0, 1) \) we have

\[
(5.15) \quad S^*_K(\lambda)S^*_K(\mu) \leq \frac{C(\alpha)}{(1 - \lambda)\alpha(1 - \mu)\alpha},
\]

where \( C(\alpha) > 0 \) is a constant depending only on \( \alpha \). □

Note that there is no restriction on \( \lambda \) or \( \mu \), in contrast to Theorem 3.4. It is also interesting to note that with \( \alpha = 1 \) and \( \lambda = \mu = 1 - \frac{1}{\log n} \) i.e for sections of almost full dimension, one has

\[
S^*_K(\lambda)S^*_K(\lambda) \leq c \log^{2\lambda} n.
\]

We close this section with an upper estimate for \( M^*(K) \) when \( K \) is in \( M \)-position of order \( \alpha > 1/2 \) and \( |K| = |D_n| \). Our method is analogous to the one used in [D] for a second proof of J. Bourgain’s estimate [B] on the isotropic constant:

5.6 Theorem. Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) which is in \( M \)-position of order \( \alpha > 1/2 \). Then,

\[
(5.16) \quad M^*(K) \leq f(\varepsilon)n^\varepsilon,
\]

where \( \varepsilon = \alpha - 1/2 \) and \( f(\varepsilon) = c\varepsilon^{-5/4} \).

Proof: We assume that \( K \) satisfies conditions (i) and (ii). Let \( d = \text{diam}(K) \). In the proof of Theorem 5.1 we checked that

\[
(5.17) \quad d \leq c_1[\varepsilon(\alpha)]^\alpha n^\alpha,
\]

for some absolute constant \( c_1 > 0 \). From (ii) we also know that for every \( j = 0, 1, 2, \ldots \) such that \( 2^j \leq d \), we can find a subset \( N_j \) of \( K \) such that \( K \subseteq \bigcup_{y \in N_j} (y + (d/2^j)D_n) \) and \( \log |N_j| \leq c(\alpha)n2^{j/\alpha}d^{-1/\alpha} \).

22
Let \( Z_j = (N_j - N_{j-1}) \cap (3d/2^j)D_n, \ j = 1, \ldots, r_0 = \lfloor \log_2 d \rfloor \). An inductive argument shows that for every \( r \leq r_0 \), every \( y \in K \) can be written in the form

\[
y = \sum_{j=1}^{r} y_j + z_r,
\]

where \( y_j \in Z_j \) and \( z_r \in (d/2^r)D_n \) (this is known as the Dudley–Fernique decomposition of \( y \in K \)). Note also that

\[
\log |Z_j| \leq 2c(\alpha) \frac{2j/\alpha n}{d^{1/\alpha}}.
\]

Consider any \( \theta \in S^{n-1} \). Using (5.18), we easily check that

\[
\max_{y \in K} |\langle x, \theta \rangle| \leq \sum_{j=1}^{r} \max_{y \in Z_j} |\langle y, \theta \rangle| + \frac{d}{2^r},
\]

for every \( r = 1, \ldots, r_0 \). This implies that

\[
M^*(K) = \int_{S^{n-1}} \|\theta\|_K^* \sigma(d\theta) \leq 2 + \sum_{j=1}^{r_0} \int_{S^{n-1}} \max_{y \in Z_j} |\langle y, \theta \rangle| \sigma(d\theta).
\]

Every \( y \in Z_j \) can be written as \( y = \zeta(y) \overline{\gamma} \) with \( \overline{\gamma} \in S^{n-1} \) and \( |\zeta(y)| \leq 3d/2^j \). Hence, for every \( j = 1, \ldots, r_0 \) we have

\[
\int_{S^{n-1}} \max_{y \in Z_j} |\langle y, \theta \rangle| \sigma(d\theta) \leq \frac{3d}{2^j} \int_{S^{n-1}} \max_{y \in Z_j} |\langle y, \theta \rangle| \sigma(d\theta).
\]

We estimate this last integral as follows: it is easy to see that there is an absolute constant \( c_2 > 0 \) such that

\[
\int_{S^{n-1}} \exp \left( \frac{|\langle \overline{\gamma}, \theta \rangle|^2 n}{c_2^2} \right) \sigma(d\theta) \leq 2.
\]

