On an Untapped Source of Medieval Keralese Mathematics

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Dedicated to the late Rāma Varma Maru Tampurān, our scholarly helpmate and ever our inspiration

Communicated by D. T. WHITESIDE

1. Introduction

It is nowadays generally accepted[1] that medieval Keralese mathematics may be credited with a number of achievements, above all the infinite-series expansions whose modern equivalents are

$$[\mathrm{A}] \quad \theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \cdots \qquad (\tan\theta \leqq 1)$$

and

$$[\mathrm{B}_1] \quad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$[\mathrm{B}_2] \quad \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \qquad (\theta \leqq \pi/2).$$

The Keralese origin of these series is independent of the discovery in a later century of the inverse-tangent expansion [A] by JAMES GREGORY (1671) and GOTTFRIED WILHELM LEIBNIZ (1673), of the sine and cosine series by ISAAC NEWTON (1670) and LEIBNIZ (1676). The Malayāḷam work *Yukti-bhāṣa*, where these series are found, belongs unquestionably to the sixteenth century, and there is not the slightest possibility of their being a subsequent interpolation in the text, drawn from a later European source and rephrased with only a veneer of local colouring.

The *Yukti-bhāṣa* also states, in the particular case $\theta = \pi/4$ of the series [A], three successively closer approximations to $\pi/4$ of form

$$[\mathrm{C}] \quad 1 - \tfrac{1}{3} + \tfrac{1}{5} - \cdots \pm \frac{1}{n} \mp f_i(n+1), \qquad i = 1, 2, 3,$$

[1] See, for instance, pp. 168–73 of A.P. JUSHKEVICH's standard history of medieval mathematics [1] and the fuller accounts by MUKUNDA MARAR & RAJAGOPAL [2] and by RAJAGOPAL & VENKATARAMAN [3].

where

$$f_1(n)=\tfrac{1}{2}n,\quad f_2(n)=\frac{\tfrac{1}{2}n}{n^2+1}\quad\text{and}\quad f_3(n)=\frac{(\tfrac{1}{2}n)^2+1}{\tfrac{1}{2}n(n^2+4+1)}.^{2}$$

In generalisation of our expressions of $f_2(n)$ and $f_3(n)$, as continued fractions, D.T. WHITESIDE has shown (in private correspondence with us) that the correcting function $f(n)$ which renders the equation

$$\frac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\cdots\mp\frac{1}{n}\mp f(n+1)$$

exact can be represented as the infinite continued-fraction

$$\frac{1}{2}\cdot\frac{1}{n+}\frac{1^2}{n+}\frac{2^2}{n+}\frac{3^2}{n+}\cdots$$

whose first three convergents are precisely the $f_i(n)$, $i=1, 2, 3$, of the approximations $[C_i]$. This, for $n=2$, would give at once

$$\frac{2}{4-\pi}=2+\frac{1^2}{2+}\frac{2^2}{2+}\frac{3^2}{2+}\cdots,$$

an infinite continued-fraction having no European parallel till WILLIAM BROUNCKER's celebrated reworking in 1654 of JOHN WALLIS' related continued-product as

$$\frac{4}{\pi}=1+\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2^+}\cdots.$$

As a preliminary to a detailed and comprehensive survey which we are making of all that is known of the Keralese series-expansions, this article has the limited aim of introducing to modern scholars a hitherto unexplored medieval Sanskrit mansuscript, the *Tantrasaṅgraha-vyākhyā*, which contains all these results save the first rounding-off approximation given by $f_1(n)$ in [C], much as in the *Yukti-bhāṣa*, unlike any other available work of its own period. The importance of this *Vyākhyā* (or 'Commentary'), of which there is a manuscript on paper in the Government Oriental Manuscripts Library at Madras,[3] can be

[2] See [2], p. 72. The expressions for $f_2(n)$ and $f_3(n)$ are given explicitly in the *Yukti-bhāṣa*, but that for $f_1(n)$ appears there only as a step in the argument leading to $f_2(n)$. For convenience, [C] in its three instances is denoted in the sequel by $[C_i]$, $i=1, 2, 3$.

