Historia Mathematica **26** (1999), 239–267
Article ID hmat.1999.2250, available online at http://www.idealibrary.com on IDEAL®

Calculus and Analysis in Early 19th-Century Britain: The Work of William Wallace

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Scottish-born William Wallace (1768–1843) was an early exponent of the differential calculus in Britain and translator of French mathematical works. Encyclopaedias published during the early 19th century provided a valuable educational resource, to which Wallace and his colleague, James Ivory, contributed. Wallace's encyclopaedia articles on "Fluxions" and his other analytical writings are examined here, as are his relations with James Ivory, John Herschel, and others. Wallace's long 1815 article on "Fluxions" in the *Edinburgh Encyclopaedia* was the first complete account of calculus, using differential notation, to be published in English. There, he attempted an original and rigorous "doctrine of limits," which deserved more attention than it received. In 1832, while professor of mathematics in Edinburgh, he applied analysis to support the reform of taxation proposed in the Reform Bill. It is suggested that the later neglect of Wallace's achievements is attributable to a mix of personal, institutional, political, and national rivalries. © 1999 Academic Press

William Wallace né en Écosse (1768–1843) fut un interprète précoce du calcul différentiel en Grande Bretagne et traduisit des ouvrages mathématiques de langue française. Les encyclopédies publiées au début du XIXe siècle fournirent une précieuse ressource pédagogique à laquelle contribuèrent Wallace et son collègue James Ivory. Les articles encyclopédiques de Wallace sur les "Fluxions" et d'autres écrits analytiques font l'objet du présent article; et aussi ses relations avec James Ivory, John Herschel et d'autres mathématiciens. Le long article de Wallace sur les "Fluxions" paru dans l'*Edinburgh Encyclopaedia* de 1815 fut le premier exposé complet du calcul, utilisant la notation différentielle, à être publié en anglais. Dans cet article, il mit en oeuvre une "doctrine des limites" originale et rigoureuse, qui méritait plus d'attention qu'elle ne reçut. En 1832, alors qu'il était professeur de mathématiques à Edimbourg, il utilisa l'analyse pour soutenir la réforme fiscale proposée dans le Projet de loi de Réforme. Cet article suggère que l'oubli postérieur de l'oeuvre accomplie par Wallace doit être attribué à un mélange de rivalités personnelles, institutionnelles, politiques et nationales. © 1999 Academic Press

1. INTRODUCTION

British exponents of what we now call the differential and integral calculus were late in adopting the superior formulation of Leibniz and his successors and in finally abandoning the fluxional notation of Newton. The Cambridge Analytical Society has long been credited with introducing continental analysis to Britain [17; 65; 66; 68]. Even recently, the eminent I. Bernard Cohen could write that "[i]n Britain, the mathematical establishment had doggedly clung to the notation of Newton. Whilst still an undergraduate, together with John Herschel and George Peacock, Babbage founded the Analytical Society which flowered briefly during 1812–14: its great and lasting achievement was the introduction of the continental notation for the calculus that enabled a renaissance of British mathematics" [26, 16].

This, however, is now recognized as a distortion. Certainly, in propounding French analytical achievements through their English translation of Lacroix's *Traité élémentaire* [47; 48],

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the Analytical Society eventually influenced mathematics at Cambridge and beyond [30; 91]. But Niccolò Guicciardini [39] clearly demonstrates that the Analytical Society members were not alone in appreciating the importance of the works of the French mathematicians. Before them, Robert Woodhouse in Cambridge, John Playfair in Edinburgh, John Toplis of Cambridge and Nottingham, Charles Hutton at the Royal Military Academy in Woolwich, James Ivory and William Wallace at the Royal Military College in Marlow (later Sandhurst), John Brinkley and Bartholomew Lloyd in Dublin, and the solitary William Spence in Greenock had all been advocates of the French analysts.[1] John Toplis (1774–1857) had been one of the first to complain of the decline of British mathematics [74]; and his 1814 English translation [50] of the first book of Laplace's *Mécanique céleste* was published two years before the appearance of the Cambridge Analytical Society's translation of Lacroix. Even earlier, in 1809, William Spence had published the first of his impressive analytical essays [69]. Also, the little-known John West (1756–1817), who had emigrated from Scotland to Jamaica, left impressive analytical manuscripts influenced by Lagrange and Arbogast. Though probably written by about 1810, these were not published till long after his death [87; 26].

Playfair, West, and Ivory received their early education at the University of St Andrews in Scotland. So, too, did John Leslie, an accomplished physicist who replaced Playfair in the mathematics chair of Edinburgh University. Leslie's successor in that chair was Wallace, who had earlier studied under Playfair and been a colleague of Ivory. Though Wallace and Ivory were early pioneers of analytical methods in Britain, Wallace was also a stout supporter of the geometrical tradition in education. Wallace's broader educational concerns and his rivalry in Edinburgh with John Leslie will be examined elsewhere.

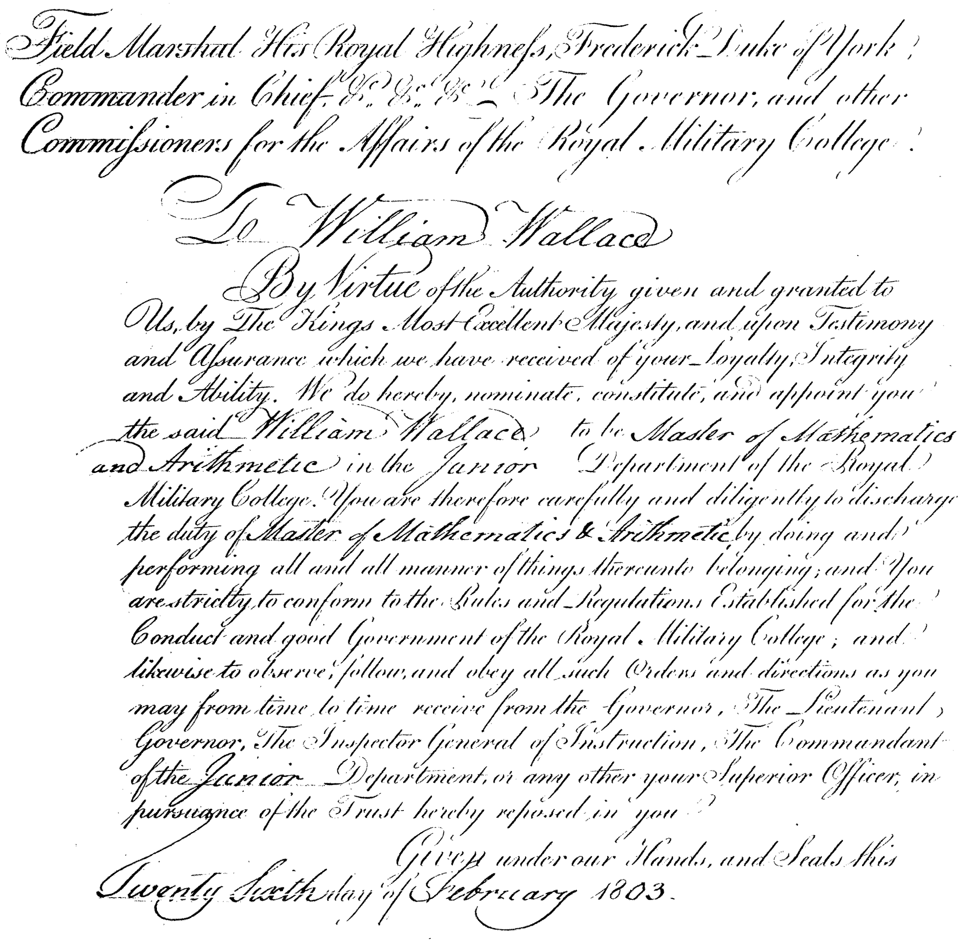
This paper centres on Wallace's analytical work and draws attention to his attempt at a rigorous "doctrine of limits" as well as to some work on infinite series. Later, as professor of mathematics in Edinburgh, Wallace took part in the debate over the proposed Reform Bill. His two mathematical papers in support of the taxation proposals are described. The uneasy personal relationship of Wallace and Ivory and their writing for encyclopaedias are explored, and their good relations with Herschel and Babbage are outlined. Finally, the later neglect of the work of Wallace and Ivory by mainly pro-Cambridge reviewers and historians is examined.

2. WALLACE AND IVORY

William Wallace (1768–1843) was born in Dysart, Fife on 23 September 1768, and received no formal schooling after the age of eleven. His father was a leather manufacturer, and William first worked as a bookbinder's apprentice. When his father moved to Edinburgh, William "became bookseller's shopman, acting as a private teacher and attending classes at the university" [71], where he was encouraged by John Robison and by Playfair. In 1794, he was appointed assistant teacher of mathematics in Perth Academy, then moved in 1803 to the Royal Military College, Marlow (which later transferred to Sandhurst); see Fig. 1. There, he was joined a year later by James Ivory.

At Marlow, he and Ivory quickly became involved with Thomas Leybourn's *Mathematical Repository*, which was produced by Military College staff. In addition, they wrote articles for the *Encyclopaedia Britannica* and prepared textbooks for their students. Wallace also

[1] On the Irish contribution, see also [35].



Field Marshal His Royal Highness, Frederick Duke of York; Commander in Chief &c. &c. &c. — The Governor, and other Commissioners for the Affairs of the Royal Military College:

To William Wallace

By Virtue of the Authority given and granted to Us, by The Kings Most Excellent Majesty, and upon Testimony and Assurance which we have received of your Loyalty, Integrity and Ability. We do hereby, nominate, constitute, and appoint you the said William Wallace to be Master of Mathematics and Arithmetic in the Junior Department of the Royal Military College. You are therefore carefully and diligently to discharge the duty of Master of Mathematics & Arithmetic by doing and performing all and all manner of things thereunto belonging; and You are strictly to conform to the Rules and Regulations Established for the Conduct and good Government of the Royal Military College; and likewise to observe, follow, and obey all such Orders and directions as you may from time to time receive from the Governor, The Lieutenant Governor, The Inspector General of Instruction, The Commandant of the Junior Department, or any other your Superior Officer, in pursuance of the Trust hereby reposed in you.

