Mean width and diameter of proportional sections of a symmetric convex body

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1 Introduction

Let K be a symmetric convex body in $\mathbb{R}^n$. The purpose of this paper is to provide upper and lower bounds for the diameter of a random $[\lambda n]$–dimensional central section of K, where the proportion $\lambda \in (0,1)$ is arbitrary but fixed. There are several aspects of our approach to this question that should be clarified right away:

(1) We are interested in bounds expressed in terms of *average parameters* of the body K which can be efficiently computed in a simple way, therefore being useful from the computational geometry point of view.

(2) The dimension n is understood to tend to infinity. Then, we say that our bounds hold for a *random* $[\lambda n]$–dimensional section of K if they are satisfied by all $K \cap \xi$ where ξ is in a subset of the appropriate Grassmannian with Haar probability measure greater than $1 - h(\lambda, n)$, and $h(\lambda, n) \to 0$ as n tends to infinity.

(3) We say that our estimates are *tight* for a class of bodies and a fixed proportion λ if the ratio of our upper and lower bounds depends only on λ. It is clear that one cannot obtain tight bounds for the class of all symmetric convex bodies: it is not hard to describe almost degenerated bodies in $\mathbb{R}^n$ (for example, an ellipsoid with highly incomparable semiaxes) for which the diameter of $[\lambda n]$–sections does not concentrate around some value. So, it is an important question to see under what conditions on K the estimates obtained by a method are tight.

We use the standard notation of the asymptotic theory of finite dimensional normed spaces (which can be found in [MS]): We consider a fixed Euclidean structure in $\mathbb{R}^n$ and write $|\ .\ |$ for the corresponding Euclidean norm. We denote the Euclidean unit ball and the unit sphere by D_n and S^{n-1} respectively, and we write σ for the rotationally invariant probability measure on S^{n-1}.

If W is a symmetric convex body in $\mathbb{R}^n$, then W induces in a natural way a norm $\|\ .\ \|_W$ to $\mathbb{R}^n$. As usual, the polar body $\{y \in \mathbb{R}^n : \max_{x\in W} |\langle y, x\rangle| \leq 1\}$ of W is denoted by W^o. An important average parameter of W is the average of the norm $\|\ .\ \|_W$ on S^{n-1}, defined by

$$M(W) = \int_{S^{n-1}} \|\theta\|_W \, \sigma(d\theta). \tag{1.1}$$

With this notation, the quantity $M^*(W) := M(W^o)$ has a natural geometric meaning: it is half the mean width of W.

Suppose that K is a symmetric convex body in $\mathbb{R}^n$ such that $\frac{1}{b}D_n \subseteq K \subseteq aD_n$. Let $\lambda \in (0,1)$ and set $k = [\lambda n]$. If $G_{n,k}$ is the Grassmannian of all k–dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,k}$ and if $\xi \in G_{n,k}$, we have

$$M^*(K \cap \xi) = \int_{S(\xi)} \|\theta\|_{(K\cap\xi)^o} \sigma_\xi(d\theta) = \int_{S(\xi)} \max_{x \in K\cap\xi} |\langle x, \theta\rangle| \sigma_\xi(d\theta), \tag{1.2}$$

and we can naturally define the function $S_K^* : (0,1) \to (0,\infty)$ with

$$S_K^*(\lambda) = \int_{G_{n,k}} M^*(K \cap \xi)\ \nu_{n,k}(d\xi). \tag{1.3}$$

In other words, $S_K^*(\lambda)$ gives the average mean width of the $[\lambda n]$–dimensional central sections of K. It is not hard to check that S_K^* is increasing in λ. In particular, $S_K^*(\lambda) \leq M^*(K)$ for every $\lambda \in (0,1)$. We view $S_K^*(\lambda)$ as an average parameter of the body K, although it is computationally more complex than the single quantity $M^*(K)$: the empirical distribution method (described in a similar setting in [BLM]) shows that given any δ and ζ in $(0,1)$, a random choice of $N = [c\frac{\log(\frac{2}{\delta})}{\zeta^2}] + 1$ points $x_1, \ldots, x_N$ in S^{n-1} satisfies

$$|M^*(K) - \frac{1}{N}\sum_{i=1}^{N} \|x_i\|_{K^o}| < \zeta M^*(K) \tag{1.4}$$

with probability at least $1-\delta$, where $c > 0$ is an absolute constant. Therefore, $M^*(K)$ can be efficiently "computed" with high probability to any given degree of accuracy. The computation of $S_K^*(\lambda)$ is more complicated and depends on whether the values of the function $M^*(K\cap\xi)$ on $G_{n,k}$ are concentrated around their mean value $S_K^*(\lambda)$. We shall deal with this question in Section 4.

The function S_K^* is clearly related to our problem on the diameter of the sections of K: for every $\xi \in G_{n,k}$ we have $\mathrm{diam}(K\cap\xi) \geq 2M^*(K\cap\xi)$. Therefore, if we define the *average diameter function*

$$D_K(\lambda) = \int_{G_{n,k}} \mathrm{diam}(K \cap \xi)\ \nu_{n,k}(d\xi), \tag{1.5}$$

we have the obvious lower bound $2S_K^*(\lambda) \leq D_K(\lambda)$ for every $\lambda \in (0,1)$. In Section 3 we assume that our body K satisfies a polynomial condition of the form $ab \leq n^t$ for some fixed $t > 0$ and show that an upper bound for $D_K(\lambda)$ in terms of S_K^* is also possible:

Theorem A. *Let $\frac{1}{b}D_n \subseteq K \subseteq aD_n$ and $ab \leq n^t$ for some $t > 0$. If $\lambda \in (0,1)$, we have*

$$2S_K^*(\lambda) \leq D_K(\lambda) \leq 5S_K^*(\lambda/\theta)/(1-\theta)^{1/2},$$

for all $\theta \in (\lambda, 1)$ with $1-\theta \geq c\lambda^{-1}t\log n/n$, where $c > 0$ is an absolute constant.

The point in the statement above is that θ can be chosen to be very close to 1, provided that the dimension n is large and t is fixed. The polynomial condition $ab \le n^t$ is mild and, roughly speaking, prevents the body K from being degenerated. For example, all the well–known natural representatives of any affine class of symmetric convex bodies satisfy a condition of this type with a small value of t: when the ellipsoid of maximal or minimal volume or the distance ellipsoid of K is a ball we have $ab \le \sqrt{n}$, when K is in the isotropic position, in the ℓ–position, or in M–position of any order $\alpha > \frac{1}{2}$ we also have $ab \le n^t$ for a suitable $t > 0$ independent from n.

The double–sided estimate given by Theorem A determines the average diameter $D_K(\lambda)$ for "most" values of $\lambda \in (0,1)$. As a consequence of the polynomial condition $ab \le n^t$, our function S_K^* is forced to increase in a regular way on most of $(0,1)$ and this implies that the bounds of Theorem A are tight: one has the a–priori information that $S_K^*(\lambda) \simeq D_K(\lambda)$ for most values of λ up to a constant depending only on λ and t.

The duality relation

$$S_K^*(\lambda) S_{K^o}^*(\mu) \le \frac{c}{1-(\lambda+\mu)} \tag{1.6}$$

holds true for every body K satisfying a polynomial condition and every $\lambda, \mu \in (0,1)$ with $\lambda + \mu < 1$, provided that n is large enough. The proof of this inequality is based on the second named author's "distance lemma" (see Theorem 3.4).

In Section 4 we study the question of the diameter of a "random" $[\lambda n]$-section of K. Passing from the average diameter $D_K(\lambda)$ to the diameter of most sections requires some strong concentration of the function $M^*(K \cap \xi)$ on $G_{n,[\lambda n]}$ around its expectation $S_K^*(\lambda)$. We study the behavior of $M^*(K \cap \xi)$ and show that it satisfies a certain Lipschitz estimate. The resulting deviation inequality is relatively weak, however in the important case where ab is roughly speaking $o(\sqrt{n})$ (or more generally $bM^*(K) = o(\sqrt{n})$) we prove an analogue of Theorem A for random sections (see Section 4 for variations of this result and more precise conditions on ab):

Theorem B. *Let K satisfy $D_n \subseteq K \subseteq \gamma(n) D_n$ with $\gamma(n) = o(n^{1/2})$. Let $\lambda \in (0,1)$ and $k = [\lambda n]$. Then, for every $\theta \in (\lambda, 1)$ we have*

$$c_1 S_K^*(\lambda\theta) \le \mathrm{diam}(K \cap \xi) \le c_2 S_K^*(\lambda/\theta)/(1-\theta)^{1/2}$$

for most $\xi \in G_{n,k}$, provided that n is large enough (depending on θ).

Using this fact one can determine the diameter of a random $[\lambda n]$-section of a body K with $ab = o(n^{1/2})$ for many values of λ. More precisely, we consider any λ-flag of subspaces $\mathbb{R}^n = E_0 \supset E_1 \supset \ldots \supset E_s$ with $\dim E_j = [\lambda^j n]$, $s = s(\lambda) \simeq \log[(1-\lambda)n]/\log(1/\lambda)$ and prove that for most orthogonal transformations $T \in O(n)$ and most values of j, the diameter of $K \cap T(E_j)$ is determined by

$$\mathrm{diam}(K \cap T(E_j)) \simeq S_K^*(\lambda^j)$$

up to a constant depending only on λ. (Theorem 4.8). This is of interest being a statement for a generic body K whose minimal/maximal volume or distance ellipsoid is a Euclidean ball.

