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The Calculus of the Trigonometric Functions

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The trigonometric functions entered "analysis" when Isaac Newton derived the power series for the sine in his *De Analysi* of 1669. On the other hand, no textbook until 1748 dealt with the calculus of these functions. That is, in none of the dozen or so calculus texts written in England and the continent during the first half of the 18th century was there a treatment of the derivative and integral of the sine or cosine or any discussion of the periodicity or addition properties of these functions. This contrasts sharply with what occurred in the case of the exponential and logarithmic functions. We attempt here to explain why the trigonometric functions did not enter calculus until about 1739. In that year, however, Leonhard Euler invented this calculus. He was led to this invention by the need for the trigonometric functions as solutions of linear differential equations. In addition, his discovery of a general method for solving linear differential equations with constant coefficients was influenced by his knowledge that these functions must provide part of that solution. © 1987 Academic Press, Inc.

Les fonctions trigonométriques sont entrées dans l'analyse lorsque Isaac Newton a obtenu une série de puissances pour le sinus dans son *De Analysi* de 1669. Par contre, aucun manuel jusqu'à 1748 n'a porté sur le calcul de ces fonctions. C'est-à-dire que, dans aucun des douzaines d'écrits portant sur le calculs publiés en Angleterre ou sur le continent pendant la première moitié du XVIII^e siècle, il n'y avait pas d'étude de la dérivée et de l'intégrale du sinus ou du cosinus, ni d'examen des propriétés de périodicité ou d'addition de ces fonctions. Cela contraste fortement avec ce qui est arrivé pour les fonctions exponentielle et logarithme. Nous essayons d'expliquer ici pourquoi les fonctions trigonométriques ne sont entrées dans l'analyse qu'aux environs de 1739. D'ailleurs, en cette année, Leonhard Euler a inventé ce calcul. Il a été conduit à cette découverte par la nécessité d'utiliser les fonctions trigonométriques comme solutions des équations différentielles linéaires. En outre, sa découverte d'une méthode générale de résolution des équations différentielles linéaires à coefficients constants a été influencée par sa connaissance que ces fonctions doivent fournir une partie de cette solution. © 1987 Academic Press, Inc.

Die trigonometrische Funktionen traten in die "Analysis" ein, als Isaac Newton die Potenzreihe der Sinusfunktion in seiner Arbeit *De Analysi* von 1669 herleitete. Andererseits betrachtete kein Lehrbuch vor 1748 den Kalkül dieser Funktionen. Das heisst, man findet weder eine Behandlung der Ableitung und des Integrals vom Sinus oder Cosinus noch eine Behandlung der Periodizitäts- oder Additionseigenschaften dieser Funktionen in irgendeinem Lehrbuch über Differential- und Integralrechnung aus der ersten Hälfte des achtzehnten Jahrhunderts. Hierin liegt ein deutlicher Gegensatz zum Falle der Exponentialfunktion und der logarithmischen Funktionen. Im vorliegenden Aufsatz versuchen wir zu erklären, weshalb die trigonometrischen Funktionen bis um 1739 rechnerisch nicht behandelt wurden. In diesem Jahr erfand Leonhard Euler den betreffenden Kalkül. Er wurde zu dieser Erfindung durch den Bedarf an trigonometrischen Funktionen als Lösungen linearer

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Differentialgleichungen geführt. Zusätzlich beeinflußte sein Wissen darum, daß diese Funktionen einen Teil der Lösung linearer Differentialgleichungen mit konstanten Koeffizienten liefern müssen, seine Entdeckung eines allgemeinen Lösungsverfahrens für solche Gleichungen.

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The trigonometric functions entered "analysis" with Isaac Newton. It is well known that in *De Analysi* [1669], Newton derived the power series for the sine by inverting the power series for the arcsine; the latter he had derived from the binomial theorem and geometrical considerations. Within the next decade, these series showed up in various places, including the correspondence between Newton and Leibniz in 1676. Leibniz noted in particular that the sine series could be derived from the cosine series by term-by-term integration since "the sum of the sines of the complement to the arc . . . is equal to the right sine multiplied by the radius, as is known to geometers" [Turnbull 1960, 65]. Nevertheless, as we will see, the calculus of the trigonometric functions did not come into existence until 1739. That is, until that date there was no sense of the sine and cosine being expressed, like the algebraic functions, as formulas involving letters and numbers, whose relationship to other such formulas could be studied using the developing techniques of the calculus. Since such was not the case for the other large class of what we call the transcendental functions, the exponential and logarithmic functions, this 70-year gap calls out for explanation. Not only will we attempt that explanation here, but we will also see why and how the trigonometric functions finally did enter calculus.

