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NOTE

Riemann and the Cauchy–Hadamard Formula for the Convergence of Power Series

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The Cauchy–Hadamard formula for the radius of convergence of a power series was stated and proved by Riemann in his lectures of November 1856. This discovery revises the widespread opinion that, after Cauchy's publication in 1821, the formula was ignored until its rediscovery by Hadamard around 1890. © 1994 Academic Press, Inc.

Riemann hat die Cauchy-Hadamard-Formel für den Konvergenzradius einer Potenzreihe in seinen Vorlesungen im November 1856 aufgestellt und bewiesen. Diese Entdeckung führt zur Revision der allgemeinen Meinung, die Formel sei nach ihrer Veröffentlichung durch Cauchy im Jahre 1821 bis zu ihrer Wiederentdeckung durch Hadamard um 1890 unbeachtet geblieben. © 1994 Academic Press, Inc.

Riemann a présenté et démontré dans ses cours, au mois de novembre 1856, la formule de Cauchy–Hadamard sur le rayon de convergence d'une série entière. Cette découverte infirme l'opinion très répandue selon laquelle la formule serait restée dans l'oubli après sa publication par Cauchy en 1821 jusqu'à sa redécouverte par Hadamard vers 1890. © 1994 Academic Press, Inc.

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1. THE FORMULA IN RIEMANN'S NOTES OF 1856

In his *Cours d'analyse* Cauchy [1821, 151, 286 *et passim*] stated and proved that a power series $\Sigma_{n=0}^{\infty} a_n z^n$ has the radius of convergence ρ, where ρ^{-1} equals "the limit or the greatest of the limits" of the sequence $|a_n|^{1/n}$. The well-known story is that this result passed unnoticed until the 1890's, when it was rediscovered by Hadamard [Pringsheim 1898, 81, note 168; Knopp 1922, 148, note 1; Bottazzini 1986, 116]. Therefore, we were surprised to find the formula and outlines of a proof in notes of Riemann's from the years 1855 and 1856. The more advanced of these notes were written when Riemann was preparing the introductory part of his lectures on complex analysis in November 1856. They have been preserved

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in the Riemann Nachlass at the University Library in Göttingen (Niedersächsische Staats- und Universitätsbibliothek Göttingen [= NSUB Göttingen]) in the file Cod. Ms. Riemann 23.4. For facsimile reproductions, see [Neuenschwander 1987, 100–104].

In his lectures of November 1856, Riemann had already shown, by the Cauchy Integral Formula, that $f(z)$ has a power series development which converges in the largest open disk around z_0 containing no singularity, and that it diverges for $|z - z_0| > \rho$ if ρ is the radius of the disk. Then he observed that the value of ρ depends on the behaviour of $|a_n|^{1/n}$ for growing values of n, and that the series does not converge if these quantities grow to infinity.

He stated: If the sequence $|a_n|^{1/n}$ remains bounded, let $f(m)$ denote the least upper bound of $|a_n|^{1/n}$ for $n > m$. The function $f(m)$ is non-increasing. Let $\delta = \lim_{m=\infty} f(m)$. Then, his proposition is that the series converges for $|z| < 1/\delta$, and diverges for $|z| > 1/\delta$. For the original German text, see Appendix [A].

The outline of the proof in his notes contains not a single word. In the following passage we reproduce the mathematical symbols of Riemann's notes, linked by our own English text and a few additions in brackets. Riemann writes mod z for $|z|$.

[*Proof.*] Let $z = re^{\varphi i}$.

Suppose first that $r > 1/\delta$. Since $f(m) \geqq \delta$, $f(m)r \geqq r\delta > 1$. [It follows that some $n > m$ exists such that]

$$\text{mod } (a_n z^n) \geqq (r\delta)^n \ [> 1]. \qquad (*)$$

The series diverges, since its members do not converge to 0. Suppose now that $r < 1/\delta$, or $r\delta < 1$. [Choose a such that] $r\delta < a < 1$, [and μ such that] $rf(m) < a$ for all $m > \mu$. Since

$$(\text{mod } a_n)^{1/n} r < rf(m) \qquad \text{for all } n > m \qquad (**)$$

we obtain

$$(\text{mod } a_n)^{1/n}\, r < a \qquad \text{for all } n > \mu$$

and, finally,

$$\text{mod } (a_n z^n) < a^n \qquad \text{for all } n > \mu.$$

[Since $a < 1$, the series converges.]

