Записки научных
семинаров ПОМИ
Том 457, 2017 г.

G. Paouris, P. Pivovarov, P. Valettas

GAUSSIAN CONVEX BODIES: A NON-ASYMPTOTIC APPROACH

Abstract. We study linear images of a symmetric convex body $C \subseteq \mathbb{R}^N$ under an $n \times N$ Gaussian random matrix G, where $N \geqslant n$. Special cases include common models of Gaussian random polytopes and zonotopes. We focus on the intrinsic volumes of GC and study the expectation, variance, small and large deviations from the mean, small ball probabilities, and higher moments. We discuss how the geometry of C, quantified through several different global parameters, affects such concentration properties. When $n = 1$, G is simply a $1 \times N$ row vector and our analysis reduces to Gaussian concentration for norms. For matrices of higher rank and for natural families of convex bodies $C_N \subseteq \mathbb{R}^N$, with $N \to \infty$, we obtain new asymptotic results and take first steps to compare with the asymptotic theory.

§1. Introduction

In this paper we study random convex sets that arise as linear images of Gaussian matrices. Specifically, let $G = G(n, N)$ be an $n \times N$ random matrix with independent columns $g_1, \ldots, g_N$ distributed according to the standard Gaussian measure γ_n on $\mathbb{R}^n$. We view $G = [g_1, \ldots, g_N]$ as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$. If $C \subseteq \mathbb{R}^N$ is a compact convex set, then the image of C under G is a random convex set in $\mathbb{R}^n$ given by

$$GC = \left\{ \sum_{i=1}^{N} c_i g_i : c = (c_i) \in C \right\}. \tag{1}$$

We call GC a *Gaussian convex body.* In this way, one can generate random convex hulls, Minkowski sums and a variety of other random convex sets.

Key words and phrases: intrinsic volumes, Gaussian matrices, deviation inequalities, higher moments.

G. P. is supported by the NSF CAREER-1151711 grant; P. P. and P. V. are supported by the NSF grant DMS-1612936. The paper was completed while the authors were in residence at the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, supported by NSF grant DMS-1440140. The hospitality of MSRI and of the organizers of the program on Geometric Functional Analysis and Applications is gratefully acknowledged.

Indeed, if $C = \operatorname{conv}\{e_1, \dots, e_N\}$, i.e., the convex hull of the standard unit vector basis, then one gets a Gaussian polytope

$$P_N := G \operatorname{conv}\{e_1, \dots, e_N\} = \operatorname{conv}\{g_1, \dots, g_N\}. \tag{2}$$

Similarly, when C is the crosspolytope, i.e., $C = B_1^N = \operatorname{conv}\{\pm e_1, ..., \pm e_N\}$, then one gets a symmetric Gaussian polytope

$$K_N := GB_1^N = \operatorname{conv}\{\pm g_1, \dots, \pm g_N\}. \tag{3}$$

When C is the cube, i.e., $C = B_\infty^N = [-1,1]^N$, then GC is just the zonotope generated by the symmetric line segments $[-g_i, g_i] = \{\lambda g_i : |\lambda| \leqslant 1\}$, i.e.,

$$Z_N := G[-1,1]^N = \sum_{i=1}^N [-g_i, g_i]. \tag{4}$$

Random sets of the form (1) arise naturally in several fields, even if they are studied from different perspectives. In stochastic geometry, Gaussian polytopes have been studied extensively and the asymptotic behavior of various functionals is now well-understood. The expectation of the k-th intrinsic volume $V_k(P_N)$, $1 \leqslant k \leqslant n$, satisfies

$$\mathbb{E}V_k(P_N) = \binom{n}{k} \frac{\omega_n}{\omega_{n-k}} (\log N)^{k/2}(1 + o(1)), \tag{5}$$

as $N \to \infty$, which is due to Affentranger [2]. Recently, major advances have been made in understanding the variance. Calka and Yukich [12] proved that

$$\lim_{N\to\infty} \operatorname{var}(V_k(P_N))(\log N)^{\frac{n+3}{2}-k} = c_{n,k}, \tag{6}$$

where $c_{n,k}$ is a finite constant that depends on n and k; while $c_{n,n}$ was proved to be positive, the authors left open the possibility that $c_{n,k} = 0$ if $k < n$. Subsequently, Bárány and Thaele [7] proved that indeed $c_{n,k} > 0$. This provides a complete resolution of the asymptotic behavior of the variance, sharpening previous bounds due to Hug and Reitzner [17] and Bárány and Vu [9]. The latter authors have also proved a central limit theorem for the volume, namely, as $N \to \infty$,

$$\frac{V_n(P_N) - \mathbb{E}V_n(P_N)}{\sqrt{\operatorname{var}(V_n(P_N))}} \xrightarrow{d} N(0,1).$$

With the recent progress on variance of $V_k(P_N)$, central limit theorems for other intrinsic volumes also follow, as explained in [9, 12]. While our focus here is on intrinsic volumes, there is a fruitful line of research connecting the facial structure of Gaussian polytopes and random projections of simplices, e.g., [4,8]; see [12] for further history and references. We also focus on Gaussian measure as opposed to other forms of randomness such as random polytopes in convex bodies; see, e.g., [6,11] and the references therein for the limiting theory and, e.g., [13] and the references therein for a non-asymptotic framework.

In the study of high-dimensional convex bodies, Gaussian matrices, or random orthogonal projections, play an essential role. For example, they arise in probabilistic proofs of Dvoretzky's Theorem. V. Milman's random version of the latter [24] asserts that given any $\varepsilon > 0$, N and convex body $C \subseteq \mathbb{R}^N$, there is a critical dimension $k_*(C)$ and a constant $c = c(\varepsilon)$ such that whenever $k \leq c(\varepsilon)k_*(C)$, "most" rank-$k$ projections of C are essentially Euclidean balls, i.e.,

$$(1-\varepsilon)w(C)P_E B_2^N \subseteq P_E C \subseteq (1+\varepsilon)w(C)P_E B_2^N; \tag{7}$$

here $w(C)$ denotes half of the mean width of C, B_2^N is the Euclidean ball, P_E is the orthogonal projection onto E and the inclusion holds with high probability (with respect to the Haar probability measure $\nu_{N,k}$ on the Grassmannian manifold $G_{N,k}$ of k-dimensional subspaces $E \subseteq \mathbb{R}^N$). The focus in this study is on phenomena that hold for arbitrary convex bodies C, the dimension N is large and the critical dimension $k_*(C)$ grows with N. For example, if C is in Löwner's position, i.e., B_2^N is the minimal volume ellipsoid containing C, then $k_*(C) \geqslant c_1 \log N$, where c_1 is an absolute constant. The latter is sharp for $C = B_1^N$, while for $C = B_\infty^N$ the parameter $k_*(B_\infty^N)$ is proportional to N. For a detailed introduction to this fundamental result and its influence in Asymptotic Geometric Analysis, we refer to the recent book by Artstein-Avidan, Giannopoulos, and V. Milman [1].

Variants of Gaussian polytopes, when N is linear in n, also arise as counter-examples, e.g., Gluskin's theorem which exhibits convex bodies of nearly extremal Banach–Mazur distance [16]; see also the survey [27] for much related work in this direction. Whereas these relate to the shape of such bodies, our interest here is on the intrinsic volumes.

Despite the fact that such sets have been studied from different points of view and in different asymptotic regimes, there are some common underlying probabilistic characteristics. Our aim is to place a family of problems on intrinsic volumes of such sets in a general framework. The goal is to determine how the probabilistic behavior of $V_k(GC)$ reflects the geometry of $C \subseteq \mathbb{R}^N$ and vice-versa. We study the expectation, variance, concentration around the mean, small and large deviations and small ball probabilities. We address these topics in a non-asymptotic setting with precise study of the dependence on C, k,n and N.

All of the above topics are meaningful even when $n=1$. Then G is simply a $1\times N$ row vector, say $G=g$, and $GC=\{\langle g,c\rangle : c\in C\}\subseteq\mathbb{R}$. Sudakov's seminal work on infinite-dimensional Gaussian processes [40] gives as a special case,

$$\mathbb{E}\sup_{x\in C}\langle x,g\rangle=\frac{1}{\sqrt{2\pi}}V_1(C). \tag{8}$$

The latter connects the first intrinsic volume $V_1(C)$ (suitably normalized mean width of C) with the supremum of an associated Gaussian process indexed by C. The aforementioned concentration properties are non-trivial even for the support function of C; for example for the ℓ_p^N-norm when $p=p(N)$, the variance has only been understood in the last several years [21, 32, 35].

For $n>1$, even less is known about higher order concentration properties of GC. Concerning the expectation of $V_n(GC)$, Tsirelson [42] extended Sudakov's identity (8) and it can be formulated as

$$\mathbb{E}V_n(GC)=\mathbb{E}\det(GG^*)^{1/2}\int\limits_{G_{N,n}}V_n(P_EC)d\nu_{N,n}(E). \tag{9}$$

Since the righthand side is a multiple of the n-th intrinsic volume of C, the latter identity is sometimes called the Gaussian representation of intrinsic volumes, e.g., [44]. When $n=1$, then (9) amounts to (8). The latter provides a direct connection between GC and random orthogonal projections of C.

Milman's random version of Dvoretzky's theorem (7) and Tsirelson's identity (9) together imply that the quantity $\mathbb{E}V_k(GC)$, up to normalizing constants, behaves like $w(C)^k$, provided k does not exceed the critical dimension $k_*(C)$. Moreover, for families of convex bodies $C_N\subseteq\mathbb{R}^N$, with $N\to\infty$ and k fixed, the assumption $k\leqslant k_*(C_N)$ is trivially satisfied if $k_*(C_N)\to\infty$, as $N\to\infty$. Thus the above reasoning readily implies laws of large numbers for $V_k(GC_N)/w^k(C_N)$ whenever $k_*(C_N)\to\infty$, $G=G(n,N)$ and $N\to\infty$; this occurs, e.g., when each C_N is in Löwner's position. However, since (7) concerns set inclusions, there is no reason to expect that the rates of convergence for $V_k(GC_N)$ should be precisely determined from (7) alone. Indeed, a more detailed analysis of $V_k(GC)$

involves parameters other than $k_*(C)$. In [29], we show that Alexandrov's fundamental inequality for intrinsic volumes can be reversed beyond what can be expected by the Dvoretzky number. An important parameter in [29], and in previous study of Gaussian concentration of norms as in [32,33,35], is $\beta_*(C)$ defined by

$$\beta_*(C) = \frac{\mathrm{Var}[h_C(g)]}{(\mathbb{E}[h_C(g)])^2}, \tag{10}$$

where h_C denotes the support function of C and g is a standard N-dimensional Gaussian vector in $\mathbb{R}^N$. In this paper, we use these recent tools to prove concentration properties that are sharper than those following from random versions of Dvoretzky's theorem.