[3] The authors are indebted to the Curator of the Library for allowing them access to the manuscript, catalogued R 2505, and especially to T.A.K. VENKATACHARIAR, for useful observations upon it. An edition of the *Tantrasaṅgrahā-vyākhyā*, based on four known manuscripts, is to be brought out shortly by K.V. SARMA of the Vishveshvaranand Institute of Sanskrit and Indological Studies, Panjab University. In this article we have largely relied on Professor SARMA's invaluable bibliographic research in the field of medieval Keralese mathematics, collected in [4] and scattered otherwise in Sanskrit works which he had edited, to supply the dates of the several medieval Keralese works and authors whom we cite.

fully appreciated only in its context. The *Yukti-bhāṣa*, now accepted as a work of JYEṢṬHADEVA (c. 1500–c. 1610) of Paraṅṅoṭṭu (the Malayāḷam adaptation of Sanskrit house-name Parakroḍa),[4] professes to concern itself with the mathematics of a classic work, the *Tantrasaṅgraha*, composed c. 1500, with fair certainty by the versatile NĪLAKAṆṬHA (c. 1445–c. 1545). It is therefore surprising that the series [A], [B] and [C] are not to be found in well-known versions of the *Tantrasaṅgraha* itself.[5] Indeed it may seem to be doubtful that there ever had existed any more comprehensive original version of the *Tantrasaṅgraha* including [A], [B] and [C], which had been orally transmitted in the traditional fashion by a succession of teachers to their pupils but never actually committed to writing. But such doubts are robbed of importance by the *Tantrasaṅgrahavyākhyā*, composed by an unknown student of JYEṢṬHADEVA (who is referred to as a Brahmin of Parakroḍa house), admittedly embodying the teaching of that master and very likely written down during his lifetime. This *Vyākhyā* presents not only the texts of the series [A], [B] and [C] but also that of an alternative form of [B] denoted by [B′] in the sequel, which is not found in the *Yukti-bhāṣa* but proves to be of crucial significance in determining the authorship of [B], and which scholars have been hitherto able to locate in its entirety only in an anonymous work, the *Karaṇapaddhati*, of the mid-eighteenth century.[6] The *Tantrasaṅgraha-vyākhyā*, too, is anonymous, but, by its very title and its repeated claim (in chapter endings) that its content is derived from JYEṢṬHADEVA, reveals that the younger mathematicians of the NĪLAKAṆṬHA era looked upon all these series as being germane to the *Tantrasaṅgraha*.

2. The series [A], [B] and [C] as enunciated in the *Tantrasaṅgraha-vyākhyā*

There are several noteworthy features of the series [A], [B] and [C] as they are presented in the text of the *Tantrasaṅgraha-vyākhyā*.[7] For one thing, the series are there more conveniently arranged, with [C] coming after [A] along with the variant transformations of the series for $\pi/4$ from [A] into more rapidly convergent forms which we call [C_1'], [C_2'] in the sequel, bearing in mind their genesis from [C_1], [C_2][8]; the two mutually related series [B] occur apart, preceded by their essentially alternative forms [B′] which would seem to have been more widely known among medieval Keralese mathematicians (as witness the pointed reference to them in the verses enunciating [B]). Again, the enunciations given of series [A] and [B] are fuller than in other works: the *Vyākhyā* adds the sufficient conditions for series [A] to be valid, $\tan\theta < 1$ explicitly and $\tan\theta = 1$ implicitly, which is not present in the *Karaṇapaddhati*

[4] See [4], pp. 59–60].

[5] A variorum text, based on ten manuscripts, has been published in the Trivandrum Sanskrit Series (No. 188, Trivandrum, 1958).

[6] See, for instance, [3], p. 3.

[7] Our following references to this work are keyed to the pages of the Madras manuscript referred to in footnote 3.

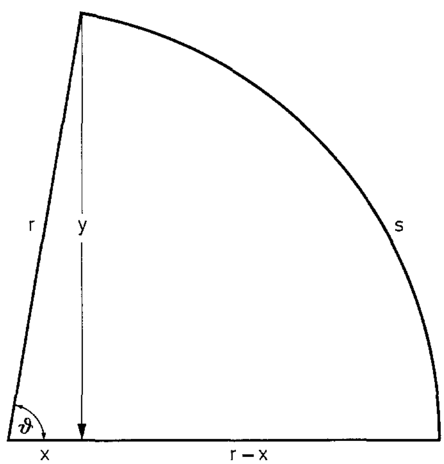
[8] As explained in [2], p. 72.

version[9]; and it explicitly states the series [B] for the sine and cosine functions which, in our previous quotation,[10] are defined only implicitly in the sequence of constructions of the individual terms of the series.

Terminology. In the following Sanskrit verses (and our translations of them into English), a *cāpam* or circular arc less than a quadrant has three elements which define it: the *bhujajyā*, the *koṭijyā* and the *śaram* or *utkramajyā* which are respectively the sine, the cosine and *sagitta* or versine of the arc, where the last named element may be replaced by the *trijyā* or the radius of the arc in an alternative definition. *Bhujajyā* and *koṭijyā* are shortened respectively to *jyā* and *koṭi* as a rule, and, as Professor T.S. KUPPANNA SASTRY has pointed out to us, to *dorjyā* (where *doḥ* = *bhuja* in its sense of 'arm') and *kojyā* on occasion. Sometimes *jyā* and *kojyā* are considered together as the two *jyās* of an arc, with implied recognition of their interchanged roles for the complement of the arc.

