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Topological and algebraic genericity and spaceability for an extended chain of sequence spaces

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Abstract

Given a pair of topological vector spaces $X, \ Y$ where X is a proper linear subspace of Y it is examined whether $Y \setminus X$ is residual in Y (topological genericity), whether $Y \setminus X$ contains a dense linear subspace of Y except 0 (algebraic genericity) and whether $Y \setminus X$ contains a closed infinite dimensional subspace of Y except 0 (spaceability). In the present paper the spaces X and Y are either sequence spaces or spaces of analytic functions on the unit disc regarded as sequence spaces via the identification of a function with the sequence of its Taylor coefficients. For the spaces under consideration we give an affirmative answer to each of these questions providing general proofs which extend previous results.

Keyword Topological genericity · Algebraic genericiy · Spaceability · Baire's theorem · ℓ^p spaces

Dedicated to the memory of Professor Dimitris Gatzouras.

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1 Introduction

In [4, 7], the chain of spaces $\cap_{p>\alpha}\ell^p$ ($\alpha \geq 0$), ℓ^p ($0 < p < +\infty$), c_0, ℓ^∞ was considered and, for any pair (X, Y) with $X \subsetneq Y$ belonging to this chain, topological and algebraic genericity and spaceability were investigated, extending previous results. We recall the definitions. Given a pair (X, Y) where Y is a topological vector space and X is a proper linear subspace of Y, we say that we have topological genericity if X is contained in an F_σ - meager subset of Y, equivalently if $Y \setminus X$ is residual in Y. This is always the case for the pairs of sequence spaces belonging to the above chain and then a question that arises naturally is whether X is indeed equal to an F_σ subset of Y or not. Furthermore, we say that we have algebraic genericity for the pair (X, Y) if there exists a linear subspace G of Y, dense in Y, such that G is contained in $(Y \setminus X) \cup \{0\}$. Finally, we have spaceability if $(Y \setminus X) \cup \{0\}$ contains a closed infinite dimensional subspace of Y.

In the present paper we extend the above chain by adding the space $A^\infty(\mathbb{D})$ which is contained in $\cap_{p>0}\ell^p$ and the spaces $H(\mathbb{D}) \subset \mathbb{C}^{\mathbb{N}_0}$ which contain ℓ^∞. The spaces $A^\infty(\mathbb{D})$ and $H(\mathbb{D})$ are spaces of holomorphic functions on the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, but they can also be seen as sequence spaces via the identification of any holomorphic function on $\mathbb{D}$ with the sequence of its Taylor coefficients. More precisely, for $f = \sum_{n=0}^{\infty} a_n z^n$, we have that f belongs to $A^\infty(\mathbb{D})$ if and only if, for every $k = 1, 2, \ldots$, it holds that $n^k a_n \to 0$ as $n \to +\infty$, while f belongs to $H(\mathbb{D})$ if and only if $\limsup \sqrt[n]{|a_n|} \leq 1$.

For all pairs of spaces X, Y with $X \subsetneq Y$ belonging to this extended chain, we examine topological and algebraic genericity and spaceability, completing thus the results of [7] and [4]. The same questions may be considered in the future for several chains of topological vector spaces such as Hardy spaces, Bergman spaces of analytic functions, Bloch spaces or intersections of the above spaces. We also mention the remarkable papers [6] and [8] which are related to this work.

For algebraic genericity and spaceability we refer the reader to [5] and [1]. For topological genericity we refer to [3].

2 Topological genericity

We begin with the following

Proposition 2.1 *Let $\mathbb{C}^{\mathbb{N}_0}$ be the set of sequences $(a_n)_{n=0}^{\infty}$ with $a_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$, endowed with the usual operations, pointwise addition, scalar multiplication $+, \cdot$. Let X, Y be two F-spaces which are linear subspaces of $\mathbb{C}^{\mathbb{N}_0}$. We assume that convergence of a sequence $a^m = \left(a_n^m\right)_{n=0}^{\infty}$ in either X or Y implies pointwise convergence, that is, if $a^m \xrightarrow{m\to\infty} a$ in X or in Y, then $a_n^m \xrightarrow{m\to\infty} a_n$ for all $n = 0, 1, 2, \ldots$ If $X \subset Y$ then the inclusion map $I : X \to Y$, $I(a) = a$, is continuous.*

Proof This follows immediately from the closed graph theorem.

