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Convergence and formal manipulation in the theory of series from 1730 to 1815

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Abstract

In the early calculus mathematicians used convergent series to represent geometrical quantities and solve geometrical problems. However, series were also manipulated formally using procedures that were the infinitary extension of finite procedures. By the 1720s results were being published that could not be reduced to the original conceptions of convergence and geometrical representation. This situation led Euler to develop explicitly a more formal approach which generalized the early theory. Formal analysis, which was predominant during the second half of the 18th century despite criticisms of it by some researchers, contributed to the enlargement of mathematics and even led to a new branch of analysis: the calculus of operations. However, formal methods could not give an adequate treatment of trigonometric series and series that were not the expansions of elementary functions. The need to use trigonometric series and introduce nonelementary functions led Fourier and Gauss to reject the formal concept of series and adopt a different, purely quantitative notion of series.

Sommario

Nella prima fase del calcolo, i matematici usavano serie convergenti per rappresentare quantità geometriche e risolvere problemi geometrici. Tuttavia le serie erano manipolate formalmente, cioè per mezzo di procedure che erano l'estensione infinitaria di procedure finite. Dagli anni venti del diciottesimo secolo, furono trovati molti nuovi risultati che non potevano essere ridotti alla nozione originale. Di fronte a tale situazione Euler fornì una nuova e più formale interpretazione del concetto di serie che permetteva di generalizzare la teoria originaria. Tale più formale approccio, nonostante alcune critiche, dominò la seconda parte del secolo. Esso contribuì alla crescita della conoscenze matematiche e, perfino, condusse alla nascita di una nuova teoria: il calcolo delle operazioni. Tuttavia, esso non fu in grado di fornire un adeguato trattamento delle serie trigonometriche e delle serie che non erano lo sviluppo di funzioni elementari. Così la necessità di usare le serie trigonometriche e di introdurre non

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elementari funzioni in analisi condusse Fourier e Gauss a rifiutare l'approccio formale e a sostenere una differente, puramente quantitativa concezione di serie.

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1. Introduction

This paper is the sequel to an article entitled "Convergence and formal manipulation of series from the origins of calculus to about 1730" [Ferraro, 2002], concerning the evolution of series theory from the origins of the calculus to the second decade of the 18th century. My aim here is to concentrate on the relationship between convergence and formal manipulation from the third decade of the 18th century to the early 19th century. Both of these papers are part of a broader research project concerning the complex of implicit and explicit principles and procedures underlying the employment of series from the beginning of the calculus to the 20th century.

In Section 2 of the present article I will clarify that while the early theory of series was based upon an intuitive (but still fairly clear) idea of convergence, series were manipulated by extending rules that were valid for the finite to the infinite. This gave rise to an interplay between the quantitative aspect, based upon convergence, and formal manipulations, based upon the infinitary extension of finite rules. By the 1720s, this way of conceiving series yielded several findings that could not be reduced to the original series conception. Indeed results concerning asymptotic series, recurrent series, and other infinite processes (continued fractions and infinite products) increasingly stressed the formal aspect.

In Section 3, I highlight that Euler offered a unitary interpretation of the complex of such results, which allowed the acceptance of those findings that did not form part of the early theory. This gave rise to a more formal approach, which generalized the early theory. This approach was predominant during the second part of the 18th century, although some mathematicians criticized it.

I will also clarify that this more formal concept of series contributed to the growth of mathematics and even led to a new branch of analysis: the calculus of operations (Section 4.1). However, it was not capable of providing an adequate treatment of trigonometric series and of series that were not expansions of elementary functions (Sections 4.2 and 5). Thus the need to use trigonometric series and introduce nonelementary functions into analysis led Fourier and Gauss to reject the formal concept of series and to introduce a different notion of series (Section 6).

Before beginning my analysis, I would like to stress that I will attempt to reconstruct the technical reasons that led to the rise and fall of the formal concept of series. Of course, other reasons (and, in particular, reasons related to applied sciences—which at first contributed to the success of certain approaches and later made necessary an enlargement of the domain of analysis—and philosophical reasons connected with the meaning of analysis) should be examined to obtain a more complete picture of the evolution of the theory of series; however, these will not be dealt with in the present paper.