The last formula is underlined. In the formula preceding (**), $<$ should be replaced by $\leqq$. Also, (*) should be slightly modified. Riemann is not content with $|a_n z^n| < a^n$, but adds $a_n z^n = b_n + c_n i$, $b_n < a^n$ for $n > \mu$, and, omitting absolute values, writes down estimates of the real parts

$$b_\nu + b_{\nu+1} + b_{\nu+2} + \dots + b_{\nu'} < a^{\nu+1} + a^{\nu+2} + \dots + a^{\nu'} = a^{\nu+1}\frac{1 - a^{\nu'-\nu}}{1 - a}.$$

The introductory remarks, including the definition of $f(m)$, are on fol. 115 verso dated (on the recto) 22 November 1856, which was a Saturday. Fol. 96 recto contains proposition and proof. On fol. 97 recto, dated 28 November, the definition of $f(m)$ is briefly repeated in the margin.

An earlier attempt, presumably dating from the summer of 1855, can be found in Cod. Ms. Riemann 23.4, fol. 198 recto. Riemann says that $\sqrt[n]{\pm a_n}$ cannot become infinitely large, or grow beyond any limit, and defines p as the greatest value of this expression if $n > \nu$. He says that p, as a function of ν, is non-increasing, and lets $\rho = \lim p$. Then, the series $\Sigma a_n x^n$ will converge if $x < 1/\rho$. The obvious proof, by comparison with a geometric series, is sketched (see Appendix [B]). This is obviously confined to real a_n and $x > 0$, and is less than what he could have learnt from [Cauchy 1821]. On the other hand, the introduction of the function $p(\nu)$, or $f(m)$ in 1856, is a valuable step toward a clarification of the lim-sup concept.

2. THE MINOR ROLE OF THE FORMULA WITHIN RIEMANN'S APPROACH

The lectures of winter 1856/1857 were announced for Friday, 12–1, and Saturday, 11–1. Apparently, Riemann prepared his notes shortly before the lectures. In general, he gave very careful summaries of the preceding lectures, and statements of his further plans, written down—with some corrections—in a meticulous form. In contrast to this, the formal contents of his lectures are quite often only briefly sketched.

Riemann did, indeed, present the statement of the formula and its proof in his lectures of 1856. There exist notes by his students E. Schering (Cod. Ms. Riemann 37, pp. 203–205) and R. Dedekind (NSUB Göttingen, Cod. Ms. Dedekind I,15, pp. 17 f.) which are virtually identical with our reconstruction, including separate estimates for real and imaginary parts of the series at the end. Also, there is a very brief discussion of the series for the derivative. The formula does not appear in Riemann's publications, nor did it become a standard part of his lectures.

Riemann never gave a *theory* of real or complex power series, in contrast to [Cauchy 1821]. His treatment of series in the lectures seems to have been cursory, and some of the copies made by his students display errors even in fundamental statements: for example, as a general condition for convergence they have that $s_{n+m} - s_n$ must converge to 0 for $n \to \infty$, for each value of m. Here, s_n denotes the partial sum [Neuenschwander 1987, 35].

A careful statement of the Cauchy convergence criterion can be found on an undated sheet (Cod. Ms. Riemann 23.4, fol. 18): "That an infinite series has a value can be said only if the sum s_n of its first n terms depends on n in such a way that $s_{n'} - s_n$ will become infinitely small if both n and n' become infinitely large; i.e. if any positive nonzero quantity δ be given, one can always find a quantity ν such that, if both n and n' are greater than ν, that is, $n' > \nu$, $n > \nu$, then, if abstraction is made of the sign, $s_{n'} - s_n < \delta$." (See Appendix [C].)