Theorem 1.1. *Let C be a symmetric convex body in $\mathbb{R}^N$,*

$$u(x) = \sqrt{x\log(e/x)}, \quad p > 0, \quad \text{and} \quad 2 \leqslant k \leqslant n \leqslant N.$$

If $2 \leqslant k \leqslant c_1\beta_*^{-1}(C)$, then

$$\frac{(\mathbb{E}V_k^p(GC))^{\frac{1}{kp}}}{(\mathbb{E}V_k(GC))^{\frac{1}{k}}} \leqslant \sqrt{1 + c\max\left\{u(k\beta_*(C)), \frac{kp}{nk_*(C)}\right\}}, \tag{11}$$

where c, c_1 are absolute constants. Moreover, if $2 \leqslant k \leqslant c_1\beta_*^{-1}(C)$ and $0 < p \leqslant \frac{c_2}{k\beta_*(C)}$, then

$$\frac{\left(\mathbb{E}V_k^{-p}(GC)\right)^{-\frac{1}{kp}}}{(\mathbb{E}V_k(GC))^{\frac{1}{k}}} \geqslant 1 - c_3\max\{u(k\beta_*(C)), pk\beta_*(C)\}, \tag{12}$$

where c_2, c_3 are absolute constants.

The latter theorem gives reverse Hölder inequalities for $\mathbb{E}V_k(GC)^p$ for both positive and negative powers. By standard arguments, these lead to deviation inequalities, which we state in Subsection 3.3. We also note that the latter theorem provides an immediate counterpart to (5) for general Gaussian convex bodies. Indeed, suppose that $(C_N)_{N=n}^{\infty}$ is a sequence of symmetric convex bodies with each $C_N \subseteq \mathbb{R}^N$, $N = n, n+1, \ldots$, in Löwner's position. If k, n and p are fixed and $N \to \infty$ and we write $G = G(n,N)$, then Theorem 1.1 implies

$$(\mathbb{E}V_k^p(G(n,N)C_N))^{\frac{1}{p}} = \binom{n}{k}\frac{\omega_n}{\omega_{n-k}}N^{k/2}w(C_N)^k(1-o(1)), \tag{13}$$

since, as mentioned, $\mathbb{E}V_k(GC_N)$ is of order $w(C_N)^k$, up to explicit constants. Now, plugging $C_N = B_1^N$ and $p = 1$ into (13) gives the symmetric analogue of (5) and explains why a logarithmic term appears: it corresponds to the mean width of B_1^N. One can check that $w(B_1^N) = \sqrt{\frac{2\log N}{N}}(1-o(1))$; see, e.g., [20] for further references and related interesting questions. In this way, asymptotic expansions for $(\mathbb{E}V_k^p(G(n,N)C_N))^{1/p}$ are directly reduced to those of the mean width of C_N. Recently, Kabluchko and Zaporozhets [19] studied asymptotics for expected intrinsic volumes of K_N, P_N and Z_N; in particular, they further develop connections between Tsirelson's identity (9) and explicit expressions for the intrinsic volumes of the cross-polytope and simplex from [5] and the cube; see also [28]. The expansion in (13) is complementary: it follows from a non-asymptotic approach, for arbitrary symmetric convex bodies C_N and (fixed) powers p. For simplicity, the focus of this paper is symmetric convex bodies C; one can use a regular simplex inscribed in the sphere S^{N-1} to model the Gaussian polytopes P_N but some non-trivial technical steps in our approach need to be modified appropriately.

In the non-asymptotic setting, relations between k, p, n and N affect the limiting behavior. While (13) concerns $N \to \infty$ with k, n and p fixed, we also study when p grows with N. For example, we prove the following comparison of higher moments.

Theorem 1.2. *Let $C \subseteq \mathbb{R}^N$ be a convex body in Löwner's position and $q > p > cN\log N$. Then*

$$c_1\sqrt{\frac{q}{p}} \leqslant \frac{(\mathbb{E}V_k(GC)^q)^{\frac{1}{kq}}}{(\mathbb{E}V_k(GC)^p)^{\frac{1}{kp}}} \leqslant c_2\sqrt{\frac{q}{p}}, \tag{14}$$

where c_1, c_2 are absolute constants.

The next proposition shows that the variance of $V_k(GC)$ can be estimated using the variance of $h_C(g)$, which we state in terms of $\beta_*(C)$.

Theorem 1.3. *Let $C \subseteq \mathbb{R}^N$ be a convex body and let $u(x) = \sqrt{x\log(e/x)}$, $0 < x \leqslant 1$. Then*

$$\mathrm{Var}[V_k(GC)] \leqslant c_1^k N^k \max\left\{\frac{k^2}{nk_*(C)}, ku(k\beta_*(C))\right\} V_k(C)^2,$$

whenever $1 \leqslant k \leqslant \min\{n, c\sqrt{nk_*(C)}, c(\beta_*(C)\log\frac{1}{\beta_*(C)})^{-1/3}\}$*, where* c, c_1 *are absolute constants. Moreover, for all* $1 \leqslant k \leqslant n$*, we have*

$$\mathrm{Var}[V_k(GC)] \geqslant \frac{c_2^k k^2 N^{k-1} V_k(C)^2}{n},$$

where c_2 *is an absolute constant.*

The latter estimates do not appear to be sharp, even when C is the cube. For comparison, we discuss asymptotic results for zonotopes in Section 4. At present, the asymptotic and non-asymptotic regimes are not completely comparable. We feel it is of interest to better understand the transition from the non-asymptotic setting to the asymptotic, for specific bodies like the cube and cross-polytope, but also in general. The latter theorem does, however, present the first general variance estimates, as far as we are aware, for arbitrary Gaussian convex bodies GC. Moreover, the approach that we explore here provides further non-trivial information about higher order concentration properties, which we discuss further in Section 3, including deviation inequalities and small ball probabilities.

§2. Preparatory tools

The setting is $\mathbb{R}^N$ equipped with the standard inner-product $\langle\cdot,\cdot\rangle$ and Euclidean norm $\|x\|_2 := \sqrt{\langle x, x\rangle}$ for $x \in \mathbb{R}^N$; B_2^N is the Euclidean ball of radius 1; S^{N-1} is the unit sphere, equipped with the Haar probability measure σ. For Borel sets $A \subseteq \mathbb{R}^N$, we use $V_N(A)$ or $|A|$ for the Lebesgue measure of A; ω_N for the Lebesgue measure of B_2^N. The Grassmannian manifold of all n-dimensional subspaces of $\mathbb{R}^N$ is denoted by $G_{N,n}$, equipped with the Haar probability measure $\nu_{N,n}$. For a subspace $E \in G_{N,n}$, we write P_E for the orthogonal projection onto E.

Throughout the paper we reserve the symbols $c, c_1, c_2, \ldots$ for absolute constants (not necessarily the same in each occurrence). We use the convention $S \simeq T$ to signify that $c_1 T \leqslant S \leqslant c_2 T$ for some positive absolute constants c_1 and c_2. Our results are most meaningful when N is large and we assume throughout that N exceeds a fixed absolute constant.

A convex body $C \subseteq \mathbb{R}^N$ is a compact, convex set with non-empty interior. The support function of a convex body C is given by

$$h_C(y) = \sup\{\langle x, y\rangle : x \in C\},\ y \in \mathbb{R}^N.$$

We say that C is (origin) symmetric if $C = -C$. For a symmetric convex body C the polar body C° is defined by

$$C^\circ := \{x \in \mathbb{R}^N : |\langle x, y\rangle| \leqslant 1, y \in C\}.$$

For $p \neq 0$, we define the p-generalized mean width of C by

$$w_p(C) := \left(\int\limits_{S^{N-1}} h_C^p(\theta) d\sigma(\theta) \right)^{1/p}. \tag{15}$$

The circumradius of C is defined by

$$R(C) = \max_{\theta \in S^{N-1}} h_C(\theta) = \max_{x \in C} \|x\|_2.$$

Note that $R(C) = w_\infty(C) := \lim_{p\to\infty} w_p(C)$. In addition, we denote by $r(C)$ the inradius of K, i.e. $r(K) = \min_{\theta\in S^{N-1}} h_C(\theta)$. Again, we have: $r(C) = w_{-\infty}(C) := \lim_{p\to\infty} w_{-p}(C)$. Note that $r(C^\circ) = 1/R(C)$.

The intrinsic volumes of a convex body $C \subseteq \mathbb{R}^N$ can be defined via the Steiner formula for the outer parallel volume of C:

$$V_N(C + tB_2^N) = \sum_{k=0}^{N} \omega_k V_{N-k}(C) t^k, \quad t > 0.$$

Here V_k, $k = 1, \ldots, N$, is the k-th intrinsic volume of C (we set $V_0 \equiv 1$); V_N is volume, $2V_{N-1}$ is surface area and $\frac{\omega_{N-1}}{N\omega_N} V_1 = w = w_1$ is half of the mean width (cf. (15)). Intrinsic volumes are also referred to as quermassintegrals (under an alternate labeling and normalization). For further background, see [38, Ch. 4]. Here we prefer to work with a different normalization, similar to that used in [15, 28]. As in the introduction, for a convex body $C \subseteq \mathbb{R}^N$ and $1 \leqslant k \leqslant N-1$, we write

$$W_{[k]}(C) := \left(\frac{1}{\omega_k} \int\limits_{G_{N,k}} V_k(P_E C)\, d\nu_{N,k}(E) \right)^{1/k}.$$

We will need the following generalization of this definition: for $p \neq 0$ we write

$$W_{[k,p]}(C) := \left(\frac{1}{\omega_k^p} \int\limits_{G_{N,k}} V_k(P_E C)^p\, d\nu_{N,k}(E) \right)^{\frac{1}{pk}}.$$

Note that by Kubota's integral formula,

$$V_k(C) = \binom{N}{k} \frac{\omega_N}{\omega_{N-k}} W_{[k]}^k(C). \tag{16}$$

We also set $W_{[N]}(C) = \mathrm{vrad}(C) := \left(\frac{V_N(C)}{V_N(B_2^N)}\right)^{1/N}$, which is the volume radius of C, i.e., the radius of a Euclidean ball with the same volume as C. For ease of reference, we will also explicitly state Alexandrov's inequalities, i.e., for $1 \leqslant n \leqslant N$,

$$w(C) = W_{[1]}(C) \geqslant \cdots \geqslant W_{[n]}(C) \geqslant \cdots \geqslant W_{[N]}(C) = \mathrm{vrad}(C). \tag{17}$$

Recall the Dvoretzky number $k_*(C)$ of $C \subseteq \mathbb{R}^N$ is the maximum dimension k such that a "random" subspace $F \in G_{N,k}$ has the property that $P_F C$ is 4-Euclidean, i.e. $\frac{1}{2} a P_F B_2^N \subseteq P_F C \subseteq 2a P_F B_2^N$, for some $a > 0$. Milman's formula (see [26]) states that

$$k_*(C) \simeq N \frac{w(C)^2}{R(C)^2}. \tag{18}$$

We also recall a definition of Klartag and Vershynin from [18]. For a symmetric convex body $C \subseteq \mathbb{R}^n$, let

$$d_*(C) = \min(-\log \sigma\{\theta \in S^{n-1} : 2\|\theta\|_C \leqslant w(C)\}, n).$$

Let C be a convex body in $\mathbb{R}^N$ with support function $h_C(\cdot)$. We define $\beta_*(C)$ to be the normalized variance of the support function of C with respect to the standard Gaussian measure in $\mathbb{R}^N$, i.e.

$$\beta_*(C) := \frac{\mathrm{var}(h_C(g))}{(\mathbb{E} h_C(g))^2}, \tag{19}$$

where g is an N-dimensional standard Gaussian vector.