To obtain familiar modern forms of the formulae verbally expressed in the Sanskrit verses, we may suppose, as in the sketch below, that θ (in radians) is the acute angle subtended at the centre of a circle of radius r by the circular arc s and hence substitute for the arc, its sine y and its cosine x:

$$s = r\theta, \qquad y = r\sin\theta, \qquad x = r\cos\theta.$$



[A] (= p. 103, ll. 13–17): Series for an arc in terms of its *jyā* and *koṭi*

इष्टज्यात्रिज्ययोर्घातात्कोट्याप्तं प्रथमं फलम्

ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम्

[9] See, for instance, [2], p. 77.

[10] See [3], p. 2. This quotation, as from a version of the *Tantrasaṅgraha*, was furnished by the late RĀMA VARMA MARU TAMPURĀN (mentioned again in footnote 14), who was one of the Princes of Cochin at the time [2] was written and referred to as such in [2]. By his generous supply of manuscript material, then practically unknown, he was, in fact, the prime mover behind [2] and [3]. A search for extant manuscripts of the *Tantrasaṅgraha* which contain some at least of the series [A], [B] and [C] would, though difficult to make, be of great value in its potential findings.

प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहुः
एकत्र्याद्योजसंख्याभिर्भक्तेष्वेतेष्वनुक्रमात्
ओजानां संयुते त्यक्ते युग्मयोगे धनुर्भवेत्
दोःकोट्योरल्पमेवेष्टं कल्पनीयमिह स्मृतम्
मुहुः कृतेऽपि लब्धानां नान्यथा वसतिर्भवेत् ।

Translation. The first result(ing term) is the product of the *trijyā* and the *jyā* of the desired arc (= *iṣṭajyā*) divided by the *koṭi* of the arc. The succeeding terms are gained by a process of iteration where the first term is repeatedly multiplied by the square of the *jyā* and divided by the square of the *koṭi*. All the terms are then divided in order by the odd numbers. And the arc is obtained by adding and subtracting [respectively] the terms of odd rank and those of even rank. Of the *jyā* and the *koṭi* (which are thought of together as the two *jyās*, *doḥ* and *koṭi*, in the Sanskrit text at this point) the former is understood to be the smaller here.[11] Otherwise the terms will not be bounded [ultimately] obtained though they be by our iterative process.

[This gives, in modern notation:

$$r \operatorname{arc} \tan \frac{y}{x} = \frac{1}{1} \cdot \frac{ry}{x} - \frac{1}{3}\frac{ry^3}{x^3} + \frac{1}{5}\frac{ry^5}{x^5} - \cdots, \quad \text{where } \frac{y}{x} \leqq 1 \Big]$$

[C_2] (= p. 109, l. 1↑–p. 110, l. 6): a finite-series approximation to circumference

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते
त्रिशरादिविषमसंख्याभक्तमृणं पृथक्क्रमं कुर्यात्
यत्संख्ययात्र हरणे कृते निवृत्ता हृतिस्तु जामितया
तस्या ऊर्ध्वगतायाः समसंख्यातद्दलं गुणोऽन्ते स्यात्
तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः
प्राग्वत्ताभ्यामाप्तं स्वमृणे कृते धनक्षेप एव करणीयः
लब्धः परिधिः सूक्ष्मो बहुकृत्वो हरणेऽतिसूक्ष्मः स्यात् ।

[11] y (the *jyā*) and x(the *koṭi*) are conditioned by $y<x$ expressly and $y \leqq x$ by implication. For, the 'otherwise' of the next sentence is the case $y>x$ where the terms of the series for the arc form an unbounded sequence. Furthermore, the proof of the series for arc tan y/x in the *Yukti-bhāṣa*, referred to in footnote 22, is recognized to be applicable to both the cases $y<x$ and $y=x$.

Translation. Let four times the diameter be succeeded by terms, alternately subtracted and added, got by dividing this quadrupled diameter in order by odd numbers 3, 5, ... Let this process of division terminate at any particular [odd] number. Then let four times the diameter be multiplied by the next even number halved and divided by unity added to that number squared. This quotient is to be added to, or subtracted from, the product of the earlier procedure according as the last term of it was subtracted from, or was added to, the last but one term of it. The final result so obtained is the circumference reckoned more exactly than by going on with the first procedure [so as to include a few more terms].

