On the Value Equivalent to π in Ancient Mathematical Texts. A New Interpretation

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1.

Studying the ancient history of mathematics, one sometimes comes across calculations of the circumference or the area of a circle or the area or the volume of a sphere or of some part of them. To us such calculations are approximations only. In works on history of mathematics those calculations can be found "translated" into modern wording, which means that the now well known formulas $2\pi R$, πR^2, $4\pi R^2$ and $\frac{4}{3}\pi R^3$ are employed. As a result of such translation one meets with an assertion running something like this: "the calculation is equivalent to the assumption that π has the value ...". But when stated in this absolute way, such an assertion can give rise to misunderstanding, for the constant π occurs in different formulas.

Thus if we learn, for instance, that the circumference of a circle is found by taking three times the diameter, we might conclude that π has the value 3. However, we must understand that such an assertion refers to calculation of the circumference only, with not even a hint that the area of a circle should be calculated by taking three times the square on the radius.

In what follows we first discuss an example of such a false interpretation, together with its consequences, from MORITZ CANTOR'S *Vorlesungen über Geschichte der Mathematik*. We next propose a modified notation for some formulas in which the constant π is met with. This will enable us to arrive at a more adequate interpretation and so to avoid ambiguity. With the help of this notation and also of the distinction of meanings of π in the formulas, we shall examine some ancient calculations concerning the squaring of the circle, the rectification of its perimeter, and the cubature of the sphere.

2.

In his *Vorlesungen über Geschichte der Mathematik* M. CANTOR mentions the following three similar quadratures of a circle[1]:

a) In a manuscript *De iugeribus metiundis* (on surveying) dating from the 9^{th}—10^{th} century but going back on the Roman art of surveying, the area of a circle is

[1] M. CANTOR (b), on pp. 591, 836—837 and 876—877, respectively. For full titles see the list of literature.

taken as that of a square the sides of which are the fourth part of the circumference of the circle:

> und so gelangt er in der nächsten Aufgabe, ein rundes Feld von 80 Ruthen Umfang zu messen, zu der Methode, den vierten Theil von 80 mit sich selbst zu vervielfachen[2].
> (and so in the next problem, to find the area of a round field with a circumference of 80 rods, he came to the method of multiplying the fourth part of 80 by itself.)

b) In the manuscript *Propositiones ad acuendos iuvenes* (problems for the quickening of the mind of the young) dating from the 10th century, the contents of which is ascribed to ALCUIN, in problem 25 the area of a round field with the circumference 400 is found as $(\frac{400}{4})^2$:

> Propositio de campo rotundo. Est campus rotundus, qui habet in gyro perticas CCCC. Dic quot aripennos capere debet?
> Solutio. Quarta quidem pars huius campi, qui CCCC includitur perticis est C, hos si per semetipsos multiplicaveris, id est, si centies duxeris, X milia fiunt, ...[3].
> (Rule for a round field. There is a round field with a circumference of 400 "perticae". Tell me how many "aripenni" it has?
> Solution. The fourth part of this field, which is included by 400 "perticae", is 100; when you have multiplied these by themselves, that is, a hundredfold, that makes 10,000.)

c) FRANCO OF LIÈGE, in a treatise on the quadrature of the circle dating from about 1050, informs us:

> Preaterea existunt, qui ambitum circuli in .iiii. distrahunt partes ex quibus quadratum struunt, quem aiunt illi circulo aequalem[4].
> (Then there are also those who divide the perimeter of the circle in four parts, from which they construct a square, which they say equals the circle.)

FRANCO mentions here only a method by which others effect the quadrature of the circle but which he himself refuses.

In the first of these three cases M. CANTOR calculates a value he considers equivalent to π; he finds $\pi=4$.[5] In cases b) and c) he also refers to this result.

Before commenting upon this we consider the following. If R denotes the radius of a circle, then its circumference is $6.28R$ approximately. A quarter part of this is $1.57R$, and consequently is the area of a circle in the three cases above

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[2] M. CANTOR (a), p. 136. The 400 square-rods are reduced to 1 "Juchart" (=288 square-rods), $1\frac{1}{2}$ "Viertel" ($=1\frac{1}{2}$ times a quarter part of a Juchart) and 4 rods (obviously square-rods).

[3] J. P. MIGNE, pp. 1151—1152. See also A. P. JUSCHKEWITSCH, pp. 337, 338. The result 10,000 is still divided by 12^2 to change over from square-perticae to aripenni. Obviously a pertica is a linear measure and an aripennus a square one. According to DUCANGE a pertica may be 10, 12, 16, 20, 22, 24 or 25 feet. To get any idea of its magnitude let us take a good $16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards or 1 pole. Then an aripennus nearly equals an acre (=160 square-poles).

[4] WINTERBERG, p. 145. A. J. E. M. SMEUR, pp. 16 and 45.

[5] M. CANTOR (a), p. 136: Diese Annahme, mathematisch ausgedrückt die Annahme $\pi=4$ (This assumption, mathematically expressed the assumption $\pi=4$,). M. CANTOR (b), p. 591: Mathematisch gesprochen lief dieses Verfahren vermöge $\left(\frac{2\pi r}{4}\right)^2=\pi r^2$ auf $\pi=4$ hinaus (Mathematically expressed this method leads to $\pi=4$ as a consequence of $\left(\frac{2\pi r}{4}\right)^2=\pi r^2$).

$2.465\,R^2$ approximately. If we put this equal to πR^2, we have to say, in the calculation of the area, that this is equivalent to the value 2.465 for π in the formula πR^2, so the calculated area is much smaller than the real one.

But even without this examination it is clear that the area found by the method mentioned above will be too small. For the circumference of the square is equal to the circumference of the circle, so the circle and the square are isoperimetric, and consequently the circle is greater in area.

We remark also that in these three cases the quadrature of the circle seems to be inferred from the assumption that a circle and a square of equal circumference have equal areas. Quite old is the supposition of a linear relation between circumference and area. Thus the historian THUCYDIDES (about 400 B.C.) estimated the extent of an isle as being proportional to the time required to navigate around it, presumably proportional to its circumference[6]. But ZENODORUS (between 200 B.C. and 90 A.D., probably close to the former date), who wrote on isoperimentric geometrical figures, proved some propositions of which the most important is "A circle is greater than any regular polygon of equal contour"[7]. M. FABIUS QUINTILIANUS (about 35—95) criticizes the idea that geometrical figures of equal circumference also have equal areas and that the extent of an isle can be deduced from the time necessary to navigate around it. Of geometrical figures with equal circumferences a more perfect one, he says, is greater in area. So, in particular, a circle is greater in area than a square of equal circumference[8].

We now consider what led M. CANTOR to the value $\pi = 4$. As we have already observed, he started from the formula $2\pi R$ for the circumference. Then the area of the square is $\left(\frac{2\pi R}{4}\right)^2$, which should be equal to πR^2. From this follows indeed $\pi = 4$. Though this reduction seems to be obvious, we are sure that the value for π is much too great. Moreover, substituting this value in the formulas employed to find it, we see that the circumference should equal $8\,R$ and the area $4\,R^2$, or, in other words, a reduction like that of M. CANTOR would lead to the suggestion that the circumference and the area of a circle are equal, respectively, to the circumference and the area of the circumscribed square. This, of course, is unacceptable.

3.

The quadrature of the circle as performed in the three examples mentioned before does hold the essential idea that the area of a circle is proportional to the square of a linear measure of that circle and thus to the square of the radius, of the diameter, or of the circumference. If we reduce the method to the more general form

$$\text{area of a circle} = \left(\frac{\text{circumference}}{4}\right)^2 = \frac{1}{16}\,(\text{circumference})^2 = \alpha\,(\text{circumference})^2,$$

[6] M. CANTOR (b), pp. 172ff. T. HEATH (b) II, p. 207.

[7] M. CANTOR (b), pp. 356, 357. T. HEATH (b) II, pp. 206—213. B. L. VAN DER WAERDEN, pp. 268, 269, remarks that the results of ZENODORUS hold good only for polygons and a circle, thus not for geometrical figures in general.