If C is a convex body in $\mathbb{R}^N$, then

$$k_*(C) \leqslant \frac{c_1}{\beta_*(C)} \leqslant c_2 d_*(C), \tag{20}$$

where c_1, c_2 are absolute constants. For the above inequalities, see [33] or [34]. In particular, when $k_*(C) \simeq N$, all quantities in (20) are equivalent. The values of $k_*(C)$, $\beta_*(C)$ and $d_*(C)$ are discussed for particular examples in Subsection 3.4.

§3. Gaussian convex bodies

3.1. Gaussian representation of intrinsic volumes and extensions. We recalled Kubota's integral formula in (16). There is a version of the latter formula that uses Gaussian random matrices, rather than orthogonal projections and integration on the Grassmannian, due to Tsirelson [42], sometimes called the Gaussian representation of intrinsic volumes, see the work of Vitale [44]. Throughout this section, we assume that $G = (g_{ij})$ is an $n \times N$ matrix with independent standard Gaussian entries. Then the n-th intrinsic volume of $C \subset \mathbb{R}^N$ is given by

$$V_n(C) = \frac{(2\pi)^{n/2}}{\omega_n n!} \mathbb{E} V_n(GC). \tag{21}$$

An extension of the previous representation involving $W_{[n,p]}(C)$ is proved in [28]. In particular it was shown that if $C \subset \mathbb{R}^N$ is a convex body and $p > -(N-n+1)$, then

$$(\mathbb{E} V_n(GC)^p)^{\frac{1}{p}} = (\mathbb{E} \det (GG^*)^{\frac{p}{2}})^{\frac{1}{p}} W^n_{[n,p]}(C)\omega_n. \tag{22}$$

The latter is based on the fact that

$$V_n(GC) = \det(GG^*)^{1/2} V_n(P_E C),$$

where $E = \mathrm{Im}(G^*)$. Moreover, E is distributed uniformly on $G_{N,n}$ and $\det(GG^*)$ and $V_n(P_E C)$ are independent; see [42] or [30]. The main result of this subsection is the following generalization of (22).

Proposition 3.1. *Let $1 \leqslant k \leqslant n \leqslant N$ and let Γ be a $k \times N$ matrix with independent standard Gaussian entries. Let $C \subset \mathbb{R}^N$ be a convex body. Then for all $p > -(N-k+1)$,*

$$\mathbb{E} W^{kp}_{[k,p]}(GC) = \mathbb{E} \det(\Gamma\Gamma^*)^{\frac{p}{2}} W^{kp}_{[k,p]}(C). \tag{23}$$

In particular,

$$\mathbb{E} \det(\Gamma\Gamma^*)^{p/2} = \mathbb{E} W^{kp}_{[k,p]}(GB_2^N). \tag{24}$$

Note that when $k = n$, (23) amounts to (22).

Proof. Let $g_1, \dots, g_N$ denote the columns of G. For any subspace $E \in G_{n,k}$, define Γ_E by

$$\Gamma_E := P_E G = [P_E g_1 \cdots P_E g_N]$$

and note that Γ_E and Γ have the same distribution. Thus using Fubini's theorem and (22), we have

$$\begin{aligned}\mathbb{E}W_{[k,p]}^{kp}(GC) &= \omega_k^{-1}\mathbb{E}\int\limits_{G_{n,k}} V_k(P_E GC)^p d\nu_{n,k}(E)\\ &= \omega_k^{-1}\int\limits_{G_{n,k}} \mathbb{E}V_k(\Gamma_E C)^p d\nu_{n,k}(E)\\ &= \int\limits_{G_{n,k}} \mathbb{E}\det(\Gamma_E\Gamma_E^*)^{\frac{p}{2}} d\nu_{n,k}(E) W_{[k,p]}^{kp}(C)\\ &= \mathbb{E}\det(\Gamma\Gamma^*)^{\frac{p}{2}} W_{[k,p]}^{kp}(C). \qquad \square\end{aligned}$$

The random matrix GG^* is distributed according to the Wishart distribution. Formulae for the expectation of $\det(GG^*)$ are well known (see e.g. [3, Chapter 7]). We will make use of the following concentration inequality for $\det(GG^*)$; a proof is given for the reader's convenience.

Proposition 3.2. *Let $n \leqslant N/2$ and $0 < p \leqslant \frac{N}{4}$. Then*

$$1 + \frac{c_1 p}{N} \leqslant \left(\mathbb{E}[\det(GG^*)]^{\frac{p}{2}}\right)^{\frac{1}{pn}} \left(\mathbb{E}[\det(GG^*)]^{-\frac{p}{2}}\right)^{\frac{1}{pn}} \leqslant 1 + \frac{c_2 p}{N}, \tag{25}$$

where $c_1, c_2 > 0$ are absolute constants. Furthermore, for $p \geqslant 2$,

$$\sqrt{1 + \frac{c_1 p}{N}} \leqslant \frac{(\mathbb{E}\det(GG^*)^{p/2})^{\frac{1}{pn}}}{(\mathbb{E}\det(GG^*)^{1/2})^{\frac{1}{n}}} \leqslant \sqrt{1 + \frac{c_2 p}{N}}. \tag{26}$$

Moreover, we have

$$1 - \frac{n}{N} \leqslant \frac{(\mathbb{E}\det(GG^*))^{\frac{1}{n}}}{\mathbb{E}\|g\|_2^2} \leqslant 1. \tag{27}$$

Finally, we have

$$\mathbb{E}(\det(GG^*)^{1/2})^{1/n} \simeq \sqrt{N}, \quad \mathrm{Var}[(\det(GG^*)^{1/2})^{1/n}] \simeq 1. \tag{28}$$

Proof. Let $d \geqslant 1$, $q \in [-\frac{d}{2}, \infty)$ and let $a_{d,q} := (\mathbb{E}\|g\|_2^q)^{\frac{1}{q}}$, where g is a d-dimensional Gaussian vector. A straightforward computation shows that

$$a_{d,q} := \sqrt{2}\left(\frac{\Gamma(\frac{d+q}{2})}{\Gamma(\frac{d}{2})}\right)^{\frac{1}{q}}.$$

Thus, for $0 < q \leqslant \frac{d}{2}$,[1]

$$\frac{(\mathbb{E}\|g\|_2^q)^{\frac{1}{q}}}{\left(\mathbb{E}\|g\|_2^{-q}\right)^{-\frac{1}{q}}} = \frac{a_{d,q}}{a_{d,-q}} = \left(\frac{\Gamma(\frac{d+q}{2})\Gamma(\frac{d-q}{2})}{(\Gamma(\frac{d}{2}))^2}\right)^{\frac{1}{q}} = 1 + \Theta\left(\frac{q}{d}\right). \tag{29}$$

Let $h_1, \cdots, h_n$ be the columns of G^*. Let $H_0 = \{0\}$ and for $k = 1, \cdots, n-1$, set $H_k := \operatorname{span}\{h_1, \cdots, h_k\}$. Then, as in e.g. [3, Chapter 7], we may write:

$$\det(GG^*)^{\frac{p}{2}} = \prod_{k=1}^{n} \|P_{H_{k-1}^{\perp}} h_k\|_2^p.$$

Integrating first with respect to h_n, then h_{n-1}, and so forth we get that:

$$\mathbb{E}\det(GG^*)^{\frac{p}{2}} = \prod_{k=1}^{n} a_{N-k+1,p}^p. \tag{30}$$

Hence using (29) for $q = p$ and $d := N - k + 1 \geqslant \frac{N}{2}$ we obtain:

$$\begin{aligned}
\frac{\left(\mathbb{E}\det(GG^*)^{\frac{p}{2}}\right)^{\frac{1}{pn}}}{\left(\mathbb{E}\det(GG^*)^{-\frac{p}{2}}\right)^{-\frac{1}{pn}}} &= \left(\prod_{k=1}^{n} \frac{a_{N-k+1,p}^p}{a_{N-k+1,-p}^p}\right)^{\frac{1}{pn}} \\
&= \left(\prod_{k=1}^{n} \left\{1 + \Theta\left(\frac{p}{N-k+1}\right)\right\}^p\right)^{\frac{1}{pn}} \\
&= 1 + \Theta\left(\frac{p}{N}\right).
\end{aligned}$$

This proves (25). Arguing similarly, we have for $q \geqslant 2$,

$$\left(\frac{a_{d,q}}{a_{d,1}}\right)^2 = 1 + \Theta\left(\frac{q}{d}\right).$$

Taking into account the above estimate and (30) we also get (26).

[1]The last asymptotic estimate follows from Taylor's theorem for the function $h \mapsto G_x(h) := \psi(x+h) + \psi(x-h) - 2\psi(x)$, i.e. there exists $0 < \xi_h < h$ such that

$$G_x(h) = \frac{h^2}{2}\left[\psi''(x+\xi_h) + \psi''(x-\xi_h)\right],$$

where $x = d/2$, $h = q/2$, $\psi(x) := \log\Gamma(x)$ and $(\log\Gamma)''(x) = \sum_{j=0}^{\infty}(x+j)^{-2} \simeq 1/x$ for $x \gg 1$.

In order to prove (27) we note that (30) for $p=2$ implies that

$$\mathbb{E}\det(GG^*) = \prod_{k=1}^{n}(N-k+1).$$

So,

$$\frac{(\mathbb{E}\det(GG^*))^{\frac{1}{n}}}{\mathbb{E}\|g\|_2^2} = \left(\frac{\prod_{k=1}^{n}(N-k+1)}{N^n}\right)^{\frac{1}{n}}.$$

The result follows.

□

3.2. General variance estimates. It is proven in [34, See Claim in the proof of Lemma 6.1] that for any convex body C in $\mathbb{R}^N$,

$$\mathrm{Var}[w(GC)] \leqslant \min\left\{\mathrm{Var}[h_C(Z)], \frac{R(C)^2}{n}\right\}.$$

In [34] it is stated for symmetric convex bodies but an inspection of the proof shows that the symmetry assumption is not essential. Moreover, we have the following lower bound:

$$\mathrm{Var}[w(GC)] \geqslant \frac{c_1 w(C)^2}{n}.$$

Indeed, integration in polar coordinates and Cauchy–Schwarz inequality yields

$$\mathrm{Var}[w(GC)] \geqslant \left(\frac{\mathbb{E}w(GC)}{\mathbb{E}\|G\|_{\mathrm{HS}}}\right)^2 \mathrm{Var}[\|G\|_{\mathrm{HS}}].$$

Recall that $\mathbb{E}\|G\|_{\mathrm{HS}} \simeq \sqrt{nN}$, $\mathbb{E}w(GC) \simeq \sqrt{N}w(C)$ and $\mathrm{Var}(\|G\|_{\mathrm{HS}}) \simeq 1$.