[This is the approximation:

$$\pi d \approx 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots \mp \frac{4d}{n} \pm 4d \frac{(n+1)/2}{(n+1)^2+1}, \qquad \text{where } d = 2r.]$$

[C_3] (= p. 112, ll. 5–8): An approximation better than [C_2]

एभ्यः सूक्ष्मतरोऽन्यो विधिरुच्यते कश्चनापि संस्कारः
अन्ते समसंख्यादलवर्गस्सैको गुणःस एव पुनः
युगगुणितो रूपयुतः समसंख्यादलहतो भवेद्धारः
त्रिशरादिविषमसंख्याहरणात्परमेतदेव वा कार्यम् ।

Translation. Another method more refined than the preceding is now given. [We retain] the first procedure inolving division of the diameter by odd numbers. [But] we subsequently add or subtract the quotient where there is [attached to four times the diameter] a multiplier which is unity added to the next even number halved and squared and a divisor which is unity added to four times the preceding multiplier [with this] multiplied by the even number halved.

[That is, more accurately than [C_2]:

$$\pi d \approx 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots \mp \frac{4d}{n} \pm 4d \frac{\{(n+1)/2\}^2+1}{\{(n+1)^2+4+1\}(n+1)/2},$$

where $d = 2r$.]

[C_1'] (= p. 111, ll. 5–4): The circumference as a series from [C_1]

व्यासाद्वारिधिनिहतात् पृथगाप्तं त्र्याद्ययुग्विमूलघनैः
त्रिघ्नव्यासे स्वमृणं क्रमशः कृत्वापि परिधिरानेयः ।

(*N.B.* The manuscript reading निघ्न at the beginning of the second line is obviously a misprint for त्रिघ्न .)

Translation. The circumference is likewise obtained when four times the diameter is [successively] divided by the cubes of the odd numbers, beginning with 3, diminished by these numbers themselves, and the [respective] quotients are alternately added to, and subtracted from, the diameter multiplied by 3.

$$\left[\pi d = 3d + \frac{4d}{3^3-3} - \frac{4d}{5^3-5} + \frac{4d}{7^3-7} - \cdots,\ d = 2r.\right]$$

[C_2'] (= p. 111, ll. 11–13): Circumference as a series from [C_2]

समपञ्चाहतायागा रूपाद्ययुजाञ्चतुर्घ्नमूलयुताः
ताभिः षोडशगुणिताद्व्यासात्पृथगाहृतेषु विषमयुते
समफलयोगे त्यक्ते स्यादिष्टव्याससंभवः परिधिः ।

Translation. The fifth powers of the odd numbers [1, 3, 5, ...] are increased by 4 times themselves; 16 times the diameter is successively divided by all such numbers [so gotten]; the results [of division] of odd rank are added and those of even rank are subtracted. The circumference corresponding to the diameter is [thereby] obtained.

$$\left[\pi d = \frac{16d}{1^5+4.1} - \frac{16d}{3^5+4.3} + \frac{16d}{5^5+4.5} - \cdots,\ d = 2r.\right]$$

[B_1] (= p. 127, ll. 8–11): Series for *jyā* of an arc
[B_2] (= p. 127, ll. 12–16): Series for *śaram* of an arc

निहत्य चापवर्गेण चापं तत्तत्फलानि च
हरेत्समूलयुग्वर्गैस्त्रिज्यावर्गाहतैः क्रमात्
चापं फलानि चाधोऽधो न्यस्योपर्युपरि त्यजेत्
जीवाप्त्यै संग्रहोऽस्यैवं विद्वानित्यादिना कृतः ।
निहत्य चापवर्गेण रूपं तत्तत्फलानि च
हरेद्विमूलयुग्वर्गैस्त्रिज्यावर्गहतैः क्रमात्
किन्तु व्यासदलेनैव द्विघ्नेनाद्यं विभज्यताम्
फलान्यधोऽधः क्रमशः न्यस्योपर्युपरि त्यजेत्
शराप्त्यै संग्रहोऽस्यैवं स्तेनस्त्रीत्यादिना कृतः ।

(*N.B.* In the third line, रूपं of the MS has been corrected to चापं.)

[B_1]: *Translation.* The arc is to be repeatedly multiplied by the square of itself and is to be divided [in order] by the square of each even number increased by itself and multiplied by the square of the radius. The arc and the terms obtained by these repeated operations[12] are to be placed in sequence in a column, and any last term is to be subtracted from the next above, the remainder from the term then next above, and so on, to obtain the *jyā* of the arc. It was this procedure which was briefly mentioned in the verse starting with '*Vidvān*'.

[There results, on *inverting* the procedure,

$$y = s - s \cdot \frac{s^2}{(2^2+2)\,r^2} + s \cdot \frac{s^2}{(2^2+2)\,r^2} \cdot \frac{s^2}{(4^2+4)\,r^2} - \cdots$$
$$= s - \frac{1}{3!}\,\frac{s^3}{r^2} + \frac{1}{5!}\,\frac{s^5}{r^4} - \cdots.]$$

[B_2]: *Translation.* The radius is to be repeatedly multiplied by the square of the arc and is to be divided [in order] by the square of each even number diminished by itself and multiplied by the square of the radius. These repeated operations yield terms of which the first involves only 2.[13] All such terms are to be placed sequentially in a column, any last term is to be subtracted from the one next above, the remainder from the one then next above, and so on, to obtain the *śaram* of the arc. Only a brief instance of this procedure was set out in the verse starting with '*Stenastri*'.