One of the raisons d'être of power series in Riemann's introductory lectures is analytic continuation, and for that purpose it is useful to know that ρ is determined by the singular points of $f(z)$. There is no need to know ρ as a function of the coefficients. In the lectures of 1856, the power series expression of a function is merely instrumental. At the beginning of several of the lectures, he underlined that there was no need to have a function given by an expression. The class of analytic functions is primarily defined by the existence of $f'(z)$, and not by the existence of local power series representations; and each single function should be characterized by the locations and types of its singularities, and, again, not by some explicit expression. (See Appendix [D].)

As was pointed out in [Neuenschwander 1980, 1981, 1987], one should not conclude that Riemann tried to avoid power series as a tool. Rather, we share the following claim [Gray 1986, 32]: "Riemann was prepared to use power series or Fourier series methods to express a function locally. He described such methods in his *Theorie der Abelschen Functionen* of 1857 as standard techniques." To an even greater extent, these remarks apply to Riemann's lectures of later years.

3. WHY DID CAUCHY'S FORMULA FALL INTO OBLIVION?

No careful reader of the *Cours* can overlook the formula for the radius of convergence of power series [Cauchy 1821, 151, 286; application on p. 399; related results on pp. 132, 143, 280—the corresponding pages of *Œuvres* (2), Vol. 3 are 136, 240; 329; 121, 129, 235]. The formula for complex power series (p. 286) is a corollary to the most general version (p. 280) on series with complex terms whose absolute values are ρ_n: let $A \leqq \infty$ be the limit or the greatest of the limits of the sequence $\rho_n^{1/n}$; then the series converges (diverges) if $A < 1$ $(A > 1)$.

Riemann borrowed the *Cours* from the University library in January 1847, and he may or may not have remembered the results of Cauchy. In any case, his proof is an improvement: the auxiliary function $f(m)$, which was not in the *Cours*, helps to make the argument understandable.

We saw that Riemann had no use for the formula in his approach to complex functions. On the other hand, those who were mainly interested in tests for convergence could easily avoid the clumsy "greatest of the limits": $A < 1$ is obviously equivalent to the existence of some $q < 1$ such that $\rho_n^{1/n} \leqq q$ for all $n \geqq n_0$; and $A > 1$ is a very special case of series whose terms do not converge to 0. It follows that Cauchy's formula, and even his more general version on p. 280, are of little practical value.

Still, one might wonder why the formula was not hailed by those who—like the Weierstrassians—considered power series to be the basic class of functions in analysis. The Weierstrassians made much ado about those completeness properties of the real numbers that had been obvious banalities for Cauchy and Riemann. They really should have welcomed the formula—but they did not even look for it!

Hadamard, at the age of 22, stated and proved the formula in a paper presented by Darboux [Hadamard 1888]. His objective was to correct the proof, and generalize the result, of [Lecornu 1887]: If a_n/a_{n+1} or $a_n^{-1/n}$ has a limit z_0, then z_0 is the only singularity of the series on its circle of convergence. Hadamard's proof of the lim-sup formula is correct and straightforward, though less elegant than Cauchy's and even less general.

Neither Lecornu nor Hadamard seems to have been aware of earlier work, and even during the following four years, Hadamard did not look very carefully into earlier texts. In any case, he failed to mention Cauchy as the discoverer of the formula. In his introduction to [Hadamard 1892], he says: "Since the works of Abel and of Cauchy, one knows that to each function which is regular in a certain circle there corresponds a Taylor expansion, and vice versa." He underlines that Weierstrass and Méray *define* a function by that series expansion. We may conclude, therefore, that the *Cours* was, by the end of the 19th century, of little influence.

APPENDIX: EXCERPTS FROM RIEMANN'S NOTES

[*A*] *NSUB Göttingen, Cod. Ms. Riemann 23.4, fol. 115 verso and 96 recto*

$$a_0 + a_1 z + a_2 z^2 \ldots$$

Eine Reihe die nach Potenzen fortschreitet hört auf zu convergiren für einen *bestimmten* Werth des Moduls von z. Es hängt dies davon ab wie sich $(\text{mod } a_n)^{1/n}$ bei wachsendem n verhält. Damit die Reihe überhaupt convergirt ist es nöthig, daß diese Größen nicht in's Unendliche wachsen

$$f(m) \geqq (\text{mod } a_n)^{1/n}, \qquad n > m.$$

$$m < m' \qquad f(m) \geqq f(m'). \qquad fm \text{ nie zunimmt.}$$

$\lim^{m=\infty} f(m) = \delta$, der größte Werth den $f(m)$ nicht erreicht.