Given under our Hands, and Seals this Twenty Sixth day of February 1803.

FIG. 1. Warrant of appointment of William Wallace as "Master of Mathematics and Arithmetic in the Junior Department of the Royal Military College" dated 26 February 1803. (An identical warrant records the appointment of James Ivory on 25 January 1804.) Reproduced by permission of the Royal Military College, Sandhurst.

wrote for Brewster's *Edinburgh Encyclopaedia*, most notably the long article "Fluxions" discussed in Section 3 below, and he published various mathematical papers.

Wallace was aged 51 when appointed to the Edinburgh chair in 1819. There, he wrote further textbooks and a few more papers, contributed to popular geographical works, invented mechanical instruments, and supported the Edinburgh Observatory at a crucial time. (A partial list of his writings is given in [60]; some others are included in [75–84].) When ill health forced his retirement in 1838, the University awarded him the honorary degree of Doctor of Laws [9; 21].

James Ivory (1765–1842) was born in Dundee on 17 February 1765, the son of a watchmaker. From 1779, at St Andrews University, along with John Leslie, he was taught mathematics by John West and Nicolas Vilant [62; 25]. Both Leslie and Ivory then proceeded to

Edinburgh University, ostensibly to study divinity though neither became ordained. There, they came under the mathematical, physical, and philosophical influences of Playfair, John Robison, and Dugald Stewart; and they subsequently met Henry (later Lord) Brougham, who shared their mathematical and scientific interests.

For three years from 1786, Ivory taught mathematics and natural philosophy at an academy in Dundee. He then became partner in a pioneering mechanized flax-spinning company with water-driven mills at nearby Douglastown. There he remained, studying mathematics and writing articles in his spare time, until the company failed in 1804. Soon thereafter, he joined William Wallace as a professor of mathematics at the Royal Military College, Marlow. By then, he seems to have read and understood much of the higher analysis, and the mechanical and astronomical writings, of Laplace and Lagrange.

But Ivory's stay at the Royal Military College was not a happy one. He quarrelled with the Governor and with Wallace, and many extant letters suggest that he suffered from bouts of mental illness. Despite his difficulties, he was at this time probably Britain's most accomplished mathematician. Ill health forced his resignation in 1816, with a pension, and he moved to London, where he lived reclusively for the rest of his life. Enforced freedom from teaching duties allowed ample time for scientific and mathematical writing, both for journals and for the *Encyclopaedia Britannica*, but recurrent mental instability made him quarrel with and become suspicious of most of London's scientific community. Nevertheless, he was awarded three gold medals by the Royal Society of London, and he was one of very few British scientists of this period whose work was taken seriously by leading Continental scholars, including Lagrange, Poisson, Liouville, and Jacobi [8].[2]

While Ivory was in London, Wallace held the Edinburgh chair, but his health had broken down in 1835, when he was 65 years of age, and he apparently did not teach from that point until he retired from the post in 1838. Wallace remained bitter that, in 1821, despite support by the Commissioners of the Royal Military College, the Government had not agreed to pay him a pension in recognition of his 17 years of service. This bitterness was compounded by the fact that Ivory had been granted one on grounds of ill-health.[3] Accordingly, in 1835 and already ill, Wallace addressed a new petition to the King, requesting a pension in recognition of his services at Sandhurst and Edinburgh. This is described in a letter to the then Lord Chancellor, Lord Brougham [5h], incorporating Wallace's own assessment of his achievements. This makes plain the importance he attached to his innovatory work on analysis. This letter reinforces the sentiments of his 1833 letter to George Peacock, discovered by Maria Panteki [60], and is reproduced in the Appendix (see also Section 6).

[2] Ivory was a member of many foreign scientific societies and received an honorary degree from his *alma mater*, St Andrews University. In 1831, he was awarded a Knighthood of the Royal Guelphic Order, along with John Leslie, David Brewster, John Herschel, and Charles Bell [58, 347]. The Royal Guelphic Order was a Hanoverian knighthood, no longer extant. Even then, there was some doubt as to its status. Leslie was reluctant to accept what he thought to be a lesser award than that previously given to an Edinburgh colleague [5d]; and Brewster enquired whether or not it entitled him to use the appellation "Sir" [5a]. Ivory, in contrast, was reluctant to accept it, as being "unsuitable to [his] present circumstances," but was keen to receive an enhanced pension [5b,c]. A fuller account of Ivory's life and works is in preparation by the author.

[3] "Prevented ... from making the least provision for futurity ... [and with] a numerous family, I was obliged to spend the hours which I might have given to relaxation in hard study and in writing for the Encyclopaedia, in order to enable me to bring up and educate my family" [5e]. Wallace mentions that, while at the R. M. College, he had lived alone while his children were being educated in Edinburgh.

No doubt through Brougham's good offices, Wallace was granted a pension of £300 per annum. In 1839, his health dramatically improved:

> after more than three years seclusion, confined a great part of the time to my bed, I have appeared again ... on the stage of human existence. Though unable to walk and almost to stand, I never ceased to think [and write works on geometry and conic sections]. ... My recovery is quite a phenomenon here ... I this day accompanied by Mr Kelland my successor in the Professorship of Mathematics here ascended Arthurs Seat which is upwards of 800 feet high and jointly with him observed many angles to determine the variation of the Compass there. [4c]

(See also [5i]). But recovery was fairly brief, and he died on 28 April 1843. In these last few years of his life, he published several more mathematical papers, some perhaps on work done long before (see Panteki's list of works [60]).[4]

Figure 2 reproduces a portrait of Wallace.

3. THE ENCYCLOPAEDIAS

The immodestly titled but now little-known *Encyclopaedia Perthensis or Universal Dictionary of Knowledge Collected from Every Source and Intended to Supersede the Use of All Other English Books of Reference ... in Twenty-Three Volumes* [6], published during 1796–1806, was mainly the work of its editor, Alexander Aitchison, and its publisher, William Morison. The preface mentions that "names of the contributors are announced," but I have found no list. Perth, a Scottish "royal burgh," was and is quite a small town, and it seems unlikely that the young schoolmaster, William Wallace, was not involved in the venture. In the article "Analysis" we read:

> Analysis in mathematics, is properly the method of resolving problems by means of algebraic equations; whence we often find that these two words, *analysis* and *algebra*, are used as synonymous. Analysis, under its present improvements, must be allowed the apex or height of all human learning; It is this method which furnishes us with the most perfect examples of the art of reasoning; gives the mind an uncommon readiness at deducing and discovery, from a few data, things unknown; and, by using signs for ideas, presents things to the imagination, which otherwise seemed out of its sphere: By this, geometrical demonstrations may be greatly abridged, and a long series of argumentations, wherein the mind cannot without the utmost effort and attention, discover the connections of ideas, are hereby converted into sensible signs, and the several operations required therein effected by the combination of those signs. But, what is more extraordinary, by means of this art, a number of truths are frequently expressed by a single line, which, in the common way of explaining and demonstrating things, would fill whole volumes. Thus, by mere contemplation of one single line, whole sciences may sometimes be learnt in a few minutes time, which otherwise could scarce be attained in many years.

The article then distinguishes "Analysis of Finite Quantities" or algebra and "Analysis of Infinites, called also the New Analysis" and refers the reader to the articles "Algebra" and "Fluxions."

The above quotation shows a striking early commitment to the "New Analysis" and is in marked contrast to the reservations expressed by others, such as John Leslie, around this time. A tempting supposition is that the author was William Wallace himself, who certainly advocated analysis a few years later when at the Royal Military College. However, the above passage also appears, almost verbatim though less prominently, in the article "Analysis" of Charles Hutton's *A Mathematical and Philosophical Dictionary* [43, 1:107]. Hutton's

[4] The sale catalogue of Wallace's extensive library [86] is extant.

FIG. 2. Portrait of William Wallace. Pencil and chalk drawing by Andrew Geddes, 24.2 cm × 17.9 cm. Scottish National Portrait Gallery, Edinburgh, Cat. No. PG 195; reproduced by permission of the Trustees. (Edinburgh University possesses a formal oil portrait of Wallace by John Thomson of Duddingston.)

article, however, first gives a lengthy description of *geometrical* analysis, beginning with Pappus, that is not present in the Perth version. Later, for David Brewster's *Edinburgh Encyclopaedia* [18], the article "Analysis" was written by none other than John Leslie, who confined it *entirely* to geometrical analysis, while Wallace wrote extensively on "Fluxions."

Examination of the *Encyclopaedia Perthensis* article on "Fluxions" shows close similarities to parts of the corresponding article in the third (1797) edition of the *Encyclopaedia*

Britannica. It also erroneously copies some diagrams from the latter, which are not needed in the altered version! The historical introduction to the Perth "Fluxions" article is also very similar to that in Abraham Rees's *Cyclopaedia or Universal Dictionary of Arts, Sciences, and Literature* [62] of 1819, which began publication in 1802. The most that can be said is that Wallace *may* have been involved in selecting material from earlier sources for the mathematical articles of the *Encyclopaedia Perthensis*, and, if so, he would have been likely to choose extracts that matched his own views.

Wallace and Ivory's early work on "continental" analysis was propagated through the medium of their colleague Thomas Leybourn's *Mathematical Repository*, a periodical miscellany of articles, news, and problem competitions based at the Military College. From 1803 onwards, alongside traditional geometry, this published articles on calculus and analysis, many of them written by Ivory and Wallace, who used a variety of pseudonyms as well as their own names.[5] Among Wallace's contributions were translations of recent French memoirs, including Lagrange on the trigonometry of spherical triangles and on "numerical analysis on the transformation of fractions" and Legendre on "elliptic transcendentals" [39, 115–117; 51; 60].