Indeed, let $(a^m, I(a^m)) = (a^m, a^m) \in \mathrm{Gr}(I)$ such that $(a^m, a^m) \xrightarrow{m \longrightarrow \infty} (a, b)$. It suffices to show that $a = b$. From our assumption, $a^m \to a$ in X. It follows that $a_n^m \to a_n$ as $m \to \infty$ for every n. Similarly from the convergence in Y, we have that $a_n^m \to b_n$ as $m \to \infty$ for every $n = 0, 1, 2, \ldots$ It follows that $a = b$. □

Proposition 2.2 *If, in addition to the assumptions of Proposition 2.1, X is different from Y, then X is contained in an F_σ meager subset of Y.*

Proof This follows from Proposition 2.1 and a theorem of Banach which is a version of the open mapping theorem ([9], Theorem 2.11), since the inclusion map $I : X \to Y$, $I(a) = a$, is linear, continuous and not surjective. □

The above find application when X, Y are among the spaces ℓ^p for $0 < p < \infty$, $\bigcap_{p>\alpha} \ell^p$ for $0 \leq \alpha < \infty$, c_0 and ℓ^∞ ([7] and [4]). In the present paper, we extend this chain by adding the spaces $A^\infty(\mathbb{D}) \subset \bigcap_{p>0} \ell^p$ and $\ell^\infty \subset H(\mathbb{D}) \subset \mathbb{C}^{\mathbb{N}_0}$, where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$.

Let us first recall the definitions.

Definition 2.3 Let $H(\mathbb{D})$ be the vector space of all holomorphic functions on the open unit disc $\mathbb{D}$ and endow this space with the topology of uniform convergence on compact subsets of $\mathbb{D}$.

We consider $H(\mathbb{D})$ as a sequence space, by identifying every function $f(z) = \sum_{n=0}^\infty a_n z^n$ with the sequence $a = (a_n)_{n=0}^\infty$ of its Taylor coefficients. It is well known that $f \in H(\mathbb{D})$ if and only if $\limsup_n \left\{\sqrt[n]{|a_n|}\right\} \leq 1$.

Definition 2.4 Let $A^\infty(\mathbb{D})$ be the vector space of holomorphic functions f on the open unit disc $\mathbb{D}$ such that f and all its derivatives $f^{(l)}$ can be continuously extended on the closed unit disc $\overline{\mathbb{D}}$.

We endow $A^\infty(\mathbb{D})$ with the natural metric $d(f, g) = \sum_{i=0}^\infty \frac{1}{2^i} \frac{\|f^{(i)} - g^{(i)}\|_\infty}{1 + \|f^{(i)} - g^{(i)}\|_\infty}$.

As before, we identify every $f \in A^\infty(\mathbb{D})$ with the sequence $a = (a_n)_{n=0}^\infty$ of its Taylor coefficients. It is easy to see that $f \in A^\infty(\mathbb{D})$ if and only if $n^k a_n \xrightarrow{n \to \infty} 0$ for every k in $\mathbb{N}_0$.

Proposition 2.5 *Convergence in $H(\mathbb{D})$ implies pointwise convergence.*

Proof Let $f_m(z) = \sum_{n=0}^\infty a_n^m z^n$ be a sequence in $H(\mathbb{D})$ that converges to $f(z) = \sum_{n=0}^\infty a_n z^n$ in $H(\mathbb{D})$.

It suffices to show that for every $n \in \mathbb{N}_0$ we have $a_n^m \xrightarrow{m \to \infty} a_n$.

By the Weierstrass theorem we have that, for every $n \in \mathbb{N}_0$, $f_m^{(n)}$ converges uniformly to $f^{(n)}$ as $m \to \infty$ on each compact subset of $\mathbb{D}$. Thus, in particular, for every $n \in \mathbb{N}_0$,

$$a_n^m = \frac{f_m^{(n)}(0)}{n!} \xrightarrow{m \to \infty} \frac{f^{(n)}(0)}{n!} = a_n$$

□

Proposition 2.6 *Convergence in $A^\infty(\mathbb{D})$ implies pointwise convergence.*

Proof It is obvious that convergence in $A^\infty(\mathbb{D})$ implies uniform convergence in $\overline{\mathbb{D}}$ which implies convergence in $H(\mathbb{D})$. Thus, by Proposition 2.5, we have pointwise convergence. □

Remark 2.7 It is obvious that convergence in ℓ^∞ implies pointwise convergence, while by definition $\mathbb{C}^{\mathbb{N}_0}$ is endowed with the topology of pointwise convergence which is induced by the metric $\rho(a,b) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{|a_i - b_i|}{1+|a_i-b_i|}$.