In Section 5 we study the case of a body in M–position of order α (an α–regular body in the terminology of [P1]: see the beginning of Section 5 for the necessary definitions). In this case, we determine $S_K^*(\lambda)$ up to a constant depending only on λ and α:

Theorem C. *Let K be a symmetric convex body in $\mathbb{R}^n$, which is in M–position of order $\alpha > 1/2$. For every $\lambda \in (0,1)$,*

$$c_1 \lambda^\alpha \text{ v.rad}(K) \leq S_K^*(\lambda) \leq c_2(1-\lambda)^{-\alpha} \text{ v.rad}(K),$$

where $\text{v.rad}(K) = (|K|/|D_n|)^{1/n}$ *and $c_1, c_2 > 0$ are constants depending only on α.*

Actually, the left hand side inequality holds true in the much stronger form

$$M^*(K \cap \xi) \geq c_1 \lambda^\alpha \text{ v.rad}(K)$$

for *every* $\xi \in G_{n,[\lambda n]}$, while the right hand side inequality holds with $M^*(K \cap \xi)$ replacing $S_K^*(\lambda)$ for most $\xi \in G_{n,[\lambda n]}$. One may view Theorem C as a proportional version of Urysohn's inequality

$$M^*(K) \geq \text{v.rad}(K)$$

for bodies in M–position of order α, which turns out to be an equivalence in this case: we have $S_K^*(\lambda) \simeq \text{v.rad}(K)$ up to functions depending only on λ and α.

Using this information one determines $\text{diam}(K \cap \xi)$ for a random $\xi \in G_{n,k}$ up to a function of λ, for all $\lambda \in (0,1)$ (Theorem 5.3).

We close this paper with two upper bounds on the quantities $S_K^*(\lambda)$ and $M^*(K)$ in the case where K is in M–position of order α:

Theorem D. *Let K be a symmetric convex body in $\mathbb{R}^n$ in M–position of order $\alpha > \frac{1}{2}$, and set $\varepsilon = \alpha - \frac{1}{2}$.*

(i) *If $1 - \lambda \simeq 1/\log n$, we have*

$$S_K^*(\lambda) S_{K^o}^*(\lambda) \leq c(\alpha) \log^{2\alpha} n.$$

(ii) *For the mean width of K we have the upper bound*

$$M^*(K) \leq c\varepsilon^{-5/4} n^\varepsilon.$$

Both estimates in Theorem D should be compared to the following classical estimate of Pisier (see [L], [FT], and [P2]): Every body K has a linear image K_1 such that $M(K_1)M^*(K_1) \leq c \log n$. Part (i) of Theorem D demonstrates a regularity of the function $F(\lambda) = S_K^*(\lambda) S_{K^o}^*(\lambda)$ in the M–position case: If K is in M–position of order α (with α even far from 1/2), then the growth of F remains logarithmic in n for λ even very close to 1. Pisier's result implies that every body K has a linear image K_1 of volume $|K_1| = |D_n|$ with mean width $M^*(K_1) \leq c \log n$. Part (ii) of Theorem D shows that a body K in M–position of order α sufficiently close to $1/2$ has mean width logarithmic in n. Simple examples show that for every value of $\varepsilon = \alpha - \frac{1}{2}$, the n^ε bound for $M^*(K)$ in Theorem D(ii) cannot be improved.

A different approach to the question of the diameter of random proportional sections was proposed in [GM1]. We briefly discuss this method in Section 2. Throughout the text, we compare the two methods whenever it is possible.

In the sequel, the letters c, c', c_1 etc. stand for absolute positive constants, not necessarily the same in every occurrence. The volume, the cardinality of a finite set and the Euclidean norm are all denoted by $| \, . \, |$: this should cause no confusion.

2 The Low M^*–estimate and a first approach to the problem

This section is a survey of results from [GM1], [GM2] and describes the "M_K^*–approach" to the diameter problem.

A crucial inequality of the asymptotic theory of finite dimensional normed spaces is the second named author's Low M^*–estimate which relates the diameter of proportional sections of a symmetric convex body W in $\mathbb{R}^m$ to its mean width $M^*(W)$. Roughly speaking, one has

$$\text{diam}(W \cap \eta) \leq M^*(W)/h_1(\mu) \tag{2.1}$$

for most $\eta \in G_{m,[\mu m]}$, where h_1 is a function depending only on $\mu \in (0,1)$. For proofs of (2.1) see [M1], [PT], [Go]: it is known that it holds true with $h_1(\mu) = c(1-\mu)^{1/2}$ and that this dependence on μ is best possible. We shall make use of the precise probabilistic form of the Low M^*–estimate which can be found in [Go], [M4]:

2.1 Theorem (Low M^*–estimate) *If W is a symmetric convex body in $\mathbb{R}^m$ and if $\mu, \varepsilon \in (0,1)$, then we have*

$$\text{diam}(W \cap \eta) \leq \frac{2M^*(W)}{(1-\varepsilon)\sqrt{1-\mu}}, \tag{2.2}$$

for all η in a subset $L_{m,k}$ of $G_{m,k}$ of measure $\nu_{m,k}(L_{m,k}) \geq 1 - c\exp(-c'\varepsilon^2(1-\mu)m)$, where $k = [\mu m]$ and $c, c' > 0$ are absolute constants. □

Theorem 2.1 already shows that the diameter of a random section of proportional dimension is controlled by the mean width of the body. In [GM1] we exploit the idea of pushing the Low M^*–estimate to its limit in order to determine a reasonable "confidence interval" for the diameter of the $[\lambda n]$–sections of a body K in $\mathbb{R}^n$ using average parameters of K with the same complexity as $M^*(K)$.

To this end, we consider the function $M_K^* : \mathbb{R}^+ \to (0,1]$ defined by

$$M_K^*(r) = M^*(K \cap rD_n)/r,$$

and as a simple consequence of Theorem 2.1 we see that if $r_1 > 0$ is the solution of the equation $M_K^*(r) = h_1(\lambda) = \frac{1}{2}(1-\lambda)^{1/2}$ in r, then most $[\lambda n]$–sections of K have diameter smaller than $2r_1$ (see [GM1], Theorem 2.1).

It turns out that this same function can provide a general *lower* bound for the diameter of the $[\lambda n]$–sections of K. The main new ingredient is a conditional Low M–estimate which is in a sense dual to Theorem 2.1:

2.2 Theorem (Conditional Low M–estimate) *If K is a symmetric convex body in $\mathbb{R}^n$ and if $\lambda \in (0,1)$, then for the solution r_2 of the equation*

$$M_K^*(r) = h_2(\lambda) := 1 - c^{\frac{1}{1-\lambda}}$$

in r we can find a subset $L_{n,k}$ of $G_{n,k}$ with $\nu_{n,k}(L_{n,k}) \geq 1 - c^k$, where $k = [\lambda n]$, such that

$$\text{diam}(K^o \cap \xi) \leq \frac{10}{r_2} C^{\frac{\lambda}{1-\lambda}}$$

for all $\xi \in L_{n,k}$, where $0 < c < 1$ and $C > 1$ are absolute constants, and n is large enough. $\square$

Theorem 2.2 shows that most $[\lambda n]$–projections of K contain a Euclidean ball of radius proportional to r_2 up to a function depending only on λ. When $\lambda \in (\frac{1}{2}, 1)$, this fact combined with Borsuk's antipodal theorem gives r_2 as a lower bound for the diameter of the $[\lambda n]$–sections of K. We thus have a double sided estimate of $\text{diam}(K \cap \xi)$ in terms of the function M_K^*:

2.3 Theorem (M_K^* approach to the diameter problem) *There exist two explicit functions $h_1, h_2 : (0,1) \to (0,1)$ such that for every $\lambda \in (\frac{1}{2}, 1)$ and every symmetric convex body K in $\mathbb{R}^n$, solving the equations $M_K^*(r) = h_1(\lambda)$ and $M_K^*(r) = h_2(\lambda)$ in r we find an upper estimate r_1 and a lower estimate r_2 for the diameter of a random $[\lambda n]$–section of K.* $\square$

The important point in Theorem 2.3 is that the functions h_1 and h_2 are universal and that the statement holds true for an *arbitrary* body K, the only restriction being that n should be large enough depending on λ. Another advantage of Theorem 2.3 is that it makes use of the *global* (hence computationally simple) parameter M^* of the body. On the other hand, being so general the estimates cannot be tight in full generality. Another disadvantage of the method in [GM1] is that the use of Borsuk's theorem forces us to study only proportions $\lambda \in (\frac{1}{2}, 1)$. This first approach gives no information for small proportions.

Our method in this paper is based on the function S_K^* which was defined in Section 1. It provides lower and upper bounds for the diameter of the sections of K of *any* fixed proportion $\lambda \in (0,1)$. We also show that the estimates obtained are tight for large classes of symmetric convex bodies and for most values of λ. Some of our results were announced in [GM2].

Let us close this section with an application of the M_K^* approach: For every integer $t \geq 2$ we define the minimal circumradius of an intersection of t rotations of a body K by

$$r_t(K) = \min\{\rho > 0 : u_1(K) \cap \ldots \cap u_t(K) \subseteq \rho D_n \text{ for some } u_1, \ldots, u_t \in SO(n)\},$$

and the "upper radius" of a random n/t–dimensional central section of K by

$$R_t(K) = \min\{R > 0 : \nu_{n,n/t}(\xi \in G_{n,n/t} : K \cap \xi \subseteq RD_\xi) \geq 1 - \frac{1}{t+1}\}.$$

It is proved in [M4] that

$$r_{2t}(K) \leq \sqrt{t} R_t(K)$$

for every $t \geq 2$ and every body K. In [GM2] we prove that the local parameter $R_t(K)$ and the global parameter $r_t(K)$ are closely related in the sense that an inverse inequality is possible in full generality:

2.4 Theorem. *For every integer $t \geq 2$ and every symmetric convex body K in $\mathbb{R}^n$, $n \geq n_0(t)$, we have*

$$R_{f(t)}(K) \leq g(t) r_t(K),$$

where $g(t) = C^t$, $f(t) = [g(t)]$, and $C > 1$ is an absolute constant. $\square$

The proof of Theorem 2.4 is based on Theorems 2.1 and 2.2. The result is somehow unexpected for an arbitrary body K. The search for the best possible functions f and g in the statement above is likely to give more information and probably new ideas related to the M_K^* approach.

