# Concentration property on probability spaces

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## 1 Introduction

**1.1** The starting point of this paper is the notion of concentration for metric probability spaces. Let $(X, d, \mu)$ be a metric space with metric $d$ and diameter $\mathrm{diam}(X) \geq 1$, which is also equipped with a Borel probability measure $\mu$. We then define the *concentration function* (or "isoperimetric constant") of $X$ by

$$\alpha(X;\varepsilon) = 1 - \inf\{\mu(A_\varepsilon) : A \text{ Borel subset of } X, \mu(A) \geq 1/2\},$$

where $A_\varepsilon = \{x \in X : d(x, A) \leq \varepsilon\}$ is the $\varepsilon$-extension of $A$. A family $(X_n, d_n, \mu_n)$ of metric probability spaces is called a *Lévy family* if for every $\varepsilon > 0$

$$\alpha(X_n;\varepsilon) \to 0$$

as $n \to \infty$. A natural example of a Lévy family is given by the family $(S^{n-1}, \rho_n, \sigma_n)$, where $S^{n-1}$ is the Euclidean sphere in $\mathbb{R}^n$, $\rho_n$ is the geodesic distance, and $\sigma_n$ is the rotationally invariant probability measure on $S^{n-1}$. Lévy observed that the isoperimetric inequality on $S^{n+1}$ implies that

$$\alpha(S^{n+1};\varepsilon) \leq \sqrt{\pi/8}\exp(-\varepsilon^2 n/2),$$

a fact which is crucial for the proof of Dvoretzky's theorem and many other results of the asymptotic theory of finite dimensional normed spaces. Other important examples are given by the family of the orthogonal groups $(O(n), \rho_n, \mu_n)$ equipped with the Hilbert-Schmidt metric and the Haar probability measure, and all homogeneous spaces of $O(n)$ (for example, any family of Stiefel manifolds $W_{n,k_n}$ or any family of Grassman manifolds $G_{n,k_n}$). Discrete examples are given by the family of spaces $E_2^n = \{-1, 1\}^n$ or the groups $\Pi_n$ of permutations of $\{1, \ldots, n\}$ with the normalized Hamming distance and the normalized counting measure. In most

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cases, new and very interesting techniques were invented in order to estimate the concentration function $\alpha(X;\varepsilon)$. We refer the reader to [MS], [Mi] and [T1] for a detailed discussion and references.

Let $(X,d,\mu)$ be a metric probability space with small concentration function. Then, every 1-Lipschitz function on $X$ concentrates around its Lévy mean (see [MS]). There exists one value $L_f$ such that

$$\mu\left(x\in X : |f(x)-L_f|\geq\varepsilon\right)\leq 2\alpha(X;\varepsilon).$$

This type of concentration implies equivalence of $L_p$-norms for Lipschitz functions on $X$, that is, inverse Hölder inequalities of the form $\|f\|_{L_p(X;\mu)}\leq c(p,\mu)\|f\|_{L_1(X;\mu)}$, where the order of the constant $c(p,\mu)$ as $p\to\infty$ reflects the degree of concentration.

Such inverse Hölder inequalities appear often in the context of probability spaces. For example, linear functionals on a convex body $K$ with volume 1 satisfy the inequality

$$\|f\|_{L_p(K;dx)}\leq cp\|f\|_{L_1(K;dx)}$$

where $c>0$ is an absolute constant [GM]. More generally, Bourgain [B1] has shown that if $f:K\to\mathbb{R}$ is a polynomial of degree $m$, then $\|f\|_p\leq c(p,m)\|f\|_2$ for every $p>2$, where $c(p,m)$ depends only on $p$ and on the degree $m$ of $f$. Talagrand [T2] showed that an analogous statement holds true for the class of convex functions on $E_2^n$. In view of these results, we would like to discuss the level of concentration with respect to a given *class of functions*.

**1.2** A typical example of concentration expressed by equivalence of $L_p$-norms is the classical Khintchine inequality: There is an absolute constant $c>0$ such that for every $n\in\mathbb{N}$, $p>2$, and $a_1,\dots,a_n\in\mathbb{R}$ we have

$$\left(\text{Ave}\,\Big|\sum_{j=1}^n\pm a_j\Big|^p\right)^{1/p}\leq c\sqrt{p}\left(\sum_{j=1}^m a_j^2\right)^{1/2}\leq c\sqrt{2}\sqrt{p}\,\text{Ave}\,\Big|\sum_{j=1}^n\pm a_j\Big|.$$

For the best constants, see [Sz] and [H]. By expanding $\exp(x^2)$ into power series, we may equivalently state Khintchine's inequality in the form

$$\|\langle\varepsilon,y\rangle\|_{L_{\psi_2}(E_2^n;\mu)}\leq c\|\langle\varepsilon,y\rangle\|_{L_1(E_2^n;\mu)},\quad y\in\mathbb{R}^n,$$

where $E_2^n=\{-1,1\}^n$ with the normalized counting measure $\mu$, and

$$\|f\|_{L_{\psi_\alpha}(\Omega;\mu)}=\inf\left\{\lambda>0:\int_\Omega\exp\big((|f|/\lambda)^\alpha\big)d\mu\leq 2\right\}$$

for every probability space $(\Omega,\mu)$ and $\alpha>0$.

The following fact was observed in [Sc] (see also [BLM] for an extension from the class of linear functionals to arbitrary norms, in the spirit of Kahane's inequality):

**Fact.** *There exist constants $C > 1$ and $c > 0$ such that: for every $n \in \mathbb{N}$ and $m \geq Cn$, a random subset $A$ of $E_2^n$ with cardinality $|A| = m$ satisfies*

$$\|\langle \varepsilon, y\rangle\|_{L_2(A;\mu(A))} \leq c\|\langle \varepsilon, y\rangle\|_{L_1(A;\mu(A))},$$

*for every $y \in \mathbb{R}^n$, where $\mu(A)$ is the normalized counting measure on $A$.*

It is not clear if $C$ may be replaced by *any* $\lambda > 1$, and $c$ by a constant $c(\lambda)$ respectively. However, the fact shows that very small sets of $n$-tuples of signs may replace $E_2^n$ in the Khintchine inequality. We are thus lead to the following definition:

**Definition.** Let $p > 1$ and $M \geq 1$. A finite set $S \subset \mathbb{R}^n$ will be called a $(p, M)$-*distribution* (for linear functionals) if

$$\|\langle x, y\rangle\|_{L_p(S;\mu(S))} \leq M\|\langle x, y\rangle\|_{L_1(S;\mu(S))}, \qquad y \in \mathbb{R}^n.$$

Analogously, $S$ will be called a $(\psi_\alpha, M)$-*distribution* if

$$\|\langle x, y\rangle\|_{L_{\psi_\alpha}(S;\mu(S))} \leq M\|\langle x, y\rangle\|_{L_1(S;\mu(S))}, \qquad y \in \mathbb{R}^n.$$

This is equivalent to the fact that $S$ is a $(p, M_p)$-distribution for every $p \geq 1$, with $M_p \leq cMp^{1/\alpha}$ for an absolute constant $c > 0$. We will often talk about a $p$ or $\psi_\alpha$-distribution without specifying the constant $M$, but the estimate for $M$ will be clear in every case.

In view of these definitions, the question which arises is to determine the minimal cardinality $m(p, n)$ ($m(\alpha, n)$ respectively) for which a random subset $A \subseteq E_2^n$ with cardinality $m \geq m(p, n)$ (or, $m \geq m(\alpha, n)$) forms a $(p, M)$-distribution (or, $(\psi_\alpha, M)$ distribution) with a "good" constant $M \geq 1$, while at the same time $A$ *represents* the space in the sense that $\|\langle x, y\rangle\|_{L_1(A;\mu(A))} \simeq \|\langle x, y\rangle\|_{L_1(E_2^n;\mu)}$ for every $y \in \mathbb{R}^n$. Known results (see [BGN], [BDGJN] and [Sc]) show that one can take $M \simeq \sqrt{p}$ and $m(p, n) \simeq n^{p/2}$ if $p \geq 2$, and $m(p, n) \simeq n$ if $1 \leq p \leq 2$. This estimate is optimal.

**1.3** The purpose of this paper is to study the level of concentration with respect to the class of linear functionals by measuring the size of minimal well-distributed substructures of certain probability spaces. These substructures should exhibit a high level of concentration and, at the same time, they should *represent the original space* in an essential way. Our setting will be an arbitrary log-concave Borel probability measure $\mu$ on $\mathbb{R}^n$. Recall that $\mu$ is called log-concave if, for all compact sets $A$, $B$ and all $\lambda \in (0, 1)$ we have $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$. We say that $\mu$ is *isotropic* if

$$\int_{\mathbb{R}^n} \langle x, y\rangle^2 \mu(dx) = L_\mu^2$$

for every $y \in S^{n-1}$. We will say that $\mu$ *satisfies a $\psi_\alpha$-estimate with constant $C_\alpha \geq 1$* if

$$\|\langle x, y\rangle\|_{L_{\psi_\alpha}(\mu)} \leq C_\alpha L_\mu$$

for every $y \in S^{n-1}$. From Borell's lemma (see [MS], Appendix III) we get

$$\|\langle x, y\rangle\|_{L_{\psi_1}(\mu)} \leq C_1 \|\langle x, y\rangle\|_{L_1(\mu)}$$

for every $y \in S^{n-1}$ and every log-concave probability measure $\mu$ on $\mathbb{R}^n$, where $C_1 \geq 1$ is an absolute constant. That is, all log-concave probability measures satisfy a $\psi_1$-estimate with some uniformly bounded constant.