[The procedure yields, on inversion,

$$r - x = r \cdot \frac{s^2}{(2^2-2)\,r^2} - r\,\frac{s^2}{(2^2-2)\,r^2} \cdot \frac{s^2}{(4^2-4)\,r^2} + \cdots$$
$$= \frac{1}{2!}\,\frac{s^2}{r} - \frac{1}{4!}\,\frac{s^4}{r^3} + \cdots.]$$

Alternative versions of the series [B], in effect as series of powers of s/c where (as before) s is a given arc and c is the quadrant, are also stated, *viz.*:

$$[\mathrm{B}'] \quad \begin{aligned} y &= s - \frac{1}{3!}\left(\frac{\pi}{2}\right)^2 \frac{s^3}{c^2} + \frac{1}{5!}\left(\frac{\pi}{2}\right)^4 \frac{s^5}{c^4} - \cdots \\ r - x &= \frac{1}{2!}\left(\frac{\pi}{2}\right)\frac{s^2}{c} - \frac{1}{4!}\left(\frac{\pi}{2}\right)^3 \frac{s^4}{c^3} + \frac{1}{6!}\left(\frac{\pi}{2}\right)^5 \frac{s^6}{c^5} - \cdots. \end{aligned}$$

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[12] The terms thus obtained are successively

$$s \cdot \frac{s^2}{(2^2+2)\,r^2}, \quad s \cdot \frac{s^2}{(2^2+2)\,r^2} \cdot \frac{s^2}{(4^2+4)\,r^2},$$

and so on.

[13] After this first term $r \cdot \dfrac{s^2}{(2^2-2)\,r^2}$, we have the second one

$$r \cdot \frac{s^2}{(2^2-2)\,r^2} \cdot \frac{s^2}{(4^2-4)\,r^2}$$

followed successively by other similarly constructed terms.

In interpreting the Sanskrit verses for the series [B′], we may note that they use a *sexagesimal* angular measurement, taking a quarter circumference to be $90° = 5400'$, the corresponding radius being taken equal to $10{,}800'/\pi$ where the constant π is to be approximated by the same mode of computation used also to calculate approximately the coefficients of the powers of s/c in [B′]. RĀMA VARMA MARU TAMPURĀN has pointed out that the coefficients in these approximate numerical evaluations are assigned individual Sanskrit names (based on the *kaṭapayādi* notation)[14], *viz. vidvān* (= coefficient of s^{11}/c^{11}), *tunnabala* (= coefficient of s^9/c^9), and so on; and also *stena* (= coefficient of s^{12}/c^{12}), *stripisuna* (= coefficient of s^{10}/c^{10}), *etc.* This valuable insight allows us to view the original Sanskrit verses for [B′] divested of the apparent concreteness with which they specify the (approximate) coefficients of powers of s/c, and so take these verses in their proper generality of meaning.

[B'_1] (= p. 126, ll. 9–16): Alternative forms of [B_1]

विद्वांस्तुन्नबलः कवीशनिचयः सर्वार्थशीलःस्थिरो
निर्विद्धाङ्गनरेन्द्ररुङ्निगदितेष्वेषु क्रमात्पञ्चसु ।
आधस्त्याद्गुणितादभीष्टधनुषः कृत्वापि हृत्वान्तिम-
स्याप्तं शोध्यमुपर्युपर्यथ घनेनैवं धनुष्यन्ततः ॥

Translation. In the sequence of five numbers *vidvān*, *tunnabala*, …, which are respectively 44 [$= 44'''$], 3306 [$= 33''06'''$], 160541 [$= 16'05''41'''$], 2735747 [$= 273'57''47'''$] and 22203940 [$= 2220'39''40'''$] in this order, the first number is multiplied by the square of the given arc [in degrees] and divided by the square of 5400, and the quotient is subtracted from the second number. The result of the subtraction is next multiplied by the square of the given arc and divided by the square of 5400 and the resulting quotient is subtracted from the third number. The process of multiplication and division followed by subtraction is repeated till all the five numbers are gone through. The final result is then multiplied by the cube of the given arc and divided by the cube of 5400. If the