[8] M. CANTOR (b), pp. 549, 550.

we see that it is similar to the method known from ancient Babylonian geometry[9] in which the area of a circle is taken as $\frac{1}{12}$ (circumference)^2. The latter supplies a better approximation but is not a direct quadrature in the sense that a square equal to the circle is indicated at once[10].

If we emphasize the fact that the circumference of a circle is proportional to the diameter or the radius, and that the area is proportional to the square of a linear measure of the circle and hence to the square of the radius, of the diameter, or of the circumference, we can set up the following two relations:

$$\frac{\text{circumference of a circle}}{\text{diameter}} = \text{constant}, \tag{I}$$

$$\frac{\text{area of a circle}}{\text{square on the radius}} = \text{constant}. \tag{II}$$

Both relations can be found in ancient mathematics, accepted intuitively, without a strict mathematical proof[11]. Much deeper lies awareness of the fact that the constants in (I) and (II) are identical.

In interpreting ancient quadratures of a circle or rectifications of its perimeter, we should distinguish between the constants:
circumference of a circle $= 2\pi_1 R$,
area of a circle $= \pi_2 R^2$,

where π_1 denotes the constant of (I) and π_2 that of (II)[12].

If we make this distinction in the examples mentioned by M. Cantor, we get the relation

$$\left(\frac{2\pi_1 R}{4}\right)^2 = \pi_2 R^2 \quad \text{or} \quad \pi_1^2 = 4\pi_2.$$

M. Cantor simply put $\pi_1 = \pi_2 = \pi$ and thus found $\pi = 4$. But if we put $\pi_1 = 3.14$, we get $\pi_2 = 2.465$.

To this one could object that the value $\pi_1 = 3.14$ is not calculated first in the examples mentioned before and so the application of this value in our reduction comes rather unsuspected. It seems to me, however, that the intent was to square a circle the circumference of which was already known, by direct measuring, for instance, and I reject the idea that this circumference in its turn was found from a formula like $2\pi R$. The first example and the second of *De iugeribus metiundis*

[9] See e.g. K. Vogel (a) II, p. 74 and B. L. van der Waerden, p. 75. I do not accept, without comment, their version that the Babylonians calculated the area of a circle from $3r^2$; see below, on p. 262.

[10] In a direct quadrature of a circle the side of the square can be derived at once from the radius or the diameter. In such cases π is given as a square of a fraction. So *e.g.* the well known Egyptian approximation $(\frac{16}{9})^2$, M. Cantor (b), pp. 98, 99, K. Vogel (a) I, p. 66; $(\frac{26}{15})^2$ going back to a time earlier than the third century B. C., A. P. Juschkewitsch, p. 101; $(\frac{9}{5})^2$, applied in the 11th century, M. Cantor (b), p. 877; $(\frac{5}{4}\sqrt{2})^2$, mentioned by Franco of Liège, 11th century, $\frac{5}{4} \times 2r$ is the diagonal of the square, Winterberg, p. 145, A. J. E. M. Smeur, p. 16. Then $(\frac{7}{4})^2$ if we put $\sqrt{2} = \frac{10}{7}$, A. P. Juschkewitsch, p. 101, M. Cantor (b), pp. 641, 642.

[11] In (I) also may be read radius instead of diameter and in (II) also diameter or circumference instead of radius.

[12] Then π_3 would be right for the calculation of the volume of a sphere. A direct cubature is mentioned by A. P. Juschkewitsch, p. 273: the volume of a sphere with diameter d is $(\frac{11}{14}d)^3$, which leads to the bad value $\pi_3 = 2.91$.

and of *Propositiones ad acuendos iuvenes*, in which the circumferences are given numerically, support this interpretation. Just as the area of a rectangle is gotten by multiplying its length by its width, which are known directly by means of a tape measure, for instance, and not from a formula, in the same way the area of a circle was calculated from the circumference which in turn could be known directly, also by means of a tape measure.

The distinction indicated above by the symbols π_1 and π_2 is made also by A. P. JUSCHKEWITSCH when he suggests that the value $\pi = 3$ in ancient Chinese mathematics probably was found separately for the circumference of a circle and for its area before a relation between these two was known. Thus $\pi_1 = 3$ may result from approximating the circumference of the circle by the circumference of the inscribed regular hexagon, while $\pi_2 = 3$ may result by taking $\frac{3}{4}$ parts of the circumscribed square[13].

4.

The equality of the two constants π_1 and π_2 is proved for the first time by ARCHIMEDES (about 287—212 B.C.) in his book *Measurement of a circle*[14]. Proposition 1 in this book reads:

> The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle.

In terms of the symbols π_1 and π_2 the Proposition states that $\pi_2 R^2 = \frac{1}{2} R \times 2\pi_1 R$, and thus $\pi_1 = \pi_2$.

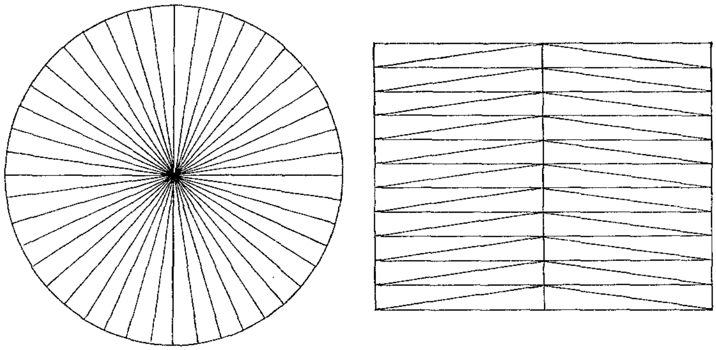
After the time of ARCHIMEDES this Proposition was illustrated again, loosely and without repeating his proof, and by several persons, including FRANCO OF LIÈGE in his treatise on the quadrature of a circle (from about 1050).[15] FRANCO starts from a circle with diameter 14 and states that its circumference is 44. Here he applies the value $\pi_1 = 3\frac{1}{7}$. For the correctness of this value he appeals to former mathematicians; he himself simply accepts this calculation of the circumference. Then, to find the area, he divides the circle in 44 equal sectors

[13] A. P. JUSCHKEWITSCH, p. 57. On p. 56 he mentions another approximation, now for a semicircle, which leads to $\pi_2 = 3$. The same supposition, that is to say that no relation between π_1 and π_2 was known is made by A. P. JUSCHKEWITSCH, p. 102, when he discusses ancient Indian calculations: "Es gibt auch keine Anzeigen dafür, daß sie um die Zusammenhänge zwischen dem Flächeninhalt eines Kreises und dem Kreisumfang gewußt oder diesen Zusammenhang gar benutzt haben." (There is also no indication for it that they were aware of a relation between the area of a circle and the circumference or even availed themselves of such a relation.) We notice that such a distinction between the constants π_1 and π_2 is lacking in J. TROPFKE IV, pp. 260ff: "Die Kreisberechnung".

[14] In the edition by T. HEATH (a), pp. 91—98. H. VON BARAVALLE, p. 484, remarks: "An outstanding contribution to the quadrature of the circle was made by Archimedes, who found that the area of a circle equals the area of a right triangle one of whose legs equals the radius and the other the circumference of the circle."

[15] The rule that the area of a circle equals the product of half the circumference and the radius can be found applied without further elucidation; see for instance A. P. JUSCHKEWITSCH, pp. 153, 218. But also with a proof, following ARCHIMEDES, idem p. 271.

with arc 1 which he pushes together into a rectangle with sides 14 and 11, as illustrated in the figure[16]:



Instead of a single triangle like ARCHIMEDES', there are now 44 triangles or, properly, sectors; the sum of their bases equals the circumference of the circle, and the height is equal to the radius[17].

In Proposition 3 of his book *Measurement of a circle* ARCHIMEDES proves that[18]

$$3\tfrac{10}{71} < \pi_1 < 3\tfrac{1}{7}.$$

By Proposition 1, the same bounds hold also for π_2.

The values are found by calculating the circumferences of the circumscribed and inscribed regular polygons of 96 sides. ARCHIMEDES states without proof that the former circumference is longer and the latter is shorter than the circumference of the circle.