With these definitions, the general formulation of our problem is the following:

**Question.** *Let $\mu$ be an isotropic log-concave Borel probability measure on $\mathbb{R}^n$, which satisfies a $\psi_\alpha$-estimate with constant $C_\alpha \geq 1$ for some $\alpha \in [1, 2]$. Find the minimal value of $m \in \mathbb{N}$ for which a random $S \subset \mathbb{R}^n$ of cardinality $|S| = m$ is a $p$-distribution or $\psi_\alpha$-distribution with a small constant $M \geq 1$, and represents $\mu$ in the sense that $\|\langle x, y\rangle\|_{L_1(\mu)} \simeq \|\langle x, y\rangle\|_{L_1(S;\mu(S))}$ for all $y \in \mathbb{R}^n$.*

Note that the isotropic condition about $\mu$ is not so restrictive: every log-concave probability measure $\mu$ whose support spans $\mathbb{R}^n$ has an image measure $T^{-1}(\mu)(A) := \mu(T^{-1}(A))$, $T \in SL(n)$, which is isotropic and log-concave. Then, every $p$ or $\psi_\alpha$-distribution of points with respect to $T^{-1}(\mu)$ corresponds to an equally good distribution of points with respect to $\mu$.

In Section 2 we study the question in full generality. Our main general result is the following:

**Theorem A.** *Let $0 < p < \infty$ and $\delta \in (0, 1)$. There exists $n_0(\delta)$ such that, for every $n \geq n_0$, every $m \geq m_0$ and every isotropic log-concave probability measure $\mu$ on $\mathbb{R}^n$, $m$ random points $x_1, \ldots, x_m \in (\mathbb{R}^n, \mathcal{B}, \mu)$ form with probability greater than $1 - \delta$ a $(p, M)$-distribution representing $\mu$, where $M = O(p)$ as $p \to \infty$, and*

$$m_0 = m_0(p, \delta) = \begin{cases} c(p, \delta) n & , \text{ if } 0 < p \leq 1 \\ c(p, \delta) n (\log n)^p & , \text{ if } 1 < p \leq 2 \\ c(p, \delta) h_{p,n} (n \log n)^{p/2} & , \text{ if } p > 2. \end{cases}$$

*The constant $h_{p,n}$ is bounded by $\min\{(p-2)^{-1}, \log n\}$, and this implies continuity of $m_0(p, \delta)$ at $p = 2$.*

One can also show that any exponential number of points is enough for a good $\psi_1$-distribution:

**Theorem B.** *Let $\mu$ be an isotropic log-concave probability measure on $\mathbb{R}^n$, and $\gamma \in (0, 1)$. If $n \geq n_0(\gamma)$ and $m \geq \exp(\gamma n)$, then $m$ points $x_1, \ldots, x_m$, chosen independently with respect to $\mu$, form with probability $> 1 - \delta$ a $(\psi_1, M)$-distribution representing $\mu$, where $M \leq c(\delta)/\sqrt{\gamma}$.*

A typical example of log-concave probability measure arises if we consider a convex body $K$ of volume 1 in $\mathbb{R}^n$. The Brunn-Minkowski inequality implies that the restriction $\mu_K$ of the Lebesgue measure onto $K$ is log-concave, therefore

$$\|\langle x, y\rangle\|_{L_{\psi_1}(K;dx)} \leq c' \|\langle x, y\rangle\|_{L_1(K;dx)}$$

for every $y \in \mathbb{R}^n$, where $c' > 0$ is an absolute constant. The $\psi_1$-estimate is best possible in full generality, but there exist bodies $K$ which allow even a $\psi_2$-estimate (the cube and the ball are such examples).

In this situation, some of the cases were previously studied: the values $0 < p \leq 1$ can be treated with the methods developped in [BLM], while the case $p = 2$ was studied by Bourgain [B2] (see also [R]). Our general approach in Section 2 uses a combination of these arguments: in particular, Bourgain's lemma 2.4 plays the key role.

**1.4** In Section 3 we follow the same geometric approach for $E_2^n$. The geometry involved is simpler here: the main advantages are the $\psi_2$-estimate for linear functionals (which comes from Khintchine's inequality), and the fact that all vertices of the cube are at distance $\sqrt{n}$ from the origin. This allows us to recover all known results on $p$-distributions, as well as optimal estimates for the minimal cardinality of $\psi_\alpha$-distributions. The following statement is true:

**Theorem C.** *Let $0 < p < \infty$ and $\delta \in (0,1)$. There exists $n_0(\delta)$ such that, for every $n \geq n_0$ a subset $A \subset E_2^n$ with $m \geq m_0$ elements forms with probability greater than $1-\delta$ a $p$-distribution representing $E_2^n$, where*

$$m_0 = m_0(p,\delta) = \begin{cases} c(p,\delta)n & , \text{ if } 0 < p \leq 2 \\ \\ c(p,\delta)n^{p/2} & , \text{ if } p > 2. \end{cases}$$

(see also [Sc], [BGN], [BDGJN]).

*Moreover, if $\gamma > 0$, $\alpha \in [1,2]$, and $n \geq n_0(\gamma,\delta)$, then a subset $A$ of $E_2^n$ with $|A| = m \geq \exp(\gamma n^{\alpha/2})$ satisfies with probability greater than $1-\delta$*

$$\|\langle \varepsilon, y\rangle\|_{L_{\psi_\alpha}(A)} \leq \frac{C(\delta)}{\gamma^{1/\alpha}}$$

*for every $y \in S^{n-1}$.*

Observe the *phase transition* at $p = 2$: For $p \in (0,2]$ we get $p$-distributions with cardinality $\simeq n$ (in the general case, up to a logarithmic term) while for $p > 2$, minimal $p$-distributions have cardinality $\simeq n^{p/2}$. The same phenomenon appears in several other questions of this nature. For example, in Section 4 we show that if $N \geq cn^{p^*}$, $p^* = \max\{1, p/2\}$, and $\{e_i\}_{i\leq N}$ is an orthonormal basis of $\mathbb{R}^N$, then for a random $E_n \in G_{N,n}$ the set $\{\sqrt{N}P_{E_n}(e_i) : i = 1,\dots,N\}$ forms a $p$-distribution on $E_n$ with $M \leq c\sqrt{p}$, $p \geq 1$. All these examples are connected with Dvoretzky's theorem for $\ell_p^N$ spaces, where a similar behavior is observed. The precise relation will be discussed throughout the paper.

Finally, in Section 5 we study a different question on random points: we fix $\gamma \in (0,1)$ and show that $m = \exp(\gamma n)$ points which are chosen uniformly and independently from a convex body $K$ with centroid at the origin in $\mathbb{R}^n$ satisfy with probability greater than $1-\delta$

$$A = \mathrm{co}\{x_1,\dots,x_m\} \supseteq c(\delta)\gamma K.$$

That is, any exponential number of random points from a convex body $K$ creates a body which "represents" $K$ in the distance sense. This question is naturally connected with the discussion in Section 2 (in particular, with Theorem B): every convex body $K$ creates a log-concave measure $\mu_K$, and a random set of $\exp(\gamma n)$ points chosen from $K$ creates a body equivalent to $K$ and, at the same time, forms a $\psi_1$-distribution for $\mu_K$.

**1.5** We assume that $\mathbb{R}^n$ is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$ and denote the corresponding Euclidean norm by $|\cdot|$. $D_n$ will be the Euclidean unit ball and $S^{n-1}$ will be the unit sphere. We also write $|\cdot|$ for the volume (Lebesgue measure) in $\mathbb{R}^n$, and for the cardinality of a finite set. The letters $c, c', c_1, c_2$ etc. will denote absolute positive constants which may change from line to line.

# 2 Log-concave probability measures satisfying a $\psi_\alpha$-estimate

In this section we study the case of a log-concave Borel probability measure $\mu$ on $\mathbb{R}^n$, which satisfies the isotropic condition

$$\int_{\mathbb{R}^n} \langle x,y\rangle^2 \mu(dx) = L_\mu^2 \quad , \quad y \in S^{n-1}$$

and a $\psi_\alpha$-*estimate with constant* $C_\alpha \geq 1$ for some $\alpha \in [1,2]$, i.e.

$$\int_{\mathbb{R}^n} \exp\big((|\langle x,y\rangle|/C_\alpha L_\mu)^\alpha\big) \leq 2$$

for every $y \in S^{n-1}$. Note that, by Borell's lemma, $\mu$ always satisfies a $\psi_1$-estimate with an absolute constant $C_1$. We first collect some Lemmas about measures with these properties. The proofs are adaptations of analogous results for isotropic convex bodies.

**2.1 Lemma.** *There exist absolute constants* $c_1, c_2 > 0$ *such that*

$$c_1 L_\mu \leq \left(\int |\langle x,y\rangle|^p \mu(dx)\right)^{1/p} \leq c_2 C_\alpha \max\{1, p^{1/\alpha}\} L_\mu,$$

*for every* $p > 0$ *and* $y \in S^{n-1}$.

*Proof:* The right hand side inequality is a direct consequence of the inequality $e^x > x^k/k!$, $x > 0$, $k = 1, 2, \ldots$. For the left hand side inequality, we use the fact that, by a result of Latala [L], there exists an absolute constant $c_1 > 0$ such that

$$c_1 L_\mu = c_1 \|\langle x,y\rangle\|_{L_2(\mu)} \leq \|\langle x,y\rangle\|_{L_p(\mu)}$$

for every $y \in \mathbb{R}^n$ and $0 < p < 2$. $\Box$

The function $x \mapsto |x|$ satisfies a better estimate:

**2.2. Lemma.** *There exists an absolute constant $A > 0$ such that*

$$\int_{nL_\mu D_n} \exp(|x|^2/A^2 n L_\mu^2)\mu(dx) \leq 2.$$

*Proof:* We follow Alesker's argument from [A]. Since $\mu$ is log-concave, it satisfies a $\psi_1$-estimate with an absolute constant $c > 0$. By Lemma 2.1, this implies that

$$\int_{\mathbb{R}^n} |\langle x, y\rangle|^p \mu(dx) \leq c_1^p p^p L_\mu^p$$

for every $y \in S^{n-1}$ and $p \geq 1$. Integrating this inequality with respect to $y$, we obtain

$$\left( \int_{\mathbb{R}^n} |x|^p \mu(dx) \right)^{1/p} \leq c_2 \sqrt{n} L_\mu \sqrt{p\left(1 + \frac{p}{n}\right)}.$$