Proposition 2.8 *The inclusion $A^\infty(\mathbb{D}) \subset \bigcap_{p>0} \ell^p$ holds, and is strict.*

Proof Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A^\infty(\mathbb{D})$, i.e. $\sum_{n=0}^{\infty} n^k |a_n| < \infty$ for all $k \geq 0$ and let $p > 0$.

Let $k \in \mathbb{N}$ be such that $kp > 1$.

We have $n^k|a_n| \to 0$ so there exists $N > 1$ such that $n^k|a_n| < 1$ for every $n \geq N$.

Thus $\sum_{n=0}^{\infty} |a_n|^p \leq \sum_{n=0}^{N-1} |a_n|^p + \sum_{n=N}^{\infty} \left(\frac{1}{n^k}\right)^p < \infty$ since $kp > 1$.

We now show that the inclusion is strict:

Consider the sequence $y = (y_s)$ where

$$y_s = \begin{cases} \sqrt{\frac{1}{s}} & \text{if } s = 2^k \text{ for some k} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

In other words, $y_{2^k} = \sqrt{\frac{1}{2^k}}$ and $y_s = 0$ elsewhere. Then $2^k y_{2^k} = 2^{k/2} \to \infty$ so $y \notin A^\infty(D)$.

On the other hand, $y \in \bigcap_{p>0} \ell^p$ since $\sum_{s=0}^{\infty} |y_s|^p = \sum_{n=1}^{\infty} \left|1/\sqrt{2}\right|^{np} < \infty$ for every $p > 0$. □

Next we prove a slightly stronger fact that will be used later.

Remark 2.9 For every infinite subset A of $\mathbb{N}_0$ we can find a sequence $y \in \bigcap_{p>0} \ell^p \setminus A^\infty(\mathbb{D})$ which is supported in A.

Proof Let $A = \{l_1, l_2, \dots\}$ where $l_1 < l_2 < \dots$. We choose $k_1 < k_2 < \dots$ such that, for every $n \in \mathbb{N}$, $l_{k_n} \geq 2^n$.

We define $y = (y_s)$ by $y_{l_{k_n}} = \sqrt{1/l_{k_n}}$ and $y_s = 0$ otherwise.

Then $l_{k_n} y_{l_{k_n}} = \sqrt{l_{k_n}} \geq 2^{n/2} \to \infty$ so $y \notin A^\infty(\mathbb{D})$, while for every $p > 0$ we have:

$$\sum_{s=0}^{\infty} |y_s|^p \leq \sum_{n=1}^{\infty} (2^{-n/2})^p < \infty$$

so $y \in \bigcap_{p>0} \ell^p$. □

Proposition 2.10 *The inclusion $\ell^\infty \subset H(\mathbb{D})$ holds and is strict.*

Proof Let $a = (a_n)_n \in \ell^\infty$. Then

$$\limsup \sqrt[n]{|a_n|} \leq \lim \sqrt[n]{||a||_\infty} \leq 1.$$

So $a \in H(\mathbb{D})$, which implies $\ell^\infty \subset H(\mathbb{D})$.

Since $\limsup \sqrt[n]{n} = 1$ it follows that the sequence $(n)_n$ is in $H(\mathbb{D})$, but not in ℓ^∞. So $\ell^\infty \subsetneqq H(\mathbb{D})$. □

Proposition 2.11 *The inclusion $H(\mathbb{D}) \subset \mathbb{C}^{\mathbb{N}_0}$ holds and is strict.*

Proof It is obvious that $H(\mathbb{D}) \subset \mathbb{C}^{\mathbb{N}_0}$.

Since $\limsup \sqrt[n]{n^{n+1}} = +\infty$ it follows that $\left(n^{n+1}\right)_n \notin H(\mathbb{D})$. Therefore, $H(\mathbb{D}) \subsetneqq \mathbb{C}^{\mathbb{N}_0}$. □

Theorem 2.12 *Consider the chain of spaces*

$$A^\infty(\mathbb{D}) \subsetneqq \bigcap_{p>0} \ell^p \subsetneqq \ell^\alpha \subsetneqq \bigcap_{q>\alpha} \ell^q \subsetneqq \ell^\beta \subsetneqq \bigcap_{p>\beta} \ell^p \subsetneqq c_0 \subsetneqq \ell^\infty \subsetneqq H(\mathbb{D}) \subsetneqq \mathbb{C}^{\mathbb{N}_0}$$

where $0 < \alpha < \beta$.