The assumptions required for these statements are found in the first of the two books *On the sphere and the cylinder* by ARCHIMEDES[19].

Assumption 1 is

Of all lines which have the same endpoints the straight line is the least.

This assumption seems to be comprehensible by intuition, for we may really assume that the distance of two points always has been measured along a straight line. Hence the perimeter of a circle is longer than that of an inscribed polygon.

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[16] WINTERBERG, pp. 152, 153. A. J. E. M. SMEUR, pp. 21, 54. To FRANCO the main difficulty is to transform the rectangle he found into a square.

[17] A similar proof is given by GANESÁ (16th century) who divides the circle in 8 equal sectors only; see A. P. JUSCHKEWITSCH, pp. 95 and 161, M. CANTOR (b), p. 656. Also LEONARDO DA VINCI has such a proof; see CH. RAVAISSON MOLLIEN, E fol. 25r.

[18] The book has only 3 Propositions. Proposition 2, no further elucidation of which is known, reads: "The area of a circle is to the square on its diameter as 11 to 14", so it is clear that this Proposition had to follow after Proposition 3.

[19] T. HEATH (a), p. 3 Assumption I and p. 4 Assumption II. E. J. DIJKSTERHUIS I, p. 119.

Assumption 2 is

> Of other lines in a plane which have the same endpoints, [any two] are unequal which are concave in the same sense and are such that one of them is either wholly included between the other and the straight line which connects the same endpoints, or is partly included by, and is partly common with, the other; and the line which is thus included is the lesser.

Hence the circumference of a circle is shorter than that of any circumscribed polygon[20]. But the correctness of the latter statement is based merely on Assumption 2, and I hesitate to claim that it is comprehensible by intuition too[21].

We remark only that ARCHIMEDES could have avoided these difficulties by considering not the circumferences but the areas of the inscribed and circumscribed polygons of 96 sides and then comparing these with the area of the circle. For in that case it will be clear that the former is smaller in area than the circle and the latter is greater[22]. As to the circumscribed regular polygon, in each isosceles triangle with the vertex in the centre of the circle and one side coinciding with one side of the polygon, the height is equal to the radius of the circle, and so the upper limit $3\frac{1}{7}$ can be maintained, but now as a limit for π_2. Of the inscribed regular polygon the height of such a triangle is but slightly smaller than the radius, *viz.* $h = R \cos \frac{2\pi}{192} = 0.9995\, R$, and so the lower limit should be somewhat less than $3\frac{10}{71}$. By Proposition 1, the same limits hold for π_1 also.

5.

In the *Elements* of EUCLID (about 300 B.C.) nothing is disclosed regarding the length of the perimeter of a circle. Definition 15 in Book I is that of a circle:

> A circle is a plane figure enclosed by a single line such that all straight lines falling upon it from one point from those lying within the figure are equal to each other[23].

We see that a circle is defined as a part of a plane and not as a curve. After this, in Definition 17, follows the term "perimeter" ($\pi\varepsilon\varrho\iota\varphi\acute{\varepsilon}\varrho\varepsilon\iota\alpha$) of a circle.

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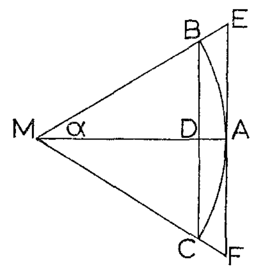
[20] Proposition 1; T. HEATH (a), p. 5.

[21] In fact it is stated that $BC < \text{arc}\, BAC < EF$, so $BD < \text{arc}\, BA < EA$ or $\sin\alpha < \alpha < \operatorname{tg}\alpha$. In teaching basic geometry one meets with the difficulties we mentioned. One then has to appeal to results of higher analysis, *viz.* the convergence of bounded increasing or decreasing series.



[22] Something like such a method was used in ancient China; see A. P. JUSCHKEWITSCH, pp. 57, 58 and our discussion, on. 262.

[23] E. J. DIJKSTERHUIS I, p. 112. Our quotation is a free translation from the Greek text. See also E. J. DIJKSTERHUIS II, p. 26 for Book III, Def. 1; p. 225 for Book XII, Prop. 2 and p. 247 for Book XII, Prop. 18.

In Definition 1 of Book III, on the circle, two circles are called "equal" if their diameters or radii are equal. Thus "equal" circles are congruent. While these have equal areas, we learn nothing about the absolute magnitude of such an area. In Book XII, which treats areas and volumes, we find some more. In Proposition 2, "circles are to one another as the squares on their diameters"[24], which means that

$$\frac{\text{area of a circle}}{(\text{diameter})^2} = \text{constant},$$

but we learn nothing about the value of that constant. In a similar way Proposition 18 of Book XII states that

$$\frac{\text{volume of a sphere}}{(\text{diameter})^3} = \text{constant}.$$

But we learn nothing in EUCLID about the ratio of the circumference of a circle to its diameter, nor does EUCLID have a relation between the two constants just mentioned.

6.

However, the fact that π_1 and π_2 are identical may perhaps have been known before EUCLID's time and thus surely also before the proof of ARCHIMEDES. This at least we may deduce from communications with respect to the so called "quadratrix", a curve used to solve the problem of the squaring of the circle. The curve in question originated with HIPPIAS OF ELIS (born about 460 B.C.). We may be sure that it was intended originally to divide an angle into a given number of equal parts and so also, among other things, to solve the trisection problem, but we do not know whether HIPPIAS himself also used the curve to square a circle. According to PAPPUS, DINOSTRATUS (about 350 B. C., so shortly before EUCLID) made use of the quadratrix to perform the quadrature of a circle. For an elaborate discussion about the reliability of PAPPUS' dates and also those of PROCLUS we refer to others[25]. What is important here is only this: using the quadratrix, given a circle with radius R, one can construct a line segment AB for which

$$AB : R = R : \text{ a quarter of the circumference of the circle}$$[26].

Then it is possible to rectify the perimeter, and for this reason the name "rectificatrix" would suit better than "quadratrix".

Besides this, the reader must be acquainted with Proposition 1 of ARCHIMEDES' *Measurement of a circle,* for only with the help of this Proposition can the area

[24] According to ARCHIMEDES this Proposition had been proved by mathematicians who lived before his time; see T. HEATH (a), pp. xlvii, xlviii. Probably by HIPPOCRATES OF CHIOS? In connection with this ARCHIMEDES refers to the *Elements* in *On the sphere and the cylinder I,* Prop. 6; see T. HEATH (a), p. 9.

[25] T. HEATH (b) I, pp. 225ff. E. J. DIJKSTERHUIS I, pp. 3ff.

[26] SPORUS (end of the third century) made the objection that in the construction the end B of the line segment is not defined; see T. HEATH (b) I, p. 230 who subscribes to this and other objections of SPORUS, and B. L. VAN DER WAERDEN, p. 192, who does not subscribe to them.

be found from the circumference[27]. We must remark that for the mere rectification of the circumference still further suppositions have to be made. T. HEATH[28] mentions them after furnishing a possible reconstruction of the way the quadratrix could have been used for rectification of the fourth part of the perimeter. These suppositions are

a) The arcs of a circle are to each other as the angles subtended at the centre. This is Proposition 33, Book VI, of EUCLID'S *Elements*. This supposition is necessary for the construction of the quadratrix.

b) The ratio of the circumference to the radius is constant. This supposition is a fundamental one, for it asserts the existence of the constant π_1. As we have mentioned before, it is just this important relation that is missing in EUCLID'S *Elements*.

c) and d) These are the same as the Assumptions of ARCHIMEDES in *On the sphere and the cylinder I*, which we have mentioned before. As we have remarked, they are equivalent to the inequalities $\sin\alpha < \alpha < \operatorname{tg}\alpha$.

ARCHIMEDES does not talk about the "quadratrix" or "squaring". He only produces a triangle equal to the circle. But a real quadrature, after the rectification has been performed, requires the construction of a square equal to a triangle. For that purpose it is necessary to be familiar with

a) Proposition 45, Book I of the *Elements*, on the transformation of a polygon into a parallelogram of equal area and with an angle equal to a given angle.

b) Proposition 14, Book II of the *Elements*, on the transformation of a rectangle into a square or, what is equivalent, the construction of the mean proportional between two given line segments[29].