This means that, for $p \leq n$, we have

$$\left( \int_{\mathbb{R}^n} |x|^p \mu(dx) \right)^{1/p} \leq c_3 \sqrt{p}\sqrt{n} L_\mu.$$

On the other hand, if $p > n$ we obviously have

$$\left( \int_{nL_\mu D_n} |x|^p \mu(dx) \right)^{1/p} \leq n L_\mu \leq \sqrt{p}\sqrt{n} L_\mu.$$

It follows that there is a constant $A > 0$ such that

$$\begin{aligned} \int_{nL_\mu D_n} \left( e^{|x|^2/A^2 n L_\mu^2} - 1 \right) \mu(dx) &= \sum_{p=1}^{\infty} \int_{nL_\mu D_n} \left( \frac{|x|}{A\sqrt{n} L_\mu} \right)^{2p} \\ &\leq \sum_{p=1}^{\infty} \frac{p^p}{p!} \left( \frac{\max\{1, c_3\}}{A} \right)^{2p} \leq 1. \quad \square \end{aligned}$$

We will also make use of the following simple lemma:

**2.3. Lemma.** *Let $0 < p \leq \alpha$ and $y \in S^{n-1}$. Then,*

$$\| |\langle x, y\rangle|^p / L_\mu^p \|_{L_{\psi_1}(\mu)} \leq 2 C_\alpha^p.$$

*Proof:* For every $s \geq 1$ we have

$$\int_{\mathbb{R}^n} \frac{|\langle x, y\rangle|^{\alpha s}}{(C_\alpha L_\mu)^{\alpha s}} \mu(dx) \leq 2\Gamma(s+1).$$

Now, if $0 < p \leq \alpha$,

$$\int_{\mathbb{R}^n} \exp\left(\frac{|\langle x,y\rangle|^p}{2C_\alpha^p L_\mu^p}\right)\mu(dx) \;=\; 1+\sum_{k=1}^{\infty}\frac{1}{k!}\int_{\mathbb{R}^n}\frac{|\langle x,y\rangle|^{pk}}{2^k C_\alpha^{pk} L_\mu^{pk}}\mu(dx)$$
$$\leq\; 1+\sum_{k=1}^{\infty}\frac{\Gamma(\frac{pk}{\alpha}+1)}{k!2^k}$$
$$\leq\; 1+\sum_{k=1}^{\infty}\frac{1}{2^k}=2$$

since $\Gamma(\frac{pk}{\alpha}+1) \leq k!$. $\square$

In what follows, $c_\alpha(\delta)$ denotes a positive constant bounded by $c\big(\log(2/\delta)\big)^{1/\alpha}$ for $\delta \in (0,1)$, where $c > 0$ is an absolute constant, not necessarily the same in each occurrence.

**2.4. Lemma.** [B2] *Let $\delta \in (0,1)$, and $x_1,\dots,x_m$ be random points in $(\mathbb{R}^n,\mathcal{B},\mu)$. If $m \leq c\delta\exp(\sqrt{n})$, then, with probability greater than $1-\delta$ we have*

$$|x_j| \leq c_2(\delta)\sqrt{n}L_\mu\sqrt{\log m}$$

*for all $j \in \{1,\dots,m\}$, and*

$$|\sum_{i\in E} x_i| \leq c_2(\delta)L_\mu\sqrt{\log m}\sqrt{|E|}\sqrt{n}+c_\alpha(\delta)C_\alpha L_\mu(\log m)^{1/\alpha}|E|$$

*for all $E \subseteq \{1,\dots,m\}$.*

*Proof:* Since $\mu$ is isotropic, we have

$$\int_{\mathbb{R}^n}|x|^2\mu(dx)=nL_\mu^2.$$

From Markov's inequality we get $\mu(4\sqrt{n}L_\mu D_n) \geq 15/16$, and Borell's lemma shows that $\mu(4t\sqrt{n}L_\mu D_n) > 1 - c\exp(-t)$ for every $t > 1$. It follows that, if $m \leq c\delta\exp(\sqrt{n})$, then $m$ random points $x_1,\dots,x_m$ satisfy with probability $> 1-\frac{\delta}{4}$

$$x_i \in nL_\mu D_n, \qquad i=1,\dots,m.$$

Observe now that

$$\text{Prob}\left(x \notin At\sqrt{n}L_\mu D_n \mid x \in nL_\mu D_n\right) \leq 2e^{-t^2}/\mu(nL_\mu D_n) \leq ce^{-t^2}$$

since $\mu(nL_\mu D_n) \geq c'\mu(4\sqrt{n}L_\mu D_n)$ for an absolute constant $c' > 0$ (if $\sqrt{n} \geq 4$ this is clear, otherwise it follows by the log-concavity of $\mu$). Since the $x_j$'s are chosen independently, we conclude that if $t \geq c_2(\delta)\sqrt{\log m}$, then

$$\text{Prob}\left(\forall j \leq m, x_j \in At\sqrt{n}L_\mu D_n \mid \forall j \leq m, x_j \in nL_\mu D_n\right) > 1-\frac{\delta}{4}.$$

Hence, with probability $> 1 - \frac{\delta}{2}$ we have

$$|x_j| \le c_2(\delta)\sqrt{n} L_\mu \sqrt{\log m}, \quad j = 1, \ldots, m.$$

Let $E \subseteq \{1, \ldots, m\}$. We write

$$\begin{aligned} |\sum_{i\in E} x_i|^2 &= \sum_{i\in E} |x_i|^2 + \sum_{i\neq j\in E} \langle x_i, x_j\rangle \\ &\le c_2^2(\delta) L_\mu^2 n(\log m)|E| + \sum_{i\neq j\in E} \langle x_i, x_j\rangle. \end{aligned}$$

If $\delta_j$ takes the values 0 or 1 with probability $1/2$, then

$$\mathbb{E}_{\vec{\delta}}\langle \sum_{i=1}^m \delta_i x_i, \sum_{j=1}^m (1-\delta_j) x_j\rangle = \frac{1}{4}\sum_{i\neq j\in E} \langle x_i, x_j\rangle.$$

Therefore, we can find $E_1, E_2 \subset E$ with $|E_1| \ge |E_2|$, $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = E$, such that

$$\begin{aligned} \sum_{i\neq j\in E} \langle x_i, x_j\rangle &\le 4\langle \sum_{i\in E_1} x_i, \sum_{j\in E_2} x_j\rangle \\ &\le 4\sum_{i\in E_1} |\langle x_i, \sum_{j\in E_2} x_j\rangle|. \end{aligned}$$

Rewrite this last sum in the form

$$\sum_{i\in E_1} |\langle x_i, \sum_{j\in E_2} x_j\rangle| = |\sum_{j\in E_2} x_j| \sum_{i\in E_1} |\langle x_i, y_{E_2}\rangle|,$$

where

$$y_{E_2} = \frac{\sum_{j\in E_2} x_j}{|\sum_{j\in E_2} x_j|} \quad , \quad |y_{E_2}| = 1.$$

Observe that the set $\{x_i\}_{i\in E_1}$ is independent from $y_{E_2}$, since $E_1 \cap E_2 = \emptyset$. If we fix $|E_1| = k$, the number of possible $E_1$'s is bounded by $m^{ck}$, therefore, the $\psi_1$-estimate on linear functionals implies that

$$\mathrm{Prob}\left(\vec{x} \in (\mathbb{R}^n)^m : \exists E_1 \subset E, |E_1| = k, \sum_{i\in E_1} |\langle x_i, y_{E_2}\rangle| > tkC_\alpha L_\mu\right) < m^{ck} e^{-ckt^\alpha}.$$

This probability will be smaller than $\delta/2m$ if $t \simeq (\log m)^{1/\alpha}$. Doing this for $k = 1, \ldots, m$, we see that $(x_1, \ldots, x_m) \in (\mathbb{R}^n)^m$ satisfies with probability greater than $1 - \frac{\delta}{2}$ the following: For every $E \subseteq \{1, \ldots, m\}$,

$$\sum_{i\neq j\in E} \langle x_i, x_j\rangle \le c_\alpha(\delta) C_\alpha L_\mu (\log m)^{1/\alpha} \max_{E_1\subset E} \{|E_1| \, \big|\sum_{j\in E\setminus E_1} x_j\big|\}.$$

To finish the proof, fix $s \in \mathbb{N}$ and write

$$A_s = \max_{|F| \leq s} |\sum_{j \in F} x_j|.$$

We have

$$|\sum_{i \in E} x_i|^2 \leq c_2^2(\delta) L_\mu^2 n (\log m)|E| + c_\alpha(\delta) C_\alpha L_\mu (\log m)^{1/\alpha} |E| A_{|E|},$$

therefore

$$A_{|E|}^2 \leq c_2^2(\delta) L_\mu^2 n (\log m)|E| + c_\alpha(\delta) C_\alpha L_\mu (\log m)^{1/\alpha} |E| A_{|E|},$$

which implies

$$A_{|E|} \leq c_2(\delta) L_\mu \sqrt{n} \sqrt{\log m} \sqrt{|E|} + c_\alpha(\delta) C_\alpha L_\mu (\log m)^{1/\alpha} |E|. \quad \square$$

*Remark:* Borell's lemma shows that, if we do not want to impose any restriction on $m$, then $m$ random points $x_1, \ldots, x_m \in (\mathbb{R}^n, \mathcal{B}, \mu)$ satisfy with probability greater than $1 - \delta$ the inequality

$$|x_j| \leq c_1(\delta) \sqrt{n} L_\mu \log m, \quad j = 1, \ldots, m.$$

Then, the proof of Lemma 2.4 gives

$$|\sum_{i \in E} x_i| \leq c_2(\delta) L_\mu \log m \sqrt{|E|} \sqrt{n} + c_\alpha(\delta) C_\alpha L_\mu (\log m)^{1/\alpha} |E|$$

for all $E \subseteq \{1, \ldots, m\}$. This observation will be useful for the proof of Theorem 2.14.