If $X \subsetneqq Y$ are two spaces from this chain then X is contained in an F_σ meager subset of Y.

Proof This follows by a combination of Propositions 2.2, 2.5, 2.6 and Remark 2.7. □

3 A constructive approach

In the previous section we showed that if $X \subsetneqq Y$ are spaces as in Theorem 2.12 then X is contained in an F_σ meager subset of Y.

In this section we examine whether X itself is an F_σ meager subset of Y. Our method will be constructive. At the same time we obtain a new proof of Theorem 2.12 without using Banach's theorem.

Proposition 3.1 *Let $X = A^\infty(\mathbb{D})$ and Y be a space from the chain of Theorem 2.12 such that $X \subsetneqq Y$. Then X is an $F_{\sigma\delta}$ subset of Y.*

Proof For $k \in \mathbb{N}_0$, $M \in \mathbb{N}$, let $F_M^k = \left\{a = (a_n) \in Y \mid n^k |a_n| \leq M \ \forall n \in \mathbb{N}_0\right\}$. It is clear that $X = A^\infty(\mathbb{D}) = \bigcap_{k=0}^\infty \bigcup_{M=1}^\infty F_M^k \subset \bigcup_{M=1}^\infty F_M^1$ and it remains to show that the sets F_M^k are closed in Y. Indeed, fix k and M and let (a^m) be a sequence in F_M^k, such that $a^m \xrightarrow{m\to\infty} a$ in Y and thus $a_n^m \xrightarrow{m\to\infty} a_n$ for all $n \in \mathbb{N}_0$. Then for all $n \in \mathbb{N}$ $n^k|a_n^m| \leq M$ and by taking the limit as m goes to ∞ we have $n^k|a_n| \leq M$ which implies that $a \in F_M^k$. This completes the proof. □

Remark 3.2 The proof of Proposition 3.1 can be used to give a new proof of the fact that $A^\infty(\mathbb{D})$ is contained in an F_σ meager subset of Y. It suffices to show that $\bigcup_{M=1}^\infty F_M^1$ has empty interior in Y. Indeed, it is obvious that $\bigcup_{M=1}^\infty F_M^1$ is a linear subspace of Y.

To see that it is a proper subspace, notice that the sequence $y = (y_n)$ of Proposition 2.8 is in $\bigcap_{p>0} \ell^p \subset Y$ and $y \notin \bigcup_{M=1}^{\infty} F_M^1$ because (ny_n) is not bounded. It follows that $\bigcup_{M=1}^{\infty} F_M^1$ has empty interior in Y.

We mention that all cases of Theorem 2.12 can be derived by the method of sect. 3 without using Banach's Theorem. We will not insist on this point.

Proposition 3.3 *Let $X = \ell^p$ for $p > 0$ and Y be a space from the chain of Theorem 2.12 such that $X \subsetneqq Y$. Then X is an F_σ meager subset of Y.*

Proof It suffices to write $\ell^p = \bigcup_{M=1}^{\infty} \{a = (a_n)_{n=0}^{\infty} \in Y \mid \sum_{n=0}^{N} |a_n|^p \leq M \; \forall N \in \mathbb{N}\}$ as in [7]. Thus the set ℓ^p, being a proper linear subspace of Y, has empty interior in Y and is equal to a countable union of closed sets in Y. □

Proposition 3.4 *Let $X = \bigcap_{p>c} \ell^p$ for $c \geq 0$ and Y be a space from the chain of theorem 2.12 such that $X \subsetneqq Y$. Then X is an $F_{\sigma\delta}$ subset of Y.*

Proof Let $p_n = c + \frac{1}{n}$. We have $\bigcap_{p>c} \ell^p = \bigcap_{n=1}^{\infty} \ell^{p_n}$. Since ℓ^p is F_σ in Y, it follows that X is $F_{\sigma\delta}$ in Y. □

Remark 3.5 Obviously c_0 is closed in ℓ^∞.

Proposition 3.6 *Let $X = c_0$ and $Y = H(\mathbb{D})$ or $\mathbb{C}^{\mathbb{N}_0}$. Then X is $F_{\sigma\delta}$ in Y.*

Proof $X = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} F_n^k$ where $F_n^k = \{a = (a_s) \in Y : |a_s| \leq \frac{1}{k} \; \forall s \geq n\}$.