In the case we consider a) is needed only in the special case of the construction of a rectangle equal to a triangle. It is rather easy to see that this can be done by keeping the base and halving the height. As to b), the construction of the mean proportional, this seems to have been known much earlier, for it is applied by HIPPOCRATES (about 430 B.C.) and ARCHYTAS (about 390 B.C.)[30].

On the basis of the presuppositions mentioned by T. HEATH as being required for rectification of the perimeter, and on the basis of the further presuppositions I have mentioned as being necessary to complete the quadrature[31], when such

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[27] Quite rightly T. HEATH (b) I, p. 182, says that the quadratrix was used "for squaring the circle, or rather for finding the length of any arc of a circle".

[28] T. HEATH (b) I, p. 229.

[29] See E. J. DIJKSTERHUIS I, pp. 201, 202, Prop. 45, Book I, and II, p. 19, Prop. 14, Book II.

[30] See B. L. VAN DER WAERDEN, pp. 118, 134, 149ff.; T. HEATH (b) I. pp. 193—195 and 246—249.

[31] *Viz.* familiarity with Proposition 1 of *Measurement of a circle* of ARCHIMEDES or, what is quite the same, the insight that π_1 and π_2 are equal. In my opinion this is a rather far-reaching supposition. E.W. HOBSON, pp. 4, 5 writes, "The fact was well known to the Greek geometers that the problems of the quadrature and the rectification of the circle are equivalent problems. It was in fact at an early time established that the ratio of the length of a complete circle to the diameter has a definite value equal to that of the area of the circle to that of a square of which the radius is side ... The problem of 'squaring the circle' is roughly that of constructing

a rectification is known, I think that the quadrature of the circle with the help of the "quadratrix" in an era before ARCHIMEDES' is somewhat questionable. Proofs that the quadratrix was indeed applied to that purpose are lacking. The only evidence regarding this is as follows:

a) IAMBLICHUS (4th century), who is not quite trustworthy, mentions the quadratrix without ascribing it to HIPPIAS OF ELIS, and he ascribes the quadrature of the circle with the help of that curve to NICOMEDES, who lived about 180. Since NICOMEDES lived after ARCHIMEDES, this evidence does not impugn my objection[32].

b) PAPPUS OF ALEXANDRIA (end of the 3rd century) mentions that DINOSTRATUS (about 350 B.C.), the brother of MENAECHMUS, who was a pupil of EUDOXUS, performed the quadrature of a circle with the help of the quadratrix. Again an ascription of this curve to HIPPIAS is lacking. As to DINOSTRATUS, who lived before ARCHIMEDES, my objections remain[33].

c) PROCLUS (410 till 485) only mentions the quadratrix of HIPPIAS and NICOMEDES.

From all this I remark that the curve may indeed originate with HIPPIAS[34], but if so, he can be credited with no more than the trisection of an angle. It is not until after his time, and probably not before ARCHIMEDES', that the curve may have been applied to square a circle. From then on the curve got the name "quadratrix", and thus possibly the squaring of a circle with the help of that curve came to be ascribed to an earlier era.

7.

We next consider some ancient Indian calculations. In the second chapter of *Āryabhaṭīya* of ĀRYABHAṬA, entitled *Gaṇitapāda* (mathematics) can be found the following rules[35]:

Rule 7. Half of the circumference multiplied by half the diameter is the area of a circle.

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a square of which the area is equal to that enclosed by a circle. This is then equivalent to the problem of rectification of the circle, i.e. of the determination of a straight line, of which the length is equal to that of the circumference of the circle". Now I wonder what E.W. HOBSON means by "at an early time". All he says is no more than an assertion, with no nuances and lacking any proof.

[32] See T. HEATH (b) I, p. 225, also as for PAPPUS and PROCLUS. E. J. DIJKSTERHUIS I, p. 9.

[33] T. HEATH (b) I, pp. 225, 229, 251. E. J. DIJKSTERHUIS I, pp. 3, 98, 99 ann. 239. B. L. VAN DER WAERDEN, pp. 191—193 and T. HEATH (b) I, p. 229 stick to the quadrature by DINOSTRATUS, and consequently they state that he, obviously, must have known the repeatedly mentioned Proposition 1 of ARCHIMEDES, that is to say at least the contents of that Proposition.

[34] T. HEATH (b) I, p. 225. E. J. DIJKSTERHUIS I, p. 3.

[35] W. E. CLARK, pp. 27, 28. His reference to *Bibliotheca mathematica IX*, p. 196, for a discussion of the inaccurate value given in the second part of Rule 7 is incorrect, for *Bibliotheca mathematica* 9_3, 1908—1909, pp. 196—199, has: "Eine indische Methode der Berechnung der Kugeloberfläche" by H. SUTER, on a calculation in *Siddhānta Śiromani* of Bhāskara. We remark that the first part of Rule 7 is equivalent to Proposition 1 of ARCHIMEDES' *Measurement of a circle*.

This area multiplied by its own square root is the exact volume of a sphere[36].
Rule 10. Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000.

It is clear that the latter rule provides a value for the constant π_1, *viz.* 3.1416. But it is only after Rule 7 that this value is communicated. Are we allowed now to make use of this value when practising Rule 7? Or have we to interpret that rule as being practised or practicable only in those cases where the circumference was known from direct measurement? But the value 3.1416 is such a good approximation that in practice it hardly could make any difference if the circumference were calculated according to Rule 10, using $\pi_1 = 3.1416$, or were found from a direct measurement. In the first case we have also to put $\pi_2 = 3.1416$ as a consequence of Rule 7, 1[37].

The second part of Rule 7 needs close analysis. One can find it interpreted as being equivalent to the assumption $\pi = \frac{16}{9}$, a value which is exceptionally small and totally different from the value 3.1416 of Rule 10. It is easy to see what led to this assumption. If A denotes the area of a great circle of a sphere and V its volume, then Rule 7, 2 states that $V = A\sqrt{A}$. Writing this as $\frac{4}{3}\pi R^3 = \pi R^2 \sqrt{\pi R^2}$, we find $\pi = \frac{16}{9}$.[38]

But this interpretation is hard to maintain. For substitution of the value $\pi = \frac{16}{9}$ in $A = \pi R^2$, a formula used to find that value, should deliver the area of a circle as $\frac{16}{9} R^2$, which is less than the area of the inscribed square. In the same way substitution of $\pi = \frac{16}{9}$ in $\frac{4}{3}\pi R^3$, also a formula used to find that value, would lead to $\frac{64}{27} R^3$ as the volume of a sphere, which is less than the volume of the greatest inscribed cylinder[39]. Of course, all this is complete nonsense, and we hardly need to remark, in regard of the area of a circle, that it is also in contradiction to Rules 10 and 7, 1 as well as to what would be found from direct measurement.

Using the value $\pi = 3.1416$ of Rule 10 and the statement of Rule 7, 1, we have to say that the volume of a sphere with radius R was calculated from $3.1416 R^2 \sqrt{3.1416 R^2}$. If, in modern symbols but with a necessary distinction of meanings of π, we write the volume of a sphere as $\frac{4}{3}\pi_3 R^3$, then π_3 equals $\frac{3}{4} \times 3.1416 \sqrt{3.1416}$ or 4.1761 approximately. This value, of course, is much too large.

It does not seem to be difficult to set up an acceptable hypothesis for what could have been the origin of the formula $V = A\sqrt{A}$. If a denotes the side of

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[36] This is the translation of W. E. Clark. After we have discussed it, we shall consider an alternative interpretation and translation by K. Elfering.

[37] A. P. Juschkewitsch, pp. 93, 153.

[38] A. P. Juschkewitsch, p. 153. D. E. Smith I, p. 156, with the remark, out of place, that $\frac{16}{9}$ is possibly an error for the ancient Egyptian value $(\frac{16}{9})^2$.

[39] A. P. Juschkewitsch, who has $\pi = \frac{16}{9}$, found obviously from $\frac{4}{3}\pi R^3 = \pi R^2 \sqrt{\pi R^2}$, makes in fact a similar mistake as M. Cantor in $\left(\frac{2\pi R}{4}\right)^2 = \pi R^2$; see above, p. 252.