Our tool from probability theory will be several versions of Bernstein's inequality:

**2.5 Lemma.** [BLM] *Let $\{f_j\}_{j \leq m}$ be independent random variables with mean 0 on some probability space $(\Omega, \mu)$.*

(i) *If $\|f_j\|_1 \leq 2$ and $\|f_j\|_\infty \leq B$, then, for every $\varepsilon \in (0,1)$,*

$$\mathrm{Prob}\left( |\sum_{j=1}^m f_j| > \varepsilon m \right) \leq 2 \exp(-\varepsilon^2 m / 8B).$$

(ii) *If $\|f_j\|_{L^{\psi_1}(\mu)} \leq A$, $j = 1, \ldots, m$, then, for every $0 < \varepsilon < 4A$,*

$$\mathrm{Prob}\left( |\sum_{j=1}^m f_j| \geq \varepsilon m \right) \leq 2 \exp(-\varepsilon^2 m / 16A^2).$$

(iii) *If* $\|f_j\|_{L^{\psi_2(\mu)}} \leq A$, $j = 1, \ldots, m$, *then, for every* $\varepsilon > 0$,

$$\mathrm{Prob}\left( \left| \sum_{j=1}^{m} f_j \right| \geq \varepsilon m \right) \leq 2\exp(-\varepsilon^2 m / 8A^2). \quad \square$$

We first study the cardinality of $p$-distributed sets for small values of $p > 0$:

**2.6. Proposition.** *Let $\mu$ be an isotropic log-concave probability measure on $\mathbb{R}^n$, which satisfies a $\psi_\alpha$-estimate with constant $C_\alpha$ for some $\alpha \in [1,2]$. Let $\delta \in (0,1)$ and $0 < p \leq \alpha$. Assume that $m \geq c_1(\delta) p^{-2} C_\alpha^{2p} n$. Then, $m$ random points $x_1, \ldots, x_m \in (\mathbb{R}^n, \mathcal{B}, \mu)$ satisfy with probability greater than $1 - \delta$*

$$\Gamma_1 L_\mu \leq \left( \frac{1}{m} \sum_{j=1}^{m} |\langle x_j, y \rangle|^p \right)^{1/p} \leq \Gamma_2 L_\mu,$$

*for all $y \in S^{n-1}$, where $\Gamma_1, \Gamma_2 > 0$ are absolute constants.*

*Proof:* Let $\theta \in (0,1)$ to be determined later, and consider a $\theta$-net $\mathcal{N}$ for $S^{n-1}$ with cardinality $|\mathcal{N}| \leq (3/\theta)^n$.

Fix $y \in \mathcal{N}$ and define

$$f(x) = \frac{|\langle x, y \rangle|^p}{L_\mu^p}.$$

We set $f_j(x_1, \ldots, x_m) = f(x_j) - \int f$ on $(\mathbb{R}^n)^m$. Since $\int f \leq 1$, Lemma 2.3 shows that

$$\mathbb{E} f_j = 0 \quad , \quad \|f_j\|_{L_{\psi_1}} \leq 4 C_\alpha^p.$$

Hence, Lemma 2.5(ii) implies that

$$\mathrm{Prob}\left( \left| \frac{1}{m} \sum_{j=1}^{m} |\langle x_j, y \rangle|^p - L_\mu^p \int f \right| \geq \varepsilon L_\mu^p \right) \leq 2\exp(-\varepsilon^2 m / c C_\alpha^{2p}),$$

if $0 < \varepsilon < 1$. This probability is smaller than $\delta / |\mathcal{N}|$, provided that

$$m \geq c_1(\delta, \theta) \varepsilon^{-2} C_\alpha^{2p} n.$$

Then, choosing $\varepsilon = c_1^p p/4$, for all $y \in \mathcal{N}$ we have

$$c_1^p (1 - p/4) L_\mu^p \leq \frac{1}{m} \sum_{j=1}^{m} |\langle x_j, y \rangle|^p \leq (1 + c_1^p p/4) L_\mu^p,$$

which implies

$$c_3 L_\mu \leq \left( \frac{1}{m} \sum_{j=1}^{m} |\langle x_j, y \rangle|^p \right)^{1/p} \leq c_4 L_\mu.$$

To complete the proof, we choose $\theta = c\delta$ and employ a standard successive approximation argument. □

**2.7. Corollary.** *Let $\mu$ be an isotropic log-concave probability measure on $\mathbb{R}^n$, $\delta \in (0,1)$ and $p \geq 1$. If $m \geq c_1(\delta)n$, then $m$ random points $x_1, \dots, x_m \in (\mathbb{R}^n, \mathcal{B}, \mu)$ satisfy with probability greater than $1-\delta$*

$$\Gamma_1 L_\mu \leq \left(\frac{1}{m}\sum_{j=1}^m |\langle x_j, y\rangle|^p\right)^{1/p},$$

*for all $y \in S^{n-1}$.*

*Proof:* Obvious from Proposition 2.6, since every log-concave probability measure $\mu$ satisfies a $\psi_1$-estimate with a uniformly bounded constant $C_1 \geq 1$, and the quantity

$$\left(\frac{1}{m}\sum_{j=1}^m |\langle x_j, y\rangle|^p\right)^{1/p}$$

is an increasing function of $p$. □

Proposition 2.6 settles our Question for $0 < p \leq 1$: the minimal cardinality of a random $p$-distribution for $\mu$ ($0 < p \leq 1$) is proportional to $n$. Also, by Corollary 2.7 we only need to consider upper bounds when we ask about $p$-distributions with $p \geq 1$: the lower bound holds with probability $> 1-\delta$ if $m \geq c_1(\delta)n$.

In order to examine the case $\alpha < p$, we follow Bourgain's argument:

**2.8. Lemma.** *Let $\mu$ be an isotropic log-concave probability measure on $\mathbb{R}^n$ which satisfies a $\psi_\alpha$-estimate with constant $C_\alpha$ for some $\alpha \in [1,2]$, and $p \geq 1$. Fix $\delta \in (0,1)$ and $B > 0$. If $m \geq c_1(\delta)n(\frac{B}{c_1 L_\mu})^p$, then $m$ random points $x_1, \dots, x_m$ satisfy with probability greater than $1-\delta$*

$$\left(\frac{1}{m}\sum_{\{j:|\langle x_j, y\rangle| \leq B\}} |\langle x_j, y\rangle|^p\right)^{1/p} \leq 2c_2 p^{1/\alpha} C_\alpha L_\mu$$

*for all $y \in S^{n-1}$.*

*Proof:* Let $\theta \in (0,1)$, to be determined. There exists a $\theta$-net $\mathcal{N}$ for $S^{n-1}$ with cardinality $|\mathcal{N}| \leq (3/\theta)^n$. Fix $y \in \mathcal{N}$ and let $I_p(y) = (\int |\langle z, y\rangle|^p \mu(dz))^{1/p}$. We define

$$f(x) = \frac{1}{I_p^p(y)}|\langle x, y\rangle|^p \chi_{\{z:|\langle z,y\rangle| \leq B\}}(x)$$

on $\mathbb{R}^n$, and set $f_j(x_1, \dots, x_m) = f(x_j) - \int f$ on $(\mathbb{R}^n)^m$. Since $I_p(y) \geq c_1 L_\mu$, we have

$$\|f_j\|_1 \leq 2 \quad , \quad \mathbb{E}f_j = 0 \quad , \quad \|f_j\|_\infty \leq \left(\frac{B}{c_1 L_\mu}\right)^p.$$

Applying Lemma 2.5(i) we get: for every $\varepsilon \in (0,1)$,

$$\mathrm{Prob}\left(\frac{1}{m}\sum_{j=1}^{m} f(x_j) - \int f > \varepsilon\right) \leq \exp\left(-\varepsilon^2 m/8(B/c_1 L_\mu)^p\right) < \delta/|\mathcal{N}|,$$

provided that $m \geq c_1(\delta,\theta)\varepsilon^{-2} n \left(\frac{B}{c_1 L_\mu}\right)^p$. This means that with probability greater than $1-\delta$,

$$\left(\frac{1}{m}\sum_{\{j:|\langle x_j,y\rangle|\leq B\}} |\langle x_j,y\rangle|^p\right)^{1/p} \leq (1+\varepsilon)^{1/p} I_p(y)$$

for all $y \in \mathcal{N}$. Choosing $\varepsilon = \theta = 1/4$, using successive approximation for an arbitrary $y \in S^{n-1}$, and taking into account Lemma 2.1 we conclude the proof. $\square$

**2.9. Lemma.** *Let $\mu$ be an isotropic log-concave probability measure on $\mathbb{R}^n$ which satisfies a $\psi_\alpha$-estimate with constant $C_\alpha$, let $\delta \in (0,1)$, and $x_1,\ldots,x_m$ satisfying Lemma 2.4. If $B \geq 4c_\alpha(\delta) C_\alpha L_\mu (\log m)^{1/\alpha}$, then*

$$\sum_{\{j:|\langle x_j,y\rangle|>B\}} |\langle x_j,y\rangle|^p \leq \begin{cases} c_2^2(\delta) h_p B^{p-2} L_\mu^2 n\log m & , \text{ if } 0<p<2 \\ c_2^p(\delta) h_p L_\mu^p (n\log m)^{p/2} & , \text{ if } p>2 \\ c L_\mu^2 n \log m \log n & , \text{ if } p=2. \end{cases}$$

*The constant $h_p$ satisfies $1 \leq h_p \leq \max\{2, \min\{|p-2|^{-1}, \log n\}\}$.*