3 Average Mean Width and Diameter of Proportional Sections of a Symmetric Convex Body Satisfying Polynomial Bounds

Let K be a symmetric convex body in $\mathbb{R}^n$. Recall that the "average diameter" function $D_K : (0,1) \to (0,\infty)$ is defined by

$$D_K(\lambda) = \int_{G_{n,k}} \text{diam}(K \cap \xi)\, \nu_{n,k}(d\xi),$$

where $k = [\lambda n]$. Since $2M^*(K \cap \xi) \leq \text{diam}(K \cap \xi)$ for every $\xi \in G_{n,k}$, we immediately compare $D_K(\lambda)$ with $S_K^*(\lambda)$:

$$2S_K^*(\lambda) \leq D_K(\lambda) \qquad (3.1)$$

for every $\lambda \in (0,1)$. Using Theorem 2.1 we shall give an upper bound for $D_K(\lambda)$ in terms of S_K^*. This is possible if we assume that K satisfies a polynomial condition:

3.1 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$, such that $\frac{1}{b}D_n \subseteq K \subseteq aD_n$ with $ab \leq n^t$. For every $\lambda \in (0,1)$, we have*
(3.2)
$$2S_K^*(\lambda) \leq D_K(\lambda) \leq 5\inf\{S_K^*(\lambda/\theta)/(1-\theta)^{1/2} : \theta \in (\lambda,1), 1-\theta \geq c_1 t\lambda^{-1}\log n/n\},$$

where $c_1 > 0$ is an absolute constant.

[It is clear that given any $\lambda \in (0,1)$ the set in (3.2) will be non-empty provided that $n \geq n_0(\lambda, t)$.]

Proof: Let $\theta \in (\lambda, 1)$ with $1-\theta \geq c_1 t\lambda^{-1}\frac{\log n}{n}$, and fix a subspace η with $\dim\eta = (\lambda/\theta)n$. There exists a subset $L_{(\lambda/\theta)n,\lambda n}(\eta)$ of $G_{(\lambda/\theta)n,\lambda n}(\eta)$ with measure $\nu(L_{(\lambda/\theta)n,\lambda n}(\eta)) \geq 1 - c\exp(-c'\frac{\lambda}{\theta}(1-\theta)n)$ such that for every $\xi \in L_{(\lambda/\theta)n,\lambda n}(\eta)$

$$\mathrm{diam}(K\cap\xi) \leq \frac{4M^*(K\cap\eta)}{\sqrt{1-\theta}}.$$

Integrating over $G_{(\lambda/\theta)n,\lambda n}(\eta)$ we get:

$$\int_{G_{(\lambda/\theta)n,\lambda n}(\eta)} \mathrm{diam}(K\cap\xi)\,\nu(d\xi) \leq \frac{4M^*(K\cap\eta)}{\sqrt{1-\theta}} + abM^*(K\cap\eta)\exp(-t\log n), \tag{3.3}$$

if $c_1 > 0$ is chosen suitably large, where we made use of the fact that for every ξ we have $\mathrm{diam}(K\cap\xi) \leq 2a \leq 2abM^*(K\cap\eta)$. Since $ab \leq n^t$, it follows that

$$\int_{G_{(\lambda/\theta)n,\lambda n}(\eta)} \mathrm{diam}(K\cap\xi)\,\nu(d\xi) \leq 5\,\frac{M^*(K\cap\eta)}{\sqrt{1-\theta}}, \tag{3.4}$$

Now, integrating (3.4) over $G_{n,(\lambda/\theta)n}$ and recalling (3.1) we obtain

$$2S_K^*(\lambda) \leq D_K(\lambda) \leq 5\,\frac{S_K^*(\lambda/\theta)}{\sqrt{1-\theta}},$$

and the proof is complete. □

3.2 Remark. If one has some information on the way $S_K^*(\lambda)$ increases as a function of λ, then Theorem 3.1 can be useful in order to determine the average diameter of the λn-dimensional sections of K. It is however clear that the lower and upper bounds provided by Theorem 3.1 will be "close" only if $S_K^*(\lambda)$ increases in a regular way.

When K satisfies a polynomial condition $ab \leq n^t$, then there will be many intervals of regularity for $S_K^*(\lambda)$. To make this more precise, let us fix some $\lambda \in (0,1)$ and consider the finite sequence $k_j = [\lambda^j n]$, $j = 0, 1, \ldots, s(\lambda)$. The length $s(\lambda)$ of the sequence is the smallest positive integer s for which $[\lambda^s n] = [\lambda^{s+1}n]$. It is easy to check that $s(\lambda) \simeq \log((1-\lambda)n)/\log(1/\lambda)$.

Since S_K^* is increasing in λ, we have

$$M^*(K) = S_K^*(k_0) \geq S_K^*(k_1) \geq \ldots \geq S_K^*(k_{s(\lambda)}).$$

We set $d_j = k_{j-1}/k_j$, $j = 1, \dots, s(\lambda)$. Given any small $\delta \in (0,1)$ and any $\zeta > 1$, consider the set $J_\zeta = \{j \leq s(\lambda) : d_j \geq \zeta\}$. Then, $|J_\zeta| \leq t \log n / \log \zeta \leq \delta s(\lambda)$ if ζ satisfies the condition $\log \zeta \geq c_1 \frac{t}{\delta} \log(\frac{2}{1-\lambda}) \log(\frac{1}{\lambda})$.
Choose any $j \in J_\zeta$. Then, Theorem 3.1 implies that

$$2S_K^*(\lambda^j) \leq D_K(\lambda^j) \leq 5(1-\lambda)^{-1/2} S_K^*(\lambda^{j-1}) \leq 5(1-\lambda)^{-1/2} \zeta S_K^*(\lambda^j). \tag{3.5}$$

Thus, $D_K(\lambda^j) \simeq S_K^*(\lambda^j)$ for all $j \leq s(\lambda)$ in a set of cardinality greater than $(1-\delta)s(\lambda)$, up to a function depending only on λ, t, and δ.

This observation has a meaning from the computational point of view, since $S_K^*(\lambda)$ can be computed in contrast to $D_K(\lambda)$. The degree of efficiency of this method clearly depends on the a-priori information one has for the concentration of the function $M^*(K \cap \xi), \xi \in G_{n,[\lambda n]}$ around $S_K^*(\lambda)$ (see next section).

It is reasonable to expect that S_K^* increases faster as $\lambda \to 1^-$. If true, this would imply that S_K^* increases regularly on every interval $[0, \lambda_0]$, $\lambda_0 < 1$, when K satisfies a polynomial condition and $n \geq n_0(\lambda_0, t)$. In particular, the bounds given by Theorem 3.1 would be tight for *all* "small" values of λ. Thus, we are lead to the following:

QUESTION: Is it true that S_K^* is a "convex" function of λ on $(0,1)$?

It is also interesting to note some duality relations which are satisfied by S_K^*: If $F : (0,1] \times (0,1] \to \mathbb{R}^+$ is defined by

$$F(\lambda, \mu) = S_K^*(\lambda) S_{K^o}^*(\mu),$$

then one has upper bounds for $F(\lambda, \mu)$ which are independent of K (assuming that ab is polynomial in n), provided that $\lambda + \mu < 1$. We start with a simple lemma:

3.3 Lemma. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that $\frac{1}{b}D_n \subseteq K \subseteq aD_n$, with $ab \leq n^t$. Then, if $\lambda, \varepsilon \in (0,1)$, and if r is the solution of the equation $M^*(K \cap rD_n) = (1-\varepsilon)(1-\lambda)^{1/2} r$, we have*

$$S_K^*(\lambda) \leq 2r,$$

provided that $n/\log n \geq c_1 t/\varepsilon^2(1-\lambda)$, where $c_1 > 0$ is an absolute constant.

Proof: Let $k = [\lambda n]$. It is clear that $r > 1/b$, hence for every $\xi \in G_{n,k}$ we have the obvious estimate $M^*(K \cap \xi) \leq a \leq abr \leq n^t r$. On the other hand, by the low M^*-estimate we know that $K \cap \xi \subseteq rD_\xi$ for all ξ in a subset $L_{n,k}$ of $G_{n,k}$ with measure exceeding $1 - c\exp(-c'\varepsilon^2(1-\lambda)n)$. Therefore,

$$\begin{aligned} S_K^*(\lambda) &= \int_{G_{n,k}} M^*(K \cap \xi) \nu_{n,k}(d\xi) \leq \nu_{n,k}(L_{n,k}^c) n^t r + \nu_{n,k}(L_{n,k}) r \\ &\leq (c\exp(-c'\varepsilon^2(1-\lambda)n) n^t + 1) r \leq 2r, \end{aligned} \tag{3.6}$$

if n is large enough. □

We will also need the Distance Lemma from [M3]:

3.4 Lemma. *Let W be a symmetric convex body in $\mathbb{R}^n$ with $\rho D_n \subseteq W \subseteq rD_n$. Assume that $(M^*(W)/r)^2 + (M(W)\rho)^2 = s > 1$. Then,*

$$\frac{r}{\rho} \leq \frac{1}{s-1}. \qquad \square \tag{3.7}$$

It is an obvious consequence of Hölder's inequality that for every symmetric convex body K in $\mathbb{R}^n$ the inequality

$$M(K)M^*(K) \geq 1 \tag{3.8}$$

holds true. Moreover, this inequality is in general far from being sharp: it holds as an equality if and only if K is a multiple of the Euclidean ball. On the other hand, a well–known sequence of results of Figiel–Tomczak [FT], Lewis [L] and Pisier [P2] states that for every K we can find a linear image $\overline{K}$ of K for which

$$M(\overline{K})M^*(\overline{K}) \leq c \log n, \tag{3.9}$$

where $c > 0$ is an absolute constant.

It is not hard to check that $S_K^*(\lambda)S_{K^o}^*(\mu) \geq 1$ for every $\lambda, \mu \in (0,1)$: If for example $\lambda \geq \mu$ we have

$$S_K^*(\lambda)S_{K^o}^*(\mu) \geq S_K^*(\mu)S_{K^o}^*(\mu) \geq 1$$

by the monotonicity of S_K^* and Hölder's inequality. Using Lemmas 3.3 and 3.4 one can see that for bodies satisfying a polynomial condition a weaker version of (3.9) is always true:

3.5 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ such that $\frac{1}{b}D_n \subseteq K \subseteq aD_n$, with $ab \leq n^t$. If $\lambda, \mu, \kappa \in (0,1)$ and $\lambda + \mu = 1 - \kappa$, then*

$$S_K^*(\lambda)S_{K^o}^*(\mu) \leq \frac{8}{\kappa}, \tag{3.10}$$

provided that n is large enough (depending on t, λ and μ).