We remark again that the formula $2\pi R$ is not employed. But anyone judging from the mere communication $\pi = \frac{16}{9}$ could get the idea that this value was also used in $2\pi R$. This, however, would mean that the circumference is less than double the diameter.

a square equal to the circle, so $A=a^2$, then for $V=A\sqrt{A}$ two interpretations are possible.

a) $V=A\sqrt{A}=a^2\sqrt{a^2}=a^3$; thus a cubature is performed analogous to the quadrature of a circle and, moreover, with the edge equal to the side of the square[40].

b) $V=A\sqrt{A}=A\sqrt{a^2}=A\,a$. In this A is the area of a great circle according to the formulation of Rule 7, 2. Then $A\,a$ is the volume of a cylinder circumscribed about the sphere but with the height $\sqrt{3.1416R^2}$ ($\approx 1.77R$) which is less than the diameter of the sphere. But this height is considerably more than $\frac{4}{3}R$, and so it is easy to understand that practice of $V=A\sqrt{A}$ gives a volume much too large.

Summarizing, I conclude that *Āryabhaṭīya* has $\pi_1=\pi_2=3.1416$ and $\pi_3=4.1761$.

In our discussion we started from the supposition that W. E. CLARK'S translation is correct. Recently K. ELFERING has proposed an alternative interpretation and translation of the second parts of Rules 6 and 7. The second part of Rule 6 has always been read as an incorrect formula for the volume of a pyramid, *viz.* half the product of the height and the area of the base. It lies outside our theme to consider K. ELFERING'S conception here, but to me it seems too studied[41].

As to the second part of Rule 7, it appears from K. ELFERING'S analysis that the original text reads: "this multiplied by its own basis (origin) is the exact ... (?)". He now relates the word "this" to "half the diameter" in the first sentence of Rule 7 and not to "the area". Next he sees "own basis" or "own origin" not as the square root of the area but as the basis of a hemisphere, which in his interpretation is the circumference of a great circle. Thus what is calculated is $R\times 2\pi R$, the exact area of a hemisphere[42].

This conception indeed leads to a correct result. I cannot say which translation is preferable, but K. ELFERING'S arguments do not wholly convince me. If W. E. CLARK'S is right, then, as I have said before, we must conclude a value 4.1761 for π_3 and not $\frac{16}{9}$. Although this is an incorrect value, we may say after all that the idea of a cubature by means of $V=A\sqrt{A}=a^3$ is easy to understand[43].

In the *Sulba-sūtras*, a collection of rules the most ancient of which date from 800 B.C.,[44] can be found a quadrature of a circle by means of $a=(1-\frac{2}{15})\times 2R$, so $\pi_2=3\frac{1}{225}$; another with $a=\left\{1-\frac{1}{8}+\frac{1}{8\times 29}-\frac{1}{8\times 29}\left(\frac{1}{6}-\frac{1}{6\times 8}\right)\right\}\times 2R$, so $\pi_2=3.088$; and the inverse operation, to find a circle equal to a square by taking $2R=\left(1+\frac{\sqrt{2}-1}{3}\right)\times a$, so $\pi_2=18\times(3-2\sqrt{2})\approx 3.088$.[45]

[40] M. CANTOR (b), p. 646, also has this hypothesis.

[41] K. ELFERING, pp. 60—63.

[42] K. ELFERING, pp. 63—64.

[43] There is a striking analogy with the two-dimensional quadrature a^2, and the same holds for Rule 6: the volume of a pyramid is half the product of the area of the base and the height, in analogy with the two-dimensional calculation of the area of a triangle.

[44] A. P. JUSCHKEWITSCH, p. 92.

[45] A. P. JUSCHKEWITSCH, pp. 101, 102.

8.

Ancient Chinese mathematics has come down to us in *Nine Chapters on the Mathematical Art*, which contain mathematical knowledge from a time as early as 1000 B.C. The work was edited in 175 and again, with a commentary, by LIU HUI in 263.[46]

In Chapter I can be found calculations of areas, including that of a circle by multiplying half the circumference by half the diameter[47]. This is equivalent to Proposition I of ARCHIMEDES' *Measurement of a circle*. The calculation assumes that the circumference equals 3 times the diameter, so $\pi_1 = 3$. We may conclude then, from the way in which the area is calculated, that $\pi_2 = 3$ also. A. P. JUSCHKEWITSCH regards it likely that both values (*i.e.* π_1 and π_2) were probably first found separately[48].

If c denotes the circumference and d the diameter, then the area A was calculated from $A = \frac{c}{2} \times \frac{d}{2} = \frac{c\,d}{4} = \frac{d\,d}{4} \times 3$ or $\frac{c\,c}{12}$.

As could be expected, the way in which the volume of a cylinder, and also that of a cone and a truncated cone, were calculated leads to the same value of π.[49] But as to the volume of a sphere, we find only the relation[50] $d = \sqrt[3]{\frac{16}{9} V}$. If we write it as $V = \frac{9}{16} d^3$, we may say that the calculation is equivalent to the value $\pi_3 = 3\frac{3}{8}$.[51]

J. E. HOFMANN has[52]

> π wird bald gleich 3, bald gleich $3\frac{3}{8}$ gesetzt.
> (π at times equals 3, then again $3\frac{3}{8}$).

From this bare pronouncement one might gain the obviously false idea that the circumference of a circle, for instance, was found by taking $3\frac{3}{8}$ times the diameter. Thus we again have reason to emphasize the importance of specifying the calculation in question.

ZHANG HENG (78 to 139) states that the square of the circumference of a circle is to the square of the circumference of the circumscribed square of that circle as 5:8, or $(2\pi R)^2 : (8R)^2 = 5:8$.[53] From this follows $\pi_1 = \sqrt{10}$. It is clear that π_1 is in question, and it is deceiving here to state vaguely that $\pi = \sqrt{10}$.

Important is the calculation of the area of a circle by LIU HUI. $ABCD\ldots$ is a polygon inscribed in a circle, while $ABba$, $BCdc$, $CDfe$ and so on are rectangles circumscribed on segments.

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[46] For ancient Chinese mathematics we refer to A. P. JUSCHKEWITSCH, p. 23 and pp. 55—62. Especially the circle-squaring is treated by YOSHIO MIKAMI.

[47] K. VOGEL (b), pp. 14—16, Chapter I, Problems 31, 32, 33, 34, 37, 38. Also p. 41, Chapter IV, Problems 17 and 18 (calculation of the circumference, the area being given).

[48] A. P. JUSCHKEWITSCH, p. 57. See also before, p. 253.

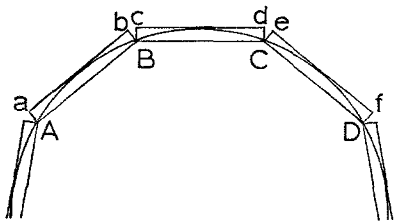
[49] K. VOGEL (b), pp. 47—54, Chapter V, Problems 9, 11, 13, 20, 23, 24, 25, 28.

[50] K. VOGEL (b), p. 43, Chapter IV, Problems 23, 24.

[51] A. P. JUSCHKEWITSCH, pp. 61, 62, supposes that the calculation probably originates from taking $\frac{3}{4}$ parts of the volume of a cylinder circumscribed to the sphere. LIU HUI found a more accurate result, *viz.* $3 < \pi < 3\frac{3}{8}$.

[52] J. E. HOFMANN, p. 74.

[53] A. P. JUSCHKEWITSCH, p. 57.



LIU HUI states that the circle is greater in area than the inscribed polygon but less than the same polygon augmented with all the circumscribed rectangles. This is evident. Because of the fact that he calculates the area and not the circumference, he evades the difficulty we have pointed out above in discussing the calculation by ARCHIMEDES. LIU HUI'S calculation of the area of the inscribed polygon (regular and with 192 sides) is not wholly correct, his result being a little too large, but nevertheless he finds two good limits and a good approximation for what is very clearly π_2, *viz.*[54] $3.141024 < \pi_2 < 3.142704$.