*Proof:* For every $\beta \geq \beta_0 = 4c_\alpha(\delta) C_\alpha L_\mu (\log m)^{1/\alpha}$ and $y \in S^{n-1}$, we define

$$E_\beta(y) = \{j \leq m : |\langle x_j,y\rangle| > \beta\}.$$

We can estimate the size of $E_\beta(y)$ as follows:

$$\begin{aligned} \beta|E_\beta| &\leq \sum_{j\in E_\beta} |\langle x_j,y\rangle| \leq \max_{\varepsilon_j=\pm1, j\in E_\beta} |\sum \varepsilon_j x_j| \\ &\leq 2\max_{F\subseteq E_\beta} |\sum_{j\in F} x_j| \\ &\leq 2c_2(\delta) L_\mu \sqrt{\log m}\sqrt{n}\sqrt{|E_\beta|} + 2c_\alpha(\delta) C_\alpha L_\mu (\log m)^{1/\alpha} |E_\beta| \\ &\leq 2c_2(\delta) L_\mu \sqrt{\log m}\sqrt{n}\sqrt{|E_\beta|} + \frac{\beta}{2}|E_\beta|, \end{aligned}$$

from where we deduce that

$$\beta^2 |E_\beta| \leq c_2^2(\delta) L_\mu^2 n \log m.$$

Note that this estimate is independent from the choice of $y \in S^{n-1}$. It follows that, if $B \geq \beta_0$ then

$$\begin{aligned}
\sum_{\{j:|\langle x_j,y\rangle|>B\}} |\langle x_j,y\rangle|^p &= \sum_{k=0}^{k_0-1} \sum_{\{j:2^k B<|\langle x_j,y\rangle|\leq 2^{k+1}B\}} |\langle x_j,y\rangle|^p \\
&\leq \sum_{k=0}^{k_0-1} |E_{2^k B}|(2^{k+1}B)^p \\
&\leq c_2^2(\delta)L_\mu^2 n(\log m) \sum_{k=0}^{k_0-1} \frac{(2^{k+1}B)^p}{(2^k B)^2} \\
&\leq 2^p c_2^2(\delta)L_\mu^2 n(\log m)B^{p-2} \sum_{k=0}^{k_0-1} 2^{(p-2)k},
\end{aligned}$$

where $k_0$ is the least integer for which $c_2(\delta)\sqrt{n}L_\mu\sqrt{\log m} \leq 2^{k_0}B$. Since $B \geq 4c_\alpha(\delta)C_\alpha L_\mu\sqrt{\log m}$, we have $k_0 \leq c\log n$. We now conclude the proof by distinguishing cases about $p$:

If $0 < p < 2$, the result follows with

$$h_p = \sum_{k=0}^{k_0-1} 2^{(p-2)k}.$$

If $p > 2$, then setting $\Delta^2 = c_2^2(\delta)L_\mu^2 n\log m$ we have

$$\begin{aligned}
\sum_{\{j:|\langle x_j,y\rangle|>B\}} |\langle x_j,y\rangle|^p &\leq 2^p\Delta^2 B^{p-2}\frac{2^{k_0(p-2)}}{2^{p-2}-1} \\
&\leq \frac{2^{p-2}}{2^{p-2}-1}(2\Delta)^2 B^{p-2}\left(\frac{2\Delta}{B}\right)^{p-2} \\
&=: h_p(2\Delta)^p. \quad \square
\end{aligned}$$

Our first result covers the case $\alpha < p < 2$, where $\mu$ satisfies a $\psi_\alpha$-estimate:

**2.10. Proposition.** *Let $\mu$ be an isotropic log-concave probability measure which satisfies a $\psi_\alpha$-estimate with constant $C_\alpha$, and $\alpha < p < 2$. Assume that $m \geq c_1^3(\delta)h_pC_\alpha^p n(\log n)^{p/\alpha}$. Then, $m$ random points $x_1,\ldots,x_m$ satisfy with probability $> 1-\delta$*

$$\left(\frac{1}{m}\sum_{j=1}^m |\langle x_j,y\rangle|^p\right)^{1/p} \leq 3L_\mu$$

*for all $y \in S^{n-1}$.*

*Proof:* Choose $B = 4c_\alpha(\delta)C_\alpha L_\mu(\log m)^{1/\alpha}$. From Lemma 2.8 we know that if $m \geq c_1(\delta)h_p C_\alpha^p n(\log m)^{p/\alpha}$, then $m$ random points $x_1, \ldots, x_m$ satisfy with probability greater than $1-\delta$

$$\left(\frac{1}{m}\sum_{\{j:|\langle x_j,y\rangle|\leq B\}} |\langle x_j,y\rangle|^p\right)^{1/p} \leq 2L_\mu$$

for all $y \in S^{n-1}$. On the other hand, from Lemma 2.9 we have

$$\begin{aligned}\frac{1}{m}\sum_{\{j:|\langle x_j,y\rangle|>B\}} |\langle x_j,y\rangle|^p &\leq \frac{1}{m}c_2^2(\delta)h_p c_\alpha^{p-2}(\delta)C_\alpha^{p-2}L_K^{p-2}(\log n)^{\frac{p-2}{\alpha}}L_\mu^2 n\log m \\ &\leq (\log n)^{\frac{\alpha-2}{\alpha}}L_\mu^p \leq L_\mu^p,\end{aligned}$$

for suitably chosen $c_3(\delta)$. Adding, we see that

$$\left(\frac{1}{m}\sum_{j=1}^m |\langle x_j,y\rangle|^p\right)^{1/p} \leq 3L_\mu$$

for all $y \in S^{n-1}$. $\square$

We now come to the case $p = 2$:

**2.11. Proposition.** *Let $\mu$ be an isotropic log-concave probability measure on $\mathbb{R}^n$. If $m \geq c_1^2(\delta)n(\log n)^2$, then $m$ random points $x_1, \ldots, x_m$ satisfy with probability $> 1-\delta$*

$$c_1 L_\mu \leq \left(\frac{1}{m}\sum_{j=1}^m \langle x_j,y\rangle^2\right)^{1/2} \leq c_2 L_\mu$$

*for all $y \in S^{n-1}$.* $\square$

*Proof:* We choose $B = 4c_2(\delta)L_\mu \log m$, and combine the estimates from Lemmas 2.8 and 2.9. $\square$

*Remark:* In this case ($p = 2$) we can actually replace $c_1$ and $c_2$ by $1-\varepsilon$, $1+\varepsilon$ respectively, if we choose $m \geq c(\delta)\varepsilon^{-2}n(\log n)^2$ and repeat the argument in a suitable way (this is the question originally studied by Bourgain [B2] and Rudelson [R] for convex bodies: note that Bourgain's method combined with Lemma 2.2 is enough for Rudelson's estimate $m = c(\varepsilon,\delta)n(\log n)^2$).

The case $p > 2$ can be treated in a similar way. The estimate in Lemma 2.9 forces us to choose $m \geq c_2^p(\delta)h_p(n\log n)^{p/2}$, and if $B = 4c_2(\delta)L_\mu \log m$, then the hypotheses of Lemma 2.8 are satisfied, provided that $n \geq n_0$. Therefore, we have the following result about the minimal cardinality of a $p$-distribution of points for $\mu$:

**2.12. Proposition.** *Let $\mu$ be an isotropic log-concave probability measure on $\mathbb{R}^n$ which satisfies a $\psi_\alpha$-estimate with constant $C_\alpha$ for some $\alpha \in [1,2]$, and $p > 2$. If $c\delta \exp(\sqrt{n}) \geq m \geq c_2^p(\delta) h_p (n \log n)^{p/2}$, then $m$ random points $x_1, \dots, x_m \in (\mathbb{R}^n, \mathcal{B}, \mu)$ satisfy with probability $> 1-\delta$*

$$c_1 L_\mu \leq \left( \frac{1}{m} \sum_{j=1}^m |\langle x_j, y\rangle|^p \right)^{1/p} \leq c_2 C_\alpha p^{1/\alpha} L_\mu. \quad \square$$

*Remark:* One may interpret all these results as giving random embeddings of $\ell_2^n$ into $\ell_p^N$, where $N \simeq (n\log n)^{p/2}$ when $p > 2$. The precision of Dvoretzky's theorem is somehow lost: the subspaces are $p^{1/\alpha}$-isomorphic to $\ell_2^n$ and the dependence on $n$ is worse because of the logarithmic term. But, the notion of "randomness" is different from the usual one. We obtain subspaces which are random with respect to the given log-concave measure.

From the above, we have the following general estimates for an isotropic log-concave probability measure $\mu$:

**2.13. Theorem.** *Let $0 < p < \infty$ and $\delta \in (0,1)$. There exists $n_0(\delta)$ such that, for every $n \geq n_0$, every $m \geq m_0$ and every isotropic log-concave probability measure $\mu$ on $\mathbb{R}^n$ which satisfies a $\psi_\alpha$-estimate with constant $C_\alpha$ for some $\alpha \in [1,2]$, $m$ random points $x_1, \dots, x_m \in (\mathbb{R}^n, \mathcal{B}, \mu)$ form with probability greater than $1-\delta$ a $(p, M)$-distribution representing $\mu$, where*

$$m_0 = m_0(p,\delta) = \begin{cases} c(\delta) p^{-2} n & , \text{ if } 0 < p \leq 1 \\ c(\delta) h_p n (\log n)^p & , \text{ if } 1 < p \leq 2 \\ c_2^p(\delta) h_p (n \log n)^{p/2} & , \text{ if } p > 2. \quad \square \end{cases}$$

*Here, $M$ is bounded by an absolute constant in the first two cases, while in the third one we may have $\simeq p^{1/\alpha}$ under the restriction $m \leq c\delta\exp(\sqrt{n})$, or $\simeq p$ with no upper restriction on $m$.*

Finally, we study the cardinality of a random $\psi_1$-distribution with respect to $\mu$:

**2.14. Theorem.** *Let $\mu$ be an isotropic log-concave probability measure on $\mathbb{R}^n$, and $\gamma \in (0,1)$. If $n \geq n_0(\gamma)$ and $m \geq \exp(\gamma n)$, then $m$ random points $x_1, \dots, x_m$ satisfy with probability $> 1-\delta$*

$$\frac{1}{m} \sum_{j=1}^m e^{\frac{\sqrt{\gamma}|\langle x_j, y\rangle|}{c_1(\delta) L_\mu}} \leq 2$$

*for every $y \in S^{n-1}$.*

*Proof:* Let $M = c_1'(\delta) L_\mu / \sqrt{\gamma}$ (where the constant $c_1'(\delta)$ is to be chosen) and $B \geq 4c_1(\delta) C_1 L_\mu \log m$. Keeping the notation of Lemma 2.9 and taking into account the