First, however, we want to review what was known about the sine and cosine in the last quarter of the 17th century and then briefly discuss their rare appearance in the first calculus textbooks of the early 18th century. Given that the sine and cosine are the most familiar examples of periodic functions, one might expect that they would make an appearance whenever there was any discussion of a periodic physical phenomenon. In fact, they did, but in ways so geometrical that there was no development of the analytic ideas. For example, in 1678 Hooke's law appeared in print in the published version of his Cutlerian lecture [Hooke 1678]. In an effort to describe the motion of a weight on a stretched spring based on his law, Hooke drew a rather complex diagram and showed that the velocity of this weight is as certain ordinates in a circle; these ordinates may be thought of as the sines of the arcs cut off. He also drew a curve which represented the time for the weight to be in any given location; this curve is in fact an arccosine curve. Hooke, however, does not use these trigonometric terms; he is content with the geometry of the situation.

A few years later, a more explicit result appears in Newton's *Principia* as Proposition XXXVIII, Theorem XII:

> Supposing that the centripetal force is proportional to the altitude or distance of places from the centre, I say, that the times and velocities of falling bodies, and the spaces which they describe, are respectively proportional to the arcs, and the right and versed sines of the arcs. [Newton 1687]

It may be, as Truesdell says in referring to this theorem, that "for Newton, simple harmonic motion was a familiar and completely mastered concept" [Truesdell 1960]. The theorem, however, occurs in the section entitled "Concerning the Rectilinear Ascent and Descent of Bodies" and Newton makes no reference there to motions repeating themselves. He simply describes the motion of bodies moving on certain curves subject to various types of forces. Newton's proof of the theorem involves taking the limit of a body moving on an arc of an ellipse which is an affine transform of a circle, as the ellipse is squeezed onto its diameter. But his diagram to the theorem shows only one quadrant of a circle; the right sine of the statement is, as is usual for that time, simply a line from the circle's diameter to its circumference. To a modern reader, the calculus of the sine and cosine is only a hair's breadth away from Newton's discussion; but Newton himself says no more about it and there is little reference to this idea in any other work over the next 30 years.

One might also expect the sine and the cosine to appear as the solution to a simple differential equation, in particular as the solution to $y'' = -ky$. Again, one does find, in effect, this equation. Leibniz in [1693] derives from his differential method the infinitesimal relation between the arc and its sine in a circle of radius a: $a^2dy^2 = a^2dx^2 + x^2dy^2$ (Fig. 1), assuming dy is constant. Leibniz takes the differential of this equation to get $2a^2dxd^2x + 2xdxdy^2 = 0$ or $a^2d^2x + xdy^2 = 0$. We would write this equation as $d^2x/dy^2 = -x/a^2$, the standard differential equation for $x = \sin(y/a)$. Leibniz does in fact derive this solution, by his method of undetermined coefficients, and writes it as a power series. Again, we wonder why Leibniz did not go further and discuss the properties of this series. But neither he nor Johann Bernoulli, who discussed the same differential equation and power series in a paper of the following year, moved any closer to the calculus of these functions.

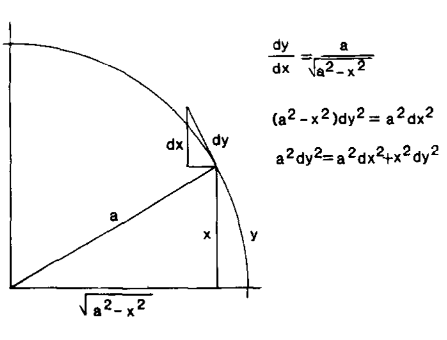

FIGURE 1

A physical problem leading to a differential equation which could also have produced the calculus of the sine function was the vibrating string problem first attacked by Brook Taylor [1713]. Taylor in fact needed to find the fluent of $c\dot{y}/\sqrt{c^2 - y^2}$; he showed that this expression was the fluxion of the circular arc whose sine is y with radius c. And since Taylor was most interested in the periodic time of the motion of the string, he did not write this in terms of the sine itself. In fact, as we will see, it was quite common in later works to deal with what we call the arcsine function rather than the sine.