9.

Regarding calculations of the circumference and the area of a circle by the Babylonians B. L. VAN DER WAERDEN writes[55]

> They took the area of a circle of the radius r to be equal to $3r^2$, the perimeter $6r$.

This is too simple a way to express what is known about the subject, as will be shown in what follows.

K. VOGEL is more correct[56]:

> Die Babylonier rechneten den Kreisumfang u als den dreifachen Durchmesser $3d$, somit $\pi = 3$, wobei sie sich sicher bewußt waren, daß es nur eine praktische Näherung war. ... Der Kreisinhalt ist (mit $\pi = 3$) dann $\frac{u^2}{12} = u^2 \times 0;5$.
> (The Babylonians found the circumference u of a circle by taking 3 times the diameter $3d$, so $\pi = 3$, in which they surely were aware that this was only an approximation for practical purposes. Then the area is (using $\pi = 3$) $\frac{u^2}{12} = u^2 \times 0;5$).

I cannot agree, however, in concluding the last sentence.

From the cuneiform texts known at this time we learn that the circumference of a circle always was found by taking three times the diameter, and conversely

[54] In fact, according to A. P. JUSCHKEWITSCH, p. 58, he found for the area A of the circle $314\frac{64}{625} < A < 314\frac{169}{625}$. This obviously concerns a circle with the radius 10. Thus it is very confusing that A. P. JUSCHKEWITSCH writes "With d (diameter) = 100 units of length was found ... $314\frac{64}{625} < A < 314\frac{169}{625}$". This would be correct if A denoted the circumference. We have at our disposal only the text of A. P. JUSCHKEWITSCH; it is clear that his rendering is absolutely incorrect. The same holds for J. E. HOFMANN, p. 76. Neither in P. L. VAN HEE nor in K. VOGEL (b) is any indication of this calculation of LIU HUI. See also D. J. STRUIK, p. 427.

[55] B. L. VAN DER WAERDEN, p. 75.

[56] K. VOGEL (a) II, p. 74. 0; 5 is a notation for the sexagesimal fraction $\frac{5}{60}$.

the diameter as a third part of the circumference. The relative texts are in BM 85194, dating from about 2000 B.C.:[57]

probl. 4	circumference 1.0°;	diameter	20′	times	1.0° = 20°
	diameter 30°;	circumference	3	times	30° = 1.30°
probl. 16	diameter 13′20″;	circumference	3	times	13′20″ = 40′
	diameter 20′;	circumference	3	times	20′ = 1°
probl. 19	circumference 40′;	diameter	20′	times	40′ = 13′20″.

From this it is clear that we have to put $\pi_1 = 3$.[58]

The same value is employed in the *Old Testament, viz.* in 1 *Kings* 7, 23 where 30 ells are given as the circumference of a circle with a diameter of 10 ells.[59]

The area of a circle always was found as 5′ times (this is $\frac{1}{12}$ part) the square of the circumference. Texts relative to this are[60]:

BM 85194			
probl. 4	circumference 1.30°;	area	$5' \times (1.30°)^2 = 11.15°$
probl. 14	circumference 4°;	area	$5' \times (4°)^2 = 1°.20'$
BM 85196			
probl. 2	same numbers as probl. 14, BM 85194		
probl. 16	circumference 30′;	area	$5' \times (30')^2 = 1'15''$
YBC 7302	circumference 3°;	area	$5' \times (3°)^2 = 45'$
YBC 11120	circumference 1°30′;	area	$5' \times (1°30')^2 = 11'15''$
YBC 7997	same numbers as the preceding		
VAT 7848(4)	circumference 1.0.0°;	area	$5' \times (1.0.0°)^2 = 5.0.0.0°$.

From this it is clear that the area was considered as proportional to the square of a linear measure of the circle. However, it is not a direct quadrature, for the side of a square equal to the circle is not calculated. According to the calculation of the circumference we have to take it as $6R$. Then the calculation of the area amounts to $\frac{1}{12}$ part of $(6R)^2$, so $3R^2$. Are we allowed now to infer that $\pi_2 = 3$? We do not know a direct relation between the area and the diameter or radius, although $3R^2$ seems to be simpler than $\frac{1}{12}$ (circumference)^2. Thus in Problem 4 of BM 85194 the area of a circle with a diameter of 30° is calculated from $5' \times (3 \times 30°)^2 = 5' \times (1.30°)^2$ and not from $3 \times (30' \times 30°)^2 = 3 \times (15°)^2$.[61]

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[57] BM means British Museum. See F. THUREAU DANGIN, pp. 23—25 for Problem 4; pp. 29, 30 for Problem 16; and pp. 31, 32 for Problem 19. The same text can be found in O. NEUGEBAUER (b), pp. 142—193, but he gives as source BM 95194. As to the notations, 1.0° means $1 \times 60°$; 20′ means $\frac{20}{60}$ $(= \frac{1}{3})$ and so on.

[58] See O. NEUGEBAUER (c), p. 47 for a different value for π_1.

[59] This holds for the ninth century B. C. The same text is in 2 *Chronicles* 4, 2.

[60] F. THUREAU DANGIN, pp. 23—25 (BM 85194, 4), pp. 28, 29 (BM 85194, 14), pp. 40ff. (BM 85196, of about 200 B.C., 2 and 6). O. NEUGEBAUER & A. SACHS, p. 44 (YBC 7302 and 11120), p. 99 (YBC 7997), p. 142 (VAT 7848(4)). YBC = Yale Babylonian Collection, New Haven; VAT = Vorderasiatische Abteilung, Tontafeln, Staatliche Museum (Pre-Asiatic department, clay-tablets, State museum), Berlin.

[61] Thus we have to dismiss the statement of B. L. VAN DER WAERDEN, which renders the question inaccurate.

If we accept the supposition that each calculation of the area of a circle was proceeded by a calculation of the circumference and that the latter was calculated from the diameter, with the value $\pi_1 = 3$, found by direct measurement, then we have to accept the value 3 for π_2 as well. But it is also possible, on the other hand, that the circumference was known by direct measurement, as perhaps in Problem 14 of BM 85194, where only a circumference of 4° is given. If we have to take it for granted that this value was known by means of direct measurement, then it relates to a circle with the radius $R = \frac{4^\circ}{6.28}$ and an area $\pi_2 \times \left(\frac{4^\circ}{6.28}\right)^2$. In the text is calculated $5' \times (4^\circ)^2 = 1^\circ 20'$. From $\pi_2 \times \left(\frac{4^\circ}{6.28}\right)^2 = 5' \times (4^\circ)^2$ follows the value $\pi_2 = 3.2865$.

We remark also that the area of a semicircle was found in a way different from that for a whole circle. In BM 85210, going back to about 2000 B.C.,[62] first is indicated the proportion of the length of the arc of a semicircle and the diameter, *viz.* 1°30′, so 3:2. In the problem the length of the arc is 30°, the length of the diameter is 20°. On the basis of what preceded we might expect a calculation like this: $\frac{1}{2} \times 5' \times (2 \times 30^\circ)^2 = 2.30^\circ$. On the other hand, a calculation with the help of the formula $\frac{1}{2}\pi_2 R^2$, with $\pi_2 = 3$, would give $\frac{1}{2} \times 3 \times (\frac{1}{2} \times 20^\circ)^2 = 2.30^\circ$. The text, however, calculates the product of the length of the arc of the semicircle by the length of the diameter, multiplicated by 15′ $(=\frac{1}{4})$, so $15' \times (30^\circ \times 20^\circ) = 2.30^\circ$. This conforms to $\frac{1}{2} \times \text{arc} \times \text{radius}$, which is Proposition I of ARCHIMEDES, quoted before, but it differs entirely from the method used to find the area of a whole circle. That this is not an example of an incidental case but is in accord with a general prescription, appears from a list published by E. M. BRUINS & M. RUTTEN. For the calculation of the area of a circle this list provides a fixed constant 5′; the area equals $5' \times (\text{circumference})^2$, and for the calculation of the area of a semicircle the list has 15′, by which is meant: take the quarter part $(\frac{1}{4} = 15')$ of the product of the length of the arc of the semicircle and that of the diameter[63].