[14] In his editorial notes (pp. 183–198) to the Malayālam version of the *Yukti-bhāṣa*, edited by him and A.R. AKHILESWARA IYER (Trichur: Mangalodayam Press, 1948). What is important in the present context is his identification of the individual numerical values of the coefficients of powers of s/c, given by the '*Vidvān*' and '*Stena*' verses, with the theoretically exact values in p. 96, [B′]. A discussion of the approximation of π, used in the evaluation of *vidvān*, etc., would take us beyond the purview of this article. We may notice, however, that there is a significant deficiency of $1'''$ in the value of the coefficient *mīnāngonarasimha* from its accurately rounded-off are; all the other coefficients are true to the nearest third minute.

quotient then resulting is subtracted from the given arc, what remains will be the required *jyā*.[15]

[There thus ensues (in minutes, corresponding to the value 5400′ for the quadrant c)

$$y \approx s - \left(\frac{s}{c}\right)^3 \left[u_3 - \left(\frac{s}{c}\right)^2 \left\{u_5 - \left(\frac{s}{c}\right)^2 \left(u_7 - \left(\frac{s}{c}\right)^2 \left[u_9 - \left(\frac{s}{c}\right)^2 u_{11}\right]\right)\right\}\right],$$

where $c = 5400'$ and also

$$u_{11} = vidv\bar{a}n \approx \frac{1}{10 \times 11} \left(\frac{\pi}{2}\right)^2 u_9^* = u_{11}^* \quad \text{(say)},$$

$$u_9 = tunnabala \approx \frac{1}{8 \times 9} \left(\frac{\pi}{2}\right)^2 u_7^* = u_9^*,$$

$$u_7 = kap\bar{\imath}\acute{s}anicaya \approx \frac{1}{6 \times 7} \left(\frac{\pi}{2}\right)^2 u_5^* = u_7^*,$$

$$u_5 = sarv\bar{a}rtha\acute{s}\bar{\imath}lasthira \approx \frac{1}{4 \times 5} \left(\frac{\pi}{2}\right)^2 u_3^* = u_5^*,$$

and

$$u_3 = nirviddh\bar{a}\dot{n}ganarendraru\dot{n}gnigadita \approx \frac{1}{2 \times 3} \left(\frac{\pi}{2}\right)^2 \times 5400' = u_3^*.]$$

[B′₂] (= p. 126, l. 17 – p. 127, 1.4): Alternative form of [B₂]

स्तेनः स्त्रीपिशुनः सुगन्धिनगनुद्भद्राङ्गभव्यासनो
मीनाङ्गो नरसिंह ऊनधनकृद्भूरेव षट्स्वेषु तु ।
आद्यस्याक्कृणितादभीष्टधनुषः कृत्या विहृत्यान्तिम-
स्याप्तं शोध्यमुपर्युपर्यथ फलं स्यादुत्क्रमस्यान्त्यजम् ॥

Translation. In the sequence of six numbers *stena*, *stripiśuna*, …, which are respectively 6 [=6‴], 512 [=5″, 12‴], 30937 [=3′09″37‴], 714324 [=71′43″24‴], 8720305 [=872′03″05‴] and 42410900 [=4241′09″00‴] in

[15] Here notice that, since the *jyā* is given only to the nearest third minute, the further terms in its full series cannot improve upon this degree of accuracy. Incidentally, the process for determining the *jyā* in [B′₁] and the *śaram* in [B′₂] by series (each of six terms) written backwards is readily seen to be an abbreviation of the process of determining the *jyā* and the *śaram* by alternative forms of the series in [B₁] and [B₂], written backwards from any last term. This remark, in the verses for [B₁] and [B₂], carries with it the implication that the series [B′₁] and [B′₂] in the Sanskrit text may be extended to any number of terms.

this order, the first number is multiplied by the square of the given arc (in degrees) and divided by the square of 5400, and the quotient is subtracted from the second number. The remainder after the subtraction is next multiplied by the square of the given arc and divided by the square of 5400 and the quotient thus resulting is subtracted from the third number. This process of multiplication and division followed by subtraction is repeated till all six numbers are used up. The final result will be the *utkramajyā* or *śaram*[16].

[Hence there ensues

$$r-x\approx\left(\frac{s}{c}\right)^2\left(u_2-\left(\frac{s}{c}\right)^2\left[u_4-\left(\frac{s}{c}\right)^2\left\{u_6-\left(\frac{s}{c}\right)^2\left[u_8-\left(\frac{s}{c}\right)^2\left\{u_{10}-\left(\frac{s}{c}\right)^2u_{12}\right\}\right]\right\}\right]\right),$$

where again $c=5400'$ and also

$$u_{12}=stena\approx\frac{1}{11\times 12}\left(\frac{\pi}{2}\right)^2u^*_{10}=u^*_{12}\quad\text{(say)},$$

$$u_{10}=str\bar{\imath}pi\acute{s}una\approx\frac{1}{9\times 10}\left(\frac{\pi}{2}\right)^2u^*_8=u^*_{10},$$