F_n^k are closed in Y. Indeed, fix $n, k \in \mathbb{N}$.

Let a^m, $m = 1, 2, \ldots$ be a sequence in F_n^k, such that $a^m \xrightarrow{m\to\infty} a$ in Y. According to Proposition 2.5 and Remark 2.7 we have $a_n^m \xrightarrow{m\to\infty} a_n$, for all $n \in \mathbb{N}_0$. Then, for all $s \geq n$, $|a_s^m| \leq \frac{1}{k}$ and by taking the limit as m goes to ∞ we have, for all $s \geq n$, $|a_s| \leq \frac{1}{k}$ which implies that $a \in F_n^k$. □

Proposition 3.7 *Let $X = \ell^\infty$ and Y be a space from the chain of Theorem 2.12 such that $X \subsetneqq Y$. Then X is an F_σ subset of Y.*

Proof Let $F_M = \{a = (a_n) \in Y : |a_n| \leq M$ for all $n \in \mathbb{N}_0\}$. Obviously $X = \bigcup_{M=1}^{\infty} F_M$. We will show that each set F_M is closed in Y.

Indeed, let a^m be a sequence in F_M such that $a^m \xrightarrow{m\to\infty} a$, for some $a \in Y$. Convergence in Y implies pointwise convergence, that is $a_n^m \xrightarrow{m\to\infty} a_n$ for every $n \in \mathbb{N}_0$. Since $|a_n^m| \leq M$ for all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, it follows that $|a_n| \leq M$ for all $n \in \mathbb{N}_0$. Thus, $a \in F_M$. □

Proposition 3.8 *Let $X = H(\mathbb{D})$ and $Y = \mathbb{C}^{\mathbb{N}_0}$. Then X is an $F_{\sigma\delta}$ subset of Y.*

Proof $X = H(\mathbb{D}) = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} F_k^j$, where
$F_k^j = \left\{a = (a_n) \in \mathbb{C}^{\mathbb{N}_0} \mid \sqrt[n]{|a_n|} \leq 1 + \frac{1}{j} \; \forall n \geq k\right\}$.

The sets F_k^j are closed in Y. Indeed, fix $j, k \in \mathbb{N}$.

Let a^m, $m = 1, 2, \ldots$ be a sequence in F_k^j such that $a^m \xrightarrow{m\to\infty} a$ in Y, so that $a_n^m \xrightarrow{m\to\infty} a_n$ for all $n \in \mathbb{N}_0$. Then for all $n \geq k$, $\sqrt[n]{|a_n^m|} \leq 1 + \frac{1}{j}$ and by taking the limit as m goes to ∞ we have for all $n \geq k$, $\sqrt[n]{|a_n|} \leq 1 + \frac{1}{j}$ which implies that $a \in F_k^j$. □

Remark 3.9 The proof of Proposition 3.8 gives that $X = H(D) \subseteq \bigcup_{k=1}^{\infty} F_k^1 \subseteq Y = \mathbb{C}^{\mathbb{N}_0}$ where the set $\bigcup_{k=1}^{\infty} F_k^1$ is an F_σ-meager subset of Y. In other words, $Y \setminus X$ contains the complement of $\bigcup_{k=1}^{\infty} F_k^1$ which is a G_δ-dense subset of Y. We mention that $Y \setminus X$ also contains the set of sequences (a_n) with the property that the power series $\sum_{n=0}^{\infty} a_n z^n$ has 0 radius of convergence or where $\sum_{n=0}^{\infty} a_n z^n$ is a universal power series of Seleznev. It is known that these two last sets are G_δ-dense subsets of $Y = \mathbb{C}^{\mathbb{N}_0}$ ([3]). A series $\sum_{n=0}^{\infty} a_n z^n$ is a universal power series of Seleznev if its partial sums approximate uniformly every polynomial on any compact set $K \subset \mathbb{C} \setminus \{0\}$ with connected complement ([10]).

Remark 3.10 In the cases where we show that X is an $F_{\sigma\delta}$ in Y, we believe that this result can not be improved, that is X itself is not an F_σ subset of Y. This is true in particular in the case where $X = \bigcap_{p>\alpha} \ell_p$ and Y is equal to either ℓ^β or $\bigcap_{q>\beta} \ell_q$ for some $0 < \alpha < \beta < \infty$ as shown by Gregoriades in [6].