Wallace also wrote the article "Fluxions" for the fourth (1810) and later editions of the *Encylopaedia Britannica* and for Brewster's *Edinburgh Encyclopaedia* (1808–1830). Much of his fourth-edition *Encylopaedia Britannica* article [76] is expressed in conventional Newtonian notation. However, his article "Fluxions" in the *Edinburgh Encyclopaedia* [77], which first appeared in 1815, now seems rather inappropriately titled, for it was "the first complete English treatise on the calculus written in differential notation" [39, 120], apart from John West's manuscript treatises [87; 26]. A much-revised version of Wallace's *Encyclopaedia Britannica* article on "Fluxions" eventually appeared around 1842, in the seventh edition, now entirely expressed in differential notation.[6]

Wallace regretted accepting Brewster's commissions for the *Edinburgh Encyclopaedia*, when later invited by Macvey Napier to contribute to the new *Supplement* of the well-established *Encylopaedia Britannica*:

> I was just in the middle of an article preparing for Dr. Brewster and about to begin another ... I mean the article *Fluxions* ... I do not recollect any other besides *Geometry* for which I am engaged to him (I mean of any length). You know I am bound by a matter of common honesty to fulfil my engagements with him, which were made before your work came on the field. But these being fulfulled ... I am willing to give every moment of my leisure to your work.
>
> ... Mr Ivory and myself have received an order from the Governor of our College to prepare a Course of Mathematics for its use. This requires a short treatise on Mensuration[,] a Work on Algebra and another on Arithmetic (the Geometry is already printed). I do not suppose that this can be a very serious labour, but I am sorry that it should have occurred at present.
>
> ... I repeat that independently of any pecuniary considerations I am disposed to do whatever can be expected of me. [2a]

[5] *Mathematical Repository* identifications are given in [92]. Also, the Edinburgh University copy of Leybourn's *Mathematical Repository* **3** (1814) has annotations, probably in William Wallace's hand, identifying himself as Hypatia, X, Peter Puzzle, Trigonometricus, G.V., and Edinburgensis, and M. Noble as Hermodorus.

[6] There is no updated article on "Fluxions" in the earlier *Supplement to the 4th, 5th and 6th Editions* of the *Encyclopaedia Britannica*, which appeared around 1824. But this *Supplement* contains an article on "Fluents" by Thomas Young (misleadingly identified as "S.F.") that incorporates a long and useful list of integrals. However, Young believed that Ivory would have done an even better job [2h].

In practice, the titles under which articles appeared were often determined by publication deadlines. A list of Wallace's articles is given in [60]. Wallace had failed to provide for Napier an article on "Analysis" by the due date, suggesting it be postponed to "Mathematical Analysis" [2b]. Ivory, then on less-than-friendly terms with his colleague and no doubt aware of Wallace's default, later wrote to Napier that "I was surprised to find nothing under the head *Analysis*. Yet it is such subjects, on which little is said in books in general circulation, that we should expect information in the Encyclopaedia" [2c]. Time had also been too short for James Ivory to write on "Calculus," though he considered that "there is room for an additional article to give an account of the Improvements in the New *Calculi*, and the new views that have been given of their fundamental principles" [2d].

Ivory himself failed to supply the commissioned article on "Curve Lines," having been "persecuted here by the stupid Colonel who is the head of this Institution" and so "obliged to withdraw myself from the power of this petty Tyrant. In short I am no longer a Member of this College ..." [2e]. Just one of many indications of Ivory's state of mind around this time is given in a letter from Wallace to Napier:

> There is a passage in the Conclusion of your letter which ... gave me great surprise and vexed me exceedingly ... respecting my having said something to Mr Ivory derogatory to you ....
>
> The truth is, Mr Ivory has somehow taken up an opinion that I am saying all manner of evil of himself. It is wonderful to me how such an idea could have come into his mind. For on the contrary during the last ten years I have been most anxious to serve him and have never missed an opportunity of doing every thing I could for his interest and convenience. I have remonstrated with him on the absurdity of the idea and I hope not without effect, but I am sorry to say with the greatest powers of mind and the most honourable and excellent intentions he is subject to this unhappy malady of believing that people who would be happy to see him are yet of a quite contrary disposition. I say this between ourselves and only with a view to account for what otherwise would be inexplicable .... [2b]

With uncertain health and now retired to London, Ivory was nevertheless able to complete several articles, including the important one on "Equations" [44], which he himself justly described as "affording a new view of a subject not hitherto much discussed." In this, "availing [him]self of M. Gauss's discovery" on inscribed regular polygons, he gives an elegant solution of the division of the circle, illustrating it with examples of 11 and 17 parts. Though he claims this as "an entirely new solution" [2f], he had himself much earlier given it anonymously in Leybourn's *Mathematical Repository*, and a similar proof is to be found in John West's then unpublished *Mathematical Treatises* [87, 2:36–38; 26, 62–63]. Leslie later commended this article "by my illustrious friend Mr Ivory ... as the most able and profound dissertation that has yet appeared" [52, 4th ed.: 329].

The enthusiasm with which Scottish scientists supported the encyclopaedias may partly be explained by their need to earn money. Also important was the egalitarian philosophy of the Scottish universities and the high regard for education then held by the populace at large. Parents of modest means made sacrifices to send clever sons to university, and scholarships were provided by a few philanthropists (such as the Earl of Kinnoul, who helped fund the St Andrews studies of John West, James Leslie, and others). But many could not attain a university education, and there was great popular demand for technical and scientific, as well as cultural and historical knowledge. This the encyclopaedias set out to supply, and Playfair, Brewster, Leslie, Ivory, and Wallace were all major contributors. So, too, was Thomas Carlyle, a former Edinburgh student and a reluctant teacher of mathematics, who went on to establish a high literary reputation. Many of Carlyle's early letters describe his mathematical preoccupations and ambitions [67]. These are considered both here and in a subsequent paper.

In fact, many encyclopaedia articles, and especially mathematical ones, were serious contributions to scholarship that must have gone above the heads of most readers. Even the talented Carlyle, with the benefit of an Edinburgh University education, had struggled over Wallace's "Fluxions" [56]. In his 1835 letter to Brougham (see Appendix), Wallace proudly claimed that his earlier "Treatise on Fluxions in the Encyclopaedia Britannica fourth Edition was read with avidity by the Cambridge men when it came out, who guessed it to have been written by Playfair."

The *Edinburgh Encyclopaedia* was mainly written by home-based Scots, though a few, such as Wallace and Ivory, were living in England. David Brewster did have wide contacts, though, and solicited articles from elsewhere. Non-Scottish contributors on mathematics and physics included Charles Babbage, John Herschel, George Peacock, Peter Barlow, John Pond, and (the Irish) Dionysius Lardner from England; Jakob Berzelius and Jean B. Biot from France; William Muller of Göttingen; and Hans C. Oersted from Denmark. For the *Encyclopaedia Britannica*, Playfair and Leslie not only wrote major articles, but also advised the editor, Macvey Napier, in his search for contributors and wrote letters to likely candidates [2]. Sometimes their overtures met with rebuff, as from Cambridge's Robert Woodhouse.[7] But many others, including Thomas Young, Charles Babbage, Edward ffrench Bromhead, Dominique F. J. Arago, and Jean B. Biot, accepted their invitations. Surprisingly, Herschel did not write for this Supplement, but he wrote for the *Encyclopaedia Metropolitana* and made substantial contributions to the later eighth edition of the *Encyclopaedia Britannica* [37].

4. WALLACE ON FLUXIONS

Wallace's encyclopaedia articles on "Fluxions" repay close scrutiny. Those in both the *Encyclopaedia Britannica* and the *Edinburgh Encyclopaedia* begin with extensive historical reviews of continental and British contributions. In his 1810 article in the 4th edition of the *Encyclopaedia Britannica*, his preferences are clear:

> ... in explaining the foundations of the method, we have endeavoured to show, that it rests upon a principle purely analytical, namely the theory of limiting ratios; and this being the case, the subject may be treated as a branch of pure mathematics, without having occasion to introduce any ideas foreign to geometry ....
>
> Sir Isaac Newton, however, in first delivering the principles of the method, thought proper to employ considerations drawn from the theory of motion. But he appears to have done this chiefly for the purpose of illustration, for he immediately had recourse to the theory of limiting ratios, and it has been the opinion of several mathematicians of great eminence ... that the consideration of motion was introduced ... without necessity .... [76, 8:705]

He describes the conventional Newtonian version, employing the concept of velocity, but only after giving an analytic account. In the latter, he argues that, if *u* denotes some function

[7] "I ought earlier to have acknowledged your letter, which, together with one from Prof^{r}. Playfair I received on Monday last. My tardiness, however, has not originated from hesitation; for it does not suit my Inclination and plans to engage in an undertaking such as you have spoken to me of ..." [2g].

of x, and u' that same function of $x+h$, where h is some increment, then

$$u' - u = ph + qh^2 + rh^3 + \cdots,$$

where $p, q, r, \ldots$ are functions of x that are independent of h. Accordingly, the ratio $(u'-u)/h$ "may be resolved into two parts; one of these, viz. p, is independent of the increment h, and the other, viz. $h(q+rh+\cdots), \ldots$ may become less than any assignable quantity" if h "be continually diminished." Therefore, the *limiting ratio* is simply p, the function which is the coefficient of the first power in x. But, though such a limit is the "quantity to which the ratio may approach nearer than any assignable difference, ... it cannot be considered as becoming absolutely equal" [76, 8:704]. Though the notation has similarities with that of Lagrange [49], Lagrange avoided terms like "limiting ratio," which implied a limit process. Instead, he *defined* the coefficients $p, q, \ldots$ in the series with h *finite* to be the successive derivatives of $u(x)$ (see, e.g., [32, Chap. 3]).