Proof: Let $n/\log n \geq 64c_1 t/\kappa^2(1-\lambda)$, where c_1 is the constant from Lemma 3.3. We apply Lemma 3.3 with $\varepsilon = \frac{1}{8}\kappa$: Find $r > 0$ such that $M^*(K \cap rD_n) = (1 - \kappa/8)(1-\lambda)^{1/2}r$. Then, Lemma 3.3 shows that

$$S_K^*(\lambda) \leq 2r. \tag{3.11}$$

Next, find $\rho > 0$ such that $M^*(K^o \cap \frac{1}{\rho}D_n) = (1-\kappa/8)(1-\mu)^{1/2}\frac{1}{\rho}$. Since $\frac{1}{a}D_n \subseteq K \subseteq bD_n$, Lemma 3.3 applies again to give

$$S_{K^o}^*(\mu) \leq \frac{2}{\rho}. \tag{3.12}$$

Without loss of generality we assume that $\rho \leq r$. Let $T = \mathrm{co}((K \cap rD_n) \cup \rho D_n)$. Then, $\rho D_n \subseteq T \subseteq rD_n$, $T \supseteq K \cap rD_n$, and $T^o \supseteq K^o \cap \frac{1}{\rho} D_n$, therefore

(3.13)

$$\left(\frac{M^*(T)}{r}\right)^2 + (M(T)\rho)^2 \geq \left(\frac{M^*(K \cap rD_n)}{r}\right)^2 + \left(M^*(K^o \cap \frac{1}{\rho} D_n)\rho\right)^2 = (1-\kappa/8)^2(2-\lambda-\mu)$$

$$= (1-\kappa/8)^2(1+\kappa) \geq 1 + \frac{\kappa}{2}.$$

Since $\rho D_n \subseteq T \subseteq rD_n$, the distance lemma implies that

$$\frac{r}{\rho} \leq \frac{2}{\kappa}. \tag{3.14}$$

Combining (3.11), (3.12) and (3.14), we obtain

$$S_K^*(\lambda) S_{K^o}^*(\mu) \leq \frac{8}{\kappa}. \quad \square$$

In Section 5 we shall see that in the case of a body in M–position of order α one can avoid the restriction $\lambda + \mu < 1$. For λ and μ both very close to 1, we have $F(\lambda, \mu)$ bounded by a constant independent from n.

4 On the Diameter of a Random Proportional Section

We proceed to see whether one can obtain more precise information about the diameter of a "random" $[\lambda n]$-dimensional section of K. Here, we specify "random" as follows: for every ξ in a subset $L_{n,[\lambda n]}$ of $G_{n,[\lambda n]}$ with measure $\nu_{n,[\lambda n]}(L_{n,[\lambda n]}) \geq 1 - h(\lambda, n)$, where $h(\lambda, n) = o_n(1)$. This approaches 1 for every λ as the dimension n grows to infinity.

To this end, we first study the behavior of $M^*(K \cap \xi)$ as a function of ξ on $G_{n,k}$. We consider two distances ρ and d on $G_{n,k}$, defined by

(4.1)

$$\rho(\xi, \eta) = \min\{(\sum_{i=1}^{k} |e_i - f_i|^2)^{1/2} : \{e_i\}_{i \leq k}, \{f_i\}_{i \leq k} \text{ are orthonormal bases of } \xi, \eta\}$$

and

$$d(\xi, \eta) = \max\{d_1(x, S(\eta)) : x \in S(\xi),\ d_1 \text{ the geodesic distance}\}. \tag{4.2}$$

Our first Lemma compares ρ with d. It is probably a well known fact but we did not find a convenient reference so we prove it below:

4.1 Lemma. *Let $\xi, \eta \in G_{n,k}$. Then,*

$$(2/\pi) d(\xi, \eta) \leq \rho(\xi, \eta) \leq (2k)^{1/2} d(\xi, \eta). \tag{4.3}$$

Proof: The left hand side inequality is clear: Let $\{e_i\}$ and $\{f_i\}$ be two orthonormal bases of ξ, η respectively. If $x = \sum t_i e_i \in S(\xi)$, then $|x - \sum t_i f_i| \leq (\sum |e_i - f_i|^2)^{1/2}$, therefore $(2/\pi) d_1(x, S(\eta)) \leq (\sum |e_i - f_i|^2)^{1/2}$. It follows that $(2/\pi) d(\xi, \eta) \leq \rho(\xi, \eta)$.

For the right hand side inequality, we use an inductive argument based on the following claim:

Claim: Let $E, F \in G_{n,m}$, $m \geq 2$, and $x \in S(E)$ be such that $|P_F(x)|$ is minimal. Then, for every $x_1 \in E \cap x^{\perp}$ we have $P_F(x_1) \perp P_F(x)$.

[*Proof:* Suppose that $x_1 \in S(E) \cap x^{\perp}$ with $\beta = \langle P_F(x_1), P_F(x) \rangle \neq 0$. Without loss of generality we may assume that $\beta > 0$. For every $t > 0$ we have

$$|P_F(x - tx_1)| \geq |x - tx_1| |P_F(x)|,$$

which implies that

$$2\beta \leq t(|P_F(x_1)|^2 - |P_F(x)|^2), \tag{4.4}$$

a contradiction if we let $t \to 0^+$.]

We use the claim to choose orthonormal bases $\{e_i\}$ and $\{f_i\}$ of ξ, η as follows: We choose $e_1 \in S(\xi)$ such that $|P_\eta(e_1)|$ is minimal. Observe that if $P_\eta(e_1) = 0$, then $d(\xi, \eta) = \pi/2$ and we have nothing to prove. If not, we set $f_1 = P_\eta(e_1)/|P_\eta(e_1)|$. If $\{e_i\}_{i \leq s}$ and $\{f_i\}_{i \leq s}$ have been chosen, we choose $e_{s+1} \in S(\xi) \cap \langle e_i, i \leq s \rangle^{\perp}$ with $|P_\eta(e_{s+1})|$ minimal. By the claim, $P_\eta(e_{s+1}) \perp \langle f_i, i \leq s \rangle$, so we set $f_{s+1} = P_\eta(e_{s+1})/|P_\eta(e_{s+1})|$.

With this construction,

$$|e_i - f_i| = \sqrt{2}(1 - |P_\eta(e_i)|^2)^{1/2} \leq \sqrt{2}(1 - |P_\eta(e_1)|^2)^{1/2} \leq \sqrt{2} d(\xi, \eta). \tag{4.5}$$

It follows that $\rho(\xi, \eta) \leq (2k)^{1/2} d(\xi, \eta)$. $\square$

Using Lemma 4.1, we can prove that $M^*(K \cap \xi)$ satisfies the following Lipschitz estimate:

4.2 Lemma. *Let K be a symmetric convex body in $\mathbb{R}^n$ with $\frac{1}{b} D_n \subseteq K \subseteq a D_n$, and fix $\lambda \in (0,1)$ and $k = [\lambda n]$. Then,*

$$|M^*(K \cap \xi) - M^*(K \cap \eta)| \leq 6a^2 b d(\xi, \eta) \tag{4.6}$$

for every $\xi, \eta \in G_{n,k}$.

Proof: Let $\{e_i\}_{i \leq k}$ and $\{f_i\}_{i \leq k}$ be two orthonormal bases of ξ and η respectively, such that $\rho^2(\xi, \eta) = \sum_{i=1}^k |e_i - f_i|^2$. Recall that

(4.7)
$$M^*(K \cap \xi) \simeq \frac{1}{\sqrt{k}} \int_\Omega \| \sum_{i=1}^k g_i(\omega) e_i \|_{(K \cap \xi)^o} d\omega = \frac{1}{\sqrt{k}} \int_\Omega \max_{x \in K \cap \xi} \langle x, \sum_{i=1}^k g_i(\omega) e_i \rangle d\omega,$$

where $g_1, \ldots, g_k$ are independent standard Gaussian random variables on some probability space Ω (with an analogous estimate holding for $M^*(K \cap \eta)$ and the orthonormal basis $\{f_i\}_{i \le k}$).

We define a function h on $\mathbb{R}^n$ by

$$h(x) = \max\{\langle z, x\rangle : z \in K \cap (\xi \cup \eta)\}. \tag{4.8}$$

Then, we easily see that

$$|M^*(K\cap\xi) - M^*(K\cap\eta)| \le \int_{S(\xi)} [h(x) - \|x\|_{(K\cap\xi)^o}] + \Big|\int_{S(\xi)} h(x) - \int_{S(\eta)} h(y)\Big| + \int_{S(\eta)} [h(y) - \|y\|_{(K\cap\eta)^o}]. \tag{4.9}$$

For the middle term note that, by Lemma 4.1,

$$\begin{aligned} \Big|\int_{S(\xi)} h(x) - \int_{S(\eta)} h(y)\Big| &\simeq \frac{1}{\sqrt{k}} \int_\Omega |h(\sum g_i e_i) - h(\sum g_i f_i)| d\omega \\ &\le \frac{1}{\sqrt{k}} \int_\Omega h(\sum g_i (e_i - f_i)) d\omega \\ &\le \frac{a}{\sqrt{k}} \int_\Omega |\sum g_i (e_i - f_i)| d\omega \\ &\le \frac{a\rho(\xi,\eta)}{\sqrt{k}} \le \sqrt{2} a d(\xi,\eta). \end{aligned} \tag{4.10}$$