$$u_8=sugandhinaganud\approx\frac{1}{7\times 8}\left(\frac{\pi}{2}\right)^2u^*_6=u^*_8,$$

$$u_6=bhadr\bar{a}\dot{n}gabhavy\bar{a}sana\approx\frac{1}{5\times 6}\left(\frac{\pi}{2}\right)^2u^*_4=u^*_6,$$

$$u_4=m\bar{\imath}n\bar{a}\dot{n}gonarasimha\approx\frac{1}{3\times 4}\left(\frac{\pi}{2}\right)^2u^*_2=u^*_4,$$

and

$$u_2=\bar{u}nadhanak\d{r}dbh\bar{u}reva\approx\frac{1}{1\times 2}\left(\frac{\pi}{2}\right)^2 5400'=u^*_2=4241'\,9''\,0.296'''+.]$$

3. The author of the series [A], [B] and [C] tentatively identified

The *Tantrasaṅgraha-vyākhyā* merely enunciates all three series without proof or mention of their discoverers. Demonstrations, acceptable to modern eyes, are given in the *Yukti-bhāṣa* alone.[17]

In seeking to determine the authorship of the three series, we must be guided for the present by scattered but specific statements in unexpected sources, in default of more cogent attribution. NĪLAKAṆṬHA's *Āryabhaṭīya-bhāṣya*, a late work of his, quotes the verses for [B'_1] as those of a mathematician named MĀDHAVA.[18] It is natural to suppose that [B'_2] as also [B_1] and [B_2] may be

[16] Here again, since the *śaram* is given to the nearest third minute, the further terms in its expansion are negligible to this order of accuracy.

[17] These are reproduced in [2] and [3] with certain (inessential) anachronistic touches which serve to clarify their arguments for us.

[18] See p. 113 of the printed edition of the work (Trivandrum Sanskrit Series, No. 101, Trivandrum, 1930), also p. 180 where NĪLAKAṆṬHA refers to MĀDHAVA as a '*Ganitagñācārya*'. This Sanskrit word conveys the idea of 'Practitioner and Preceptor' (of mathe-

attributed to him, especially since, as we have seen, the verses for [B'_1] and [B'_2] are said to enunciate their mathematical procedures in the same manner as, but more 'briefly' than, the essentially identical procedures in the verses for [B_1] and [B_2].[19] In a discussion of the remaining series, K.V. SARMA[20] has attributed [A], [C_2] and [C_3] to MĀDHAVA, citing in support the *Kriyākramakarī*, which is said to be a commentary on BHĀSKARA II's *Līlāvatī* and of date not later than 1534. However, we would remark that this work, though it contains [A] and [C_2], expressly attributing the latter to MĀDHAVA,[21] does not have [C_3]. It is a very plausible step from this to suppose that [C_3] and *a fortiori* the infinite series for $\pi/4$ implied by [C_2] are MĀDHAVA's discoveries. Equally plausible is the further conjecture that the series [A] for $\theta<\pi/4$, like that for $\theta=\pi/4$, is again due to MĀDHAVA, particularly since both series are proved in the *Yukti-bhāṣa*, without distinction being made between them, by a method essentially equivalent to the quadrature term by term of the series-expansion

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$

with respect to the base variable $x=\tan\theta\leq 1$, the individual integrals of the powers of x being evaluated as limit-sums, essentially as in our modern Riemannian manner.[22]

The mathematician MĀDHAVA to whom we attribute all the series [A], [B] and [C] has been identified by bibliographical researchers as the one who lived

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matics) much the same as 'Professor' in academic parlance today, but without any suggestion of professional status. However, as pointed out in [4], p. 52, it is usual in medieval Keralese literature to honour MĀDHAVA somewhat differently as a '*golavid*' (=master of global science). A justification for this may be found in his only two works (of astronomical computations) which have come down to us intact, *viz.* the *Veṇvāroha* (on the moon's motion) and the *Agaṇita* (on planetary positions). Further, the use of sexagesimals in the verses for [B′] implies an astronomical context. That the rules of approximation in the verses may indeed be a prelude to an astronomical treatise of some sort (as customarily) appears likely from the fragmentary manuscript mentioned in footnote 19, which is one of the sources of [B′].

[19] At the suggestion of Professor Y. SITARAMAN of the Mathematics Department at Kerala University, the Director of the Oriental Research Institute and Manuscripts Library there has kindly furnished a transcript of an extremely fragmentary and damaged manuscript (8358-G: *Jīvānayanam*) earlier brought to our attention by Professor K.V. SARMA and Doctor N. PARAMESWARAN UNNI of the Sanskrit Department, Kerala University. This manuscript contains a significant remark in the first person at the end of the Sanskrit verses for the series [B] and [B′]: "This procedure [embodied in the verses referred to] is briefly set down by me after a fashion *(ityatra yuktissaṅkshepāt-katañcillikhitā mayā)* so that scholars may follow it in all its ups-and-downs *(unātirekata śodyā vicārya viduṣām ganaiḥ)*". The tone of this would suggest that it is an authentic aside by the originator of [B] and [B′], which reaches out to us across the centuries.