Using the binomial theorem, which, following Lagrange, he regards as a result given by algebra alone, Wallace establishes classes of functions that may be so expanded, namely, any "rational and integer function" of the form $Ax^\alpha + Bx^\beta + Cx^\gamma + \cdots$, where A, B, C, α, β, γ are constants; the nth power of any such function; and any quotient of two such functions; also functions a^x, where a is constant. The procedure is illustrated by the examples x^2, x^3, and x^4. This, of course, sidesteps the issue of the derivation of the binomial theorem for noninteger powers, which is best obtained from Taylor's theorem. Since this would entail circular argument, the account is subject to objections similar to those raised against Lagrange's *Théorie des fonctions analytiques* [49], namely, that infinitesimals and limits cannot logically be avoided.

In fact, Wallace seems to have been aware of this deficiency, for he returns [76, 8:707] to the series expansion of $(1+v)^n$ to show that the second term is indeed nv for all rational positive or negative powers n (as Lagrange had also done), but he does not address subsequent terms, the remainder, or the convergence of the series.

Wallace's *Edinburgh Encyclopaedia* article on "Fluxions" adopts a fundamentally different approach, avoiding all use of the binomial theorem. The article is long and comprehensive, occupying 186 quarto pages with double columns of rather small print. On its publication, Wallace sent a copy to John Herschel, explaining that

> I have attempted some novelty in establishing the principles; There are certain analytical formulae towards the beginning which I believe are new at least in their form, concerning which I should not be sorry to have your opinion. You will observe that I have attempted to dispense with the Binomial theorem or any development depending on it and yet I think I have obtained general results. I have also attempted to bring Logarithmic and Exponential Formulae within the power of the Common Algebraical processes but on these I refer you to the work itself. Of course you will not expect that in a work which must be hastily composed there should be a great share of novelty. You will perhaps however find a little in an analytical solution to the problem of finding Rectifiable Curves and in an easy investigation of Fagnanis Theorem. [4a]

Herschel duly replied that

> It might appear presumptious in me to offer any opinion on the article you were so kind to send me. I can however say with perfect sincerity that I was very highly gratified with the elegant manner in which the doctrine of limits is laid down. This doctrine is perhaps the best basis for an elementary treatise

> of that kind, though I should be sorry to see it made the groundwork of a very extensive work, as it is impossible not to see the difficulties in which it involves the discussion of imaginary functions. —The expressions for the circular arcs, and logarithms are exquisite—I had seen them before in the Edinburgh Transactions (I believe) and I am very glad to find a greater degree of publicity given them, which they richly deserve .... [4d][8]

Herschel's reservations about making "the doctrine of limits ... the groundwork of a very extensive work" are in line with those expressed in the Analytical Society translation of Lacroix's work on calculus. Though Lacroix had employed a derivation using limits, the Babbage–Herschel–Peacock translation [48] incorporates additional notes giving the alternative Lagrangian derivation based on the binomial and Taylor's theorems. Their preference for this approach was doubtless influenced by Woodhouse's 1803 *The Principles of Analytic Calculation* [93].[9] But it is unclear why Herschel thought that, in the doctrine of limits, "it is impossible not to see the difficulties in which it involves the discussion of imaginary functions." Though there were still unresolved difficulties about imaginary quantities, why did Herschel believe that his own approach was superior?

The bulk of Wallace's article need not be examined in detail. Suffice to say that it gives a rather comprehensive account of differential and integral calculus, with many examples involving algebraic, trigonometric, and logarithmic functions; differential equations; and calculation of arc lengths, areas, and volumes. In this respect, it is not much different from other French works that Wallace cites. After an historical introduction, he "mention[s], with regret, that there is not a book in the English language from which anything like a tolerable knowledge of the fluxional or differential calculus, in its present improved state, can be obtained. Such as wish to study this science beyond its mere elements, must have recourse to the writings of Euler, or to the French treatises" [77, 388]. In particular, he notes that a second edition of Lacroix's three-volume treatise [46] "is now printing" and gives a comprehensive page-long list of other works [77, 388–389]. The earliest of these are John Craig's 1685 *Methodus* and Newton's 1687 *Principia*; others include works by Maclaurin, Euler, Landen, Lagrange, Cousin, and Bossut. His most recent references, from 1811, are works on calculus by Du Bourguet, Garnier, and Legendre.

Here, we focus mainly on section 1 (pp. 390–401) of Wallace's article, entitled "Fundamental Principles of the Theory of Fluxions." This is the section that he considered to be the most novel, of which Herschel commended the "elegant manner," as quoted above. Jettisoning the approach of his earlier *Encyclopaedia Britannia* article, Wallace sets out to avoid the binomial theorem. By careful, and quite lengthy, arguments based on algebra and strict use of inequalities, he solves his "Fundamental Problem" of finding upper and lower

[8] The "exquisite" results for "the circular arcs, and logarithms" appear on pp. 407–408. The former involves expansion in terms of tangents with repeatedly halved arguments, which contrasts with the more usual expansions in increasing multiples of the argument. These originate from Wallace's 1812 article in the *Transactions of the Royal Society of Edinburgh* [75], where he had shown that

$$\frac{1}{a} - \frac{1}{\tan a} = \tfrac{1}{2}\tan\tfrac{1}{2}a + \tfrac{1}{4}\tan\tfrac{1}{4}a + \tfrac{1}{8}\tan\tfrac{1}{8}a + \cdots,$$

together with certain series for $(\log x)^{-1}$ and $(\log x)^{-2}$. But, as he only later discovered, Euler had obtained such results before him. I have not found the above series in any modern compilation, but it may be recovered from the often-quoted infinite product $\Pi_{k=1}^{\infty}\cos(a/2^k) = (\sin a)/a$ on taking logarithms and then differentiating.

[9] See also Becher [13; 15].

bounds for the function $(v^p - 1)/(v^q - 1)$, where p and q are any whole numbers, and both q and the variable v are positive. He shows that, both when v is less and when it is greater than unity, "we have always

$$\frac{1 - v^p}{1 - v^q} = \frac{p}{q} v^{z(p-q)}, \tag{A}$$

and here z is a certain quantity, which in every case is of an intermediate magnitude between 0 and 1; and in the case of p being a negative number, is contained between the narrower limits of $-p/(-p+q)$ and $q/(-p+q)$" [77, 392]. These results extend an idea introduced in Wallace's earlier paper [75, 319–321].

Wallace's "novelty in establishing the principles" of the calculus went unappreciated by Thomas Carlyle, who commented unfavourably on this, in letters to his friend Robert Mitchell written on 19 November 1817 and 16 February 1818:

> His introduction, it must be confessed, is ponderous and repulsive. His horrour of the binomial theorem leads him into strange bye-paths. But he demonstartes [sic] with great rigour. The worst of it is, we are led to his conclusion, as it were thro a narrow lane—often, by its windings shutting from our view the object of our search—and never affording us with a glimpse of the surrounding country[10]—I wish I had it in my head—But, unless I quit my historical pursuits, it may be doubted whether this will ever happen. [67, 1:112]
>
> I have found his demonstrations circuitous but generally rigorous. Yet I must except the proof of Maclaurin's theorem in pag[e] 4 [14?]—which, if I were not a little man & Wallace a great, I should have small hesitation to pronounce unsatisfactory not to say absurd. [67, 1:120]

Wallace's "Fundamental Problem" is doubtless one of the "strange bye-paths" that Carlyle complained of, but Wallace puts it to good use, applying the substitution $(x + h)/x = v^q$ to obtain the ratio of increments

$$\frac{(x + h)^n - x^n}{h} = x^{n-1} \frac{v^{nq} - 1}{v^q - 1},$$

where n is "any quantity whatever." He then claims that

> [a]s we are at liberty to suppose q to be any [whole] number we please, it may evidently be assumed such a positive integer that nq shall be a positive or negative integer. Let $nq = p$, and then by formula (A), ...
>
> $$\frac{v^p - 1}{v^q - 1} = \frac{p}{q} v^{z(p-q)} = n(v^q)^{z(n-1)},$$
>
> and consequently,
>
> $$\frac{(x + h)^n - x^n}{h} = nx^{n-1} \left\{ \frac{x + h}{x} \right\}^{z(n-1)};$$
>
> ... [where] z denotes some function composed of x, h, and n, which is always an intermediate magnitude between 0 and 1. [77, 392]

[10] Carlyle's sentiments here show clear influence of the Scottish "Common Sense" school of philosophy (see, e.g., [27; 59]). His mathematical training under John Leslie was strongly geometrical, but he also had some familiarity with calculus.

It follows that the desired ratio of increments must lie between nx^{n-1} and $n(x+h)^{n-1}$. Accordingly, as h decreases, the limit of the ratio of increments is nx^{n-1}. Wallace then gives the alternative form

$$\frac{(x+h)^n - x^n}{h} = n(x+h')^{n-1},$$

where h' lies between 0 and h. Also, as $(x+h')^{n-1} = x^{n-1} + (n-1)(x+h'')^{n-2}h'$, where h'' lies between 0 and h', he deduces that [77, 392–393]

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + n(n-1)(x+h'')^{n-2}h'.$$

If repeatedly extended, this process does *not* readily yield the binomial series, due to the appearance of even more unknown quantities h', h'', etc. Wisely, Wallace does not pursue this line. The above results are rigorously derived and, as Wallace claimed in the letter to Herschel quoted above, "new at least in their form." Though the final results resemble special cases of Lagrange's remainder formula—and also a much earlier geometrical one of Maclaurin [33, 399]—Wallace's approach *via* his "Fundamental Problem" is his own.