4 Algebraic genericity

In continuation to the previous project [7] we examine whether we have algebraic genericity for the couple of spaces (X, Y), where $X \subsetneq Y$ are vector spaces belonging to the chain of Theorem 2.12.

We recall the definition:

Definition 4.1 Let Y be an F-space and let X be a proper linear subspace of Y. We say that we have algebraic genericity for the couple (X, Y) if there is a linear subspace G of Y, dense in Y, such that $G \setminus \{0\} \subset Y \setminus X$.

The main result is that if X and Y are two spaces belonging to the chain of theorem 2.12 then we have algebraic genericity for the couple (X, Y). Here we deal with the case $Y \neq \ell^\infty$. When $Y = \ell^\infty$ the proof, due to Papathanasiou [8], follows a different method since ℓ^∞ is non separable.

Lemma 4.2 *Let Y be a sequence space which is an F-space and let X be a linear subspace of Y satisfying the following properties:*

1. $c_{00} \subset X \subset Y \subset \mathbb{C}^{\mathbb{N}_0}$, $X \neq Y$
2. *If $A \subset \mathbb{N}_0$ is infinite then there exists $y \in Y \setminus X$ supported in A.*
3. c_{00} *is dense in Y*
4. *For every $a \in X$ and $A \subset \mathbb{N}_0$ the product $a\chi_A$ belongs to X.*

Then we have algebraic genericity for the pair (X, Y).

Proof Since c_{00} is dense in Y it follows that $c_{00} \cap (\mathbb{Q} + i\mathbb{Q})^{\mathbb{N}_0}$ is dense in Y.

Let $\{x_j : j \in \mathbb{N}\}$ be an enumeration of $c_{00} \cap (\mathbb{Q} + i\mathbb{Q})^{\mathbb{N}_0}$ and let $(A_j)_{j\in\mathbb{N}}$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$. By condition 2, for every $j \in \mathbb{N}$ there exists $y_j \in Y \setminus X$, y_j supported in A_j.

Since Y is a topological vector space, for every $j \in \mathbb{N}$, there exists $c_j \in \mathbb{C} \setminus \{0\}$ such that $c_j y_j \in B_Y(0, \frac{1}{j})$. Let $f_j = x_j + c_j y_j$ for every j. From $d_Y(f_j, x_j) < \frac{1}{j}$ and the fact that Y does not have isolated points it follows that $\{f_j : j \in \mathbb{N}\}$ is dense in Y. This proves that $G = \langle f_1, f_2, \dots \rangle$ is dense in Y. Also, $f_j \notin X$ because $y_j \notin X$.

It remains to show that $G \cap X = \{0\}$.

Suppose that there exists $\sum_{j=1}^{M} t_j f_j \in X \setminus \{0\}$, $t_j \in \mathbb{C}$ for all $j = 1, 2, \dots, M$. Since $x_1, x_2 \dots, x_M \in c_{00}$, there exists $N \in \mathbb{N}$ such that $x_j(n) = 0$ for all $j = 1, 2, \dots, M$ and $n \geq N$. Let $j_0 \in \{1, 2, \dots, M\}$ be such that $t_{j_0} \neq 0$.

Then from assumption 4 we have that $\sum_{j=1}^{M} t_j f_j \chi_{A_{j_0} \cap [N,\infty)} \in X$, so that

$$\sum_{j=1}^{M} t_j f_j \chi_{A_{j_0} \cap [N,\infty)} = t_{j_0} y_{j_0} \chi_{A_{j_0} \cap [N,\infty)} = t_{j_0} y_{j_0} \chi_{[N,\infty)} = t_{j_0} y_{j_0} - t_{j_0} y_{j_0} \chi_{[0,N)} \in X.$$

Since $t_{j_0} y_{j_0} \chi_{[0,N)} \in c_{00} \subset X$, X is a vector space and $t_{j_0} \neq 0$, it follows that $y_{j_0} \in X$, which is a contradiction. □

Remark 4.3 Using the terminology of [2] (Definition 2.1), the assumptions of our Lemma 4.2 imply that c_{00} is dense lineable in Y, $Y \setminus X$ is lineable and $Y \setminus X$ is stronger than c_{00}. Thus, one can also use Theorem 2.2 of [2] to obtain the result of the previous lemma. We mention that although Theorem 2.2 of [2] is stated for Banach spaces, it can easily be generalized to F-spaces.