Thus we can be sure that the Babylonians were not familiar with a formula like $A = \pi R^2$. We have to admit that separate prescriptions existed for the calculation of the circumference of a circle, the area of a whole circle and the area of a semicircle, and that the Babylonians surely at least in the beginning, were not aware of any relation between the numbers 3, 5′ and 15′, and certainly not that those numbers were connected by one and the same factor of proportionality, our number π.

10.

Finally we consider the Egyptian circle calculations. In the Rhind papyrus, dating from about 1800 B.C., we meet with five problems on this subject[64].

In Problem 50 the area of a round field with a diameter 9 is calculated in this way: $(\frac{8}{9} \times 9)^2 = 64$. In fact the area is taken as $(d - \frac{1}{9}d)^2$, where d is the diameter. This is equivalent to $\pi_2 = (\frac{16}{9})^2 = 3.1605 \ldots$, a very good approximation.

[62] F. THUREAU DANGIN, pp. 50, 51.

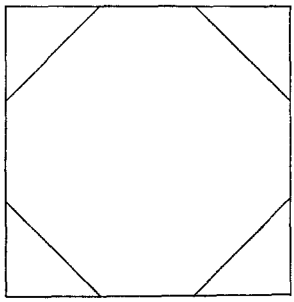
[63] E. M. BRUINS & M. RUTTEN, pp. 27, 28.

[64] A. B. CHASE I, p. 92 (Problem 50), p. 86 (Problems 41 and 42), p. 87 (Problem 43) and p. 91 (Problem 48).

In Problem 41 the volume of a cylinder is calculated. The diameter is 9 and the area of the bottom is $(\frac{8}{9} \times 9)^2 = 64$, just as in Problem 50.

Problem 42 differs from Problem 41 only in that the diameter of the cylinder is 10. The calculation is performed in the same way.

Worth mentioning is Problem 48, in which are compared the areas of a circle and the circumscribed square. Again the diameter is 9. The area of the square is 81, that of the circle $8^2 = 64$. The latter value fits the formula $(\frac{8}{9} \times \text{diameter})^2$. But a figure is given which, while drawn quite roughly, one has to consider as a square with four triangles in the vertices. In the middle of the figure is the



demotic sign for 9. The area of the octagon that remains after removing the four triangles is $\frac{7}{9} \times 9^2 = 63$. This has been seen as a possible explanation of the formula $(\frac{8}{9} \times d)^2$ for $\frac{7}{9} = \frac{63}{81}$ which is nearly as much as $\frac{64}{81} = (\frac{8}{9})^2$. The octagon should be considered as a first approximation to the circle inscribed in the square[65].

Here we meet with a direct quadrature; $\frac{8}{9}$ parts of the diameter is the side of the square that equals the circle. It is clear that the value $(\frac{16}{9})^2$ holds good only for π_2. In the preserved Egyptian texts occurs not a single calculation of the circumference[66], so the mere remark that π equals $(\frac{16}{9})^2$, without any comment, is deceptive.

The calculations in Problem 43 and in a problem of the Kahun papyrus, which shows some resemblance to it, are not entirely clear. Probably they concern the calculation of a volume with the help of a value $\pi_3 = 3.2$. [67]

In Problem 10 of the Moscow papyrus is performed a calculation in which again we find $(\frac{8}{9})^2$, so we can be sure that it concerns an area of an object which has to do with a circle in some way. The text is damaged and is illegible at important points, so it is unclear exactly what are the data of this problem, but the whole calculation is legible, *viz.* $9 \times \frac{8}{9} \times \frac{8}{9} \times 4\frac{1}{2} = 32$, or, as we may say in referring to the significance of $(\frac{8}{9})^2$, discussed before, $9 \times \frac{\pi_2}{4} \times 4\frac{1}{2}$.

[65] K. VOGEL (a) I, p. 66; E. M. BRUINS (c), p. 8 and (b), pp. 207, 208. Also O. NEUGEBAUER (a), p. 124, who has some doubts, however. M. SIMON, p. 43, on the other hand, sees the origin of the formula as merely experimental. E. W. HOBSON, p. 13, has "probably empirical".

[66] O. NEUGEBAUER (a), p. 124, suggests that $(\frac{16}{9})^2$ was used also for the calculation of the circumference, *viz.* in Problem 10 of the Moscow papyrus; also W. W. STRUVE, pp. 167 and 177, 178, and T. E. PEET (see O. NEUGEBAUER). I definitely disagree with this conception, as will become clear from the discussion that follows.

[67] L. BORCHARDT, pp. 150—152. M. CANTOR (b), p. 99. M. SIMON, pp. 43—45. T. HEATH (b) I, pp. 125, 126.

W. W. STRUVE, who published the text of the Moscow papyrus, supposes the object to be a basket shaped like a hemisphere with a diameter $4\frac{1}{2}$.[68] He then concludes that the Egyptians knew the right formula for the area of a sphere, $4\pi R^2$, some 1500 years before ARCHIMEDES[69]. The same conclusion has been reached by R. J. GILLINGS, who recently re-examined the text of Problem 10 of the Moscow papyrus[70]. But such a conclusion is out of place. What has come down to us of Egyptian geometry are some calculations for practical purposes, with no theoretical background worth mentioning. Thus the insinuation that the Egyptians knew the area of a sphere to be exactly four times that of a great circle is not allowed. Indeed, if the problem in question regards the area of a hemisphere, then we only may say that they were lucky in finding such a good approximation. In any case we remark that according to the formula $2\pi R^2$ the calculation should have been $2\times(\frac{8}{9}\times 4\frac{1}{2})^2$ instead of $9\times\frac{8}{9}\times\frac{8}{9}\times 4\frac{1}{2}$.

T. E. PEET interpreted the damaged preliminary text in a way that differs from STRUVE'S[71]. He looks upon the basket as a semicylinder (axis horizontal) with $4\frac{1}{2}$ as the length of the axis and with also $4\frac{1}{2}$ as the diameter of the semicircle. Then $9\times(\frac{8}{9})^2$ should be the length of the arc of the semicircle and $9\times(\frac{8}{9})^2\times 4\frac{1}{2}$ the area of the semicylinder. O. NEUGEBAUER prefers this interpretation to that of W. W. STRUVE, but we can make some important objections at once.

In the first place $(\frac{8}{9})^2$ is now considered as $\pi_1/4$. This, however, is so much against the nature of the magnitude $(\frac{8}{9})^2$, which really is intended for the calculation of an area, as to make me refuse the proposed interpretation if for this reason alone, since otherwise we should have to admit that the Egyptians had some idea of the identity of the two constants π_1 and π_2, or, differently expressed, that they knew that

$$\frac{\text{circumference of a circle}}{\text{diameter}} = \frac{\text{area of a circle}}{\text{square on the radius}}.$$

But as we have seen above, this is equivalent to a Proposition of ARCHIMEDES. In this connection it is worth remarking that O. NEUGEBAUER, who thinks it improbable that the Egyptians should have known more than a thousand years before ARCHIMEDES his Proposition on the area of a sphere, and in this I agree with him, on the other hand admits, as I do not, that they knew the far-reaching statement $\pi_1 = \pi_2$, which was proved for the first time by ARCHIMEDES[72].

68 W. W. STRUVE, p. 157.

69 To O. NEUGEBAUER (a), p. 129, this interpretation seems to be too improbable. He rejects it as, in imitation of him, does B. L. VAN DER WAERDEN, pp. 33, 34. In (c), p. 78, O. NEUGEBAUER writes merely "It has even been claimed that the area of a hemisphere was correctly found in an example of the Moscow papyrus, but the text admits also a much more primitive interpretation which is preferable".

70 R. J. GILLINGS, p. 116.

71 See the elaborate analysis in O. NEUGEBAUER (a), pp. 129—137, which I have borrowed.

72 See R. J. GILLINGS, pp. 114 and 116, who also points to this difficulty: "but nowhere in the mathematical papyri do we find the circumference of a circle found by taking $(\frac{8}{9})^2\times 4d$, which is equivalent to writing $c=\pi d$. If this were in fact known to the Egyptians, as PEET assumed that it was, then we are led inevitably to the conclusion that the Egyptians antedated the Greek DINOSTRATUS by more than 1,400 years in thus evaluating the circumference of a circle in terms of the diameter".