In 1696 there appeared the first textbook on the new calculus of Newton and Leibniz, the *Analyse des infiniment petits* by the Marquis de l'Hospital [1696]. It was followed in France by the *Analyse demontrée* of Charles Reyneau [1708] and in England by a number of texts including works by George Cheyne [1703], Charles Hayes [1704], Humphrey Ditton [1706], John Craig [1718], Edmund Stone [1730], James Hodgson [1736], John Muller [1736], and Thomas Simpson [1737]. What do we find in these texts on the calculus of the trigonometric functions? Essentially, we find nothing. As we already noted, this is in contrast to the situation with exponential and logarithmic functions. For even though these functions are not treated as inverses of one another, their derivatives and integrals are dealt with. In most of these texts we find some sort of derivation of the basic result that the derivative (differential, fluxion) of the logarithm of a quantity is the derivative (differential, fluxion) of the quantity divided by the quantity itself. We also usually find an extensive treatment of the derivatives of expressions of the type a^x where the exponent x is a variable and where a is either a constant or a variable. In fact, most of the authors deal with even more complicated exponential expressions.

What does appear about trigonometric functions? If there is anything at all, it is only a discussion of the relationship between the sine or tangent and the arc; this treatment is then carried out in the manner of Newton's original work via power series. The only one of the authors cited who has anything more is Thomas Simpson. In the course of solving a problem dealing with spherical triangles, he proves geometrically the result that "the Fluxion of any circular arch is to the Fluxion of its Sine, as Radius to the Cosine" [Simpson 1737, 179]. This proof uses, in effect, the differential triangle of Leibniz, and its similarity with the triangle whose hypotenuse is the radius of the circle and whose legs are the sine and cosine of the intercepted arc (Fig. 2). We can of course translate the theorem into the standard calculus result that the derivative of the sine is the cosine.

The cited proof was not, however, original to Simpson. It appeared some 15 years earlier when the manuscripts of Roger Cotes were published 6 years after his untimely death at the age of 34. Cotes proved the result at the beginning of a tract *On the Estimation of Errors* in which he analyzed the errors which occurred in astronomical observations. The particular lemma was stated by Cotes as "the small variation of any arc of a circle is to the small variation of the sine of that arc, as the radius to the sine of the complement" [Gowing 1983]. The "small variations" can be considered as fluxions or as differentials; in any case Cotes uses

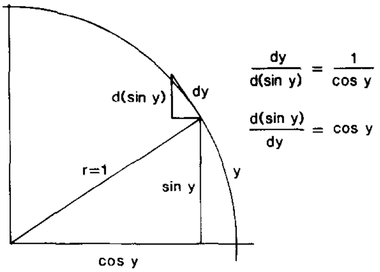

FIGURE 2

essentially the same diagram as Simpson did later and gives the same proof. (The same result with the same proof is also found in a paper of Euler's colleague at St. Petersburg, F. C. Maier [1727].) Cotes followed this lemma with two others in which he proved by similar methods results equivalent to the theorems that the derivative of the tangent is the square of the secant and the derivative of the secant is the product of the secant and the tangent.

Cotes' manuscripts contain other work related to the calculus of the trigonometric functions. As part of his *Logometria* he calculates the integrals of the tangent and secant functions. Here I use the word function in the modern sense, because Cotes in fact sketches several periods of both of these functions and shows how these curves are derived from the geometric definitions.