Proposition 4.4 *If X, Y are vector spaces from the chain of Theorem 2.12 such that $X \subsetneqq Y$, and $Y \neq \ell^\infty$ then conditions 1, 2, 3, 4 of Lemma 4.2 are satisfied.*

Proof Let X, Y be spaces from the chain of Theorem 2.12 such that $X \subset Y$, $X \neq Y$. It is obvious that condition 1 holds. We now prove that condition 2 holds. Let $X = \ell^p, \bigcap_{p>\alpha} \ell^p, c_0$ or ℓ^∞. Since the inclusion $X \subset Y$ is strict, we can choose $a \in Y \setminus X$. Let A be an infinite subset of $\mathbb{N}$. We can spread out the elements a_n in such a way that the support of a is contained in A. To be more precise, let $A = \{i_1, \dots, i_k, \dots\}$ be an enumeration of A such that $i_k < i_{k+1}$ for all $k \in \mathbb{N}$. Set:

$$b_n = \begin{cases} a_k, & n = i_k,\ k \in \mathbb{N} \\ 0, & n \notin A \end{cases}$$

Then, $y = (b_n)_n \in Y \setminus X$ and has support in A. This proves that condition 2 holds for these spaces.

If $X = A^\infty(\mathbb{D})$ then condition 2 follows from Remark 2.9.

If $X = H(\mathbb{D})$ then we construct a sequence supported in $A = \{l_1 < l_2 < l_3 < \cdots\}$:

$$c_n = \begin{cases} n^n & \text{if} \quad n = l_k \text{ for some k} \\ 0 \text{ otherwise} \end{cases}$$

Then, $(c_n)_n \in Y \setminus X$ and has support in A.

We now prove that condition 3 holds.

If $Y = \ell^p, c_0, \mathbb{C}^{\mathbb{N}_0}$ then it is obvious that c_{00} is dense in Y.

Let $Y = \bigcap_{p>\alpha} \ell^p$ and consider $y \in \bigcap_{p>\alpha} \ell^p$. The fact that the sequence (y_n) of c_{00} with $y_n = y\chi_{[0,n]}$, $n \in \mathbb{N}_0$, converges to y in each space ℓ^p $(\alpha < p < \infty)$ implies that (y_n) converges to y in $\bigcap_{p>\alpha} \ell^p$. This proves that c_{00} is dense in $\bigcap_{p>\alpha} \ell^p$.

Let $Y = H(\mathbb{D})$. Every $a \in c_{00}$ can be identified with a complex polynomial. It is well known that every holomorphic $f \in H(\mathbb{D})$ can be approached by polynomials, uniformly on the compact subsets of $\mathbb{D}$. It follows that c_{00} is dense in Y.

We now prove that condition 4 holds.

Let A be a subset of $\mathbb{N}_0$ and $a = (a_n)_n \in X$. Then $|a_n\chi_A| \leq |a_n|$ for all $n \in \mathbb{N}_0$ and from this inequality condition 4 is obvious for the spaces $\ell^p, \bigcap_{q>\alpha} \ell^q, 0 \leq \alpha < \infty, c_0$. If $(a_n)_n \in H(\mathbb{D})$, equivalently $\limsup_n \left\{\sqrt[n]{|a_n|}\right\} \leq 1$, then $\limsup_n \left\{\sqrt[n]{|a_n\chi_A|}\right\} \leq 1$, which proves that $a\chi_A \in H(\mathbb{D})$. Similarly, if $(a_n)_n \in A^\infty(\mathbb{D})$, equivalently $n^k a_n \xrightarrow{n\to\infty} 0$ for every $k \in \mathbb{N}$, then $n^k a_n\chi_A \xrightarrow{n\to\infty} 0$ for every $k \in \mathbb{N}$, which implies that $a\chi_A \in A^\infty(\mathbb{D})$. □

Theorem 4.5 *If X, Y are vector spaces from the chain of Theorem 2.12 with $X \subsetneq Y$ and $Y \neq \ell^\infty$, then we have algebraic genericity for the couple (X, Y).*

Proof It follows from Lemma 4.2 and Proposition 4.4. □

If $Y = \ell^\infty$ then $X \subset c_0$. According to Papathanasiou [8] there exists a linear subspace F of ℓ^∞ dense in ℓ^∞ such that $F \setminus \{0\} \subset \ell^\infty \setminus c_0 \subset \ell^\infty \setminus X$. Thus, we have algebraic genericity for the couple (X, ℓ^∞). Combining this with Theorem 4.5 we obtain:

Theorem 4.6 *Let (X, Y) be vector spaces from the chain of Theorem 2.12 with $X \subsetneq Y$. Then we have algebraic genericity for the couple (X, Y).*

5 Spaceability

In the final section we examine whether we have spaceability for the couple of spaces (X, Y), where X, Y are vector spaces from the chain of Theorem 2.12.