There is just one fly in the ointment. Wallace's assertion that there exists a q such "that nq shall be a positive or negative integer" holds only for *rational* powers n. It seems that Wallace appreciates this limitation, but that it does not greatly bother him, for, a little later, when discussing logarithms, he briefly dismisses a similar restriction (see below). Though perhaps intuitively aware that extension from rational to all real values could be accomplished by considering a further limiting process, he does not do so.

He uses the same fundamental result (A) to deal with logarithms in a rather unexpected way. Defining $x = b^u$, where u is the logarithm of x to the base b, he seeks upper and lower bounds for u. To do so, he lets m, n denote any two given numbers. "Then, whatever be the values of the quantities x and b, provided they are both greater than unity, it will always be possible to find two whole numbers p and q, and a positive quantity v, such that $v^p = x^{1/n}$, $v^q = b^{1/m}$. This requires $p/q = (m\log x)/(n\log b)$, and the right-hand side, though "incommensurable, ... may be expressed by numbers as near to perfect accuracy as we choose" [77, 394]. With this proviso, he deduces from his result (A) that

$$u = \frac{b^{z/m}}{x^{z/n}} \cdot \frac{n(x^{1/n}-1)}{m(b^{1/m}-1)},$$

where z lies between 0 and 1. It follows that bounds for $u = \log_b x$ are

$$\frac{n(x^{1/n}-1)}{m(b^{1/m}-1)} \quad \text{and} \quad \frac{b^{1/m}}{x^{1/n}} \cdot \frac{n(x^{1/n}-1)}{m(b^{1/m}-1)},$$

which "are remarkable on account of their involving two arbitrary quantities m and n which have no apparent connection with the function they serve to express" [77, 394]. Further, taking "m and n to be as great as we please, we can make z/m and z/n each nearly $= 0$ ....

In effect, therefore, we may consider, that

$$\log.x = \frac{n(x^{1/n} - 1)}{m(b^{1/m} - 1)}$$

provided we do not limit the magnitude of the m and n, but regard them as greater than any assignable numbers," a result that is "valuable, because it identifies logarithmic and exponential expressions with common algebraic quantities" [77, 393–394]. A rather similar discussion of logarithms appears also in Wallace's article [75, 321–323]. Of course, such results occur in much earlier work of Euler, but Wallace's method seems novel. Wallace then demonstrates, with a numerical example, that such limits do, in fact, exist as m, n increase indefinitely. Only then does he go on to derive

$$\frac{\log.(x+h) - \log.x}{h} = \frac{1}{B(x+h')} = \frac{1}{Bx} + H,$$

where "log." denotes logarithm to the base b, H is a quantity that vanishes when h' or $h = 0$, and $B = \log_e b$.

He similarly treats the increment of a^x. Then, invoking trigonometric results from the article "Arithmetic of Sines," he derives corresponding results for $\sin x$ and $\cos x$. Thus, Wallace gives satisfactory derivations, as limits, of the derivatives of x^r (with rational powers r), $\log x$, a^x, $\sin x$, and $\cos x$. He then rather airily notes that extension readily follows to all sums, quotients, and products of these "five elementary functions ... so we may regard it as belonging to all functions whatever of a variable quantity x;" and "in every case,

$$\frac{f(x+h) - f(x)}{h} = p + H;$$

where p is a new function of x, which is independent of h; and H is a function of x and h, which has the property of vanishing when $h = 0$: so that p is a *limit* to which the function which expresses the ratio continually approaches, as h decreases" [77, 399–400]. This, of course, will not do, but he was nevertheless in the good company of Lagrange and Ampère in believing that "all functions" could be treated in this way. It was left to Cauchy, Bolzano, and Abel to put matters right [16; 29; 32; 36].

Next, Wallace gives a brief account of Newton's method of fluxions, incorporating motion, but concluding that "[i]t seems, therefore, to have been entirely without necessity, that motion was ever employed in explaining the theory of fluxions; ... we shall not hesitate to reject the cumbersome apparatus of reasoning, as well as the incommodious notation hitherto employed in this country, and adopt the more legitimate theory and convenient notation of the foreign writers" [77, 401]. The section that follows, "Of the Direct Method of Fluxions," gives a quite readable account of differential calculus, with many examples, employing where necessary the results already established.

It is clear that Wallace cared about establishing the theory on a rigorous basis and that he was unhappy about assuming the binomial theorem, for arbitrary powers, as a way of doing this. Only much later in his article, on pp. 413–414, does he derive the binomial theorem as a special case of Taylor's theorem. "[Taylor's] theorem has considerably excited the attention of mathematicians ever since the late M. Lagrange proposed to make it the basis of the fluxional or differential calculus" [77, 413]. Wallace acknowledges that his demonstration of the theorem "is taken from a memoir of M. Ampère, published in ... the *Journal de l'Ecole Polytechnique*" [7].[11] Thus, the remainder after n terms of the binomial expansion of $(x + h)^r$ is expressed in terms of the nth x-derivative of a certain function $X(x, k)$. Here, $k \equiv x + h$ is regarded as independent of x, and $X(x, k) = p(x) + H(x, h)$, as defined above, but with k rather than h now an independent variable. Then, still following Ampère, the argument is generalized to the Taylor expansion of "any function" $f(x + h)$.

This is unsatisfactory, as Carlyle rightly noted in the above quotation [67, 1:120], but whether Carlyle fully appreciated the reasons is open to doubt. The Lagrangian theory [49] *defines* the derivatives by the coefficients of the (assumed) power-series expansion, but to *generate* the power series, one still needs the binomial theorem. Furthermore, it is necessary to show the equivalence of this formal definition with the concepts of rate of change and of tangents to curves. Finally, as Cauchy and others later pointed out, not all functions have convergent power-series expansions, and different functions can have the same expansion (see [16; 32]). Accordingly, the limit concept is still required, though Lagrange and Ampère did not seem to think so.

These difficulties are mostly avoided by Wallace's rigorous derivation, as limits, of the derivatives of the elementary functions. When combined with Ampère's version of Taylor's theorem, this gives a logically sound treatment, at least for the class of functions whose derivatives exist and can be found by the established rules. Wallace's work has gone largely unremarked, however, and it seems to have had no influence on any later rigorous derivations of the calculus.

Wallace's older article on "Fluxions" for the *Encyclopaedia Britannia* [76] was reprinted virtually unchanged in the 5th and 6th editions, and it was only in the 7th edition of 1842 [79] that the old notation was replaced by differentials. There, Wallace again based his account on the binomial theorem. He may simply have done so from inertia, for that was the basis of his article in the 4th edition. Alternatively, he may have done so for didactic reasons, or because he had been discouraged by Herschel's faint praise, quoted above.

The Cambridge Analytical Society flourished briefly between 1812 and 1814, and their translation of Lacroix's *Calculus* [48] did not appear until 1816, the year after Wallace's *Edinburgh Encyclopaedia* article on "Fluxions." Wallace's work was quickly overshadowed by the Lacroix translation. The format of the latter was certainly far more convenient than that of an article in a multi-volume encyclopaedia.[12] Moreover, Carlyle would certainly not have been the only reader repelled by Wallace's rigorous, but "circuitous," section on "Fundamental Principles." No publisher would have been interested in publishing Wallace's work in book form without guaranteed sales or financial commitment from the author, and Wallace would not have been in a position to promise either.

Undoubtedly, the neglect of Wallace's "doctrine of limits" was partly due to its being published in an encyclopaedia, rather than as a book. It is perhaps more significant that his approach was at variance with the Lagrangian one espoused by Babbage and Herschel and that their work temporarily won the day in Britain. Later, however, the Lagrangian view

[11] Discussions of Ampère's paper are given in [32, 127–132; 36].

[12] Edward Bromhead advised Babbage that "The true faith will never flourish till a Book has been published in *English*, in *octavo*, on the plan of Woodhouse's Anal. Calc. & in a *compact* & *tangible* shape" [3f; 15].

was abandoned, and the doctrine of limits assumed its rightful place at the heart of calculus, following its rigorization by Cauchy and others [16; 29; 32; 36].[13]

Wallace surely deserves credit not only for being the first British writer to publish a comprehensive work on the calculus in differential notation, but also for confronting thorny issues on its foundations, and for giving a novel and mostly quite rigorous discussion of the "doctrine of limits."

5. LATER ANALYTICAL WORK

In October 1817, two years after his *Edinburgh Encyclopaedia* article on *Fluxions* had appeared, Wallace accepted an invitation to visit Herschel at Slough (perhaps accompanied by Thomas Leybourn) and was "much gratified in the prospect of meeting with Mr Babbage" [4]. He then went on to describe to Herschel some new work on series, which he apparently never published.[14]

Wallace, Ivory, Herschel, and Babbage were all analytically productive at this time. Only a few months earlier, on 30 January 1817, Herschel had enthusiastically written to Babbage that

> I want to see you particularly just now for I have at last got all my analytical mania returned glowing hot from its perihelion. Spence's papers have set me mad. After wading through immense heaps of trash and beginning to despair of finding anything important enough for publication, or at least anything strikingly new—I the day before yesterday struck upon an unfinished Essay full of the most beautiful properties of strange transcendents of the form

$$\int \frac{dx}{x} \int \frac{dx}{x} \int \frac{dx}{x} \cdots \int dx\, \Phi(x)$$

[13] Not all the later British critics of the Lagrangian approach carried conviction. For instance, in 1838, William Whewell vaguely wrote that "[t]he attempt made by Lagrange to evade the use of this idea [of limits], however ingenious, is quite incapable of being realised .... The temporary favour which the project found in the eyes of some mathematicians, arose, as I conceive, from their persuasion that mathematical truths are exhibited in their most genuine shape when they are made to depend upon definitions alone; an opinion of which I hope I have made the falsity apparent [in another work entitled *Mechanical Euclid*] " [89, xii]. For Whewell, a staunch Newtonian, this meant a return to intuitive geometrical and mechanical concepts.