By symmetry, it remains to estimate the first term in (4.9). Given $x \in S(\xi)$, suppose that $h(x) = \langle z, x\rangle$ for some $z \in K\cap\eta$ with $\|z\| = 1$ (if the max was attained for some $z \in K \cap \xi$ we would simply have $h(x) - \|x\|_{(K\cap\xi)^o} = 0$). We can find $x_0 \in |z|S(\xi)$ with $|z - x_0| \le |z| d(\xi,\eta)$. If $x_0 \in K \cap \xi$, then $h(x) - \|x\|_{(K\cap\xi)^o} \le \langle z - x_0, x\rangle \le |z - x_0| \le ad(\xi,\eta)$. Assume that $\|x_0\| > 1$. Then, $\|x_0\| \le \|z\| + \|x_0 - z\| \le 1 + abd(\xi,\eta)$, and we write

$$\begin{aligned} h(x) - \|x\|_{(K\cap\xi)^o} &\le \langle z - \frac{x_0}{\|x_0\|}, x\rangle \le \langle z - x_0, x\rangle + \langle (1 - \frac{1}{\|x_0\|}) x_0, x\rangle \\ &\le ad(\xi,\eta) + \frac{abd(\xi,\eta)}{1 + abd(\xi,\eta)} |x_0| \\ &\le \left[a + \frac{a^2 b}{1 + abd(\xi,\eta)}\right] d(\xi,\eta) \le 2a^2 b d(\xi,\eta). \end{aligned} \tag{4.11}$$

Inserting this information into (4.9) we conclude the proof. □

Lemma 4.2 and a well–known deviation inequality for a Lipschitz function on $G_{n,k}$ (see [MS], Chapter 6 and Appendix V) give us the following estimate:

4.3 Lemma. *Let K be a symmetric convex body in $\mathbb{R}^n$ with $\frac{1}{b}D_n \subseteq K \subseteq aD_n$, and fix $\lambda \in (0,1)$ and $k = [\lambda n]$. Then,*

(4.12)
$$\nu_{n,k}\left(\{\xi \in G_{n,k} : |M^*(K\cap\xi) - S_K^*(\lambda)| \geq \frac{1}{2}S_K^*(\lambda)\}\right) \leq \exp(-c\frac{n}{a^4b^2}[S_K^*(\lambda)]^2),$$

where $c > 0$ is an absolute constant. □

Lemma 4.3 provides a rather weak concentration of the values of the function $M^*(K\cap\xi)$ around its expectation $S_K^*(\lambda)$: when applied directly, it is practically useful only if ab is considerably small. However, as a first step we can make use of this information in a quite interesting case: when $D_n \subseteq K \subseteq aD_n$ with $a = o(\sqrt{n})$.

In analogy to $M_K^*(r)$ we define the auxiliary function

$$S_K^*(r,\lambda) = \int_{G_{n,k}} M^*(K\cap rD_n\cap\xi)\,\nu_{n,k}(d\xi), \tag{4.13}$$

where $r > 0$, $\lambda \in (0,1)$, and $k = [\lambda n]$. This is a function increasing in r and λ. For fixed $\lambda \in (0,1)$, the obvious inequality $\|\theta\|_{(K\cap rD_n\cap\xi)^o} \leq \|\theta\|_{(K\cap rD_n)^o}$ for $\theta \in S(\xi)$ shows that

$$S_K^*(r,\lambda) \leq M^*(K\cap rD_n) = rM_K^*(r) \tag{4.14}$$

for all $r > 0$. Furthermore, one has the following additional information:

4.4 Lemma. *The functions $S_K^*(r,\lambda)$ and $rM_K^*(r) = M^*(K\cap rD_n)$ are concave in r.*

Proof: We first show that $M^*(K\cap rD_n)$ is concave. Let $r_1, r_2 > 0$ and $0 < \beta < 1$. Given $\theta \in S^{n-1}$, there exist $x_i \in K\cap r_iD_n$, $i = 1,2$, such that $\max_{x\in K\cap r_iD_n}\langle x,\theta\rangle = \langle x_i,\theta\rangle$. Then, $\beta x_1 + (1-\beta)x_2 \in K\cap(\beta r_1 + (1-\beta)r_2)D_n$, and

$$\max_{x\in K\cap(\beta r_1+(1-\beta)r_2)D_n}\langle x,\theta\rangle \geq \langle\beta x_1 + (1-\beta)x_2,\theta\rangle$$
$$= \beta\max_{x\in K\cap r_1D_n}\langle x,\theta\rangle + (1-\beta)\max_{x\in K\cap r_2D_n}\langle x,\theta\rangle.$$

Integrating over S^{n-1} we get

$$M^*(K\cap(\beta r_1 + (1-\beta)r_2)D_n) \geq \beta M^*(K\cap r_1D_n) + (1-\beta)M^*(K\cap r_2D_n).$$

In exactly the same way we show that $M^*(K\cap rD_n\cap\xi)$ is concave in r for every $\xi \in G_{n,k}$, and integrating on $G_{n,k}$ we see that $S_K^*(r,\lambda)$ is concave too. □

Let $\{a_n\}$ be a sequence satisfying $a_n/\sqrt{n} \to 0$ as $n \to \infty$. Using Lemma 4.3 and the function $S_K^*(r,\lambda)$ we have the following information about the diameter of random proportional sections:

4.5 Proposition. *Suppose that $D_n \subseteq K \subseteq a_nD_n$. Let $\lambda, \theta \in (0,1)$ and $k = [\lambda n]$. We denote by $r_K^\theta(\lambda)$ the solution of the equation*

$$S_K^*(r,\lambda) = (1-\theta)^{1/2}r/3 \tag{4.15}$$

in r. *Then, if* $\lambda < \theta < 1$ *and* $n/a_n^2 \geq C(\lambda, \theta)$ *we have*

$$(1-\theta)^{1/2} r_K^\theta(\lambda)/3 \leq \mathrm{diam}(K \cap \xi) \leq 2r_K^\theta(\lambda/\theta), \tag{4.16}$$

for a random $\xi \in G_{n,k}$.

Proof: Given $\lambda \in (0,1)$, we find $r_K^\theta(\lambda)$ solving (4.15) and then apply Lemma 4.3 to the body $K \cap r_K^\theta(\lambda) D_n$: we can find a subset L_1 of $G_{n,k}$ with measure

$$\nu_{n,k}(L_1) \geq 1 - \exp(-c\frac{(1-\theta)n}{9r^2}) \geq 1 - h(\theta, n)$$

where $h(\theta, n) = o_n(1)$, such that for every $\xi \in L_1$,

$$(1-\theta)^{1/2} r_K^\theta(\lambda)/6 < M^*(K \cap r_K^\theta(\lambda) D_n \cap \xi) < (1-\theta)^{1/2} r_K^\theta(\lambda)/2. \tag{4.17}$$

The left hand side inequality clearly implies that for all $\xi \in L_1$ we have

$$\mathrm{diam}(K \cap \xi) \geq (1-\theta)^{1/2} r_K^\theta(\lambda)/3. \tag{4.18}$$

On the other hand, the right hand side of (4.17) shows that there exists a subset L_2 of $G_{n,[(\lambda/\theta)n]}$ with measure $\geq 1 - h(\theta, n)$, such that

$$M^*(K \cap r_K^\theta(\lambda/\theta) D_n \cap \eta) < (1-\theta)^{1/2} r_K^\theta(\lambda/\theta)/2 \tag{4.19}$$

for every $\eta \in L_2$, and the Low M^*-estimate implies that for most $\xi \in G_{[(\lambda/\theta)n],k}(\eta)$ we have

$$\mathrm{diam}(K \cap r_K^\theta(\lambda/\theta) D_n \cap \xi) \leq 4\frac{M^*(K \cap r_K^\theta(\lambda/\theta) D_n \cap \eta)}{\sqrt{1-\theta}} < 2r_K^\theta(\lambda/\theta), \tag{4.20}$$

which shows that

$$\mathrm{diam}(K \cap \xi) \leq 2r_K^\theta(\lambda/\theta), \tag{4.21}$$

for all $\xi \in L_3 \subseteq G_{n,k}$ with $\nu_{n,k}(L_3) \geq 1 - ch(\theta, n)$. By (4.18) and (4.21), (4.16) holds true with probability greater than $1 - c_1 h(\theta, n)$. □

4.6 Remark. Let $\gamma(\theta, n) = (c(1-\theta)n/18 \log n)^{\frac{1}{2}}$, where c is the constant from Lemma 4.3. Assume that $D_n \subseteq K \subseteq aD_n$, where $a < \gamma(\theta, n)$. Then, a careful reading of the proof of Proposition 4.5 shows that it holds true with $h(\theta, n) = n^{-2}$. It is then not hard to compare the solution $r_K^\theta(\lambda)$ of the equation $S_K^*(r, \lambda) = (1-\theta)^{1/2} r/3$ with the function S_K^* itself. We clearly have $S_K^*(r, \lambda) \leq S_K^*(\lambda)$ for every $r > 0$, and hence

$$r_K^\theta(\lambda) \leq 3S_K^*(\lambda)/(1-\theta)^{1/2} \tag{4.22}$$

for all $\lambda, \theta \in (0,1)$. On the other hand, assuming that $D_n \subseteq K \subseteq \gamma(\theta, n) D_n$, by Proposition 4.5 for every $\lambda < \theta < 1$ we can find $L \subseteq G_{n,k}$ with $\nu(L^c) \leq cn^{-2}$ such that $\mathrm{diam}(K \cap \xi) \leq 2r_K^\theta(\lambda/\theta)$ for all $\xi \in L$. Since $r_K^\theta(\lambda/\theta) \geq 1$, a simple estimate gives

$$(4.23) \quad S_K^*(\lambda) \leq \frac{1}{2}\int_{G_{n,k}} \mathrm{diam}(K \cap \xi) d\xi \leq r_K^\theta(\lambda/\theta)[1 + c\gamma(\theta, n)n^{-2}] \leq c' r_K^\theta(\lambda/\theta)$$

where $c' > 0$ is an absolute constant. Therefore, we obtain an analogue of Proposition 4.5 in which the process of "solving the equation in r" is avoided:

4.7 Theorem. *Let $\lambda, \theta \in (0,1)$ with $\lambda < \theta$, and $k = [\lambda n]$. For every symmetric convex body K in $\mathbb{R}^n$, $n \geq n_0(\theta)$, satisfying $D_n \subseteq K \subseteq \gamma(\theta, n)D_n$, there exists a subset $L_{n,k}(\theta)$ of $G_{n,k}$ with measure greater than $1 - cn^{-2}$, such that*

$$(4.24) \qquad c_1 S_K^*(\lambda\theta) \leq \mathrm{diam}(K \cap \xi) \leq c_2 S_K^*(\lambda/\theta)/(1-\theta)^{1/2}$$

for all $\xi \in L_{n,k}(\theta)$, where $c_1, c_2 > 0$ are absolute constants. □

It is clear that the new point here lies in the left hand side inequality. The right hand side of (4.24) is a consequence of the Low M^*–estimate (under milder assumptions on K: see section 3).