Lemma 3.3. *Let C be a symmetric convex body in $\mathbb{R}^N$ and let $p>0$. Then*

$$(\mathbb{E}[w(GC)]^p)^{\frac{1}{p}} \leqslant \sqrt{1+\frac{c_1 p}{nk_*(C)}}\,\mathbb{E}w(GC) = \sqrt{1+\frac{c_1 p}{nk_*(C)}}\,\mathbb{E}\|g\|_2 w(C), \tag{31}$$

where g is a standard N-dimensional Gaussian vector.

Proof. We will need the following:

Claim. For any $n\times N$ matrices T_1, T_2 with rank n we have

$$|w(T_1C) - w(T_2C)| \leqslant \frac{R(C)}{\sqrt{n}}\|T_1 - T_2\|_{\mathrm{HS}}. \tag{32}$$

Proof of Claim. Since $w(TC) = \int_{S^{n-1}} h_C(T^*\theta)\,d\sigma(\theta)$, by the triangle inequality we get

$$\begin{aligned}|w(T_1C) - w(T_2C)| &\leqslant \int_{S^{n-1}} h_C((T_1^* - T_2^*)\theta)\,d\sigma(\theta)\\ &\leqslant R(C)\int_{S^{n-1}} \|(T_1^* - T_2^*)\theta\|_2\,d\sigma(\theta)\\ &\leqslant \frac{R(C)}{\sqrt{n}}\|(T_1 - T_2)^*\|_{\mathrm{HS}},\end{aligned}$$

which proves the claim.

By the claim, the function $\mathbb{R}^{Nn} \ni T \mapsto w(TC)$ is Lipschitz with constant $R(C)/\sqrt{n}$. By [29, Proposition 3.3] and the standard Gaussian concentration, we get the lemma. □

Proposition 3.4. *Let C be a symmetric convex body in $\mathbb{R}^N$ and let $1 \leqslant k \leqslant n \leqslant N$.*

i. *For all $1 \leqslant k \leqslant n$,*

$$\mathrm{Var}[W_{[k]}^k(GC)] \leqslant \mathrm{Var}[W_{[k]}^k(\Gamma C)] = \mathrm{Var}[\mathrm{vrad}(\Gamma C)^k],$$

where Γ is a Gaussian $k \times N$ matrix.

ii. *For $2 \leqslant k \leqslant \min\{n, c\sqrt{nk_*(C)}, c(\beta_*(C)\log\frac{1}{\beta_*(C)})^{-1/3}\}$, we have*

$$\mathrm{Var}[W_{[k]}^k(GC)] \leqslant \max\left\{\frac{k^2}{nk_*(C)}, ku(k\beta_*(C))\right\} W_{[k]}^{2k}(C)t_{N,k},$$

where $t_{N,k} := (\mathbb{E}\det(\Gamma\Gamma^*)^{1/2}]^2 = 2^k[\Gamma(\frac{N-k+2}{2})/\Gamma(\frac{N}{2})]^2$ so that $t_{N,k}^{1/k} \simeq N$ and $u(x) = \sqrt{x\log(e/x)}$, $0 < x \leqslant 1$.

iii. *For all $1 \leqslant k \leqslant n$, we have*

$$\mathrm{Var}[W_{[k]}^k(GC)] \geqslant W_{[k]}^{2k}(C)\frac{k^2}{nN}t_{N,k}.$$

Proof. Part (i) follows from the Cauchy–Schwarz inequality. Indeed, if G, G' are independent $n \times N$ Gaussian matrices, then

$$2\omega_k^2 \mathrm{Var} W_{[k]}^k(GC) = \mathbb{E}\left(\int_{G_{n,k}} V_k(P_F GC)\, d\nu_{n,k}(F) - \int_{G_{n,k}} V_k(P_F G'C)\, d\nu_{n,k}(F)\right)^2$$
$$\leqslant \int_{G_{n,k}} \mathbb{E}\left(V_k(P_F GC) - V_k(P_F G'C)\right)^2 \, d\nu_{n,k}(F).$$

But for fixed $F \in G_{n,k}$ the matrices $P_F G$ and $P_F G'$ are independent and each has the same distribution as a Gaussian $k \times N$ matrix Γ. Thus, we may write:

$$2\omega_k^2 \mathrm{Var}[W_{[k]}^k(GC)] \leqslant \int_{G_{n,k}} \mathbb{E}\left(V_k(\Gamma C) - V_k(\Gamma' C)\right)^2 \, d\nu_{n,k}(F) = \mathrm{Var}[V_k(\Gamma C)].$$

(ii) Using Alexandrov's inequality (17) and Lemma 3.3, we have

$$\mathbb{E} W_{[k]}^{2k}(GC) \leqslant \mathbb{E}(w(GC))^{2k} \leqslant \left(1 + \frac{ck}{nk_*(C)}\right)^k (\mathbb{E} w(GC))^{2k}$$
$$= \left(1 + \frac{ck}{nk_*(C)}\right)^k (\mathbb{E}\|g\|_2)^{2k} w(C)^{2k},$$

where $g \sim N(0, I_N)$.

Next, we also use our recent reverse form of Alexandrov's inequality [29, Theorem 1.1], which states that $W_{[k]}(C)$ is very close to $w(C)$ for all k up to $c\beta_*^{-1}(C)$. In particular, it was proved that

$$W_{[k]}(C) \geqslant (1 - cu(k\beta_*(C)))\, w(C), \tag{33}$$

where $u(x) := \sqrt{x \log(e/x)}$. Thus

$$\mathbb{E} W_{[k]}^k(GC) = \mathbb{E}\det(\Gamma\Gamma^*)^{1/2} W_{[k]}^k(C)$$
$$\geqslant \mathbb{E}\det(\Gamma\Gamma^*)^{1/2} (1 - cu(k\beta_*(C)))^k w(C)^k,$$

as long as $k \leqslant c/\beta_*(C)$. It follows that

$$\frac{\mathbb{E}[W_{[k]}^{2k}(GC)]}{(\mathbb{E} W_{[k]}^k(GC))^2} \leqslant \left(1 + c\max\left\{\frac{k}{nk_*(C)}, u(k\beta_*(C))\right\}\right)^k \frac{(\mathbb{E}\|g\|_2)^{2k}}{(\mathbb{E}\det(\Gamma\Gamma^*)^{1/2})^2}.$$

Finally, we may check that

$$\frac{(\mathbb{E}\|g\|_2)^{2k}}{(\mathbb{E}\det(\Gamma\Gamma^*)^{1/2})^2} = \frac{a_{N,1}^{2k}}{\prod_{s=1}^k a_{N-s+1,1}^2}$$
$$= \left[\frac{\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2})}\right]^{2k}\left[\frac{\Gamma(\frac{N-k+1}{2})}{\Gamma(\frac{N+1}{2})}\right]^2 \leqslant \left(1+\frac{ck}{N}\right)^k,$$

for all $1 \leqslant k \leqslant n$. Putting all inequalities together we obtain:

$$\frac{\mathbb{E}[W_{[k]}^{2k}(GC)]}{(\mathbb{E}W_{[k]}^k(GC))^2} \leqslant \left(1 + c\max\left\{\frac{k}{nk_*(C)}, u(k\beta_*(C))\right\}\right)^k$$
$$\leqslant 1 + c_1\max\left\{\frac{ck^2}{nk_*(C)}, ku(k\beta_*(C))\right\},$$

as long as $\max\{\frac{k^2}{nk_*(C)}, ku(k\beta_*(C))\} \leqslant 1$. Hence,

$$\mathrm{Var}[W_{[k]}^k(GC)] = (\mathbb{E}W_{[k]}^k(GC))^2\left(\frac{\mathbb{E}[W_{[k]}^{2k}(GC)]}{(\mathbb{E}W_{[k]}^k(GC))^2} - 1\right)$$
$$\leqslant c_1\max\left\{\frac{k^2}{nk_*(C)}, ku(k\beta_*(C))\right\} W_{[k]}^{2k}(C)[\det(\Gamma\Gamma^*)^{1/2}]^2,$$

for all $1 \leqslant k \leqslant \min\left\{n, c\sqrt{nk_*(C)}, c\sqrt[3]{\frac{1}{\beta_*(C)}/\log\frac{1}{\beta_*(C)}}\right\}$.

(iii) We apply Hölder's inequality to get

$$\mathrm{Var}[W_{[k]}^k(GC)] = (\mathbb{E}W_{[k]}^k(GC))^2\left(\frac{\mathbb{E}[W_{[k]}^{2k}(GC)]}{(\mathbb{E}W_{[k]}^k(GC))^2} - 1\right)$$
$$\geqslant \mathbb{E}W_{[k]}^k(GC))^2\left(\frac{\mathbb{E}\|G\|_{\mathrm{HS}}^{2k}}{(\mathbb{E}\|G\|_{\mathrm{HS}}^k)^2} - 1\right)$$
$$= W_{[k]}^{2k}(C)\frac{k^2}{nN}[\det(\Gamma\Gamma^*)^{1/2}]^2. \qquad \square$$

3.3. Concentration properties. In this section we discuss concentration properties of the intrinsic volumes of Gaussian convex bodies. In addition to [29, Theorem 1.1], which we stated in (33), we will also need the following stronger statement (see Proposition 4.2 in [29]).

Proposition 3.5. *Let C be a symmetric convex body in $\mathbb{R}^n$ and let $2 \leqslant k \leqslant c_1\beta_*^{-1}(C)$. Then for all $0 < p \leqslant c_2(k\beta_*(C))^{-1}$,*

$$W_{[k,-p]}(C) \geqslant (1 - c_3 \max\{u(k\beta_*(C)), pk\beta_*(C)\})\, w(C), \tag{34}$$

where $u(x) := \sqrt{x\log(e/x)}$.