As a final example of Cotes' use of the trigonometric functions in calculus, we may cite his extensive table of integrals. Cotes notes in effect that for many integrands involving sums or differences of powers, a change of sign in the integrand changes the integral (or fluent) from a logarithm to an inverse trigonometric function. In fact, Cotes uses the same notation for both, stating that one or the other is meant depending on the sign of a certain quantity. In other words, the inverse trigonometric functions, at least, had the same standing in Cotes' mind as the logarithmic functions. He could deal with one as easily as with the other. This "equivalence" of the two types of functions led Cotes to a result close to Euler's famous expression of $e^{i\theta} = \cos\theta + i\sin\theta$. Cotes' result was stated, though, in terms of logarithms.

Unfortunately for the development of the calculus of the trigonometric functions, Cotes died before he could formulate his results in a systematic way. Some of the English textbook writers referred briefly to Cotes' work, but no one developed it much further.

Why do the sine and cosine not appear in these texts? We will give two explanations, both of which will be justified by the work of the man who unquestionably was responsible for the ultimate introduction of these functions into calculus, Leonhard Euler. First of all, though sine tables existed in abundance, the sine was not considered as a "function," even to the extent that logarithms or exponentials were. It was thought of geometrically as a certain line in a circle of a given radius;

one did not, in general, draw a graph of such a function so there was no question of finding tangent lines or areas. That is not to say that graphs of the sine function did not occur at all. There are several early examples of the appearance of one or more arches of the sine in the mid-17th-century works of Roberval and Wallis [Boyer 1947]. But the idea of the sine as a "function" did not enter the mathematical mainstream at this time.

As another indication of this, we cite the unpublished manuscript of Euler's *Calculus Differentialis*, which Yushkevich [1983] believes to have been written about 1727 for use as a text in the university at St. Petersburg. In this text, Euler gave a definition of function similar to the one he was later to give in his *Introductio in analysin infinitorum*; he then proceeded to subdivide all functions into two classes, algebraic and transcendental. But this latter class, he noted, consists solely of the exponential and logarithmic functions. Euler gave a complete treatment of the differential calculus of these latter functions, but he did not mention the trigonometric functions at all.

A second reason for their nonappearance may be related to a statement of Edmund Stone's in a somewhat different context. Stone, commenting on the absence even of exponential functions from the work of L'Hospital and also from his own work, noted,

> As the illustrious author [L'Hospital] has omitted the exponential calculus, or manner of finding the fluxions of exponential quantities, such as $x^x = a$, $x^x = y^y$, etc. where the index's of the variable quantities are also variable, thinking, as I suppose, this branch of doctrine to be of very little or no use, so I have been silent in this matter also; which it is much better to be, than take up the reader's time in learning what is only mere speculation. [Stone 1730, Introduction]

That is, trigonometric functions may have been avoided because no one saw any reasonable use for them as yet. As a possible confirmation of this reason, we will again cite Euler. In the abstract to a paper he published in 1754 which dealt extensively with the calculus of the sine and cosine, Euler claimed this invention for himself. As he, or the journal's editor, wrote,

> In addition to the logarithmic and exponential quantities there occurs in analysis a very important type of transcendental quantity, namely the sine, cosine and tangent of angles, whose use is certainly most frequent. Therefore this type rightly merits, or rather demands, that a special calculus be given, whose invention in so far as the special signs and rules are comprised, the celebrated author of this dissertation is able rightly to claim all for himself, and of which he gave examples in his *Introduction to Analysis* and *Institutions of the Differential Calculus*. Numerous examples stand out in his work on the motion of the moon and on the perturbations of the motion of Saturn and Jupiter, in which this type of calculus is frequently used in the investigations, and without the help of which these were scarcely able to be performed. Therefore, for this new calculus, which is called the calculus of sines, not only does Euler present the first principles and reveal the highest uses in various parts of mathematics through the most impressive examples, but also he continues to enrich it by new inventions, which the present article splendidly demonstrates. [Euler 1754, 543]

One could thus say that by 1754 there were definitely uses for these functions; in 1727 Euler himself saw none.