[20] See [4], footnotes on pp. 20, 24.

[21] See p. 379 of the Sanskrit text edited by K.V. SARMA (Vishveshvaranand Indological Series, No. 66, Hoshiarpur, 1975).

[22] See [2], p. 68.

about 1340–1425 at Saṅgamagrāma, the modern Irinjālakkuḍa, near Cochin. This MĀDHAVA, even if he be credited with only the discoveries of the series [A] and [B] at so unexpectedly early a date, assuredly merits a permanent place among the great mathematicians of the world. There were before him many who had made notable contributions to algebra and geometry. But it was he who, in India at least, took the decisive step onwards from the finite procedures of 'ancient' mathematics to treat their limit-passage to infinity which is the kernel of modern classical analysis.

The authors are indebted to Dr. D.T. WHITESIDE for his improvement of their original manuscript in several significant points of detail.

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Added: 20 September 1977. The information in our article about the date and authorship of the *Yukti-bhāṣa*, and about the date of the anonymous *Karaṇapaddhati is* at variance with the information in papers [2], [3] cited under References, but essentially in accord with later bibliographic research as presented, for instance, in a paper by

K.V. SARMA: Jyeṣṭhadeva and his identification as the author of the *Yukti-bhāṣa*, *Adyar Library Bulletin*-vol. 22 (1958), pp. 35–40.

and in the Introduction in English to the

Karaṇapaddhati edited by S.K. NAYAR, Madras Government Oriental Series No. 98 (1956).

Further relevant information may be gathered from a paper by

K. KUNJUNNI RAJA: Astronomy and mathematics in Kerala, *Adyar Library Bulletin* – vol. 27 (1963), pp. 118–167.

A pioneer effort in the same direction, of special interest even today, is the paper by

CHARLES M. WHISH: On the Hindú Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four Śástras, the

Tantra Sangraham, Yucti Bháshá, Carana Padhati, and *Sadratnamála, Transactions of the Royal Asiatic Society of Great Britain and Ireland*—vol. 3 (1835) pp. 509–523.

K.V. SARMA's edition of the *Tantrasaṅgraha-vyākhyā* is Part (ii) of the complete work comprising the (i) *Tantrasaṅgraha,* (ii) *Yuktidīpikā* (iii) *Laghuvivṛti* (Panjab University Indological Series—10, 1977, Vishveshvaranand Vishva Bhandu Institute of Sanskrit and Indological Studies, Panjab University, Hoshiarpur). He makes the following positive affirmations in his Introduction. (a) The *Tantrasaṅgraha-vyākhyā* is professedly related to the *Yukti-bhāṣa* because it presents in lucid and succinct texts the mathematics expounded in the later work, this exposition being (as usually regarded) of the basic mathematics needed for the computation of planetary motions as in the *Tantrasaṅgraha* (p. xlv); and the author of the *Tantrasaṅgraha-vyākhyā* is actually referring to the *Yukti-bhāṣa,* not to the *Tantrasaṅgraha,* when he speaks of his work being concerned with the "exposition [80] well stated by the Parakroḍa brahmin" (p. xxxix). (b) There is nevertheless a less obvious but significant relation between the *Tantrasaṅgraha-vyākhyā* and the *Tantrasaṅgraha* in that the former rearranges the material presented in the *Yukti-bhāṣa* in a sequence of groups of verses, in the same order, and headed by the same key words, as verses in the latter (p. xlvi). (c) The author of the *Tantrasaṅgraha-vyākhyā,* to whom is due also the early major part of the *Kriyākramakarī* and the entire *Laghuvivṛti,* is ŚANKARA VĀRIYAR (p. xlix); and the date of the *Tantrasaṅgraha-vyākhyā* is inferentially prior to c. 1534 (p. lxi). SARMA's negative conclusion is that "it is impossible even to visualize the possibility of a larger version of the *Tantrasaṅgraha*" (p. xliv) containing the texts of significance for us. He, however, concedes that the *Tantrasaṅgraha-vyākhyā* is "a repository of the mathematical and astronomical tradition prevalent in Kerala about 1500" (p. lxxvi). We may therefore conclude finally: even if SARMA's position be fully granted, viz. that both WHISH and RĀMA VARMA mistook for the complete *Tantrasaṅgraha* that portion of the work generally named as such today, plus some suggested or actual interpolations from the *Tantrasaṅgraha-vyākhyā,* the statements made by us, about the origin, and especially the authorship, of some major Keralese achievements in mathematical analysis, neither gain nor lose in credibility.

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