Therefore, for every \( t \geq 1 \) we have

\[
|\{ \theta : \max_{y \in Z_j} |\langle \overline{\gamma}, \theta \rangle| > c_2 t \left( \frac{\log |Z_j|}{n} \right)^{1/2} \}| \leq |Z_j| \left| \{ \theta : \exp \left( \frac{|\langle \overline{\gamma}, \theta \rangle|^2 n}{c_2^2} \right) \geq |Z_j|^t \} \right| \leq |Z_j|^{1-t^2},
\]

which implies that for all \( j \leq r_0 \),

\[
\int_{S^{n-1}} \max_{y \in Z_j} |\langle \overline{\gamma}, \theta \rangle| \sigma(d\theta) \leq c_3 \left( \frac{\log |Z_j|}{\sqrt{n}} \right)^{1/2} \leq c_3 [c(\alpha)]^{1/2} \frac{2j/\alpha}{d^{1/\alpha}}.
\]
Going back to (5.21) and adding the estimates, we obtain

\begin{equation}
M^\ast(K) \leq 2 + c_4[c(\alpha)]^{1/2}d^{1-\frac{1}{2\alpha}} \sum_{j=1}^{r_0} \frac{1}{2^{j/(1-\frac{1}{2\alpha})}}
\end{equation}

\leq c_5 \frac{|c(\alpha)|^{1/2}}{\alpha - 1/2} d^{1-\frac{1}{2\alpha}}.

Setting \( \varepsilon = \alpha - \frac{1}{2} \), we have \( c(\alpha) \leq c_6/\sqrt{\varepsilon} \) and \( d^{1-\frac{1}{2\alpha}} \leq c_7 n^\varepsilon \), thus (5.24) takes the form

\begin{equation}
M^\ast(K) \leq \frac{c}{\varepsilon^{3/4}} n^\varepsilon. \quad \Box
\end{equation}

5.7 Remark. It is easy to see that if \( \text{diam}(K) \) is the diameter of a symmetric convex body \( K \) in \( \mathbb{R}^n \), then \( M^\ast(K) \geq c|\text{diam}(K)|/\sqrt{n} \). On the other hand, given any \( \alpha > \frac{1}{2} \) it is not hard to construct a body \( K \) in \( M \)-position of order \( \alpha \) with \( \text{diam}(K) \geq cn^\alpha \).

Consider for example the body \( K_1 = \text{co}\{D_n, \pm n^\varepsilon e_n\} \) and normalize it to receive a body \( K \) of volume 1. It then follows that \( M^\ast(K) \geq cn^\varepsilon \) where \( \varepsilon = \alpha - \frac{1}{2} \). This shows that the estimate provided by Theorem 5.5 is exact.

The same example, combined with Theorem 5.1 shows that even in this very natural \( M \)-position, the function \( S^\ast_K \) may increase in an irregular way. It has logarithmic growth up to \( \lambda = 1 - \frac{\log n}{n} \) while \( S^\ast_K(1) \approx n^\varepsilon \).

5.8 Remark. Choosing \( \varepsilon \approx 1/\log n \) in Theorem 5.6, we get that every symmetric convex body \( K \) in \( \mathbb{R}^n \) has a linear image \( K' \) with the properties:

(i) \( |K'| = |D_n| \) and \( M^\ast(K') \leq c(\log n)^{5/4} \).

(ii) \( N(K, tD_n) \leq \exp(cn\sqrt{\log n/t^2}) \) for every \( t \geq 1 \).

This should be compared with the \( \ell \)-position of \( K \): It is a well-known fact (see [P2]) that there exists a linear image \( K_1 \) of \( K \) such that \( |K_1| = |D_n|, M^\ast(K_1) \leq \log n \), and by Sudakov’s inequality \( N(K_1, tD) \leq \exp(cn \log^2 n/t^2) \) for every \( t \geq 1 \).

Of course, the existence of bodies in \( M \)-position of order \( \alpha \) inside every affine class and for every \( \alpha > 1/2 \) depends heavily on Pisier’s estimate about the \( \ell \)-position.

References


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