We have already mentioned on several occasions Euler's famous text *Introductio in analysin infinitorum* [1748c]. In this text Euler provides a complete treatment of what we may call the precalculus of the trigonometric functions. That is, he defines them numerically, not as lines in a circle, discusses their various properties including addition formulas and periodicity, and develops their power series. He also draws the graphs of some of these functions. Since this text was not designed to be a calculus text, there is no mention of derivatives. This lack was only temporary, however. Euler was already writing his differential calculus text, which appeared 7 years later [Euler 1755]. In this text, of course, there was a derivation, using power series, of the standard rules for the derivatives of the trigonometric functions. As we shall see, Euler was in full command of these rules much earlier. The question remains, then, since Euler claimed the invention of the calculus of trigonometric functions, what were the circumstances under which this invention took place, or, more specifically, how did these functions enter calculus?

A consideration of Euler's papers before 1740 provides an answer. The trigonometric functions entered calculus via the study of differential equations. Not only did this study give the sine and cosine the status of "function" in our sense, and give them an equal status with the exponential and logarithmic functions, but it also provided the necessary uses for these functions. The study of differential equations was not just the cause of the sine and cosine functions entering calculus, however. It was Euler's knowledge of these functions which led him, I believe, to the development of the standard method of solving linear differential equations with constant coefficients. The remainder of this paper will be devoted to convincing the reader of the truth of these assertions.

In some of Euler's earliest papers, in the late 1720s, Euler needs to integrate equations of the form $dt = dx/\sqrt{a^2 - x^2}$. As was the custom of the time, he gave as the solution that t was the arc whose sine was x with radius a. In fact, Euler probably learned this solution from his teacher Johann Bernoulli. Bernoulli himself, in a paper dealing with vibrations [1728], studied an object which moves subject to a force proportional to its distance from a given point. He in effect set up the equation $d^2y/dt^2 = a - y$, where a is a constant, and solved it using two integrations. We will present the essence of his solution, which is similar to one worked out a decade earlier by Jakob Hermann. We first set $v = dy/dt$. From $dv/dt = a - y$, we then derive the result that $dv/dy = (a - y)/v$ or $vdv = (a - y)dy$. An integration of this equation leads to $v^2 = 2ay - y^2$ or $v = \sqrt{2ay - y^2}$ or finally $dy/dt = \sqrt{2ay - y^2}$. Bernoulli, however, does not put the equation in that form; he rewrites it as $dt = dy/\sqrt{2ay - y^2}$ and proceeds to integrate once more to solve for t by use of the arcsine. That is, he, like Taylor earlier, is interested in the time of the motion, not the motion itself. Bernoulli does get a sine curve out of this problem, but it is simply the shape of a string; he does not deal with sinusoidal motion. Over the next several years, Euler dealt with similar integrals in the same way as Bernoulli, by solving for the arcsine.

On the other hand, during this same time Euler was certainly familiar with the periodicity properties of the sine function. In a well-known paper [1735b] he used the fact that the power series for sin s, $s - s^3/3! + s^5/5! - \ldots$ had nonzero roots at $\pm\pi$, $\pm 2\pi, \ldots$ to "factor" $1 - s^2/3! + s^4/5! - \ldots$ as $(1 - s^2/\pi^2)(1 - s^2/4\pi^2) \ldots$ and derive his result that $1 + 1/2^2 + 1/3^2 + \ldots = \pi^2/6$. Similarly, in another paper [1736], in which he integrated a differential equation of the above type to get the arc of a circle whose cosine is y/a, he stated the various values of the arc at which y would be 0 or a.

Nevertheless, Euler did not always recognize the sine function as the solution of a differential equation. Daniel Bernoulli wrote to Euler on May 4, 1735, to discuss a problem on the vibrations of an elastic band. As part of this problem he had to solve the differential equation $k^4\,(d^4y/dx^4) = y$. He wrote, "This matter is very slippery. . . . The logarithm satisfies the equation . . . but no such logarithm is general enough for the present business" [Truesdell 1960, 166]. By "logarithm," Bernoulli of course meant the exponential function. Euler dealt with the same problem and the same equation in a paper later that year and was also unable to find a complete finite solution. He was, however, able to solve it using power series. But since he incorporated the initial conditions into that solution, he did not recognize that there was a sine or cosine hidden in the series he finally obtained [Euler 1735a].