[14] "Having had occasion to seek for some very elementary way of finding the area of a Conic section I found the series on the opposite page which applies also to the Calculation of Logarithms ....

Let a be any arc of a circle, put

$$n \text{ for } 1 + \cos a + 2\, \cos \tfrac{1}{2} a$$

$$n' \text{ for } 1 + \cos \tfrac{1}{2} a + 2\, \cos \tfrac{1}{4} a$$

$$n'' \text{ for } 1 + \cos \tfrac{1}{4} a + 2\, \cos \tfrac{1}{8} a$$

$$n''' \text{ for } 1 + \cos \tfrac{1}{8} a + 2\, \cos \tfrac{1}{16} a$$

etc. Then

$$a = \sin a + (1 - \cos a) \sin a \left\{ \frac{1}{n} + \frac{1}{nn'} + \frac{1}{nn'n''} + \frac{1}{nn'n''n'''} + \cdots \right\}.\text{''} \quad [4b]$$

A geometrical construction readily follows if desired. This series gives rapid convergence, and it is a quite easy numerical algorithm to apply, using $\cos(x/2) = 2^{-1/2}(\cos x + 1)^{1/2}$ for repeatedly halved angles. I have not found it stated elsewhere.

> analogous to the general properties of log^c transcend^t[.] I devoured the Essay with avidity—the field it opens is immense ....[15]
>
> We must get another vol. of the Anal. Soc^y out, you must write a paper—so must Bromhead—we will ask Peacock, and get Ivory or Wallace to Present us a paper. —It will be capital sport. —What do you say? Will you take part—shall we revive the Anal. Soc. in a more moderate way, & in good *earnest* and get Ivory & those people for members—never mind the nonsense of having weekly or monthly meetings—an annual or quarterly dinner in Town will do just as well. Write me how you like the scheme but let us at all events get out another volume. [4f]

Though his suggestion of a revived, more broadly based and more "earnest" Analytical Society never bore fruit, it shows both a spirit of openness and Herschel's clear regard for the talents of Ivory and Wallace. Much analytical interest then centered on "functional equations." Following on from earlier work of Arbogast [10], Babbage, Herschel,Wallace, John Herapath (1790–1871), and others all published papers; and Ivory and E. F. Bromhead knowledgably corresponded with Babbage (see [28; 34]). As early as 1815, Herschel wrote to Wallace that "Mr Babbage has been making pretty considerable advances in the theory of a class of equations which may be best designated by the name of functional equations," but cautiously observed that "[w]hether its further improvement will ultimately, as he seems to imagine, alter the face of analytical science, time and hard labour alone can decide" [4d].

But, despite a brief flowering, analysis did not then continue to flourish in Britain. Babbage became increasingly obsessed with his pioneering but ultimately unsuccessful work on calculating engines. From 1825 onwards, Herschel committed himself to completing the astronomical catalogue begun by his late father, Sir William Herschel, and spent the period 1834–1838 at the Cape of Good Hope Observatory. Despite playing a major part in modernizing the Cambridge Mathematical Tripos, George Peacock published little, and that mainly on algebra, drawing on earlier work of Woodhouse [13]. Though Ivory maintained a stream of mathematical publications, mainly concerning Laplacian problems on the shape of rotating self-gravitating bodies and on atmospheric refraction, many were obsessively repetitious, polemical, and unoriginal and did his reputation more harm than good [73]. The eccentric and disillusioned Herapath turned to editing a railway magazine [23].

William Wallace, too, ran out of steam or lost his incentive and published little original mathematics while professor in Edinburgh from 1819 to 1838. Apart from the "boroughs" papers mentioned below, two papers appeared on trigonometric functions and their logarithms, one on mechanical drawing instruments, one on geometrical applications in geodesy, and one on Kepler's problem in astronomy. However, after his retirement and recovery from illness, he published two textbooks and four more geometrical and analytical papers. The most substantial of these concerned a functional equation and its applications. All these publications are listed in [60, 129–130]. It seems likely that these late papers mostly describe work that he had done years earlier.

Did Wallace's contact with Ivory act as a necessary catalyst for his best analytical work? By the time he moved to Edinburgh, he was aged over fifty. It is, perhaps, unsurprising that he there devoted much of his energy to teaching, to astronomy, to designing mechanical instruments, and to writing textbooks and popular works.

[15] Herschel edited and published the posthumous analytical manuscripts of the isolated and then little-known Scottish mathematician, William Spence (1777–1815) [70]. Both Wallace and Ivory would surely have commended Herschel's selfless action. In contrast, John Leslie failed to perform a similar service for his old teacher, John West [26].

Certainly, Babbage, Herschel, and Brewster considered that, by 1830, there had been no marked improvement in British mathematics and science [12; 19]. Likewise, in 1826, Ivory condemned the continued lack of interest in theoretical mechanics—or, perhaps more accurately, the lack of interest in his own achievements. In the introduction to one of his many publications on the figure of the planets, he wrote that

> [s]peculations of this kind are entirely out of fashion, or rather they are discouraged and undervalued as much as possible. They seem even to be excluded from what is popularly called the inductive philosophy, forgetting that they form a part of the noblest and most successful induction that, we may venture to predict, will ever do honour to the human intellect. What has occupied the attention of Maclaurin and Simpson, of D'Alembert, Lagrange and Laplace, is now utterly condemned as useless, with a degree of levity that will not easily be believed. [45, 81]

The actual regeneration of British mathematics, and its applications to the physical world, were rather longer in coming.

6. WALLACE AND THE REFORM BILL: ANALYSIS APPLIED

In 1832, under his Latinised initials "G.V.," Wallace published two articles in the *Philosophical Magazine* [81a,b][16] that give a novel, though simple, application of analysis to the taxation of boroughs. Its political background was the Reform Bill, then under discussion in Parliament. This required that English and Welsh boroughs be ranked in order of economic importance, measured by their number of houses and total assessed taxes for the previous year. But the manner in which these two criteria were to be applied had caused heated debate. The proposals incorporated in the draft Bill had been drawn up by a Lieutenant Thomas Drummond of the Royal Engineers and were as follows: 1. Find the average number H of houses per borough, and divide the actual number h in each borough by this average, to obtain h/H. 2. Similarly, find the average T of assessed taxes per borough, and divide the actual assessed taxes t of each borough by the average, getting t/T. 3. Then the sum $(h/H)+(t/T)$ gives a set of numbers which measure the relative importance of each borough. Though quite simple, this scheme was criticized as too complicated, and a counter-proposal was made that, instead, the geometric mean $(ht)^{1/2}$ should be the appropriate measure. Wallace's article was written on this "matter ... of the highest importance, ... to place it beyond further controversy, by discussing it on principles strictly mathematical" [81a, 219].

Wallace had clearly done so at the request of Drummond, for it was the latter who wrote to the editor of *Philosophical Magazine* "enclos[ing] a paper from Professor Wallace as a Supplement to the one formerly sent" [1].[17] Wallace also wrote to Lord Brougham

> that besides the brief popular view ... which was published in the Scotsman Newspaper, I afterwards sent to the Philosophical Magazine a short article ... My paper however can only be understood by such as are tolerably well acquainted with the principles of the Differential Calculus, for, understanding that some of the *Cambridge Men* now in the House of Commons were to oppose my friend Drummond's principle, I suited my reasoning to the present mode of discussion of abstract questions by the Mathematicians of Cambridge. And besides my very learned Colleague Sir John Leslie having (I suppose) understood that the Article in the Scotsman was by me, he of course formed an opposite opinion and gave it out ... that Drummond was wrong ... [5f]

[16] These works were not listed by Panteki in [60].
[17] The enclosed paper is clearly [81b].

Wallace added that "[p]erhaps you will suppose that it was a waste of *power* to apply so powerful an engine as the Differential and Integral Calculus to so simple a question ... [but] there were two distinct opinions ... as to the *Form* of the *Function* which should express the relative importance of a Borough ... I have shown by strict demonstration that there is only one *form* which the function can have" [5f].

In his articles, Wallace begins with three principles: 1. Boroughs with the same number of houses, and paying the same sum in taxes, are equal in importance. 2. A borough containing double the number of houses and paying twice as much in taxes as another, is twice as important; and similarly for any other fixed multiple. 3. If a town contains as many houses as several boroughs together, and pays the same in taxes as the total paid by these boroughs, then the town's importance "will be equal to the united importance of all these boroughs" [81a, 218]. Introducing $x = h/H$, $y = t/T$, and b/B as the ratios of houses, taxes, and importance of any given borough to those of a particular town or borough, he considers the form that must be taken by the function $b/B = f(x, y)$. From his three principles, Wallace shows that, for any two boroughs, represented by x, x', etc.,

$$f(x, y) + f(x', y') = f(x + x', y + y'),$$

from which it follows that $df(x, y)/dx = df(x', y')/dx'$ and $df(x, y)/dy = df(x', y')/dy'$. Accordingly, since these are *any* two boroughs, these derivatives must be constant and so $b/B = m(h/H) + n(t/T)$, where m, n are undetermined constants. This is consistent with Drummond's rule, which corresponds to $m = n = 1$, but it is inconsistent with the rival geometric-mean proposal.

Some fairly trivial objections were made by a Dr. M'Intyre [81c], to which Wallace's second article was a reply. Confusion for M'Intyre, and perhaps Wallace himself, was caused by the unaccountable fact that *H*, *T*, and *B* were not defined as overall averages but as values for a particular borough. Though this is hardly exciting mathematics, these papers are early instances of its innovative application to the sociopolitical sciences.[18]

7. WALLACE AND THE REVIEWERS

Maria Panteki [60] discusses some of Wallace's writings and reports her discovery of a previously unknown letter from Wallace to George Peacock, one of the Analytical Society's founders. The letter, written in 1833, takes issue with Peacock's "Report on Certain Branches of Analysis" made to the meeting of the British Association for the Advancement of Science [61], held that year in Cambridge. In this, Peacock had extolled the contributions of several Cambridge mathematicians but had failed to mention the early work of Wallace and Ivory.