Remark after Lemma 2.4, we estimate as follows:

$$\begin{aligned}\sum_{\{j:|\langle x_j,y\rangle|>B\}}\exp(|\langle x_j,y\rangle|/M) &= \sum_{k=0}^{k_0-1}\sum_{\{j:2^kB<|\langle x_j,y\rangle|\le 2^{k+1}B\}}\exp(|\langle x_j,y\rangle|/M)\\ &\le c_2^2(\delta)L_K^2 n\log^2 m\sum_{k=0}^{k_0-1}\frac{\exp(2^{k+1}B/M)}{(2^kB)^2}\\ &\le \frac{c_2^2(\delta)L_\mu^2 n\log^2 m}{B^2}\exp\left(\frac{2c_2(\delta)\sqrt{n}L_\mu\log m}{M}\right)\\ &\le cn\exp(\sqrt{\gamma n\log m/2}).\end{aligned}$$

It follows that, if $n\ge n_0(\gamma)$ and $m\ge\exp(\gamma n)$, then

$$\frac{1}{m}\sum_{\{j:|\langle x_j,y\rangle|>B\}}\exp(|\langle x_j,y\rangle|/M)\le\frac{1}{2}.$$

On the other hand, by Lemma 2.1, $\int\exp\big(|\langle x,y\rangle|/M\big)\le 5/4$ for every $y\in S^{n-1}$. For every $B>0$, we define

$$f(x)=\frac{\exp(|\langle x,y\rangle|/M)}{\int\exp\big(|\langle x,y\rangle|/CL_\mu\big)}\chi_{\{x:|\langle x,y\rangle|\le B\}}(x)$$

and following the proof of Lemma 2.8 we get

$$\frac{1}{m}\sum_{\{j:|\langle x_j,y\rangle|\le B\}}\exp\big(|\langle x_j,y\rangle|/M\big)\le 3/2$$

for all $y\in S^{n-1}$, provided that

$$m\ge c_1(\delta)n\exp\big(B/M\big).$$

We choose $B=4c_1(\delta)C_1L_\mu\log m$, and check that this restriction is satisfied. Adding the estimates above, we conclude the proof. □

*Remark:* Consider the case $\mu=\mu_K$, where $K$ is an isotropic convex body in $\mathbb{R}^n$. This means that $|K|=1$, and

$$\int_K\langle x,y\rangle^2dx=L_K^2$$

for every $y\in S^{n-1}$. Then, $\mu_K$ is an isotropic log-concave probability measure on $\mathbb{R}^n$, and this implies that all the results of this Section are valid for points $x_1,\dots,x_m$ chosen independently and uniformly from $K$. Moreover, all the results may be stated without the restriction $m\le c\delta\exp(\sqrt{n})$, since a result of Alesker [A] shows that $\|\langle x,y\rangle\|_{L_{\psi_2}(K;dx)}\le A\sqrt{n}L_K$ for an absolute constant $A>0$ (which is a stronger statement than Lemma 2.2).

Observe that, in this case, there exists an absolute constant $a > 0$ such that $\mu(a\sqrt{n}L_K D_n) \leq a$. Thus, for a random choice of points $S = \{x_1, \ldots, x_m\}$ in $K$ there exists $i \leq m$ for which $|x_i| \geq a\sqrt{n}L_K$. Therefore,

$$\max_{y \in S^{n-1}} \|\langle x, y\rangle\|_{L_p(S;\mu(S))} \geq \frac{|x_i|}{m^{1/p}} \geq \frac{a\sqrt{n}L_K}{m^{1/p}}$$

for every $p \geq 2$. It follows that a random $p$-distribution $S$ for $\mu_K$ must have cardinality of order at least $n^{p/2}$. Hence, the estimates in Theorem 2.13 are optimal up to the logarithmic terms. We do not know if the estimate for $m$ in Theorem 2.14 is also optimal.

# 3 Well distributed sets of vertices of the cube

Consider $E_2^n = \{-1, 1\}^n$ with the product measure $\mu(A) = |A|/2^n$, $A \subset E_2^n$, and write $\epsilon$ for an element of $E_2^n$. The analogue of Lemma 2.1 in this case is Khintchine's inequality:

**3.1. Lemma.** *There exist absolute constants $c_1, c_2 > 0$ such that*

$$c_1|y| \leq \left(\int_{E_2^n} |\langle \epsilon, y\rangle|^p \mu(d\epsilon)\right)^{1/p} \leq \max\{1, c_2\sqrt{p}\}\ |y|$$

*for every $p > 0$ and $y \in \mathbb{R}^n$.* □

Given $\delta \in (0,1)$, we ask for the minimum value of $m \in \mathbb{N}$ which satisfies the following: with probability greater than $1 - \delta$, a subset $A$ of $E_2^n$ with $m$ elements is a $p$-distribution (analogously, a $\psi_\alpha$-distribution) representing $E_2^n$. The method used in the previous section allows us better estimates in this case, because the cube satisfies a $\psi_2$-estimate and has small diameter: Using the facts that $|\epsilon| = \sqrt{n}$ for every $\epsilon \in E_2^n$ and $\|\langle \epsilon, y\rangle\|_{L_{\psi_2}} \leq c$ for every $y \in S^{n-1}$, we obtain the following analogue of Lemma 2.4:

**3.2. Lemma.** *Let $\delta \in (0,1)$ and $\epsilon_1, \ldots, \epsilon_m$ be random points in $E_2^n$. With probability greater than $1 - \delta$ we have*

$$|\sum_{i \in E} \epsilon_i| \leq \sqrt{n}\sqrt{|E|} + c_2(\delta)\sqrt{\log m}|E|$$

*for all $E \subseteq \{1, \ldots, m\}$.* □

We will first consider the case $0 < p \leq 2$:

**3.3. Proposition.** *Let $p \in (0,2]$ and $\delta \in (0,1)$. If $m \geq c_1(\delta)p^{-2}n$, then a subset $A$ of $E_2^n$ with $|A| = m$ satisfies with probability $> 1 - \delta$*

$$c \leq \left(\frac{1}{|A|}\sum_{\epsilon \in A} |\langle \epsilon, y\rangle|^p\right)^{1/p} \leq c'$$

*for every $y \in S^{n-1}$, where $c, c' > 0$ are absolute constants.*

*Proof:* For every $y \in S^{n-1}$ we have $\|\langle \epsilon, y\rangle\|_{L_{\psi_2}(E_2^n;\mu)} \leq c$ and $\|\langle \epsilon, y\rangle\|_2 = 1$. By Lemma 2.3, $\|\langle \epsilon, y\rangle\|_{L_{\psi_1}(E_2^n;\mu)} \leq c^p$. Then, we follow the proof of Proposition 2.6. $\square$

For the case $p > 2$ we need the analogue of Lemma 2.9:

**3.4. Lemma.** *Let $\epsilon_1, \ldots, \epsilon_m$ be as in Lemma 3.2. If $B \geq 4c_2(\delta)\sqrt{\log m}$, then*

$$\sum_{\{j : |\langle \epsilon_j, y\rangle| > B\}} |\langle \epsilon_j, y\rangle|^p \leq h_p(4n)^{p/2},$$

*for every $p > 2$ and $y \in S^{n-1}$.*

*Proof:* As in Lemma 2.9, we define $E_\beta(y) = \{j \leq m : |\langle \epsilon_j, y\rangle| > \beta\}$. Then, for every $\beta \geq 4c_2(\delta)\sqrt{\log m}$ and $y \in S^{n-1}$ we have $\beta^2 |E_\beta(y)| \leq n$.

Let $B \geq 4c_2(\delta)\sqrt{\log m}$, and $k_0$ be the smallest integer for which $2^{k_0}B \geq \sqrt{n}$. Then,

$$\begin{aligned}
\sum_{\{j : |\langle \epsilon_j, y\rangle| > B\}} |\langle \epsilon_j, y\rangle|^p &= \sum_{k=0}^{k_0-1} \sum_{\{j : 2^k B < |\langle \epsilon_j, y\rangle| \leq 2^{k+1}B\}} |\langle \epsilon_j, y\rangle|^p \\
&\leq \sum_{k=0}^{k_0-1} |E_{2^k B}| (2^{k+1}B)^p \leq n \sum_{k=0}^{k_0-1} \frac{(2^{k+1}B)^p}{(2^k B)^2} \\
&= 2^p n B^{p-2} \sum_{k=0}^{k_0-1} 2^{(p-2)k} \leq 2^p n B^{p-2} \frac{2^{k_0(p-2)}}{2^{p-2}-1} \\
&\leq \frac{2^p}{2^{p-2}-1} n B^{p-2} \left(\frac{2\sqrt{n}}{B}\right)^{p-2} = h_p(4n)^{p/2}. \quad \square
\end{aligned}$$

On the other hand, an adaptation of the proof of Lemma 2.8 gives: If $m \geq c(\delta) n \left(\frac{B}{c_1}\right)^p$, then a subset $A$ of $E_2^n$ with $|A| = m$ satisfies with probability $> 1 - \delta$

$$(*) \qquad \left(\frac{1}{|A|} \sum_{\{\epsilon \in A : |\langle \epsilon, y\rangle| \leq B\}} |\langle \epsilon, y\rangle|^p\right)^{1/p} \leq c_4 \sqrt{p}$$

for all $y \in S^{n-1}$. If we choose $B = 4c_2(\delta)\sqrt{\log m}$ and assume that $n \geq n_0(\delta)$, then any $m \geq h_p(4n)^{p/2}$ satisfies our condition for $(*)$. Therefore, $(*)$ and Lemma 3.4 imply

$$\left(\frac{1}{|A|} \sum_{\epsilon \in A} |\langle \epsilon, y\rangle|^p\right)^{1/p} \leq (c_4 + 1)\sqrt{p}$$

for every $y \in S^{n-1}$. The lower bound is clear from Proposition 3.3 and the monotonicity of our average in $p$. We summarize as follows:

**3.5. Theorem.** *Let $0 < p < \infty$ and $\delta \in (0,1)$. There exists $n_0(\delta)$ such that, for every $n \geq n_0$ a subset $A \subset E_2^n$ with $m \geq m_0$ elements forms a $p$-distribution with probability greater than $1-\delta$, where*

$$m_0 = m_0(p,\delta) = \begin{cases} c(\delta)p^{-2}n & \text{, if } 0 < p \leq 2 \\ \\ h_p(4n)^{p/2} & \text{, if } p > 2. \quad \Box \end{cases}$$

The next two lemmas will allow us to estimate the size of a $\psi_\alpha$-distribution in $E_2^n$:

**3.6. Lemma.** *Let $\epsilon_1, \ldots, \epsilon_m$ be as in Lemma 3.2. Let $\alpha \in [1,2]$ and $M > 0$. If $B \geq 4c_2(\delta)\sqrt{\log m} > 2M$, then*

$$\sum_{\{j:|\langle \epsilon_j, y\rangle| > B\}} \exp\left(\frac{|\langle \epsilon_j, y\rangle|^\alpha}{M^\alpha}\right) \leq \exp\left(\frac{2^\alpha n^{\alpha/2}}{M^\alpha}\right),$$

*for all $y \in S^{n-1}$.*

*Proof:* Keeping the notation of Lemma 3.4, we estimate as follows:

$$\begin{aligned} \sum_{\{j:|\langle \epsilon_j, y\rangle| > B\}} \exp\left(\frac{|\langle \epsilon_j, y\rangle|^\alpha}{M^\alpha}\right) &= \sum_{k=0}^{k_0-1} \sum_{\{j:2^k B < |\langle \epsilon_j, y\rangle| \leq 2^{k+1}B\}} \exp\left(\frac{|\langle \epsilon_j, y\rangle|^\alpha}{M^\alpha}\right) \\ &\leq \sum_{k=0}^{k_0-1} |E_{2^k B}| \exp\left(\frac{(2^{k+1}B)^\alpha}{M^\alpha}\right) \\ &\leq \frac{n}{B^2} \sum_{k=0}^{k_0-1} 2^{-2k} \exp\left(\frac{(2^{k+1}B)^\alpha}{M^\alpha}\right) \\ &\leq \frac{n}{2^{2k_0}B^2} \exp\left(\frac{(2^{k_0}B)^\alpha}{M^\alpha}\right), \end{aligned}$$

since $B > 2M$ guarantees that the sum is dominated by the last term. On observing that $2^{k_0}B \leq 2\sqrt{n}$, we conclude the proof. $\Box$

**3.7. Lemma.** *Let $\alpha \in [1,2]$, $\delta \in (0,1)$ and $B > 0$. If $m \geq c(\delta)n\exp\big((B/c)^\alpha\big)$, then $m$ random points $\epsilon_1, \ldots, \epsilon_m \in E_2^n$ satisfy with probability greater than $1-\delta$*

$$\frac{1}{m} \sum_{\{j:|\langle \epsilon_j, y\rangle| \leq B\}} \exp\big((|\langle \epsilon_j, y\rangle|/c)^\alpha\big) \leq 3/2$$

*for all $y \in S^{n-1}$.*

*Proof:* There exists $c > 0$ such that $\int_{E_2^n} \exp\big((|\langle \epsilon, y\rangle|/c)^\alpha\big) \leq 5/4$ for every $y \in S^{n-1}$ and $\alpha \in [1,2]$. We define

$$f(x) = \frac{e^{\left(\frac{|\langle \epsilon, y\rangle|}{c}\right)^\alpha}}{\int_{E_2^n} \exp\big((|\langle \epsilon, y\rangle|/c)^\alpha\big)} \chi_{\{\zeta:|\langle \zeta, y\rangle| \leq B\}}(\epsilon)$$

and follow the proof of Lemma 2.8. $\square$

**3.8. Theorem.** *Let $\gamma > 0$, $\delta \in (0,1)$ and $\alpha \in [1,2]$. If $n \geq n_0(\gamma, \delta)$, then a subset $A$ of $E_2^n$ with $|A| = m \geq \exp(\gamma n^{\alpha/2})$ satisfies with probability greater than $1-\delta$*

$$\|\langle \epsilon, y\rangle\|_{L_{\psi_\alpha}(A)} \leq \frac{C(\delta)}{\gamma^{1/\alpha}}$$

*for every $y \in S^{n-1}$.*

*Proof:* We choose $B = 4c_2(\delta)\gamma^{\frac{1}{2}} n^{\frac{\alpha}{4}}$, and $M = 8c(\delta)/\gamma^{\frac{1}{\alpha}}$, where $c_2(\delta), c(\delta)$ are the constants in Lemmas 3.6 and 3.7 respectively. By Lemma 3.6,

$$\sum_{\{j : |\langle \epsilon_j, y\rangle| > B\}} \exp\left(\frac{\gamma |\langle \epsilon_j, y\rangle|^\alpha}{(8c(\delta))^\alpha}\right) \leq \exp\left(\frac{\gamma n^{\alpha/2}}{(4c(\delta))^\alpha}\right) \leq m/2,$$

if $n \geq n_0(\delta, \gamma)$. We may also assume that $\exp(\frac{\gamma}{2} n^{\alpha/2}) \geq c(\delta) n$, therefore the condition for Lemma 3.7 becomes

$$\frac{\gamma}{2} n^{\alpha/2} \geq \frac{1}{2^\alpha} \gamma^{\frac{\alpha}{2}} n^{\frac{\alpha^2}{4}},$$

which is obviously satisfied since $\alpha \leq 2$. Hence, Lemma 3.7 gives

$$\sum_{\{j : |\langle \epsilon_j, y\rangle| \leq B\}} \exp\bigl(\gamma |\langle \epsilon_j, y\rangle| / (8c(\delta))^\alpha\bigr) \leq 3m/2.$$

Adding the estimates, we conclude the proof with $C(\delta) = 8c(\delta)$. $\square$

*Remark:* The estimates in Theorems 3.5 and 3.8 are optimal (see the Remark after Theorem 2.14).

# 4 Random projections onto $n$-dimensional subspaces

In this Section we discuss a different type of question, which reflects the same geometry. We are going to present two formulations of the problem:

$(a)$ Let $N > n$, and consider an orthonormal basis of $\mathbb{R}^N$. For every $U = (u_{ij})$ in the orthogonal group $O(N)$, define

$$v_i = \sqrt{N} P_n U^*(e_i) = (\sqrt{N} u_{ij})_{j \leq n} \quad , \quad i = 1, \ldots, N,$$

where $P_n$ denotes the orthogonal projection of $\mathbb{R}^N$ onto $\mathbb{R}^n$. Let $V = \{v_1, \ldots, v_N\}$. Using the orthogonality of $U$, we easily check that

$$\|\langle v, y\rangle\|_{L_2(V;\mu(V))} = \left(\frac{1}{N}\sum_{i=1}^N \langle v_i, y\rangle^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^N (\sum_{j=1}^n u_{ij} y_j)^2\right)^{\frac{1}{2}} = |y|$$

for every $y \in \mathbb{R}^n$. The question is: given $p \geq 1$, find the minimal value $N(p)$ of $N > n$ for which a random (with respect to $U \in O(N)$) set $V = V(U)$ as above is a $(p, M)$-distribution for some good constant $M \geq 1$.
The answer is given by the following fact:

**4.1. Theorem.** *For every $\lambda > 1$ there exists $c(\lambda) > 0$ such that: if $N \geq \lambda n$, then a random $U \in O(N)$ satisfies*

$$\|\langle v, y\rangle\|_{L_1(V,\mu(V))} \geq c(\lambda)$$

*for every $y \in S^{n-1}$. If $p > 2$ and $N \geq cn^{p/2}$, then a random $U \in O(N)$ satisfies*

$$\|\langle v, y\rangle\|_{L_p(V,\mu(V))} \leq c\sqrt{p}$$

*for every $y \in S^{n-1}$, where $c > 0$ is an absolute constant.*

*Proof:* Let $U \in O(N)$. Then, $U$ induces a random embedding of $\mathbb{R}^n$ into $\mathbb{R}^N$, given by

$$y \mapsto (\langle v_i, y\rangle)_{i \leq N} = \sqrt{N} U y.$$

For every $y \in \mathbb{R}^n$ and every $p \geq 1$ we have

$$\left(\frac{1}{N}\sum_{i=1}^{N} |\langle v_i, y\rangle|^p\right)^{1/p} = N^{\frac{1}{2}-\frac{1}{p}}\|U(y)\|_p.$$

Now, Dvoretzky's theorem for $\ell_p^N$, $p > 2$ shows that if $N \geq cn^{p/2}$, then for a random $U \in O(N)$ we will have

$$N^{\frac{1}{2}-\frac{1}{p}}\|U(y)\|_p \leq 2N^{\frac{1}{2}-\frac{1}{p}} M_p |y|$$

for every $y \in \mathbb{R}^n$, where

$$M_p = \int_{S^{n-1}} \|x\|_p \sigma(dx) \leq c\sqrt{p} N^{\frac{1}{p}-\frac{1}{2}}.$$

This shows that $\|\langle v, y\rangle\|_{L_p(V,\mu(V))} \leq c\sqrt{p}$ for every $y \in S^{n-1}$. The proof of the other inequality is analogous: we now use the fact (first proved by Kashin [Ka]) that, for every $\rho \in (0,1)$, a random $\rho N$-dimensional subspace of $\ell_1^N$ is $C(\rho)$-Euclidean (see also [STJ], or [Pi, Chapter 6]). $\square$

For every $p \geq 1$, we define $p^* = \max\{1, p/2\}$. Then, combining the two estimates in Theorem 4.1 we obtain:

**4.2. Corollary.** *Let $p \geq 1$ and $N \geq cn^{p^*}$. Then, a random $U \in O(N)$ satisfies*

$$\|\langle v, y\rangle\|_{L_p(V,\mu(V))} \leq c\sqrt{p}\ \|\langle v, y\rangle\|_{L_1(V,\mu(V))}$$

*for every $y \in \mathbb{R}^n$, where $V = \{\sqrt{N} P_n U^*(e_i) : i = 1, \ldots, N\}$.* $\square$

Observe that we have a phase transition at $p = 2$, which is a consequence of the corresponding change of behavior in Dvoretzky's theorem for $\ell_p^N$.