It is not clear that $f(m)$ denotes the *least* upper bound. This is indicated, at the end of Riemann's preparations for the preceding lecture, on fol. 115 recto:

Es hängt dies davon ab, wie sich $(\text{mod } a_n)^{1/n}$ bei wachsendem n verhält.

$f(m) \geqq (\text{mod } a_n)^{1/n}$, $f(m)$ der kleinste Werth, welcher diese Eigenschaft besitzt.

$$n > m.$$

$f(m)$ eine Function, welche nie zunimmt.

$$\lim^{m=\infty} f(m) = \delta = f\,\infty.$$

$f\,\infty$ der größte Werth, den sie nicht erreicht.

Fol. 96 recto:

Die Reihe convergirt dann für alle Werthe von z deren Modul $<1/\delta$, und divergirt für alle Werthe deren Modul $>1/\delta$.

The remainder of fol. 96 recto contains the "proof without words" as printed in Section 1.

[*B*] *Cod. Ms. Riemann 23.4, fol. 198 recto*

$$a_0 + a_1 x + a_2 x^2 + \ldots$$

$\sqrt[n]{\pm a_n}$ kann nicht unendlich groß werden / über alle Grenzen wachsen.
p = dem größten Werth von $\sqrt[n]{\pm a_n}$, während $n > \nu$.

p nimmt immer ab, oder wenigstens nie zu, wenn ν in's Unendliche wächst. $\lim p = \rho$. Dann convergirt die Reihe, wenn $x < 1/\rho$ $x\rho < 1$. δ ein Werth, so daß $1 > \delta > x\rho$. Man vergleiche die Reihe[n]

$$\begin{array}{l} a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \\ 1 + \quad \delta + \quad \delta^2 + \ldots + \quad \delta^n + \ldots \end{array}$$

so wird, in Folge von $\delta > x\rho$, zuletzt $\pm\, a_nx^n < \delta^n$ und folglich, [···] $\delta > x\sqrt[n]{+a_n}$ und in Folge von $\delta < 1$, $s_{m'} - s_m$ unendlich klein.

Riemann explains that s_n, S_n, S'_n denote the partial sums of $\Sigma\delta^m$, Σa_mx^m, $\Sigma \pm a_mx^m$ $(= \Sigma|a_mx^m|)$ and observes that:

$$\left.\begin{array}{l} S'_{m'} - S'_m \\ S_{m'} - S_m \end{array}\right\} < s_{m'} - s_m,$$

$$s_{m'} - s_m < \frac{\delta^m - \delta^{m'}}{1 - \delta}.$$

[C] *Cod. Ms. Riemann 23.4, fol. 18 recto*

Von einem Werthe einer unendlichen Reihe kann man nur reden, wenn die Summe der n ersten Glieder, s_n, sich mit n so ändert, daß $s_{n'} - s_n$ unendlich klein wird, wenn sowohl n als n' unendlich groß werden, d. h. wenn irgend eine positive Größe δ, sie sei, was sie wolle, wenn sie nur nicht Null ist, gegeben wird, so muß sich immer eine Größe ν so annehmen lassen, daß, wenn n und n' beide größer als ν also

$n' > \nu,\ n > \nu$ dann abgesehen vom Zeichen $s_{n'} - s_n < \delta$ wird.

[D] *Cod. Ms. Riemann 23.3, fol. 25 recto*

d. 5. Dec.

Um eine Function einer complexen Größe völlig zu bestimmen, ist es durchaus nicht nöthig, daß ein Ausdruck der F[unction] gegeben sei, mit Hülfe dessen man für jeden W[erth] d[er] veränderlichen Größe den zugehörigen Werth der F[unction] finden könne. Wir haben vielmehr in der vorigen Stunde gesehen, daß eine Function einer complexen Größe schon völlig bestimmt ist durch gewisse Eigenschaften, welche die Art wie sie unstetig wird, betreffen.

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