The sine and cosine do appear in other papers of Euler in the late 1730s as well as in some of his correspondence. But if the papers deal with calculus at all, these functions appear only in the contexts discussed above. Outside of calculus, we can find references to various trigonometric formulas, especially to those dealing with multiple angles. It is only in 1739 that Euler is able to put all of his knowledge of these functions together.

On March 30, 1739, Euler presented the paper *De novo genere oscillationum* [1739a] to the Academy of Sciences at St. Petersburg. The paper dealt with the motion of what we would call a sinusoidically driven harmonic oscillator; that is, Euler considered the motion of an object in which the force acting was composed of two separate parts, one proportional to the distance, the other one varying sinusoidally with the time. Euler noted in a letter to Johann Bernoulli on May 5 that "there appear . . . motions so diverse and astonishing that one is unable altogether to foresee until the calculation is finished" [Eneström 1905, 33]. Note that Euler wrote about the motions; these now become central rather than merely the period. What calculations did Euler perform to derive these motions? He began by deriving three simultaneous differential equations which the four given variables s, t, y, v had to satisfy. Here t is time measured along the arc of a circle of radius a while y is the sine of that arc. In addition, s represents the position of the object while v/a represents the square of the velocity. The equations are

$$dv = -ds\left(\frac{s}{b} + \frac{y}{g}\right), \qquad dt = \frac{a\,dy}{\sqrt{a^2 - y^2}}, \qquad \sqrt{v} = \frac{-ds\sqrt{a}}{dt}.$$

Euler's aim was to use these equations to eliminate y and v, thus getting a single equation relating s and t which he can solve. In particular, he solves the second equation, a very familiar one, in a new way. Not only does he get the usual solution $t = a \arcsin(y/a)$ but also, for the first time, he writes this in the inverse form as $y = a \sin(t/a)$. As we already mentioned, Euler is now dealing with the motion; so t becomes the independent variable. Using this result as well as the solution $v = ads^2/dt^2$ of the third equation, he easily derives the desired differential equation

$$2ad^2s + \frac{sdt^2}{b} + \frac{adt^2}{g}\sin\frac{t}{a} = 0$$

relating s and t. He now wanted to find a solution to this in finite terms.

It is not necessary here to discuss his entire solution. But there are a few major points of interest. First of all, he solves the special case where "$b = \infty$," that is, $2gd^2s + dt^2 \sin(t/a) = 0$. To do this, he obviously needs to know the differentials of the sine and cosine; so he writes them down: diff. $\sin(t/a) = (dt/a)\cos(t/a)$ and diff. $\cos(t/a) = -(dt/a)\sin(t/a)$. (There was no necessity to derive these; as we have noted, the results themselves had in one form or another been known for years.) After solving the equation in the form $s = (a^2/2g)\sin(t/a)$, he proceeds to analyze the periodicity properties of this solution. Second, in the process of solving the general case he first deletes the $\sin(t/a)$ term and considers the equation $2ad^2s + sdt^2/b = 0$. Note that this is similar to the equation of Bernoulli already mentioned. This time, however, Euler solves it by integrating twice to get first $2abds^2 + s^2dt^2 = C^2dt^2$ or $dt = -\sqrt{2ab}ds/\sqrt{C^2 - s^2}$ and then, by what he usually calls a quadrature of the circle, the result $t = \sqrt{2ab}\arccos(s/C)$, or finally, as before, $s = C\cos(t/\sqrt{2ab})$. Again, the trigonometric functions appear explicitly. Finally, Euler solves the general case by postulating a solution of the form $s = u\cos(t/\sqrt{2ab})$ where u is a new variable. He proceeds to substitute that expression into the equation and solve for u. This, of course, involves being thoroughly familiar with the calculus manipulations of the sine and cosine. After much of this type of manipulation, he demonstrates the "motions so diverse and astonishing" about which he wrote to Bernoulli.

We note that in a paper which appeared only 45 pages later in the same volume of the *Commentarii* of the St. Petersburg Academy [1739b], Euler again uses the differentials of the sine and cosine. Already, he is becoming "fluent" in their use.