Proposition 3.6. *Let C be a symmetric convex body in $\mathbb{R}^N$, $u(x) = \sqrt{x\log(e/x)}$, $p > 0$ and $1 \leqslant k \leqslant n \leqslant N$. Then*

$$\left(\mathbb{E}W_{[k]}^{kp}(GC)\right)^{\frac{1}{kp}} \leqslant \sqrt{1 + \frac{c_1 kp}{nk_*(C)}}\mathbb{E}\|g\|_2 w(C), \tag{35}$$

where g is a standard N-dimensional Gaussian vector. Moreover, if $2 \leqslant k \leqslant c_1\beta_*^{-1}(C)$, then

$$\mathbb{E}W_{[k]}(GC) \geqslant (1 - c_2 u(k\beta_*(C))\, \mathbb{E}\det(\Gamma\Gamma^*)^{\frac{1}{2k}}\, w(C), \tag{36}$$

where Γ is an $k \times N$ Gaussian matrix. In particular we have that for every $2 \leqslant k \leqslant c_1\beta_*^{-1}(C)$ and $p > 0$,

$$\frac{\left(\mathbb{E}W_{[k]}^{kp}(GC)\right)^{\frac{1}{kp}}}{\mathbb{E}W_{[k]}(GC)} \leqslant \sqrt{1 + c\max\left\{u(k\beta_*(C)), \frac{kp}{nk_*(C)}\right\}}. \tag{37}$$

Finally we have that for $2 \leqslant k \leqslant c_1\beta_*^{-1}(C)$ and $0 < p \leqslant \frac{c_2}{k\beta_*(C)}$,

$$\frac{\left(\mathbb{E}W_{[k]}^{-kp}(GC)\right)^{\frac{1}{-kp}}}{\mathbb{E}W_{[k]}(GC)} \geqslant 1 - c\max\{u(k\beta_*(C)), pk\beta_*(C)\}. \tag{38}$$

Proof. By Alexandrov's inequality (17) and Lemma 3.3 we get that

$$\left(\mathbb{E}W_{[k]}^{kp}(GC)\right)^{\frac{1}{kp}} \leqslant \left(\mathbb{E}[w(GC)]^{kp}\right)^{\frac{1}{kp}} \leqslant \sqrt{1 + \frac{c_1 kp}{nk_*(C)}}\mathbb{E}\|g\|_2 w(C),$$

which proves (35). Using Proposition 3.1, Hölder's inequality and Proposition 3.5 we get that

$$\begin{aligned}
\mathbb{E}W_{[k]}(GC) &\geqslant \mathbb{E}W_{[k,\frac{1}{k}]}(GC) \\
&= \mathbb{E}\det(\Gamma\Gamma^*)^{\frac{1}{2k}}\, W_{[k,\frac{1}{k}]}(C) \\
&\geqslant (1 - c_2 u(k\beta_*(C))\, \mathbb{E}\det(\Gamma\Gamma^*)^{\frac{1}{2k}}\, w(C),
\end{aligned}$$

which proves (36). Next, (37) follows from (36), (35) and Proposition 3.2. Finally, using Hölder's inequality, Alexandrov's inequality (17), followed by Propositions 3.1, 3.2 and 3.5, we have

$$\begin{aligned}\frac{(\mathbb{E}W_{[k]}^{-pk}(GC))^{-\frac{1}{pk}}}{\mathbb{E}W_{[k]}(GC)} &\geqslant \frac{(\mathbb{E}W_{[k,-p]}^{-pk}(GC))^{-\frac{1}{pk}}}{\mathbb{E}w(GC)} \\ &\geqslant \frac{(\mathbb{E}(\det(\Gamma\Gamma^*))^{-\frac{p}{2}})^{-\frac{1}{kp}}}{\mathbb{E}\|g\|_2}\frac{W_{[k,-p]}(C)}{w(C)} \\ &\geqslant \left(1-\frac{c\max(p,k)}{N}\right) \\ &\times (1-c_3\max\{u(k\beta_*), pk\beta_*(C)\})\, w(C). \qquad \square\end{aligned}$$

Applying the previous proposition and Markov's inequality, we get the following.

Proposition 3.7. *(Small deviations near the mean) Let C be a symmetric convex body in $\mathbb{R}^N$ and $2 \leqslant k \leqslant \frac{c}{\beta_*(C)}$. Set $u(x) = \sqrt{x\log(e/x)}$ and $r := u(k\beta_*(C))$. Then for every $\varepsilon \geqslant r$,*

$$\mathbb{P}\left(W_{[k]}(GC) \geqslant (1+\varepsilon)\mathbb{E}W_{[k]}(GC)\right) \leqslant e^{-c\varepsilon^2 nk_*(C)}. \tag{39}$$

Moreover, for every $r \leqslant \varepsilon \leqslant 1$,

$$\mathbb{P}\left(W_{[k]}(GC) \leqslant (1-\varepsilon)\mathbb{E}W_{[k]}(GC)\right) \leqslant e^{-\frac{c\varepsilon^2}{\beta_*(C)}}. \tag{40}$$

Proposition 3.8. *(Small ball probabilities below the mean) Let $1 \leqslant k \leqslant n \leqslant N$. Assume that C is a symmetric convex body in $\mathbb{R}^N$. Then for all $0 < \varepsilon < 1$,*

$$\mathbb{P}\left(W_{[k]}(GC) \leqslant c\varepsilon\mathbb{E}W_{[k]}(GC)\right) \leqslant \varepsilon^{d_*(C)}, \tag{41}$$

where c is an absolute constant.

Proof. Let $0 < p < N-k+1$. We first recall an inequality from [15, p. 1014]:

$$W_{[k,-p]}(C) \geqslant c_0 w_{-kp}(C), \tag{42}$$

where c_0 is a positive absolute constant. By Alexandrov's inequality (17),

$$\mathbb{E}W_{[k]}(GC) \leqslant \mathbb{E}w(GC) \simeq \sqrt{N}w(C). \tag{43}$$

By Hölder's inequality and Proposition 3.1, we get

$$\mathbb{E}W_{[k]}^{-kp}(GC) \leqslant \mathbb{E}W_{[k,-p]}^{-kp}(GC) = \mathbb{E}\det(\Gamma\Gamma^*)^{-p/2}W_{[k,-p]}^{-kp}(C). \tag{44}$$

Thus,

$$\mathbb{E}W_{[k]}^{-kp}(GC) \leqslant c_1^{kp} N^{-kp/2} w_{-kp}^{-kp}(C).$$

Finally, we apply Markov's inequality with $p = d_*(C)/k$.

$$\begin{aligned}\mathbb{P}\left(W_{[k]}(C) \leqslant c\varepsilon \mathbb{E}W_{[k]}(GC)\right) &= \mathbb{P}\left(W_{[k]}^{-kp}(C) \geqslant (c\varepsilon)^{-kp}(\mathbb{E}W_{[k]}(GC))^{-kp}\right) \\ &= (c\varepsilon)^{kp}(\mathbb{E}W_{[k]}(GC))^{kp}\mathbb{E}W_{[k]}^{-kp}(GC) \\ &\leqslant \varepsilon^{kp}\end{aligned}$$

for a suitable absolute constant c. □

In [28], a general method to estimate small ball probabilities for the volume of random convex sets is developed. The method applies beyond Gaussian convex bodies – to random sets similar to (1) but the randomness involves arbitrary continuous distributions with bounded densities. An essential ingredient is the use of affine quermassintegrals, which were introduced by E. Lutwak [22]. For a compact set $C \subseteq \mathbb{R}^N$ and $1 \leqslant k \leqslant N$, we define the k-th affine quermassintegral as

$$\Phi_{[k]}(C) := \left(\int_{G_{N,k}} V_k(P_F C)^{-N} d\nu_{N,k}(F) \right)^{-\frac{1}{Nk}}. \tag{45}$$

By Hölder's inequality,

$$\omega_k^{1/k} W_{[k]}(C) \geqslant \Phi_{[k]}(C). \tag{46}$$

A crucial inequality from [28], that will also be needed here, is an "isomorphic" solution to a conjecture of E. Lutwak [23], namely, for every convex body $C \subseteq \mathbb{R}^N$ and every $1 \leqslant k \leqslant N$,

$$\Phi_{[k]}(C) \geqslant c\sqrt{\frac{N}{k}} V_N(C)^{\frac{1}{N}}. \tag{47}$$

Proposition 3.9. *(Small ball probabilities below the volume) Let C be a convex body in $\mathbb{R}^N$ and let $1 \leqslant k \leqslant n \leqslant N$. Then for every $\varepsilon > 0$,*

$$\mathbb{P}\left(W_{[k]}(GC) \leqslant c\varepsilon V_N(C)^{\frac{1}{N}}\right) \leqslant (c\varepsilon)^{\frac{kN}{2}}. \tag{48}$$

Proof. As in the proof of the previous proposition, we have

$$\mathbb{E}W_{[k]}^{-kp}(GC) \leqslant \mathbb{E}\det(\Gamma\Gamma^*)^{-p/2} W_{[k,-p]}^{-kp}(C). \tag{49}$$

Next, we note that for $0 < p \leqslant N$, Hölder's inequality and (47) imply that

$$W_{[k,-p]}^{-kp}(C) \leqslant \Phi_{[k]}(C)^{-p/N} \leqslant (c\omega_k^{1/k}\, \mathrm{vrad}(C))^{-p/N}.$$

By Markov's inequality, we get the result. □

By compactness, $\omega_n W_{n,p}^n(C) \to \max_{E\in G_{N,n}} V_n(P_E C)$ as $p \to \infty$. The next proposition provides a quantitative form of this fact.

Proposition 3.10. *Let C be a symmetric convex body in $\mathbb{R}^N$ and let $1 \leqslant n \leqslant N/2$. Then, for all $p \geqslant cN\log(R(C)/r(C))$ we have:*

$$c \max_{F\in G_{N,n}} V_n(P_F C)^{1/n} \leqslant \omega_n^{1/n} W_{[n,p]}(C) \leqslant \max_{F\in G_{N,n}} V_n(P_F C)^{1/n}.$$

For the proof we will need estimates for the metric entropy on the Grassmannian which are due to Szarek [41] (see also [31] for the formulation we use below).

Lemma 3.11. *Let $1 \leqslant n \leqslant N-1$ and let σ_∞ be the invariant metric defined by:*

$$E, F \in G_{N,n}, \quad \sigma_\infty(E,F) = \inf\{\|I-U\|_{\mathrm{op}} : U(E) = F,\, U \in O(N)\}.$$

We define the "spherical cap" with respect to σ_∞ of radius $\varepsilon > 0$, centered at $F \in G_{N,n}$, as follows:

$$C_\infty(F,\varepsilon) = \{E \in G_{N,n} : \sigma_\infty(E,F) < \varepsilon\}.$$

Then, we have

$$\nu_{N,n}(C_\infty(F,\varepsilon)) \geqslant (c\varepsilon)^{n(N-n)}. \tag{50}$$

Lemma 3.12. *Let K be a symmetric convex body with $s = R(C)/r(C)$ in $\mathbb{R}^N$. Let $E, F \in G_{N,n}$ with $\sigma_\infty(E,F) = t$. Then, there exists $U \in O(N)$ such that*

$$U(E) = F \quad \textit{and} \quad U(P_E K) \subseteq (1+ts)P_F K.$$

Proof. We consider $U \in O(N)$ such that $t = \|I-U\|$. Let $\theta \in S_F$. Then, $U^*\theta = \phi \in S_E$ therefore we have $\|\theta - \phi\|_2 \leqslant t$. We may write:

$$\frac{h_{U(P_E K)}(\theta)}{h_{P_F K}(\theta)} = \frac{h_K(\phi)}{h_K(\theta)} \leqslant 1 + \frac{h_K(\theta-\phi)}{h_K(\theta)} \leqslant 1 + \frac{tR(K)}{r(P_F K)}.$$

Since $r(P_F K) \geqslant r(K)$ the result follows. □

Now we are ready to prove the aforementioned result:

Proof of Proposition 3.10. The rightmost inequality is trivial. For the leftmost inequality let $F_0 \in G_{N,n}$ such that $\max_{F\in G_{N,n}} V_n(P_F C) = V_n(P_{F_0} C)$. For any $\varepsilon \in (0,1)$ note that if $F \in C_\infty(F_0,\varepsilon)$, then by Lemma 3.12 we get $V_n(P_F C) \geqslant (1+\varepsilon s)^{-n} V_n(P_{F_0} C)$, where $s = R(C)/r(C)$. Hence, we may write:

$$\begin{aligned}\int_{G_{N,n}} V_n(P_F C)^p \, d\nu_{N,n}(F) &\geqslant \int_{C_\infty(F_0,\varepsilon)} V_n(P_F C)^p \, d\nu_{N,n}(F)\\ &\geqslant (1+\varepsilon s)^{-pn} V_n(P_{F_0} C)^p \nu_{N,n}(C_\infty(F_0,\varepsilon))\\ &\geqslant (1+\varepsilon s)^{-pn} (c\varepsilon)^{nN} V_n(P_{F_0} C)^p,\end{aligned}$$

where in the last step we have used Lemma 3.11. Choosing $\varepsilon \simeq 1/s$ we find

$$W_{[n,p]}(C) \geqslant c_1 V_n(P_{F_0} C)^{1/n} (c_2/s)^{N/p} \geqslant c_3 V_n(P_{F_0} C)^{1/n},$$

provided that $p \geqslant N \log(c's)$, which is the desired result. □

The next proposition is immediate:

Proposition 3.13. *(Higher moments) Let C be a symmetric convex body in $\mathbb{R}^N$ and let $1 \leqslant k \leqslant n \leqslant N/2$. Then, for any $p \geqslant cN\log(R(C)/r(C))$,*

$$\frac{(\mathbb{E}W_{[k]}^p(GC))^{\frac{1}{kp}}}{(\mathbb{E}W_{[k]}(GC))} \simeq \sqrt{\frac{p}{N}} \frac{\max_{F\in G_{N,k}} V_k(P_F C)^{\frac{1}{k}}}{W_{[k]}(C)}. \tag{51}$$

Proof. Using Proposition 3.1, we get

$$\frac{(\mathbb{E}W_{[k]}^p(GC))^{\frac{1}{kp}}}{(\mathbb{E}W_{[k]}(GC))} = \frac{(\mathbb{E}\det(\Gamma\Gamma^*)^{\frac{p}{2}})^{\frac{1}{pk}}}{(\mathbb{E}\det(\Gamma\Gamma^*)^{\frac{1}{2}})^{\frac{1}{k}}} \cdot \frac{W_{[k,p]}(C)}{W_{[k]}(C)}.$$

The assertion now follows from Proposition 3.10 and estimate (26). □

3.4. Gaussian polytopes and zonotopes. In the previous sections, we presented general concentration properties of the quantities $W_{[k]}(GC)$ in terms of the parameters $k_*(C)$, $d_*(C)$ and $\beta_*(C)$ and $W_{[k]}(C)$. In this section, we review bounds on all of the latter quantities when C is the cross-polytope B_1^N and the cube B_∞^N, which correspond to K_N and Z_N, respectively, as defined in the introduction. We also set

$$m_k(C) = \max_{F\in G_{N,k}} V_k(P_F C)^{\frac{1}{k}}.$$

The following table summarizes the geometric parameters of interest.

C	$k_*(C)$	$\frac{1}{\beta_*(C)}$	$d_*(C)$	$\mathrm{vrad}(C)$	$W_{[k]}(C)$	$w(C)$	$m_k(C)$
B_1^N	$\log N$	$\log^2 N$	$\geqslant N^{c_0}$	N	$\sqrt{\frac{\log(eN/k)}{N}}$	$\sqrt{\frac{\log N}{N}}$	$\frac{1}{k}$
B_∞^N	N	N	N	$\sqrt{N}$	$\sqrt{N}$	$\sqrt{N}$	$\sqrt{\frac{N}{k}}$

The table gives the order of magnitude, up to absolute constants, of each of the given parameters; for example $k_*(B_1^N) \simeq \log N$, while for $d_*(B_1^N)$, we have given just a lower bound in terms of an absolute constant $c_0 \in (0,1)$.

For the cube $C = B_\infty^N$, one has $k_*(B_\infty^N) \simeq N$, by (18). Moreover, using (20) the values of $d_*(B_\infty^N)$ and $1/\beta_*(B_\infty^N)$ are also of order N. The intrinsic volumes of the cube satisfy

$$\sqrt{N} \simeq \mathrm{vrad}(B_\infty^N) \leqslant W_{[k]}(B_\infty^N) \leqslant w(B_\infty^N) \simeq \sqrt{N}, \tag{52}$$

which can be seen by direct computation or as consequence of Alexandrov's inequality (17). The final entry in the row for B_∞^N follows from, e.g., the inclusion $B_\infty^N \subseteq \sqrt{N} B_2^N$ and (52).

For the cross-polytope B_1^N, $k_*(B_1^N) \simeq \log N$, which can be directly computed using (18). To compute $\beta_*(B_1^N)$, it is enough to compute the $\mathrm{var}(\max_{i\leqslant N}|g_i|)$, which is well-known; see, e.g., [10]. The quantity $d_*(B_1^N)$ has been estimated in [28, Props. 6.1, 6.2]. For the maximal volume k-dimensional projection of B_1^N we will need that every k-codimensional section of $\overline{B_1^N}$ has volume of order c^k, i.e. $V_{N-k}(\overline{B_1^N} \cap F^\perp)^{\frac{1}{k}} \simeq 1$ (see e.g. [25]). Then by Rogers–Shephard inequality [36], we have that for every $F \in G_{N,k}$,

$$cV_k(P_F\overline{B_1^N})^{\frac{1}{k}} \leqslant V_k(P_F\overline{B_1^N})^{\frac{1}{k}} V_{N-k}(\overline{B_1^N} \cap F^\perp)^{\frac{1}{k}} \leqslant \binom{N}{k}^{\frac{1}{k}} \leqslant \frac{eN}{k}.$$

Here $\overline{B_1^N} = V_N(B_1^N)^{-\frac{1}{N}} B_1^N$ and $V_N(B_1^N)^{\frac{1}{N}} \simeq \frac{1}{N}$ so we get that

$$\max_{F\in G_{N,k}} V_k(P_F B_1^N)^{\frac{1}{k}} \leqslant \frac{c}{k}.$$

On the other side we have that if F is the subspace spanned by $\{e_1, \cdots, e_k\}$, then $V_k(P_{F_0} B_1^N)^{\frac{1}{k}} = V_k(B_1^k)^{\frac{1}{k}} \simeq \frac{1}{k}$.

Using the above table of values and the results in the previous section, we readily get the next theorems.

Theorem 3.14. *For every* $1 \leqslant k \leqslant n \leqslant c_0 N$, *and every* $\varepsilon > c\sqrt{\frac{k}{N}\log\frac{N}{k}}$,

$$\mathbb{P}\left(W_{[k]}(Z_N) \geqslant (1+\varepsilon)\mathbb{E}W_{[k]}(Z_N)\right) \leqslant e^{-c\varepsilon^2 nN} \tag{53}$$

and for every $\varepsilon \in [\sqrt{\frac{k}{N}\log\frac{eN}{k}}, 1]$,

$$\mathbb{P}\left(W_{[k]}(Z_N) \leqslant (1-\varepsilon)\mathbb{E}W_{[k]}(Z_N)\right) \leqslant e^{-c\varepsilon^2 N}. \tag{54}$$

Moreover for every $\varepsilon > 0$,

$$\mathbb{P}\left(W_{[k]}(Z_N) \leqslant \varepsilon\mathbb{E}W_{[k]}(Z_N)\right) \leqslant (c\varepsilon)^{\frac{kN}{2}}. \tag{55}$$

Theorem 3.15. *For every* $1 \leqslant n \leqslant N$, $1 \leqslant k \leqslant \min\{\log^2 N, n\}$ *and every* $\varepsilon > c\sqrt{\frac{k}{\log N}\log\frac{e\log N}{k}}$,

$$\mathbb{P}\left(W_{[k]}(K_N) \geqslant (1+\varepsilon)\mathbb{E}W_{[k]}(K_N)\right) \leqslant e^{-c\varepsilon^2 n\log N} \tag{56}$$

and for every $\varepsilon \in [\sqrt{\frac{k}{\log N}\log\frac{e\log N}{k}}, 1]$,

$$\mathbb{P}\left(W_{[k]}(K_N) \leqslant (1-\varepsilon)\mathbb{E}W_{[k]}(K_N)\right) \leqslant e^{-c\varepsilon^2 \log^2 N}. \tag{57}$$

Moreover for every $\varepsilon > 0$,

$$\mathbb{P}\left(W_{[k]}(K_N) \leqslant \varepsilon\mathbb{E}W_{[k]}(K_N)\right) \leqslant (c\varepsilon)^{N^{c_0}}. \tag{58}$$

Finally, for every $p \geqslant N\log N$,

$$\frac{(\mathbb{E}V_k(K_N)^p)^{\frac{1}{pk}}}{(\mathbb{E}V_k(K_N))^{\frac{1}{k}}} \simeq \frac{\sqrt{pN}}{k\sqrt{\log(N/k)}}. \tag{59}$$

§4. Asymptotic results for zonotopes

In this section, we discuss properties of the limiting distributions of the intrinsic volumes of the Gaussian zonotopes $Z_N = G[-1,1]^N = \sum_{i=1}^{N}[-g_i, g_i]$ to complement the non-asymptotic results in the previous section.

A central limit theorem for the volume of Minkowski sums of random convex sets was proved by Vitale [43]. As a special case, it was shown that as $N \to \infty$,

$$\frac{V_n(Z_N) - \mathbb{E}V_n(Z_N)}{\sqrt{\mathrm{var}(V_n(Z_N)}} \to \mathcal{N}(0,1). \tag{60}$$

In [30], we used the latter fact to prove a central limit theorem for the volume of random orthogonal projections of rank n of B_∞^N, when n is fixed and $N \to \infty$. We also proved bounds for centered moments of the volume $V_n(Z_N)$. The next proposition extends this to other intrinsic volumes. The proof is a natural generalization of [30, Proposition 4.2]; detailed proofs are included here for completeness.

Theorem 4.1. *Let $1 \leqslant k \leqslant n \leqslant N$.*

(1) *For each $p \geqslant 2$,*

$$\mathbb{E}|V_k(Z_N) - \mathbb{E}V_k(Z_N)|^p \leqslant c_{n,k,p} N^{p(k-\frac{1}{2})}, \tag{61}$$

where $c_{n,k,p}$ is a constant that depends only on n, k and p.

(2) *The variance of $V_k(Z_N)$ satisfies*

$$\frac{\operatorname{var}(V_k(Z_N))}{N^{2k-1}} \to c_{n,k}, \tag{62}$$

where $c_{n,k}$ is a positive constant that depends only on n and k.