Let us first recall the definition:

Definition 5.1 Let Y be an F space and let X be a proper linear subspace of Y. We say that we have spaceability for the couple (X, Y) if there exists a closed infinite dimensional subspace G of Y such that $G \setminus \{0\} \subset Y \setminus X$.

The main result is that if X and Y are two spaces from the chain of Theorem 2.12 such that $X \subsetneq Y$ then we have spaceability for the couple (X, Y).

Lemma 5.2 *Let $Y \subset \mathbb{C}^{\mathbb{N}_0}$ be a sequence space which is an F space and let X be a proper linear subspace of Y satisfying the following:*

1. *If $A \subset \mathbb{N}_0$ is infinite then there exists $y \in Y \setminus X$ supported in A.*
2. *Convergence in Y implies pointwise convergence.*
3. *For every $a \in X$ and $A \subset \mathbb{N}_0$ the product $a\chi_A$ belongs to X.*

Then we have spaceability for the pair (X, Y).

Proof Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$. By condition 1, for every j there exists $y_j \in Y \setminus X$ supported in A_j.

Consider $G = \overline{\langle y_j | j \in \mathbb{N}\rangle}$.

It is obvious that G is a closed linear subspace of Y. Since the sets A_j are disjoint, it follows that G is infinite dimensional.

It remains to show that if $f \in G,\ f \neq 0$ then $f \notin X$.

Indeed, there exists a sequence $f^m \in \langle\{y_j | j \in \mathbb{N}\}\rangle$ such that $f^m \xrightarrow{m\to\infty} f$ in Y and by condition 2 we have $f^m(i) \xrightarrow{m\to\infty} f(i)$ for all $i \in \mathbb{N}_0$. For every m we can write $f^m = c_1^m y_1 + c_2^m y_2 + c_3^m y_3 + \ldots$ where finitely many of the c_j^m are non zero, i.e. for every m the set $\{j \in \mathbb{N} \mid c_j^m \neq 0\}$ is finite.

But $f \neq 0$, so there exists $i_0 \in \mathbb{N}$ such that $f(i_0) \neq 0$. If $i_0 \notin \bigcup_j A_j$ then $f^m(i_0) = 0$ for all m, so $f(i_0) = \lim_m f^m(i_0) = 0$, which is a contradiction. Hence, $i_0 \in A_{j_0}$ for some $j_0 \in \mathbb{N}$.

Since $A_1, A_2, \ldots$ are pairwise disjoint, we have $f^m(i) = c_{j_0}^m y_{j_0}(i)$ for all $i \in A_{j_0}$.

If $y_{j_0}(i_0) = 0$ then $f^m(i_0) = 0$ for all m, which is a contradiction as above, so $y_{j_0}(i_0) \neq 0$.

Let $c_{j_0} = \lim_m c_{j_0}^m = \lim_m \frac{f^m(i_0)}{y_{j_0}(i_0)} = \frac{f(i_0)}{y_{j_0}(i_0)} \neq 0$.

Then for all $i \in A_{j_0}$ we have

$$f(i) = \lim_m f^m(i) = \lim_m c_{j_0}^m y_{j_0}(i) = c_{j_0} y_{j_0}(i)$$

thus $f\chi_{A_{j_0}} = c_{j_0} y_{j_0} \chi_{A_{j_0}} = c_{j_0} y_{j_0} \notin X$ and by condition 3 we have $f \notin X$ as needed. □

Theorem 5.3 *Let* (X, Y) *be vector spaces from the chain of Theorem 2.12 with* $X \subsetneq Y$. *Then we have spaceability for the couple* (X, Y).

Proof It suffices to see that conditions 1-3 of Lemma 5.2 hold for any pair of spaces X, Y from the chain of Theorem 2.12 with $X \subsetneq Y$.

Conditions 1 and 3 have been proved in Proposition 4.4.

Condition 2 has been proved in Sect. 2. □

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