We have now seen how the calculus of the sine and cosine appeared as part of the process of solving an interesting differential equation. But there is more. In the same letter of May 5 to Johann Bernoulli, Euler mentions that he also solved, in finite terms, the equation $a^3d^3y = ydx^3$. He writes, "though it appears difficult to integrate, needing a triple integration and requiring the quadrature of the circle and hyperbola, it may be reduced to a finite equation; the equation of the integral is

$$y = be^{x/a} + ce^{-x/2a}\sin\frac{(f + x)\sqrt{3}}{2a}$$

. . . where b, c, f are arbitrary constants arising from the triple integration" [Eneström 1905, 31]. Euler does not state explicitly how he found this solution. But one can hazard a guess based on his statement that it required the quadrature of the circle and the hyperbola and on some of his earlier methods of solving differential equations. Namely, Euler was certainly aware that $y = e^{x/a}$ was a solution. As far back as 1728 he used the fact that for that relation, $dy = (1/a)ydx$; differentiating twice more would give that function as a solution to Euler's third order equation. Of course, using the exponential function as a solution meant using the "quadrature of the hyperbola"; Euler was aware of the relationships between the logarithm and both the hyperbola and the exponential function. Once Euler had one solution, he could use it to reduce the order of the equation. That is, he could use $e^{-x/a}$ as an "integrating factor." He had dealt with such factors in [Euler 1735c]; later he wrote a detailed study of their use in reducing the order of equations [1750]. In this particular case, if we multiply the original form $a^3d^3y - ydx^3$ by $e^{-x/a}$ and assume it is the differential of $e^{-x/a}(Ad^2y + Bdydx + Cydx^2)$, it is not difficult to show that a new solution of the original equation must also satisfy the second-order equation $a^2d^2y + adydx + ydx^2 = 0$, or, in derivative notation, $a^2(d^2y/dx^2) + a(dy/dx) + y = 0$.

How would Euler solve this second-degree equation? Using the technique of multiplication mentioned in *De novo genere oscillationum*, he could easily guess as a solution $y = ue^{\alpha x}$. If we differentiate that twice and put it into the equation, we are able to eliminate the term in $dudx$ by setting $\alpha = -1/2a$. Now using $y = ue^{-1/2a}$, we reduce the equation to $a^2d^2u + \frac{3}{4}u\,dx^2 = 0$. The latter equation is similar to several that we have already seen Euler solve by two integrations and the quadrature of the circle. The result in this case is $u = C\sin((x + f)\sqrt{3}/2a)$ as desired. The complete solution to the third-order equation then follows immediately. (We note that Euler used this method explicitly in [Euler 1743].)

In any case, since Euler has now used the sine and the exponential function together in a solution of a differential equation, it is clear that the former now has equal status with the latter insofar as the calculus is concerned; that is, the sine, and of course the other trigonometric functions as well, have now entered calculus. But we will go further. It is the introduction of these functions into the calculus which gave Euler the impetus to find the general solution to linear differential equations with constant coefficients, some special cases of which he had already solved in early 1739.

On September 15, 1739, Euler wrote to Johann Bernoulli giving this solution. Several months later, he wrote again noting that "after treating this problem in many ways, I happened on my solution entirely unexpectedly; before that I had no suspicion that the solution of algebraic equations had so much importance in this matter" [Eneström 1905, 46]. What is this "unexpected" solution? It is the standard method in use today. We replace the given differential equation

$$y + a\frac{dy}{dx} + b\frac{d^2y}{dx^2} + c\frac{d^3y}{dx^3} + \ldots = 0$$

by the algebraic equation

$$1 + ap + bp^2 + cp^3 + \ldots = 0.$$

We factor the polynomial on the left into its real linear and quadratic factors. For each linear factor $1 - \alpha p$ we take as a solution $y = Ce^{x/\alpha}$ while for each irreducible quadratic factor $1 + \alpha p + \beta p^2$ we take as solution

$$e^{-\alpha x/2\beta}\left(C \sin \frac{x\sqrt{4\beta - \alpha^2}}{2\beta} + D \cos \frac{x\sqrt{4\beta - \alpha^2}}{2\beta}\right).$$

Euler gave as his example of this process the solution to the equation

$$y - \frac{k^4 d^4 y}{dx^4} = 0.$$

Since the algebraic equation $1 - k^4p^4$ factors as $(1 - kp)(1 + kp)(1 + k^2p^2)$, Euler finds the complete solution

$$y = Ce^{-x/k} + De^{x/k} + E \sin \frac{x}{k} + F \cos \frac{x}{k}.$$

In this letter, Euler did not consider the case of multiple factors to the polynomial.