(3) *$V_k(Z_N)$ satisfies the following central limit theorem:*

$$\frac{V_k(Z_N) - \mathbb{E}V_k(Z_N)}{\sqrt{\operatorname{var}(V_k(Z_N))}} \to N(0,1). \tag{63}$$

To prove Theorem 4.1, it will be useful to recall several results on U-statistics; for background, see e.g. [14,37,39]. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with values in a measurable space $(S, \mathcal{S})$. Let $h : S^m \to \mathbb{R}$ be a measurable function. For $N \geqslant m$, the U-statistic of order m with kernel h is defined by

$$U_N = U_N(h) = \frac{(N-m)!}{N!} \sum_{(i_1,\ldots,i_m)\in I_N^m} h(X_{i_1}, \ldots, X_{i_m}), \tag{64}$$

where

$$I_N^m = \{(i_1, \ldots, i_m) : i_j \in \mathbb{N}, 1 \leqslant i_j \leqslant N, i_j \neq i_k \text{ if } j \neq k\}.$$

When h is symmetric, i.e., $h(x_1, \ldots, x_m) = h(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ for every permutation σ of m elements, we can write

$$U_N = U(X_1, \ldots, X_N) = \frac{1}{\binom{N}{m}} \sum_{1 \leqslant i_1 < \ldots < i_m \leqslant N} h(X_{i_1}, \ldots, X_{i_m}); \tag{65}$$

here the sum is taken over all $\binom{N}{m}$ subsets $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, N\}$. We recall several well-known results, which go back to Hoeffding (e.g. [39, Ch. 5]).

Theorem 4.2. *For $N \geqslant m$, let U_N be a statistic with kernel $h : S^m \to \mathbb{R}$. Set $\zeta = \mathrm{Var}\mathbb{E}[h(X_1, \ldots, X_m)|X_1]$.*

(1) *The variance of U_N satisfies $\mathrm{Var}U_N = \frac{m^2\zeta}{N} + O(N^{-2})$ as $N \to \infty$.*
(2) *If $\mathbb{E}|h(X_1, \ldots, X_m)| < \infty$, then $U_N \stackrel{a.s.}{\to} \mathbb{E}U_N$ as $N \to \infty$.*
(3) *If $\mathbb{E}h^2(X_1, \ldots, X_m) < \infty$ and $\zeta > 0$, then*

$$\sqrt{N}\left(\frac{U_N - \mathbb{E}U_N}{m\sqrt{\zeta}}\right) \stackrel{d}{\to} \mathcal{N}(0,1) \text{ as } N \to \infty.$$

We will also recall the following decoupling result for U-statistics. Assume that $h : (\mathbb{R}^n)^m \to \mathbb{R}$ satisfies $\mathbb{E}|h(X_1, \ldots, X_m)| < \infty$ and let $1 < r \leqslant m$. Following [14, Definition 3.5.1], we say that h is degenerate of order $r-1$ if

$$\mathbb{E}_{X_r,\ldots,X_m} h(x_1, \ldots, x_{r-1}, X_r, \ldots, X_m) = \mathbb{E}h(X_1, \ldots, X_m)$$

for all $x_1, \ldots, x_{r-1} \in \mathbb{R}^n$, and the function

$$S^r \ni (x_1, \ldots, x_r) \mapsto \mathbb{E}_{X_{r+1},\ldots,X_m} h(x_1, \ldots, x_r, X_{r+1}, \ldots, X_m)$$

is non-constant. If h is not degenerate of any positive order r, we say it is non-degenerate or degenerate of order 0. We will make use of the following randomization theorem, which is a special case of [14, Theorem 3.5.3].

Theorem 4.3. *Let $1 \leqslant r \leqslant m$ and $p \geqslant 1$. Suppose that $h : S^m \to \mathbb{R}$ is degenerate of order $r-1$ and $\mathbb{E}|h(X_1, \ldots, X_m)|^p < \infty$. Set*

$$f(x_1, \ldots, x_m) = h(x_1, \ldots, x_m) - \mathbb{E}h(X_1, \ldots, X_m).$$

Let $\varepsilon_1, \ldots, \varepsilon_N$ denote i.i.d. Rademacher random variables, independent of $X_1, \ldots, X_N$. Then

$$\mathbb{E}\Big|\sum_{(i_1,\ldots,i_m)\in I_N^m} f(X_{i_1}, \ldots, X_{i_m})\Big|^p \simeq_{m,p} \mathbb{E}\Big|\sum_{(i_1,\ldots,i_m)\in I_N^m} \varepsilon_{i_1}\cdots\varepsilon_{i_r} f(X_{i_1}, \ldots, X_{i_m})\Big|^p.$$

Here $A \simeq_{m,p} B$ means $C'_{m,p}A \leqslant B \leqslant C''_{m,p}A$, where $C'_{m,p}$ and $C''_{m,p}$ are constants that depend only on m and p.

Corollary 4.4. *Let $1 \leqslant k \leqslant n \leqslant N$ and let $X_1, \ldots, X_N$ be i.i.d. random vectors distributed according to an absolutely continuous probability* *measure μ on $\mathbb{R}^n$. Assume that $\mathbb{E}\|X_1\|_2^p < \infty$ for some $p \geqslant 2$. Define $f : (\mathbb{R}^n)^k \to \mathbb{R}$ by*

$$f(x_1, \dots, x_k) = \int\limits_{G_{n,k}} |\det [P_E x_1 \cdots P_E x_k]| d\nu_{n,k}(E) - \mathbb{E} \int\limits_{G_{n,k}} |\det [P_E X_1 \cdots P_E X_k]| d\nu_{n,k}(E).$$

Then

$$\mathbb{E}\Big| \sum_{1 \leqslant i_1 < \dots < i_k \leqslant N} f(X_{i_1}, \dots, X_{i_k}) \Big|^p \leqslant C_{n,k,p} N^{p(k-\frac{1}{2})} \mathbb{E}|f(X_1, \dots, X_k)|^p,$$

where $C_{n,k,p}$ is a constant that depends on n, k and p.

Proof. Since μ is absolutely continuous, $\dim(\mathrm{span}\{X_1, \dots, X_r\}) = r$ a.s. for $r = 1, \dots, k$. Moreover, $f(ax_1, \dots, x_k) = |a| f(x_1, \dots, x_k)$ for any $a \in \mathbb{R}$, hence f is non-degenerate. By Hölder and Hadamard's inequalities,

$$\mathbb{E} \left(\int\limits_{G_{n,k}} |\det[P_E X_1 \cdots P_E X_k]| d\nu_{n,k} \right)^p \leqslant \mathbb{E}\|X_1\|_2^p \cdot \ldots \cdot \mathbb{E}\|X_k\|_2^p.$$

Thus we may apply Theorem 4.3 with $r = 1$:

$$\mathbb{E}\Big| \sum_{1 \leqslant i_1 < \dots < i_k \leqslant N} k! f(X_{i_1}, \dots, X_{i_k}) \Big|^p = \mathbb{E}\Big| \sum_{(i_1, \dots, i_k) \in I_N^k} f(X_{i_1}, \dots, X_{i_k}) \Big|^p \leqslant C_{n,k,p} \mathbb{E}\Big| \sum_{(i_1, \dots, i_k) \in I_N^k} \varepsilon_{i_1} f(X_{i_1}, \dots, X_{i_k}) \Big|^p.$$

Suppose $X_1, \dots, X_N$ are fixed. Taking expectation in $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ and appling Khintchine's inequality and then Hölder's inequality, we have

$$\mathbb{E}_\varepsilon \Big| \sum_{(i_1, \dots, i_k) \in I_N^k} \varepsilon_{i_1} f(X_{i_1}, \dots, X_{i_k}) \Big|^p = \mathbb{E}_\varepsilon \Big| \sum_{i_1=1}^N \varepsilon_{i_1} \sum_{\substack{(i_2, \dots, i_k) \\ (i_1, \dots, i_k) \in I_N^k}} f(X_{i_1}, \dots, X_{i_k}) \Big|^p$$

$$\leqslant c\Big|\sum_{i_1=1}^{N}\Big(\sum_{\substack{(i_2,\dots,i_k)\\(i_1,\dots,i_k)\in I_N^k}} f(X_{i_1},\dots,X_{i_k})\Big)^2\Big|^{\frac{p}{2}}$$

$$\leqslant c\left(\binom{N-1}{k-1}(k-1)!\right)^{\frac{p}{2}}\Big|\sum_{(i_1,\dots,i_k)\in I_N^k} f(X_{i_1},\dots,X_{i_k})^2\Big|^{\frac{p}{2}}$$

$$\leqslant c\left(\binom{N-1}{k-1}(k-1)!\right)^{\frac{p}{2}}\left(\binom{N}{k}k!\right)^{\frac{p-2}{2}}\sum_{(i_1,\dots,i_k)\in I_N^n}|f(X_{i_1},\dots,X_{i_k})|^p,$$

where c is an absolute constant. Taking expectation in the X_i's gives

$$\mathbb{E}\Big|\sum_{(i_1,\dots,i_k)\in I_N^k}\varepsilon_{i_1} f(X_{i_1},\dots,X_{i_k})\Big|^p$$

$$\leqslant \quad \left(\binom{N-1}{k-1}(k-1)!\right)^{\frac{p}{2}}\left(\binom{N}{k}k!\right)^{\frac{p-2}{2}}\binom{N}{k}k!\mathbb{E}|f(X_1,\dots,X_k)|^p.$$

We complete the proof by using the estimate $\binom{N}{k}\leqslant (eN/k)^k$. □

Proof of Theorem 4.1. Note that for $\alpha_{n,k} := \binom{n}{k}\frac{\omega_n}{\omega_k\omega_{n-k}}$, we have

$$\begin{aligned} V_k(Z_N) &= V_k\left(\sum_{i=1}^{N}[-g_i,g_i]\right)\\ &= \alpha_{n,k}\int_{G_{n,k}} V_k\left(\sum_{i=1}^{N}[-P_Eg_i,P_Eg_i]\right)d\nu_{n,k}(E)\\ &= \alpha_{n,k}\sum_{|I|=k}\int_{G_{n,k}}|\det([P_Ex_1,\dots,P_Ex_k])|d\nu_{n,k}(E). \end{aligned}$$

Thus if $h:(\mathbb{R}^n)^k\to\mathbb{R}^+$ is the permutation-invariant function given by

$$h(x_1,\dots,x_k) = \alpha_{n,k}\int_{G_{n,k}}|\det([P_Ex_1,\dots,P_Ex_k])|d\nu_{n,k}(E), \tag{66}$$

then $\frac{1}{\binom{N}{k}}V_k(Z_N)$ is a U-statistic with kernel h.