$(b)$ Another interpretation of the same fact: Let $N > n$ and consider an orthonormal basis $\{e_i\}_{i\leq N}$ of $\mathbb{R}^N$. For every $n$-dimensional subspace $E_n \in G_{N,n}$, define the vectors

$$w_i = \sqrt{N} P_{E_n}(e_i) \quad , \quad i = 1, \dots, N,$$

and write $W = W(E_n)$ for the set $\{w_1, \dots, w_N\}$. Given $p \geq 1$, the question is to find $N(p)$ such that: if $N \geq N(p)$, then a random $E_n \in G_{N,n}$ satisfies

$$\|\langle w, y\rangle\|_{L^p(W,\mu(W))} \leq c\sqrt{p}\,\|\langle w, y\rangle\|_{L^1(W,\mu(W))}$$

for every $y \in E_n$. The isotropic condition is now coming from the observation that

$$\|\langle w, y\rangle\|^2_{L_2(W,\mu(W))} = \sum_{i=1}^{N} \langle P_{E_n}(e_i), y\rangle^2 = |y|^2$$

for every $E_n \in G_{N,n}$ and $y \in E_n$.

Observe that there is a natural correspondence between the sets $V(U)$ in $(a)$ and the sets $W(E_n)$ in $(b)$: in the first case we project a random orthonormal basis of $\mathbb{R}^N$ onto a fixed $n$-dimensional subspace, while in the second case we project a fixed orthonormal basis onto a random subspace. As expected, the estimates for $N(p)$ in case $(b)$ are similar to the ones in Corollary 4.2:

**4.3. Theorem.** *Let $N \geq N(p)$, where $N(p) \simeq n^{p^*}$, $p^* = \max\{1, p/2\}$. Then, a random $E_n \in G_{N,n}$ satisfies*

$$\|\langle v, y\rangle\|_{L_p(V,\mu(V))} \leq c\sqrt{p}\,\|\langle v, y\rangle\|_{L_1(V,\mu(V))}$$

*for all $y \in E_n$, where $V = \{\sqrt{N} P_{E_n}(e_i) : i = 1, \dots, N\}$.* □

(c) One can also study the minimal value $N(\alpha)$, $\alpha \in [1, 2]$, of $N > n$ for which a random set $V = V(U)$ or $W = W(E_n)$ forms a $\psi_\alpha$-distribution (in the notation of (a) and (b) respectively). The argument will be exactly as in the proof of Theorem 4.1. We will have to use Dvoretzky's theorem for $\ell_{\psi_\alpha}^N$: a direct computation of the quantity $k(\alpha, N) = n(M/b)^2$ where $M = \int_{S^{n-1}} \|x\|_{\ell_{\psi_\alpha}^N} \sigma(dx)$ and $b = \max\{\|x\|_{\ell_{\psi_\alpha}^N} : x \in S^{n-1}\}$, and the fact that $k(\alpha, N)$ determines (up to a constant) the maximal dimension for which a random subspace of $\ell_{\psi_\alpha}^N$ is 4-Euclidean, will give the relation between $N(\alpha)$ and $n$. We have $k(\alpha, N) \simeq (\log N)^{2/\alpha}$, and need

$$k(\alpha, N) \simeq n,$$

therefore $N \simeq \exp(n^{\alpha/2})$:

**4.4 Theorem.** *Let $N \geq c(\delta)\exp(n^{\alpha/2})$. Then,*

(i) *With probability greater than $1 - \delta$, an orthogonal transformation $U \in O(N)$ satisfies*

$$\|\langle v, y\rangle\|_{L_{\psi_\alpha}(V,\mu(V))} \leq c\,\|\langle v, y\rangle\|_{L_1(V,\mu(V))}$$

*for every* $y \in \mathbb{R}^n$*, where* $V = \{\sqrt{N} P_n U^*(e_i) : i = 1, \ldots, N\}$.

(ii) *With probability greater than* $1 - \delta$*, a subspace* $E_n \in G_{N,n}$ *satisfies*

$$\|\langle v, y\rangle\|_{L_{\psi_\alpha}(V,\mu(V))} \leq c\, \|\langle v, y\rangle\|_{L_1(V,\mu(V))}$$

*for all* $y \in E_n$*, where* $V = \{\sqrt{N} P_{E_n}(e_i) : i = 1, \ldots, N\}$. □

# 5 Convex hull of random points inside a convex body: distance estimates

In this Section we consider the following question: Let $K$ be a convex body with centroid at the origin in $\mathbb{R}^n$, and let $\delta \in (0,1)$. We fix $\gamma \in (0,1)$ and choose $N = \exp(\gamma n)$ points $x_1, \ldots, x_N$, uniformly and independently from $K$. The quantity we want to estimate is $\alpha = \alpha(\delta, \gamma)$, the smallest positive number for which

$$\mathrm{co}(x_1, \ldots, x_N) \supset \alpha K$$

with probability greater than $1 - \delta$. We may clearly assume that $K$ is isotropic with centroid at the origin, in which case we can make use of the fact that

$$(*) \qquad \frac{c_1}{n} D_n \subseteq \frac{c_1'}{n} L_K D_n \subseteq K \subseteq (n+1) L_K D_n \subseteq c_2 n^{3/2} D_n.$$

The support function of $K$ is defined by $h_K(y) = \max_{x \in K} \langle x, y\rangle$. We will need the following simple lemma:

**5.1. Lemma.** *Let* $K$ *be an isotropic convex body in* $\mathbb{R}^n$*, with centroid at the origin. For every* $\theta \in S^{n-1}$ *define* $f_\theta(t) = |K \cap (\theta^\perp + t\theta)|$*. Then, for every* $\varepsilon \in (0,1)$ *we have*

$$\int_{\varepsilon h_K(\theta)}^{h_K(\theta)} f_\theta(t)\, dt \geq \frac{c}{n^2}(1 - \varepsilon)^n.$$

*Proof:* By the Brunn-Minkowski inequality $f_\theta^{1/(n-1)}$ is concave, and $f_\theta(s) = 0$ for every $s > h_K(\theta)$. Therefore,

$$f_\theta(t) \geq \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f_\theta(0),$$

and, integrating on $[\varepsilon h_K(\theta), h_K(\theta)]$, we get

$$\int_{\varepsilon h_K(\theta)}^{h_K(\theta)} f_\theta(t)\, dt \geq \frac{f_\theta(0) h_K(\theta)}{n}(1 - \varepsilon)^n.$$

But $f_\theta(0) \geq \|f_\theta\|_\infty / e$ (see [MM]), and $(n+1)\|f_\theta\|_\infty h_K(\theta) \geq |K| = 1$ because $K$ has its centroid at the origin. Hence, the lemma follows. □

**5.2. Theorem.** *Let $\gamma \in (0,1)$ and $K$ be an isotropic convex body with centroid at the origin in $\mathbb{R}^n$. For every $\delta \in (0,1)$, $m = \exp(\gamma n)$ points $x_1, \ldots, x_m$ chosen uniformly and independently from $K$, satisfy with probability greater than $1-\delta$*

$$K \supset \mathrm{co}(x_1, \ldots, x_m) \supset c(\delta)\gamma K.$$

*Proof:* Let $\eta \in (0,1)$ to be determined, and consider an $\eta$-net $\mathcal{N}$ for $S^{n-1}$, with $|\mathcal{N}| \leq \exp(n \log(3/\eta))$. For every $\theta \in \mathcal{N}$ we have

$$\mathrm{Prob}\,(x \in K : \langle x, \theta \rangle < \varepsilon h_K(\theta)) < 1 - \frac{c(1-\varepsilon)^n}{n^2}$$

by Lemma 5.1. Hence, $m$ random points $x_1, \ldots, x_m$ from $K$ will satisfy

$$\max_{j \leq m} \langle x_j, \theta \rangle < \varepsilon h_K(\theta)$$

with probability smaller than

$$\left(1 - \frac{c(1-\varepsilon)^n}{n^2}\right)^m \leq \exp(-cm(1-\varepsilon)^n/n^2).$$

Therefore, if we set $A = \mathrm{co}(x_1, \ldots, x_m)$, we will have with probability greater than $1-\delta$

$$h_A(\theta) \geq \varepsilon h_K(\theta)$$

for all $\theta \in \mathcal{N}$, provided that

$$m \geq c(\delta) \log(3/\eta) n^3 \exp(2\varepsilon n).$$

Then, the triangle inequality and $(*)$ show that

$$h_A(\theta) \geq \left(\varepsilon - \frac{c_2 n^{5/2} \eta}{c_1}\right) h_K(\theta) \geq \frac{\varepsilon}{2} h_K(\theta)$$

for all $\theta \in S^{n-1}$, that is,

$$K \supset A \supset \frac{\varepsilon}{2} K,$$

provided that $\eta \simeq \varepsilon n^{-5/2}$, which gives the restriction $m \geq c(\delta) \log(3n^{5/2}/\varepsilon) n^3 e^{2\varepsilon n}$. Putting $m = \exp(\gamma n)$ and choosing the best $\varepsilon$, we conclude the proof. $\square$

An inspection of the argument above shows that if we want $A$ to be very close to $K$ in the distance sense, we still have an estimate of the number of points needed:

**5.3 Proposition.** *Let $K$ be an isotropic convex body with centroid at the origin in $\mathbb{R}^n$. For every $\delta, \varepsilon \in (0,1)$, $m$ points $x_1, \ldots, x_m$ chosen uniformly and independently from $K$, satisfy with probability greater than $1-\delta$*

$$K \supset \mathrm{co}(x_1, \ldots, x_m) \supset (1-\varepsilon)K,$$

*provided that $m \geq c(\delta)(c/\varepsilon)^n$.* $\square$

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