2. The increasing importance of formal aspects in series theory from 1730s

In Ferraro [2002] and Ferraro and Panza [2003] we clarified that the theory of series at the beginning of the calculus was based upon an intuitive idea of convergence.[1] The meaning of the equality

$$f(x)=\sum_{k=0}^{\infty} f_k(x) \tag{1}$$

was not defined in an explicit way. It seemed obvious that the left-hand side and the right-hand side of (1) denoted the same quantity. It also seemed obvious that the series $\sum_{k=0}^{\infty} f_k(x)$ and the function $f(x)$ denoted the same quantity if and only if the series $\sum_{k=0}^{\infty} f_k(x)$ converged to the function $f(x)$ on an interval I of the values of x, namely, if the sequence $\sum_{k=0}^{n} f_k(x)$ approached $f(x)$ indefinitely when n increased, for any value of x belonging to I, and it was finally equal to $f(x)$, when n was an infinite number. (I shall later say that an equality of the kind $f(x)=\sum_{k=0}^{\infty} f_k(x)$ is a *quantitative* equality if the series $\sum_{k=0}^{\infty} f_k(x)$ converges to the function $f(x)$.[2])

Although the equality between a series and a function was meant in such a sense, series were manipulated by using *formal procedures*, namely procedures or rules that were based upon the following analogical principle:

(IE)[3] If a rule R was valid for finite expressions or if a procedure P depended on a finite number n of steps $S_1, S_2, S_3, \ldots, S_n$, then it was legitimate to apply the rule R and the procedure P to infinite expressions and in an unending number of steps $S_1, S_2, S_3, \ldots$.

A formal rule was applied to (finite or infinite) analytical expressions $A(x,y,\ldots)$, $B(x,y,\ldots), \ldots,$ regardless of the actual meaning of such expressions and of the meaning of each single step in the procedure (namely regardless of whether the replacement of the indeterminate $x, y, \ldots$ by certain quantities $a, b, \ldots$ actually produced quantities $A(a,b,\ldots)$, $B(a,b,\ldots), \ldots,$ or whether the symbols $A(a,b,\ldots)$, $B(a,b,\ldots), \ldots,$ were simply without meaning).[4]

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[1] In the 18th century, referring to a series $\sum a_n$ as a convergent series usually meant that a (finite or infinite) sequence $a_k, a_{k+1}, a_{k+2}, \ldots$ of the terms of the given series tended to 0 (in absolute value). A series could first be convergent and then divergent (see, e.g., d'Alembert [1768]). To avoid confusion, I shall use the term "convergent" in the sense that is specified above.

[2] I would like to specify:

(a) In the early theory of series, function series were almost exclusively constituted by power series $\sum_{k=0}^{\infty} a_k x^k$ or, at most, by series of the type $\sum_{k=0}^{\infty} a_k x^{-k}$ or $\sum_{k=0}^{\infty} a_k x^{\alpha_k}$, where finitely many of the exponents α_k were rational numbers. The early theory of series was substantially a theory of power series. Only from the 1740s did other function series and in particular trigonometric series begin to be examined. In Section 5, we will see that the concepts and techniques originating with power series were applied to trigonometric series too and that this application was rather problematic.

(b) The domain of functions was much restricted. A function, in the strict sense of the term, was always conceived as a composition of a finite number of elementary functions (i.e., algebraic functions, logarithm, exponential and trigonometric functions). (On the concept of a function, see Fraser [1989], Panza [1996], and Ferraro [2000a].)

[3] IE is an abbreviation for "infinitary extension."

[4] Formal procedures were not invented or created freely: mathematicians only considered rules deriving from the infinitary extension of the properties that were valid for finite expressions. In Ferraro and Panza [2003, 24–27], we showed that 18th-century mathematicians expanded functions by using the Mercator expansions of fractions and square roots of polynomials, the binomial expansion for any (rational or irrational) exponent, the method of indeterminate coefficients, and the differentiation or integration of power series term by term.

The early theory of series was therefore characterized by an interplay between the quantitative and the formal. For this reason I shall later refer to this concept of series as the "*formal-quantitative concept of series*."

By the 1720s, this way of conceiving series yielded several findings that could not be reduced to the original series notion and that led mathematicians to a more formal conception. In [2000b] and [2002], I highlighted three aspects of this process:

(a) the emergence of asymptotic series,
(b) the use of recurrent series,
(c) the investigation of other infinite processes (continued fractions and infinite products).