There is still more. If $9 \times (\frac{8}{9})^2$ were the length of the arc of a semicircle, then would 9 have to be double the diameter. T. E. PEET has put it in this way, and O. NEUGEBAUER has accepted this opinion. But nowhere in the calculation does it become evident that the value 9 is found as twice $4\frac{1}{2}$, while, on the other hand, from the Egyptian calculations known at this time we are accustomed to whole operations which can be followed step by step.

Finally, as I have said before, Egyptian geometry lacks all speculation. It only supplies calculations for areas and volumes, approximately sometimes, for practical purposes. Why then suppose a method of calculation for finding the circumference of a circle when this circumference can better be measured directly and correctly?

Apart from the interpretations of W. W. STRUVE and T. E. PEET, O. NEUGEBAUER himself has proposed a third one. He sees the "basket" of the problem as a dome-shaped storehouse such as can be seen sometimes in old pictures; its form is nearly that of a beehive or paraboloid. He then takes $9 \times (\frac{8}{9})^2$ as being the circumference of the circular base with $4\frac{1}{2}$ as diameter, and he takes $4\frac{1}{2}$ as being also the length of the arc from the top to the base. In this way $9 \times (\frac{8}{9})^2 \times 4\frac{1}{2}$ should approximate the area of the paraboloid. In addition to the objections mentioned before, *viz.* that 9 should be double the diameter and that $(\frac{8}{9})^2$ is looked upon as $\pi_1/4$, there is now the further one that the calculation gives much too gross an approximation. In fact the way in which O. NEUGEBAUER calculates is right only for a cone with $4\frac{1}{2}$ as the slant height and also $4\frac{1}{2}$ as diameter of the circular base.

Besides the objections already discussed as to the interpretation of Problem 10 of the Moscow papyrus, we shall remark three other difficulties in each of those interpretations.

In the first place, what can have been the intention of this calculation, *viz.* the area of a curved surface[73]? The purpose of the calculations concerning circles in the Rhind papyrus was clear: the area of a round field (Problem 50) or the area of the bottom of a cylinder, the volume of which was to be calculated (Problems 41 and 42). O. NEUGEBAUER suggests that the area in Problem 10 of the Moscow papyrus was calculated perhaps so as to know how much material was needed for the storehouse. If this is so, then the hopes of the Egyptians were really decieved as a consequence of the gross approximation. And as to the interpretations of W. W. STRUVE and T. E. PEET, who obviously consider the basket as a small one, for carrying, a calculation of the area so as to know how much material was needed seems to be a far-fetched explanation.

A second difficulty arises from this. What were really the shape and size of an ancient Egyptian basket, putting first and foremost that the object intended was indeed a basket? In what preceded we have met with three different shapes, a hemisphere, a semicylinder and a paraboloid, or something like it, which seem to have been chosen for no better reason that one could interpret the calculations.

Then, finally, there is a third difficulty. Only one number is given in the problem, *viz.* $4\frac{1}{2}$. The calculation starts with: "Take $\frac{1}{9}$ of 9, since the (basket?) is the half of a ... (?); result 1. Take the remainder, namely 8". This is the

[73] K. VOGEL (a) I, p. 67, says: "eine Aufgabe, in der eine krumme Fläche berechnet wird" (a problem in which is calculated the area of a curved surface).

calculation of $\frac{8}{9}$ parts of 9. But why 9? The text has: "since the ... is the half of a ...". W. W. STRUVE, T. E. PEET and O. NEUGEBAUER have $4\frac{1}{2}$ as the diameter of the object in question. Why then twice this diameter although (the text has "since"!) the object is the half of something? In view of the method of calculation in the Rhind papyrus it is obvious that in Problem 10 of the Moscow papyrus we have to expect an object with diameter 9.

In the foregoing I have remarked that $(\frac{8}{9})^2$ has to be considered as a value for $\pi_2/4$ only, thus not for $\pi/4$ in general, without any special indication. For this reason I have rejected T. E. PEET's semicylinder[74] and O. NEUGEBAUER's paraboloid, in which $(\frac{8}{9})^2$ is used to calculate the circumference, thus taking $(\frac{8}{9})^2$ as a value for $\pi_1/4$. For the same reason I reject the way in which W. W. STRUVE tried to explain how the Egyptians came to the right result for the area of a hemisphere[75]. STRUVE supposes that the Egyptians, by measuring, first found the circumference of a circle to be $\frac{64}{81}$ parts of the circumference of the circumscribed square. This means, in our notation, that $\frac{64}{81}$ was first found as a value for $\pi_1/4$. He next supposes that the Egyptians reasoned the area of a circle also to have that proportion (*viz.* $\frac{64}{81}$) to the area of the circumscribed square (taking now $\frac{64}{81}$ also as a value of $\pi_2/4$). Then, experimentally, they should have found that the volume of a sphere equals $\frac{2}{3}$ of the volume of a circumscribed cylinder. Again from this they could have reasoned the area of a sphere to have that same proportion (*viz.* $\frac{2}{3}$) to the area of a circumscribed cylinder. Thus the area of a sphere equals $\frac{2}{3}\times\{2\times\{(\frac{8}{9}d)^2+d\times(\frac{8}{9})^2\times 4d\}$ or $4\times(\frac{8}{9}d)^2$, and the area of a hemisphere equals $2\times(\frac{8}{9}d)^2$. As will be clear, all these are suppositions only and, moreover not very obvious ones since they lack any indication in favour of them in the preserved texts.

Also for this same reason, that is to say that we are not allowed to interpret $\frac{64}{81}$ as a value for $\pi_1/4$, I reject E. M. BRUINS' explanation of how the Egyptians could have found the area of a hemisphere[76]. A great circle, he says, is obviously too small. On the other hand the sum of the areas of the curved surface of the circumscribed cylinder (of the hemisphere) and of a great circle is obviously too large. In E. M. BRUINS' opinion the Egyptians took the arithmetic mean of these two, which gives the correct result. But he, again, sees $\frac{64}{81}$ as a value for $\pi_1/4$ as well as for $\pi_2/4$.

Reviewing everything, I think that an alternative interpretation of T. E. PEET, that the object in question is a semicircle, is for the present the most obvious one[77]. Then $4\frac{1}{2}$ is the radius, and the diameter is 9. The beginning of the calculation is performed in the same way as that of the area of a circle[78]. We need not be

[74] R. J. GILLINGS, p. 114.

[75] W. W. STRUVE, pp. 167—169 and 176—180.

[76] E. M. BRUINS (a), XV 37a, pp. 42, 43. On pp. 40, 41 he talks about the circumference of a circle in a way which can be justified in no way by what we know from the preserved texts; the Egyptians did not "calculate" the circumference of a circle.

[77] R. J. GILLINGS, p. 114.

[78] We notice that it is very uncertain that the Egyptian "nbt" means indeed "basket" in the period of which the Moscow papyrus originates. Moreover, the text has only that the object is "the half of a ...". Here, at the most important place, the papyrus is damaged. At this time new inquiries are being made into the text of the problem by an expert Egyptologist, whose results will be published in due time.

surprised that the Egyptians (or at least the composer of the Moscow papyrus) did not calculate the area of half a circle by taking the half of that of a whole circle, for we know of an analogous fact in ancient Babylonian mathematics.

11.

Starting with a decidedly faulty interpretation by M. CANTOR of some Medieval texts, from which he came to a value $\pi = 4$, we have made a distinction between π_1 for the calculation of the circumference of a circle, π_2 for the calculation of the area of a circle, and π_3 for the calculation of the volume of a sphere. To presume that these constants always have been known to be identical and so to speak of π without any comment, sometimes leads to an interpretation which is definitely false, while in other cases it is needless or even reprehensible. With the distinction between π_1, π_2 and π_3, however, we have been able to regard several existing conceptions with more nuances. But, of course, we must observe, or we have to assume, that this distinction has sense only for texts whose authors we are sure were not familiar with ARCHIMEDES.

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