To prove (1), we note that h is non-degenerate. Thus we can apply Corollary 4.4 to the function

$$f(x_1,\dots,x_k) = h(x_1,\dots,x_k) - \mathbb{E}h(g_1,\dots,g_k). \tag{67}$$

Next, we prove (2). For fixed $E \in G_{n,k}$, we write

$$|\det[P_E g_1 \cdots P_E g_k]| = \|v_1\|_2 \|P_{F_1^\perp} v_2\|_2 \cdot \ldots \cdot \|P_{F_{k-1}} v_k\|_2, \qquad (68)$$

where $v_i = P_E g_i$, for $i = 1, \ldots, k$ and $F_r = \mathrm{span}\{v_1, \ldots, v_r\}$ for $r = 1, \ldots, k-1$, while $F_0 = \{0\}$. Denote the expectation with respect to g_r by $\mathbb{E}_r$. Note that for $r = 2, \ldots, k-1$, $\mathbb{E}_{r+1}\|P_{F_r^\perp} v_{r+1}\|$ depends only on the dimension of F_r, which is equal to r a.s. By Fubini's theorem, integrating first in g_k, then g_{k-1}, and so on, we have

$$\mathbb{E}_2 \cdot \ldots \cdot \mathbb{E}_k \|P_{F_1^\perp} v_2\|_2 \cdot \ldots \cdot \|P_{F_{k-1}} v_k\|_2 = \prod_{r=1}^{k-1} \mathbb{E}\sqrt{\chi_r^2} =: \beta_{n,k},$$

where χ_r^2 denotes a chi-squared random variable with r degrees of freedom, $r = 1, \ldots, k-1$. Moreover, the latter expression is independent of E. Thus

$$\begin{aligned}\mathbb{E}[h(g_1, \ldots, g_k)|g_1] &= \mathbb{E}_{2,\ldots,k}\left(\alpha_{n,k} \int\limits_{G_{n,k}} |\det[P_E g_1 \cdots P_E g_k]| d\nu_{n,k}(E)\right) \\ &= \alpha_{n,k}\beta_{n,k} \int\limits_{G_{n,k}} \|P_E g_1\|_2 d\nu_{n,k}(E).\end{aligned}$$

Thus $\zeta = \mathrm{var}(\mathbb{E}[h(g_1, \ldots, g_k)|g_1]) > 0$ and hence we can apply Theorem 4.2(2), to get

$$\sqrt{N}\left(\frac{V_k(Z_N) - \mathbb{E}V_k(Z_N)}{\binom{N}{k} k\sqrt{\zeta}}\right) \to N(0,1), \text{ as } N \to \infty. \qquad (69)$$

On the other hand, by part (1) we have

$$\frac{\mathbb{E}|(V_k(Z_N)) - \mathbb{E}V_k(Z_N)|^4}{N^{4k-2}} \leqslant C_{n,,k,4}.$$

This implies that the sequence $(V_k(Z_N) - \mathbb{E}V_k(Z_N)/N^{k-\frac{1}{2}})_N$ is uniformly integrable, hence

$$\frac{\sqrt{\mathrm{var}(V_k(Z_N))}}{N^{-\frac{1}{2}}\binom{N}{k} k\sqrt{\zeta}} \to 1 \text{ as } N \to \infty.$$

Part (3) now follows from (69) and Slutsky's theorem. □

References

1. S. Artstein-Avidan, A. Giannopoulos, V. D. Milman, *Asymptotic Geometric Analysis.* Part I. — Mathematical Surveys and Monographs, American Math. Soc., Providence, RI, **202** (2015), XX–451.
2. F. Affentranger, *The convex hull of random points with spherically symmetric distributions.* — Rend. Sem. Mat. Univ. Politec. Torino, **49**, No. 3 (1991), 359–383 (1993).
3. T. W. Anderson, *An Introduction to Multivariate Statistical Analysis.* Third ed: Wiley Series in Probability and Statistics, Wiley-Interscience [John Wiley & Sons], Hoboken, NJ (2003), XX–721
4. F. Affentranger, R. Schneider, *Random projections of regular simplices.* — Discrete Comput. Geom., **7**, No. 3 (1992), 219–226.
5. U. Betke, M. Henk, *Intrinsic volumes and lattice points of crosspolytopes.* — Monatsh. Math., **115**, No. 1-2 (1993), 27–33.
6. I. Bárány, M. Reitzner, *On the variance of random polytopes.* — Adv. Math., **225**, No. 4 (2010), 1986–2001.
7. I. Bárány, C. Thaele, Intrinsic volumes and Gaussian polytopes: the missing piece of the jigsaw. 2016, `https://arxiv.org/abs/1606.07945`.
8. Y. M. Baryshnikov, R. A. Vitale, *Regular simplices and Gaussian samples.* — Discrete Comput. Geom., **11**, No. 2 (1994), 141–147.
9. I. Bárány, V. Vu, *Central limit theorems for Gaussian polytopes.* — Ann. Probab., **35**, No. 4 (2007), 1593–1621.
10. S. Chatterjee, *Superconcentration and Related Topics.* Springer Monographs in Mathematics, 2014.
11. P. Calka, J. E. Yukich, *Variance asymptotics for random polytopes in smooth convex bodies.* — Probab. Theory Related Fields, **158**, No. 1-2 (2014), 435–463.
12. P. Calka, J. E. Yukich, *Variance asymptotics and scaling limits for Gaussian polytopes.* — Probab. Theory Related Fields, **163**, No. 1-2 (2015), 259–301.
13. N. Dafnis, A. Giannopoulos, A. Tsolomitis, *Quermaßintegrals and asymptotic shape of random polytopes in an isotropic convex body.* — Michigan Math. J., **62**, No. 1 (2013), 59–79.
14. V. H. de la Peña, E. Giné, *Decoupling. From Dependence to Independence.* Ser.: Springer Monographs in Mathematics. Springer Verlag. NY, 1999.
15. N. Dafnis, G. Paouris, *Estimates for the affine and dual affine quermassintegrals of convex bodies.* — Illinois J. Math., **56**, No. 4 (2012), 1005–1021.
16. E. D. Gluskin, *The diameter of the Minkowski compactum is roughly equal to n.* — Funktsional. Anal. i Prilozhen., Akademiya Nauk SSSR., **15**, No. 1 (1981), 72–73.
17. D. Hug, M. Reitzner, *Gaussian polytopes: variances and limit theorems.* — Adv. in Appl. Probab. **37**, No. 2 (2005), 297–320.
18. B. Klartag, R. Vershynin, *Small ball probability and Dvoretzky's theorem.* — Israel J. Math. **157** (2007), 193–207.
19. Z. Kabluchko, D. Zaporozhets, *Expected volumes of Gaussian polytopes, external angles, and multiple order statistics.* preprint, 2017, `https://arxiv.org/abs/1706.08092`.

20. A. Litvak, *Around the simplex mean width conjecture.* 2017, available at `http://www.math.ualberta.ca/ alexandr/OrganizedPapers/AL-simp-wv.pdf`.
21. A. Lytova, K. Tikhomirov, *The variance of the ℓ_p^n-norm of the Gaussian vector, and Dvoretzky's theorem.* available at `https://arxiv.org/abs/1705.05052`
22. E. Lutwak, *A general isepiphanic inequality.* — Proc. Amer. Math. Soc. **90**, No. 3 (1984), 415–421.
23. E. Lutwak, *Inequalities for Hadwiger's harmonic Quermassintegrals.* — Math. Ann. **280**, No. 1 (1988), 165–175.
24. V. D. Milman, *A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies.* — Funkcional. Anal. i Priložen. Akademija Nauk SSSR. **5**, No. 4 (1971), 28–37.
25. M. Meyer, A. Pajor, *Sections of the unit ball of L_p^n.* — J. Funct. Anal. **80**, No. 1 (1988), 109–123.
26. V. D. Milman, G. Schechtman, *Asymptotic Theory of Finite-Dimensional Normed Spaces.* Lecture Notes in Mathematics, vol. 1200, Springer-Verlag, Berlin, 1986, With an appendix by M. Gromov.
27. P. Mankiewicz, N. Tomczak-Jaegermann, *Quotients of finite-dimensional Banach spaces; random phenomena.* — Handbook of the geometry of Banach spaces, North-Holland, Amsterdam, **2** (2003), 1201–1246.
28. G. Paouris, P. Pivovarov, *Small-ball probabilities for the volume of random convex sets.* — Discrete Comput. Geom. **49**, No. 3 (2013), 601–646.
29. G. Paouris, P. Pivovarov, P. Valettas, *On a quantitative reversal of Alexandrov's inequality.* — Trans. Amer. Math. Soc. (to appear), 2017, `https://arxiv.org/abs/1702.05762`.
30. G. Paouris, P. Pivovarov, J. Zinn, *A central limit theorem for projections of the cube.* — Probab. Theory Related Fields, **159**, No. 3-4 (2014), 701–719.
31. G. Paouris, P. Valettas, *Neighborhoods on the Grassmannian of marginals with bounded isotropic constant.* — J. Funct. Anal. **267**, No. 9 (2014), 3427–3443.
32. G. Paouris, P. Valettas, *On Dvoretzky's theorem for subspaces of L_p.* Preprint, 2015, `https://arxiv.org/abs/1510.07289`.
33. G. Paouris, P. Valettas, *A Gaussian small deviation inequality for convex functions.* — Ann. Probab. (to appear) (2016), available at `https://arxiv.org/abs/1611.01723`.
34. G. Paouris, P. Valettas, *Variance estimates and almost Euclidean structure.* Preprint, 2017, `https://arxiv.org/abs/1703.10244`.
35. G. Paouris, P. Valettas, J. Zinn, *Random version of Dvoretzky's theorem in ℓ_p^n.* — Stochastic Process. Appl. **127**, No. 10 (2017), 3187–3227.
36. C. A. Rogers, G. C. Shephard, *The difference body of a convex body.* — Arch. Math. (Basel) **8** (1957), 220-233.
37. H. Rubin, R. A. Vitale, *Asymptotic distribution of symmetric statistics.* — Ann. Statist. **8**, No. 1 (1980), 165–170.
38. R. Schneider, *Convex bodies: the Brunn–Minkowski theory.* Encyclopedia of Mathematics and its Applications, expanded, vol. 151, Cambridge University Press, Cambridge, 2014.
39. R. J. Serfling, *Approximation Theorems of Mathematical Statistics.* Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons Inc. NY, 1980.

40. V. N. Sudakov, *Geometric problems of the theory of infinite-dimensional probability distributions.* — Trudy Mat. Inst. Steklov. Akademiya Nauk SSSR. **141** (1976), 191.
41. S. J. Szarek, *Nets of Grassmann manifold and orthogonal group.* — Proceedings of research workshop on Banach space theory (Iowa City, Iowa, 1981).Univ. Iowa, Iowa City, IA (1982), 169–185.
42. B. S. Tsirelson, *A geometric approach to maximum likelihood estimation for an infinite-dimensional Gaussian location.* II — Teor. Veroyatnost. i Primenen. **30**, No. 4 (1985), 772–779.
43. R. A. Vitale, *Symmetric statistics and random shape.* — Proceedings of the 1st World Congress of the Bernoulli Society, Vol. 1 (Tashkent, 1986), VNU Sci. Press, Utrecht (1987), 595–600.
44. R. Vitale, *On the Gaussian representation of intrinsic volumes.* — Statist. Probab. Lett. **78**, No. 10 (2008), 1246–1249.

Department of Mathematics, Mailstop 3368
Texas A&M University
College Station TX 77843-3368 USA
E-mail: grigoris@math.tamu.edu

Поступило 12 сентября 2017 г.

Mathematics Department
University of Missouri
Columbia, MO 65211 USA
E-mail: pivovarovp@missouri.edu

Mathematics Department
University of Missouri
Columbia, MO 65211 USA
E-mail: valettasp@missouri.edu