Using Proposition 4.5 we proceed to obtain a-priori information on the diameter of a random $[\lambda n]$-section of K for "most" $\lambda \in (0,1)$. To this end, for every $\lambda \in (0,1)$ we define a λ-*flag* of subspaces of $\mathbb{R}^n$ to be a finite sequence of subspaces

$$\mathbb{R}^n = E_0 \supset E_1 \supset \ldots \supset E_{s(\lambda)},$$

of dimension $\dim(E_j) = k_j = [\lambda^j n]$, $j = 0, 1, \ldots, s(\lambda)$. The length of the λ-flag is the smallest integer s for which $k_s = k_{s+1}$. As in Remark 3.2, one easily checks that $s(\lambda) \simeq \log[(1-\lambda)n]/\log(1/\lambda)$.

Let r_j, $j = 0, 1, \ldots, s(\lambda)$, be the solution of the equation

$$(4.25) \qquad S_K^*(r, k_j) = (1-\lambda)^{1/2} r/3$$

in r. In the notation of Proposition 4.5 we have $r_j = r_K^\lambda(\lambda^j)$. We shall show that for most $j \leq s(\lambda)$ the diameter of a random k_j-section of K with $D_n \subseteq K \subseteq \gamma(\lambda, n)D_n$ is equal to r_j up to a function depending only on λ:

4.8 Theorem. *Let $\lambda \in (0,1)$ and K be a symmetric convex body in $\mathbb{R}^n$ with $D_n \subseteq K \subseteq \gamma(\lambda, n)D_n$. Let $\{E_j\}_{j \leq s(\lambda)}$ be any λ-flag of subspaces of $\mathbb{R}^n$. For every $\beta \in (0,1)$ we can find a set of indices $J \subseteq \{0, 1, \ldots, s(\lambda)\}$ with $|J| \geq (1-\beta)s(\lambda)$ and a subset L of the orhogonal group $O(n)$ with Haar measure $\nu(L) \geq 1 - \frac{1}{n}$, such that*

$$(4.26) \qquad c_1 r_j \leq \mathrm{diam}(K \cap T(E_j)) \leq c_2 g(\lambda, \beta) r_j,$$

for every $T \in L$ and every $j \in J$, where $c_1, c_2 > 0$ are absolute constants and $g(\lambda, \beta)$ is an explicit function depending only on λ and β.

Proof: We first observe that the sequence r_j, $j = 0, 1, \ldots, s(\lambda)$ is decreasing: we write $f_j(r)$ for the function $S_K^*(r, \lambda^j)$. Then r_j is the solution of the equation $f_j(r) = (1-\lambda)^{1/2} r/3$ in r. Since $\lambda^{j-1} > \lambda^j$, we have $f_{j-1} \geq f_j$ on $(0, \infty)$ for every $j = 1, \ldots, s(\lambda)$. Also, by Lemma 2.1 each f_j is a concave increasing function of r. It is then clear that the unique points r_j where f_j intersect the line $y = (1-\lambda)^{1/2} r/3$ satisfy the inequality $r_{j-1} \geq r_j$ for all $j \leq s(\lambda)$.

We denote by d_j the ratio r_{j-1}/r_j, $j \leq s(\lambda)$. Since $D_n \subseteq K \subseteq \gamma(\lambda, n) D_n$, we have

$$d_1 \ldots d_{s(\lambda)} = r_0 / r_{s(\lambda)} \leq \gamma(\lambda, n). \tag{4.27}$$

It follows that for every $\zeta > 1$, if $J_\zeta = \{j \leq s(\lambda) : d_j \geq \zeta\}$ we must have $|J_\zeta| \leq \log n / \log \zeta$. This means that $|J_\zeta| \leq \beta s(\lambda)$ provided that $\log \zeta \geq \frac{c'}{\beta} \log(\frac{2}{1-\lambda}) \log(1/\lambda)$.

Consider any $j \in J_\zeta$. By Proposition 4.5, for all T in a subset L_j of $O(n)$ with measure greater than $1 - c_1 n^{-2}$ we have

$$(1-\lambda)^{1/2} r_j / 3 \leq \mathrm{diam}(K \cap T(E_j)) \leq 2 r_{j-1} \leq 2 \zeta r_j. \tag{4.28}$$

Set $L = \bigcap_{j \in J_\zeta} L_j$. Then, $\nu(L) \geq 1 - n^{-1}$ if n is large enough, and for every $j \in J_\zeta$ and every $T \in L$ we have $\mathrm{diam}(K \cap T(E_j)) \simeq r_j$ up to $\zeta/(1-\lambda)^{1/2}$. Recall that $|J_\zeta| \geq (1-\beta) s(\lambda)$ if $\zeta \geq g(\lambda, \beta) = \frac{c'}{\beta} \log(\frac{2}{1-\lambda}) \log(\frac{1}{\lambda})$, and the proof is complete. □

4.9 Remark. In view of (4.22), (4.23) and (4.25), one can replace r_j by $S_K^*(\lambda^j)$ in Theorem 4.8. An argument similar to the one in the proof of Theorem 4.8 shows that this is true for most $j \in J$.

4.10 Remark. Suppose that the maximal/minimal volume ellipsoid or the distance ellipsoid of K is a Euclidean ball. Without loss of generality we may assume that $D_n \subseteq K \subseteq a D_n$ with $a \leq \sqrt{n}$. If $\gamma(\theta, n) \leq a \leq \sqrt{n}$, we may apply the results of this section to the body $K_1 = K \cap \gamma(\theta, n) D_n$. Since $K_1 \subseteq K \subseteq c(\theta) \sqrt{\log n} K_1$, all statements will hold true up to a $\sqrt{\log n}$-factor for the body K as well.

Let us also note that an additional application of the Low M^*–estimate shows that the results of this section hold for every symmetric convex body K with $\frac{1}{b} D_n \subseteq K \subseteq a D_n$ and $b M^*(K) = o(\sqrt{n})$. It would be interesting to know if the condition can be replaced by the weaker $M(K) M^*(K) = o(\sqrt{n})$.

5 The case of a body in M-Position of order α

If A, B are symmetric convex bodies in $\mathbb{R}^n$ we define as usual the covering number $N(A, B)$ of A by B to be the smallest integer N for which we can find $y_i \in \mathbb{R}^n, i = 1, \ldots, N$ such that $A \subseteq \bigcup_{i \leq N} (y_i + B)$. It is known that given any symmetric convex body $\overline{K}$ in $\mathbb{R}^n$ and any $\alpha > \frac{1}{2}$, there exists a linear image K of $\overline{K}$ satisfying the following two conditions:

(i) The volume radius of K is 1: $|K| = |D_n|$.

(ii) $\max\{N(K,tD_n),N(D_n,tK),N(K^o,tD_n),N(D_n,tK^o)\}\leq \exp(c(\alpha)\frac{n}{t^{1/\alpha}})$, for every $t\geq 1$, where $c(\alpha)>0$ is a constant depending only on α: $c(\alpha)=O(1/(\alpha-\frac{1}{2})^{1/2})$ as $\alpha\to\frac{1}{2}$.

Condition (i) is just a normalization. We could have omitted it and replaced D_n by sD_n, where $|K|=|sD_n|$, in (ii). The fact that a body K which satisfies (i) and (ii) exists in every affine class for every $\alpha>\frac{1}{2}$ is an improvement of Pisier (see e.g [P1], Chapter 7) on previous work of Milman related to the inverse Brunn–Minkowski inequality [M2], where (ii) had been established for $\alpha=1$.

In this section we assume that K is a symmetric convex body satisfying (i) and (ii), and say that K is *in M–position of order α* (α–regular in the terminology of [P1]). One of the consequences of (i) and (ii) is the inverse Brunn-Minkowski inequality [M2] which will be used below in the following precise form: If $u_1,\dots,u_s$ are orthogonal transformations of $\mathbb{R}^n$, then $u_i(K)$ is in M–position of order α for every $i\leq s$, and

$$|\frac{1}{s}\sum_{i=1}^{s}u_i(K)|^{\frac{1}{n}}\leq c'(\alpha)s^{\alpha}|K|^{\frac{1}{n}}. \tag{5.1}$$

The constant $c'(\alpha)$ in (5.1) is related to $c(\alpha)$ in (ii) as follows: $c'(\alpha)\leq\exp(2c(\alpha))$ [P1]. Observe also that, if $r>0$ then $K\cap rD_n$ and $\mathrm{co}(K\cup rD_n)$ (normalized so that their volume will be $|D_n|$) are also in M–position of order α, with $c(\alpha)$ replaced by $c'c(\alpha)$, where c' is an absolute constant.

We shall prove that in this case $S_K^*(\lambda)$ is determined by the volume radius of K up to explicit functions depending only on λ and α:

5.1 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ which is in M–position of order $\alpha>\frac{1}{2}$. For every $\lambda\in(0,1)$ we have*

$$c_1(\alpha)\lambda^{\alpha}\left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}\leq S_K^*(\lambda)\leq\frac{c_2(\alpha)}{(1-\lambda)^{\alpha}}\left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}, \tag{5.2}$$

where $c_1(\alpha),c_2(\alpha)$ are constants depending only on α.