How did Euler happen upon his general method of solution for this class of differential equations? Euler does not tell us. But it seems that the examples dealt with in the March paper and the May letter must have been influential. It was by dealing with these examples that Euler first learned that the trigonometric functions were necessary parts of that general solution. He surely had known for years that exponential functions had to be involved. In fact, Johann Bernoulli noted in his December 9 response to Euler's letter that he had found such solutions at least 20 years earlier by assuming that $y = e^{x/p}$ was a solution and solving the resulting equation

$$e^{x/p}\left(1 + \frac{a}{p} + \frac{b}{p^2} + \frac{c}{p^3} + \ldots\right) = 0$$

for p. But Bernoulli had only dealt with a single real solution. For example, for Euler's case of

$$y - \frac{k^4 d^4 y}{dx^4} = 0$$

Bernoulli solved $p^4 - k^4 = 0$ as only $p = k$. He then stated that "the logarithm, whose subtangent $= k$, satisfies the proposed equation" [Eneström 1905, 41], that is, what we call the exponential function $e^{x/k}$. Bernoulli was not able to deal with complex solutions to the algebraic equation, or even with more than a single real solution. It is this major advance that Euler was able to take.

Once Euler knew that the trigonometric functions were in fact part of the complete solution, it was only necessary to see how the parameters of these functions could be found algebraically. And if, in fact, Euler found the solution to the third-order equation by a method similar to what I have proposed, he would have noted that the second-order equation to which the original reduced was in some sense a factor of the original. Euler's genius in dealing with formal identities would then have led him to writing down the characteristic polynomial to the differential equation and exploring the factors. It would have been clear that it was the irreducible quadratic factors which led to the trigonometric solutions.

Over the next couple of years, Johann Bernoulli debated with Euler about the validity of this method of solution in a series of letters. In essence, he did not understand how complex roots of the characteristic polynomial could lead to solutions involving the "real quadrature of the circle." Euler finally showed him in 1740 that in fact $2 \cos x$ and $e^{ix} + e^{-ix}$ were equal. Daniel Bernoulli, on the other hand, accepted Euler's solution and in an article showed that the power series solution agreed with Euler's for the equation $d^4y = ydx^4/f^4$ [Bernoulli 1741]. In any case, beginning in the early 1740s Euler was able to use the calculus of the trigonometric functions with ease; it appears in several of his papers, including a paper in which he published his method for solving linear differential equations with constant coefficients and further explained the relationship between trigonometric and complex exponential solutions [1743]. In that paper, Euler also handled the case of multiple factors of the characteristic polynomial.

It is well that Euler had invented this calculus when he did, for in the mid-1740s he became involved in several major areas of investigation in which the trigonometric functions were to play a crucial role. First of all, he needed the explicit solution of the same fourth-order differential equation in the first appendix of his work on the calculus of variations [1744]. Second, Euler introduced trigonometric series in a prize paper he wrote for the Paris Academy in 1748 on the question of the inequalities in the movements of Saturn and Jupiter [1748a]. It is worthy of mention that in this paper Euler did not yet expect his readers to be familiar with the calculus of the trigonometric functions. In the early pages of that work, Euler discussed this calculus and showed how important it was to the understanding of the topic at hand. Third, at the same time Euler also introduced trigonometric series and the necessary calculus into his initial debates with d'Alembert on the question of the vibrating string [1748b].

As we noted earlier, Euler finally published this calculus in textbook form in his *Institutiones calculi differentialis* [1755]. His methods immediately drove out the earlier geometric methods of dealing with the sine and cosine. Practically without exception, the calculus texts published in the second half of the 18th century all adopted Euler's calculus for their treatment of the trigonometric functions. And, of course, we have continued to use Euler's invention to the present day.

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