Panteki asks "Was Peacock ignorant of Wallace' existence as a mathematician, or was he simply indifferent to his work?" Confusingly, she "suspect[s] that Peacock had never heard of Wallace," but then goes on to give several reasons why "Peacock must have been aware of Wallace and his work." [60, 125–126]. The latter is surely the case. As Panteki herself points out, Herschel was a frequent correspondent with both Peacock and Wallace; and the conservative Cambridge "fluxionist," D. M. Peacock (1768–1840), criticized Wallace's

[18] One cannot resist remarking that all these participants, Brougham, Drummond, Wallace, Leslie, and M'Intyre, were Scots, playing an active part in the running of England and Wales, then as now!

work along with that of the Analytical Society. Even more compellingly, George Peacock must surely have known who occupied the prestigious Edinburgh chair of mathematics. Also, Duncan F. Gregory (1813–1844), then in Cambridge and well known to Peacock, had been taught by Wallace when in Edinburgh.[19] Surprisingly, Gregory also makes no reference to Wallace in his work on calculus [38], which was a sequel to the Analytical Society publications. Cantabrigian myopia or some animosity seems the likely explanation for George Peacock's omission. Was Gregory's omission occasioned by his close links with Peacock, or had he simply failed to appreciate the value of his former teacher's work?

Wallace begins his letter to Peacock by saying (surely with heavy irony, though it could be read as politeness): "it might well happen that the date of some of the steps by which the improvement [in analysis] has been brought about may not be exactly known, and it cannot be supposed that you should have known the efforts that were made by obscure persons living far from Cambridge, the Holy City of Mathematics ..." [60, 122]. Wallace goes on to mention his and Ivory's early contributions. In particular, he emphasizes that "about the year 1807 I had abandoned the English notation, and from that time forward employed the foreign notation ... in the Mathematical Repository ... These were, I believe, the earliest applications of that notation to our English mathematics, and I can assure you, we employed it in a Revolutionary spirit ..." [60, 122–123].

Wallace specifically mentions his English translation of Lagrange's 1797 memoir on spherical triangles, which appeared in 1806, at a time when "the Men of Cambridge ... were working with the black Triangle and sturdily setting their faces against change" [60, 123]. Regarding his encyclopedia articles, Wallace notes that "I was the first to give to the British Public a Treatise on the Differential Calculus in the notation of Euler and the foreign mathematicians" [60, 123]. He makes the same justifiable claims in his letter to Brougham, reproduced in the Appendix.

Yet English reviewers and historians—mostly based or educated at Cambridge—long continued to ignore or belittle the earlier achievements of Wallace, Ivory, and others and to ascribe to Cambridge the popularization of differential calculus in Britain. John Herschel is a rare exception, mentioning Wallace and Ivory in his article "Mathematics" for Brewster's *Edinburgh Encyclopaedia* [42]. So too is (the presumably Scottish) Alexander Macfarlane, who in 1916 wrote that "[a]t the beginning of the 19th century, there was only one mathematician in Great Britain (namely Ivory, a Scotsman) who was familiar with the achievements of the Continental mathematicians" [54, 10].

More recently, the respected Maria Boas Hall is among those to fall into the trap, describing the award of a Royal Society medal to "James Ivory, a Cambridge mathematician" [40, 28–29]; and Harvey Becher unequivocally claims that "most nineteenth-century British mathematicians and mathematical physicists graduated from Cambridge University as wranglers. The Cambridge curriculum, therefore, contoured British mathematics, mathematical physics and other scientific fields" [15, 406]. Becher's statement is contestable for the first third of the 19th century, though Cambridge graduates certainly predominated later [41]. For instance, many Military College staff were Scots, and few if any were Cambridge graduates. Also, in 1828, the 31 candidates for London University's first chair of mathematics

[19] Davie [27, 160–168] discusses D. F. Gregory's attempts to reconcile algebra and geometry, suggesting the influence of Wallace, but he rather misleadingly characterizes Wallace as "geometrically-minded."

included eight Cambridge graduates; while, among the others, seven were Scots, three were Irish, and two French [63]. However, without clearer definition of a "mathematician," further discussion of Becher's statement seems pointless.

William Whewell's *History of the Inductive Sciences, from Earliest to the Present Times* [88] even prompted David Brewster to defend the reputation of his late antagonist, John Leslie. Brewster anonymously savaged Whewell's work, first criticizing it for ignoring previous historical surveys, including those by Bossut, Playfair, and Leslie; then for systematically neglecting or downgrading the major contributions made by Scots; and finally for ignoring science in technology (including Babbage's calculating engine, balloons, steamboats, steam-guns, gas illumination, locomotive engines, and railways):

> We think we can perceive system even in his errors; and we do not hesitate to say, that the generalization which he has most successfully pursued is that of grossly neglecting the claims of the philosophers and authors of Scotland. We shall not enquire how far the animadversions of Playfair, of Leslie, of Brougham, and of other eminent Scotsmen, may have excited unfriendly feelings in our sister universities; and still less shall we enquire to what degree personal and even political feelings may have mingled their poison in this injustice towards our intellectual home. We shall content ourselves with ... a brief outline of the evidence upon which we have ventured to give utterance to this deep and painful conviction ...." [20, 147]

Many, even in Scotland, thought that the fiery Brewster had yet again gone too far (see [27, 175–187]). But this hostile review openly exposed the political, philosophical, and institutional loyalties which coloured Whewell's and Brewster's very different views of science and its history [14].[20] Whewell was a Tory; and Brewster, like Leslie, Wallace, and Ivory, was a Whig.

Brewster, always a keen-eyed journalist, was writing mainly for the Scottish market, and, around this time, Scottish sensitivity to perceived injustice at English hands was probably even greater than usual. One obvious cause was the treatment of Scottish medical graduates, many from Edinburgh, who had long been in demand in English cities because of their undoubtedly superior training. Their situation had been seriously undermined by the restrictive Apothecaries Act of 1815 and by subsequent English legal judgments of the early 1830s, which effectively debarred them from general practice in England unless they first became licentiates of the Apothecaries' Company [24, 138–146]. Another, even stronger, cause was antipathy to the first Royal Commission on the Scottish Universities. This had been set up by the Government in 1826; it reported in 1830; and a bill was brought to Parliament in 1836. Though the Scottish universities needed structural reform [57], the proposals were considered insensitive to the distinctive nature of Scottish education. Many sectors of Scottish life, including the Church of Scotland, town councils, universities, and professional bodies, strongly opposed the bill, and it had to be withdrawn [27, 26–40].

But Scottish reviewers could themselves be guilty of prejudice towards fellow Scots. For instance, certainly out of personal dislike, John Leslie did not mention William Wallace in his 1842 *Dissertation Fourth on the Progress of Mathematical and Physical Sciences*

[20] Whewell, who well knew his antagonist's identity, later loftily retorted that "The Reviewer, obviously an enemy eager to find faults, was able to detect but very few passages which are really mistakes. ... But I am compelled by justice to acknowledge that the value of this testimony is materially weakened by the Reviewer's extreme laxity and obscurity of view with regard to the nature of science; —defects which make his judgment on such subjects nearly worthless" [90, preface].

... published in the *Encyclopaedia Britannica* [53]. Neither did the reformer, James D. Forbes, in his *Dissertation Sixth* of 1853 [31]. A Tory admirer of Whewell and Cambridge, the young Forbes had been controversially chosen as Leslie's successor in the Edinburgh Natural Philosophy chair, defeating his early mentor David Brewster.[21] Though Forbes can perhaps be excused, since he specifically excluded pure mathematics from his report, Panteki comments that he disliked both Wallace and "the entire Scottish approach to mathematics" [60, 126].[22]

Despite Brewster's claims of English—or at least Cambridge—prejudice towards Scottish science, there is no evidence of any rivalry between Wallace and Ivory on the one hand and Babbage and Herschel on the other. Herschel and Ivory seem to have had relatively little contact with one other, but Babbage and Ivory remained mostly on amicable terms.[23] Wallace maintained a friendly correspondence with Herschel over many years, mainly on mathematics and astronomy, but including such unlikely topics as Herschel's favorite black cat [4]! Remarkably, Herschel and Babbage also managed to maintain mostly friendly relations with Brewster, despite occasional differences.[24]

Just how dominant the mathematical reputation of Cambridge became, by midcentury, is rather piquantly exemplified by an 1854 *Treatise on the Differential Calculus* written by the now-forgotten Thomas Miller [55]. Miller had studied mathematics at St Andrews University under Robert Haldane, and he subsequently became the first mathematics master at the St Andrews school, Madras College. There, he so distinguished himself that "in 1836 the Trustees received a strong letter from the professor of mathematics [Thomas Duncan] in the university complaining that mathematics teaching in the Madras College was attracting students from his classes and asking them to forbid university students attending Mr. Miller's classes. The difficulty was resolved when Mr. Miller left to become rector of Perth Academy" [72, 24]. The latter, of course, is the very same school where William Wallace started his teaching career.

Miller's textbook is a clear but rather pedestrian work, doubtless useful to students in its day because of its many examples. It is dedicated to Haldane, by then principal of

[21] Carlyle records that the chair had first been offered to Herschel, who had refused it [67, 6:273], and he refers to the political nature of the appointment: "Dr Brewster has been a candidate for the Professorship Leslie held, and to the shame of the Town Council (an old Tory body now on its last legs) has been *rejected*, in favour of a young lad, about three-and-twenty, the extent of whose talents, great otherwise, is as yet known only to himself!" [67, 6: 309–314].