The lower bound may be viewed as a proportional version of Urysohn's inequality. Together with the upper bound it shows that if we accept a small loss of dimension in computing the mean width of the body, then in the case of a body in M–position of order α we have an equivalence

$$S_K^*(\lambda)\simeq \mathrm{v.rad}(K)$$

up to a function depending only on λ and α. For the proof of the lower bound we shall need the following geometric Lemma which is based on measure concentration arguments:

5.2 Lemma. *Let $\gamma\geq 1,p>0$, $0<\lambda<1$, and W be a symmetric convex body in $\mathbb{R}^m$ such that*

$$N(W,tD_m)\leq\exp(\gamma\frac{m}{\lambda t^p})$$

for every $t \geq 1$. Then, there exists a subspace $\eta \in G_{m,[m/2]}$ such that

$$W \cap \eta \subseteq c\gamma^{1/p}\lambda^{-1/p} D_\eta, \tag{5.3}$$

where $c > 0$ is an absolute constant.

Proof: Let $t \geq 1$. We can find $N \leq \exp(\gamma \frac{m}{\lambda t^p})$ and $x_i \in \mathbb{R}^m,\ i = 1, \ldots, N$, such that $W \subseteq \bigcup_{i \leq N}(x_i + tD_m)$. Consider the sphere RS^{m-1}, where $R > 0$ is a constant to be chosen.

Let σ_R denote the normalized rotationally invariant measure on RS^{m-1}. It is easy to see that for every $i \leq N$ the intersection $A_i = (x_i + 2tD_m) \cap RS^{m-1}$ has measure

$$\sigma_R(A_i) \leq \sigma(B(2t/R)),$$

where $B(\varepsilon)$ denotes a cap of angular radius $\varepsilon > 0$ in S^{m-1}. We estimate $\sigma(B(2t/R))$ in a standard way:

$$\sigma(B(2t/R)) = \frac{\int_0^{\sin^{-1}(2t/R)} \sin^{m-2} s\, ds}{2\int_0^{\pi/2} \sin^{m-2} s\, ds} \leq \left(\frac{c_1 t}{R}\right)^{m-1},$$

for some absolute constant $c_1 > 0$. This implies that if we set $A = \bigcup_{i \leq N} A_i$, then

$$\sigma_R(A \cap RS^{m-1}) \leq \exp(\gamma \frac{m}{\lambda t^p}) \left(\frac{c_1 t}{R}\right)^{m-1}. \tag{5.4}$$

Assuming that R is chosen large enough, this is exponentially small in m. More precisely, since the cardinality of a t–net in $RS^{[m/2]-1}$ is bounded by $(2R/t)^{[m/2]}$, a standard argument (see [MS], Chapter 4) shows that if

$$\exp(\gamma \frac{m}{\lambda t^p}) \left(\frac{2R}{t}\right)^{\frac{m}{2}} \left(\frac{c_1 t}{R}\right)^{m-1} < 1,$$

then we can find a subspace $\eta \in G_{m,[m/2]}$ and a t-net $C(\eta)$ for $\eta \cap RS^{m-1}$ such that $A \cap C(\eta) = \emptyset$. Analyzing the condition on R, we see that it is enough to choose

$$R = c_2 t \exp(\frac{3\gamma}{\lambda t^p}), \tag{5.5}$$

for some constant $c_2 > c_1$. We can now easily show that with this choice of R we have $W \cap \eta \subseteq RD_\eta$: Suppose not. Then, we can find $x \in RS^{m-1}$ which is also in $W \cap \eta$. It follows that $|x - x_i| \leq t$ for some $i \leq N$, and $|x - y| \leq t$ for some $y \in C(\eta)$. But then, $|y - x_i| \leq 2t$, which means that $A \cap C(\eta) \neq \emptyset$, a contradiction.

We choose $t = (\gamma/\lambda)^{1/p} \geq 1$. Then, $R = c\gamma^{1/p}\lambda^{-1/p}$ and the proof is complete. □

We can now pass to the proof of the Theorem:

Lower bound: Let $\lambda \in (0,1)$, $k = [\lambda n]$, and consider any $\xi \in G_{n,k}$. The projection $P_\xi(K^o)$ of K^o onto ξ satisfies

$$N(P_\xi(K^o), tD_\xi) \leq N(K^o, tD_n) \leq \exp(c(\alpha)\frac{k}{\lambda t^{1/\alpha}}), \tag{5.6}$$

for every $t \geq 1$. We may clearly assume that $c(\alpha) \geq 1$. We apply Lemma 5.2 with $W = P_\xi(K^o)$, $m = k$, $\gamma = c(\alpha)$, and $p = 1/\alpha$: There exists $\eta \in G_{k,[k/2]}(\xi)$ for which

$$P_\xi(K^o) \cap \eta \subseteq c[c(\alpha)]^\alpha \lambda^{-\alpha} D_\eta := [c_1(\alpha)]^{-1}\lambda^{-\alpha} D_\eta. \tag{5.7}$$

Taking polars in η we see that $P_\eta(K \cap \xi) \supseteq c_1(\alpha)\lambda^\alpha D_\eta$. Recall that for every symmetric convex body W in $\mathbb{R}^m$ and every $\eta \in G_{m,s}$ the inequality $M(W \cap \eta) \leq \sqrt{m/s}M(W)$ holds, so we get
(5.8)

$$M^*(K \cap \xi) = M((K \cap \xi)^o) \geq \frac{1}{\sqrt{2}} M((K \cap \xi)^o \cap \eta) = \frac{1}{\sqrt{2}} M^*(P_\eta(K \cap \xi)) \geq c_1'(\alpha)\lambda^\alpha.$$

It is then obvious that

$$S_K^*(\lambda) \geq c_1'(\alpha)\lambda^\alpha. \quad \square$$

[It is interesting to note that the lower bound (5.8) holds true for *every* subspace $\xi \in G_{n,k}$. Observe also that $c_1'(\alpha) \geq c/(\alpha - \frac{1}{2})^{\alpha/2}$.]

Upper bound: Let $\lambda \in (0,1)$ and $k = [\lambda n]$. Find $r > 0$ such that

$$M^*(K \cap rD_n) = \frac{1}{2}(1-\lambda)^{1/2} r. \tag{5.9}$$

By the Low M^*-estimate there exists a subset $L_{n,k}$ of $G_{n,k}$ with measure $\nu_{n,k}(L_{n,k}) \geq 1 - c\exp(-c'(1-\lambda)n)$, such that

$$M^*(K \cap \xi) \leq \frac{1}{2}\mathrm{diam}(K \cap \xi) \leq r \tag{5.10}$$

for every $\xi \in G_{n,k}$. On the other hand (see [BLM]), we can find $s \leq \frac{c_1}{1-\lambda}$ and orthogonal transformations $u_1, \ldots, u_s$, satisfying

$$\frac{1}{4}(1-\lambda)^{1/2} rD_n \subseteq \frac{1}{s}\sum_{i=1}^s u_i(K \cap rD_n) \subseteq (1-\lambda)^{1/2} rD_n. \tag{5.11}$$

Set $K_1 = \frac{1}{s}\sum u_i(K \cap rD_n)$. Then, for every $\xi \in G_{n,k}$ we have

$$K_1 \cap \xi \supseteq \frac{1}{s}\sum_{i=1}^s [u_i(K \cap rD_n) \cap \xi],$$

which, together with (5.11), implies that

$$(1-\lambda)^{1/2} r \geq M^*(K_1 \cap \xi) \geq \frac{1}{s}\sum_{i=1}^s M^*[u_i(K \cap rD_n) \cap \xi],$$

and an integration over $G_{n,k}$ shows that

$$S^*_{K\cap rD_n}(\lambda) \leq (1-\lambda)^{1/2} r. \tag{5.12}$$

We give an upper bound for r using the inverse Brunn-Minkowski inequality: $K\cap rD_n$ is α-regular, therefore by (5.1) and (5.11) we obtain

(5.13)

$$\frac{1}{4}(1-\lambda)^{1/2} r \leq \left(\frac{|K_1|}{|D_n|}\right)^{\frac{1}{n}} \leq c'(\alpha) s^{\alpha} \left(\frac{|K\cap rD_n|}{|D_n|}\right)^{\frac{1}{n}} \leq \frac{c_1^{\alpha} c'(\alpha)}{(1-\lambda)^{\alpha}} \left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}.$$

Finally, we can compare $S^*_{K\cap rD_n}(\lambda)$ with $S^*_K(\lambda)$: Observe first that there is a constant $c_3(\alpha)$ such that $K\subseteq c_3(\alpha)n^{\alpha}D_n$ and $K^o\subseteq c_3(\alpha)n^{\alpha}D_n$. This follows immediately from the bounds (ii) of the covering numbers of K and K^o by large balls. Choosing $t=c_2[c(\alpha)]^{\alpha}n^{\alpha}$ for some absolute constant c_2, we can make both $N(K,tD_n)$ and $N(K^o,tD_n)$ smaller than 2. Using this information and the fact that the set $L_{n,k}$ has almost full measure, we easily check that for $W=K$ or $K\cap rD_n$,

$$\int_{L_{n,k}} M^*(W\cap\xi)\nu_{n,k}(d\xi) \simeq \int_{G_{n,k}} M^*(W\cap\xi)\nu_{n,k}(d\xi),$$

up to absolute constants. But, $M^*(K\cap\xi)=M^*(K\cap rD_n\cap\xi)$ for every $\xi\in L_{n,k}$, which implies that

$$S^*_K(\lambda) \leq c S^*_{K\cap rD_n}(\lambda). \tag{5.14}$$

Combining (5.12), (5.13) and (5.14), we get

$$S^*_K(\lambda) \leq c_2(\alpha)(1-\lambda)^{-\alpha}\left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}.$$

This completes the proof of the theorem. Observe that $c_2(\alpha)\leq c_4 c'(\alpha)\leq c_4\exp(2c(\alpha))$ for some absolute constant $c_4>0$. □

A careful reading of the proof above shows that the diameter of "most" sections $K\cap\xi$, $\xi\in G_{n,\lambda n}$, is determined up to constants depending only on λ:

5.3 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ which is in M–position of order α. Then, for every $\lambda\in(0,1)$ and for most $\xi\in G_{n,\lambda n}$ we have*

$$c_1(\alpha)\lambda^{\alpha}\left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}} \leq \mathrm{diam}(K\cap\xi) \leq \frac{c_2(\alpha)}{(1-\lambda)^{\alpha+\frac{1}{2}}}\left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}.$$

Proof: Since $\mathrm{diam}(K\cap\xi)\geq 2M^*(K\cap\xi)$, the lower estimate holds true for every $\xi\in G_{n,\lambda n}$ by (5.8). According to the proof of the upper estimate in Theorem 5.1, if r is the solution of the equation $M^*(K\cap rD_n)=(1-\lambda)^{1/2}r/2$, then

$$\mathrm{diam}(K\cap\xi) \leq 2r \leq \frac{c_2(\alpha)}{(1-\lambda)^{\alpha+\frac{1}{2}}}\left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}},$$

for all $\xi \in L_{n,\lambda n} \subseteq G_{n,\lambda n}$, where $\nu_{n,\lambda n}(L_{n,\lambda n}) \geq 1 - c\exp(-c'(1-\lambda)n)$ (see (5.10) and (5.13)). $\square$

5.4 Remark. The discussion above also shows that the M_K^* approach is equivalent to the S_K^* approach in the M–position: If $r = r(\lambda)$ is the solution of the equation $M_K^*(r) = \frac{1}{2}\sqrt{1-\lambda}$, then

$$r \simeq S_K^*(\lambda) \simeq \mathrm{diam}(K \cap \xi)$$

for all $\lambda \in (0,1)$ and for most $\xi \in G_{n,[\lambda n]}$, up to functions depending only on λ and α. This is clear from the lower bound in Theorem 5.1 and the inequalities (5.12), (5.13) and (5.14).

It should also be noted that for an α-regular body K in $\mathbb{R}^n$, as a consequence of the upper estimate in Theorem 5.1 and of the Blaschke – Santaló inequality, we have the following analogue of the duality relation given by Theorem 3.4:

5.5 Corollary. *Let K be a symmetric convex body in $\mathbb{R}^n$, which is in M–position of order $\alpha > 1/2$. For every $\lambda, \mu \in (0,1)$ we have*

$$S_K^*(\lambda) S_{K^o}^*(\mu) \leq \frac{C(\alpha)}{(1-\lambda)^\alpha (1-\mu)^\alpha}, \tag{5.15}$$

where $C(\alpha) > 0$ is a constant depending only on α. $\square$

Note that there is no restriction on λ or μ, in contrast to Theorem 3.4. It is also interesting to note that with $\alpha = 1$ and $\lambda = \mu = 1 - \frac{1}{\log^t n}$ i.e for sections of almost full dimension, one has

$$S_K^*(\lambda) S_{K^o}^*(\lambda) \leq c \log^{2t} n.$$

We close this section with an upper estimate for $M^*(K)$ when K is in M–position of order α and $|K| = |D_n|$. Our method is analogous to the one used in [D] for a second proof of J. Bourgain's estimate [B] on the isotropic constant:

5.6 Theorem. *Let K be a symmetric convex body in $\mathbb{R}^n$ which is in M–position of order $\alpha > 1/2$. Then,*

$$M^*(K) \leq f(\varepsilon) n^\varepsilon, \tag{5.16}$$

where $\varepsilon = \alpha - 1/2$ and $f(\varepsilon) = c\varepsilon^{-5/4}$.

Proof: We assume that K satisfies conditions (i) and (ii). Let $d = \mathrm{diam}(K)$. In the proof of Theorem 5.1 we checked that

$$d \leq c_1 [c(\alpha)]^\alpha n^\alpha, \tag{5.17}$$

for some absolute constant $c_1 > 0$. From (ii) we also know that for every $j = 0, 1, 2, \ldots$ such that $2^j \leq d$, we can find a subset N_j of K such that $K \subseteq \bigcup_{y \in N_j} (y + (d/2^j) D_n)$ and $\log |N_j| \leq c(\alpha) n 2^{j/\alpha} d^{-1/\alpha}$.

Let $Z_j = (N_j - N_{j-1}) \cap (3d/2^j)D_n$, $j = 1, \dots, r_0 = [\log_2 d]$. An inductive argument shows that for every $r \leq r_0$, every $y \in K$ can be written in the form

$$y = \sum_{j=1}^{r} y_j + z_r, \tag{5.18}$$

where $y_j \in Z_j$ and $z_r \in (d/2^r)D_n$ (this is known as the Dudley–Fernique decomposition of $y \in K$). Note also that

$$\log |Z_j| \leq 2c(\alpha) \frac{2^{j/\alpha} n}{d^{1/\alpha}}. \tag{5.19}$$

Consider any $\theta \in S^{n-1}$. Using (5.18), we easily check that

$$\max_{y \in K} |\langle x, \theta \rangle| \leq \sum_{j=1}^{r} \max_{y \in Z_j} |\langle y, \theta \rangle| + \frac{d}{2^r}, \tag{5.20}$$

for every $r = 1, \dots, r_0$. This implies that

$$M^*(K) = \int_{S^{n-1}} \|\theta\|_{K^o}\, \sigma(d\theta) \leq 2 + \sum_{j=1}^{r_0} \int_{S^{n-1}} \max_{y \in Z_j} |\langle y, \theta \rangle|\, \sigma(d\theta). \tag{5.21}$$

Every $y \in Z_j$ can be written as $y = \zeta(y)\overline{y}$ with $\overline{y} \in S^{n-1}$ and $|\zeta(y)| \leq 3d/2^j$. Hence, for every $j = 1, \dots, r_0$ we have

$$\int_{S^{n-1}} \max_{y \in Z_j} |\langle y, \theta \rangle|\, \sigma(d\theta) \leq \frac{3d}{2^j} \int_{S^{n-1}} \max_{y \in Z_j} |\langle \overline{y}, \theta \rangle|\, \sigma(d\theta). \tag{5.22}$$

We estimate this last integral as follows: it is easy to see that there is an absolute constant $c_2 > 0$ such that

$$\int_{S^{n-1}} \exp\left(\frac{|\langle \overline{y}, \theta \rangle|^2 n}{c_2^2} \right) \sigma(d\theta) \leq 2.$$

Therefore, for every $t \geq 1$ we have

$$\begin{aligned} |\{\theta : \max_{y \in Z_j} |\langle \overline{y}, \theta \rangle| > c_2 t \left(\frac{\log |Z_j|}{n} \right)^{1/2} \}| &\leq |Z_j|\, |\{\theta : \exp\left(\frac{|\langle \overline{y}, \theta \rangle|^2 n}{c_2^2} \right) \geq |Z_j|^{t^2} \}| \\ &\leq |Z_j|^{1-t^2}, \end{aligned}$$

which implies that for all $j \leq r_0$,

$$\begin{aligned} \int_{S^{n-1}} \max_{y \in Z_j} |\langle \overline{y}, \theta \rangle|\, \sigma(d\theta) &\leq c_3 \frac{(\log |Z_j|)^{1/2}}{\sqrt{n}} \\ &\leq c_3 [c(\alpha)]^{1/2} \frac{2^{j/2\alpha}}{d^{1/2\alpha}}. \end{aligned} \tag{5.23}$$

Going back to (5.21) and adding the estimates, we obtain

$$M^*(K) \leq 2 + c_4[c(\alpha)]^{1/2} d^{1-\frac{1}{2\alpha}} \sum_{j=1}^{r_0} \frac{1}{2^{j(1-\frac{1}{2\alpha})}} \tag{5.24}$$

$$\leq c_5 \frac{[c(\alpha)]^{1/2}}{\alpha - 1/2} d^{1-\frac{1}{2\alpha}}.$$

Setting $\varepsilon = \alpha - \frac{1}{2}$, we have $c(\alpha) \leq c_6/\sqrt{\varepsilon}$ and $d^{1-\frac{1}{2\alpha}} \leq c_7 n^{\varepsilon}$, thus (5.24) takes the form

$$M^*(K) \leq \frac{c}{\varepsilon^{5/4}} n^{\varepsilon}. \quad \square \tag{5.25}$$

5.7 Remark. It is easy to see that if $\mathrm{diam}(K)$ is the diameter of a symmetric convex body K in $\mathbb{R}^n$, then $M^*(K) \geq c\,\mathrm{diam}(K)/\sqrt{n}$. On the other hand, given any $\alpha > \frac{1}{2}$ it is not hard to construct a body K in M–position of order α with $\mathrm{diam}(K) \geq cn^{\alpha}$. Consider for example the body $K_1 = \mathrm{co}\{D_n, \pm n^{\alpha} e_n\}$ and normalize it to receive a body K of volume 1. It then follows that $M^*(K) \geq cn^{\varepsilon}$ where $\varepsilon = \alpha - \frac{1}{2}$. This shows that the estimate provided by Theorem 5.5 is exact.

The same example, combined with Theorem 5.1 shows that even in this very natural M–position, the function S_K^* may increase in an irregular way. It has logarithmic growth up to $\lambda = 1 - \frac{\log n}{n}$ while $S_K^*(1) \simeq n^{\varepsilon}$.

5.8 Remark. Choosing $\varepsilon \simeq 1/\log n$ in Theorem 5.6, we get that every symmetric convex body $\overline{K}$ in $\mathbb{R}^n$ has a linear image K with the properties:

(i) $|K| = |D_n|$ and $M^*(K) \leq c(\log n)^{5/4}$.

(ii) $N(K, tD_n) \leq \exp(cn\sqrt{\log n}/t^2)$ for every $t \geq 1$.

This should be compared with the ℓ-position of $\overline{K}$: It is a well-known fact (see [P2]) that there exists a linear image K_1 of $\overline{K}$ such that $|K_1| = |D_n|$, $M^*(K_1) \leq \log n$, and by Sudakov's inequality $N(K_1, tD) \leq \exp(cn \log^2 n/t^2)$ for every $t \geq 1$. Of course, the existence of bodies in M–position of order α inside every affine class and for every $\alpha > 1/2$ depends heavily on Pisier's estimate about the ℓ-position.

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