[22] In fact, Forbes himself was no great mathematician and complained of the trend of "modern writers, of making ... the facts of nature mere pegs on which to suspend festoons of algebraic drapery" (quoted in [59, 234]). But he proved himself an effective reformer of the Scottish system, along lines which, though influenced by Cambridge, also retained some of the traditional structures [27].

[23] In 1815–1816, Ivory proposed Babbage for fellowship in the Royal Society of London and commented favourably on Babbage's work [11] on functional equations [3a,b]. He had supported Babbage's unsuccessful candidacy both for a post at the East India College at Haileybury [3c] and for the Edinburgh mathematics chair which Ivory's own colleague, Wallace, secured [3d,e]. For the former, Ivory even persuaded Playfair and Leslie to write supporting certificates, though they and Babbage had not met. Babbage long continued to lend books to Ivory.

[24] Brewster wholeheartedly supported Babbage's 1830 diatribe on *The Decline of Science* [12; 19], and Herschel valued Brewster's editorial work. Both wrote for Brewster's *Edinburgh Encyclopaedia*, and they sent contributions to the various journals that Brewster edited. Herschel regretted that Brewster's dispute with his co-editor, Robert Jameson, and the publishers had led, in 1824, to Brewster's defection from the *Edinburgh Philosophical Journal*, "the best Scientific Journal in the country, perhaps in Europe and I am sorry to find its forces are divided" [4g]. This dispute is described in [22].

St Mary's College at St Andrews, and to Thomas Duncan.[25] In his preface, Miller notes that it is "necessary for the student to accustom himself with the notation of Leibnitz before he could read the works of Biot, Poisson, Lacroix, and Laplace, to say nothing of those of Airy, Whewell, and Pratt." [55, vi]. He makes several references to the text of Duncan F. Gregory [38], but he nowhere mentions Wallace, Ivory, or any earlier Scottish mathematician.

The work begins with a number of fullsome and revealing "Recommendatory Notices" mostly quoted from press reviews. Thomas Duncan himself wrote that "[t]his treatise has more the appearance of a Cambridge work than any one I have seen published in Scotland. It is a most respectable production," and the *Illustrated London News* reported that "The student of the exact sciences ... has chiefly looked for instruction to the *savans* on the classic margins of the Cam or the Isis. The scholar can with safety now go farther north, for on the banks of the silvery Tay Dr Miller has quietly achieved a work which has passed muster among the giants of Modern Athens." The Scottish press echoed the theme. *The Edinburgh Evening Post and Scottish Record* claims (incorrectly) that "[t]he subject is carried much farther than has ever been done in Scotland," while the *Aberdeen Journal* admires the author for "boldly entering the lists with Cambridge men ... though undignified with the title of Senior Wrangler ...." *The Perth Constitutional* commends the "admirable Treatise" on so exalted a subject, "handled in so able a manner by the studious Rector of Perth Academy"—the reviewer signing himself "Cantab" [55, 1–2]!

Wallace had died only 11 years and Ivory 12 years previously. Wallace and Ivory were surely well known to Haldane, Duncan, and Miller. Ivory was an eminent native of Dundee, where Duncan had taught for many years before becoming professor in St Andrews [64], and Ivory had been awarded a St Andrews LL.D. Miller must have known of Wallace's rise from assistant in the school where he himself was now rector to professor of mathematics at Edinburgh. It is remarkable that neither Wallace nor Ivory was mentioned, either by Miller or by the writers of his "Recommendatory Notices." Just one possible explanation may be that Haldane grudged Wallace his success in securing the Edinburgh chair, for Haldane had himself coveted it and had been defeated after a vote [85]. If this were so, then Miller may have considered it prudent to avoid citing Wallace's work.

In marked contrast to others' omissions, Lord Brougham frequently extolled the merits of both Wallace and Ivory. In his testimonial for Wallace's candidature for the Edinburgh chair, he had written that "both at home and abroad, yourself and our friend Ivory are admitted on all hands to be our first mathematicians" [85, 17]. Much later, he repeated these sentiments in his 1860 inaugural speech as first Chancellor of Edinburgh University, describing Ivory and Wallace together as "distinguished mathematicians who were both deeply imbued with the principles of modern analysis, but diligently cultivating the ancient too" [60, 126].

8. CONCLUSION

William Wallace and James Ivory were among the first in Britain to propound the "continental" calculus and its applications. They published much of this work in Thomas Leybourn's *Mathematical Repository* and in encyclopaedias. The encyclopaedias were then an important medium for general instruction, and leading Scottish scientists were prominently involved as editors and contributors. Wallace's articles on "Fluxions" were

[25] In the year before his book's appearance, Miller was awarded an honorary LL.D. by St Andrews University on the recommendation of Haldane and Duncan.

pioneering, but they have received little subsequent recognition, and his meritorious attempt to provide an improved theory of limits went largely unappreciated. His contribution to the debate on the Reform Bill was an early application of mathematics to sociopolitical studies. Though Wallace was not a creative mathematician of the highest rank, his successful early struggle to educate himself in adverse circumstances and his later rise to academic prominence attest to both a strong character and an able mind.

Cambridge became preeminent by the middle of the 19th century, and Cambridge-influenced reviewers and historians neglected the early achievements of Ivory and Wallace, exaggerating the role of the Analytical Society in popularizing "continental" analysis in Britain. Yet Herschel and Babbage respected the talents of both Scots, with whom they had mostly friendly relations. Wallace's reputation seems to have suffered from a mix of personal, institutional, political, and national antipathies: rivalry with fellow Scottish professors, John Leslie, Robert Haldane, and later James D. Forbes; Cambridge's neglect—or ignorance—of the achievements of the military colleges and the Scottish and Irish universities, together with a wish to praise its own; Tory antagonism toward those with Whiggish sympathies; and, perhaps, the continuing mutual suspicion of Scots and English.

APPENDIX

Petitioning for a pension, and in poor health, Wallace summarizes his achievements [5]:

W. Wallace to H. Brougham

13 Sergeants Inn, Fleet Street
19th May 1835

Dear Lord Brougham,

I take the liberty most respectfully to intimate to you that, following up the views of my friends, I have addressed a Petition and Memorial to the King stating such grounds as I humbly presume may induce the Government to consider my case under my present unfortunate Circumstances. These grounds are, my service of seventeen years of the prime of life in the Royal Military College with acknowledged advantage to the Institution insomuch as to have been recommended (but without effect) by the Officers directing its affairs to the Government as deserving the allowance of half pay which would have been given if, after fourteen years service, I had retired in bad health;[26] my continued exertions during the above mentioned period (and I might have added the preceding ten years),[27] also during the last sixteen years in the University of Edinburgh, in the improvement of Mathematical Science and its diffusion. I could not with propriety enumerate the various labours I have performed with a view to the extension of Mathematical Science in Britain but I may tell your Lordship, because you can appreciate my efforts, that I have written ['seven' del.] Memoirs which contain improvements and new views on seven distinct subjects in the pure Mathematics. These are contained in the Edinburgh Transactions. I have translated and been the first to introduce to the British student some of the finest speculations of Legendre and Lagrange, and I have in periodical works given original examples of Mathematical investigation as much high mathematics as would form a volume or two: and all this, the labour of years of my life, without any pecuniary advantage. I have introduced into two Scottish Encyclopaedias Treatises on the Mathematics of a different character from such as were given before in like works. I trust I may mention that a Treatise on Fluxions in the Encyclopaedia Britannica fourth Edition was read with avidity by the Cambridge men when it came out, who guessed it to have been written by Playfair. I and another person [i.e., Ivory] were the first to introduce the Notation of the Continent into Britain in our writings and the first Treatise on Fluxions in the English language in which the Notation of the great Masters

[26] James Ivory had done just this earlier.
[27] For the 10-year period in question, Wallace served as assistant teacher of mathematics at Perth.

of that Science was employed was written by me for the Edinburgh Encyclopaedia. I have invented some Mathematical Instruments, one in particular which does correctly what hitherto has been done imperfectly by the *Pantagraph*. To return to my petition, I have stated the failure of my health and the little chance there is of its being better[:] that I have three unmarried daughters and a Son Educated for the Scottish church who depend on my exertions: That a considerable part of my income has, for a year & half past, been witheld by the deranged state of affairs of the City of Edinburgh: And I have prayed that such aid might be given as might prevent my daughters from falling into indigence in the event of my demise or nonrecovery. This of course can only be prevented by the aid being given to my daughters independently of my life (now in the 67th year).

I understand that this petition requires support of Names and influence. Your Lordship is the person to whom I am most intimately known. I have ever found you ready to render service if it could be done. On this occasion I most respectfully solicit your support in whatever way you think proper.

Having now put your Lordship in possession of my case, I fear with more detail than is convenient for you to follow I shall only farther say that it would be gratifying to me to pay my respects to your Lordship personally, for which a brief space would suffice if it can be spared from your more necessary avocations.

I have the honour to be

Your Lordships obliged & most obedient servant
William Wallace.

{To the Right Hon. the Lord Brougham & Vaux & & &}

ACKNOWLEDGMENTS

I am grateful to Dr. John G. Fauvel, Professor Ivor Grattan-Guinness, and Professor Judith Grabiner for their helpful comments, and to the editor and referees for their further suggestions. I also thank Dr. Martine Henry for providing the French abstract. This work was completed during a one-year visiting research appointment at the Research Institute for Mathematical Sciences, Kyoto University, Japan. I am most grateful to the Institute staff for their hospitality. I am also grateful to the Carnegie Trust for the Universities of Scotland for their travel support.

I thank Edinburgh University Library, the British Library, the Royal Society of London, and University College, London for permission to quote from manuscript sources in their possession; also the Royal Military College, Sandhurst, and the Trustees of the Scottish National Portrait Gallery, Edinburgh, for permission to reproduce the warrant and portrait shown in Figs. 1 and 2.

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