Recurrent series[5] were power series but were not of interest for their use (when convergent) in representing quantities. Mathematicians began to study them because they offered the possibility of investigating combinations of objects. In the theory of recurrent series, questions of convergence were nonexistent: the letter x is treated as a mere sign, a placeholder, and one operated upon series in a purely *combinatorial* way, namely by combining and rearranging letters and numbers.

The heart of the theory of recurrent series was the observation that any rational function with a numerator whose degree was less than the denominator could be expanded into a series, and, vice versa, that any recurrent series could be summed and that sum was a rational function with a numerator whose degree was less than the denominator (see de Moivre [1730, 27–35]). This property allowed one to identify recurrent series with their generating functions. In his *Introductio in analysin infinitorum*, Euler [1748, 1, 175] made this explicit and called the sum of a recurrent series the function that generated it.

Other fields where the combinatorial use of series was particularly evident were combinatorics and number theory (see, for instance, Euler's *Introductio* [1748, 313–337], where series are used to count objects and the author disregards convergence completely).

As regards asymptotic series, it should be noted that they arose from an attempt to improve the rate of convergence (see Stirling [1730]). Although series theory was concerned primarily with convergent series and many of the first researches aimed at speeding up convergence, the manipulation of series led in practice to the appearance of divergent series in a rather natural and indiscernible way. It was Stirling who discovered the first example of a divergent asymptotic series in his *Methodus differentialis* [1730, 135]:

$$\log x! = \frac{1}{2}\log 2\pi + \left(x+\frac{1}{2}\right)\log\left(x+\frac{1}{2}\right) - a\left(x+\frac{1}{2}\right) - \frac{a}{2\cdot 12\left(x+\frac{1}{2}\right)} + \frac{7a}{8\cdot 360\left(x+\frac{1}{2}\right)^3} - \cdots \tag{2}$$

($a = 1/\ln 10$, $\log x$ denotes the logarithms to the base 10 of x, and $\ln x$ is the natural logarithm of x).

Later MacLaurin and Euler found the Euler–MacLaurin summation formula, a divergent series useful in computation (see MacLaurin [1742, 670–671] and Euler [1732–1733]).

5 On recurrent series, see Ferraro [2002, 195–198].

It is probable that Stirling was not aware of the divergence of series (2). However, the existence of series with anomalous properties was already clear to Euler in his [1739a], where he pointed out some difficulties with asymptotic series. In this paper Euler derived

$$\pi = \frac{1}{n} + \sum_{k=1}^{n} \frac{4n}{n^2+k^2} + \sum_{h=0}^{\infty} (-1)^h B_{4h+2} \frac{1}{2^{2h}(2h+1)n^{4h+2}}, \tag{3}$$

where B_{2j} are the Bernoulli numbers.[6] He observed that this series actually "converged" (i.e., it decreased in absolute value) only up to a certain term, after which it began to increase in absolute value [1739a, 357].

Taking (3) for $n = 1, 3, 5$, Euler determined some approximations of π. By comparing these values with a known approximation of π, he observed that one moved away from the truth (*a veritate*), which, in his opinion, was worthy of note because there was no error in the proof and formula (3) allowed π to be approximated easily. He [1739a, 359–360] thought that this departure from the truth was due to the divergence of series and advised using divergent series with caution.

Initially asymptotic series were puzzling to Euler' eyes. Until then mathematicians had thought that a series could be used to approximate a quantity if and only if convergence held so that the sum was the unique, true, ultimate value of the series. By contrast, an asymptotic series had no ultimate value and this was evidently a difficulty. In the years that followed Euler quickly overcame his initial caution, as one can see in his [1755], where asymptotic series were used in an unproblematic way.

Following Euler, most mathematicians in the second part of the century accepted asymptotic series, but this occurred as the result of a substantial shift in the concept of series. According to the formal-quantitative concept of series, the relation between a series and its sum was both quantitative and formal. However, in the case of asymptotic series the one and only justification and guarantee of the exactness of the relation was the formal derivation of the series from a certain analytical quantity.

As concerns other infinite processes, for the sake of brevity, I only briefly mention continued fractions[7] and observe that the link between series and continued fraction was viewed in a formal way, independent of the convergence of series and fractions. *A series and a continued fraction were considered to be equal if they were related to each other by the formal transformation*

$$a_1 + \frac{b_1}{a_2+} \frac{b_2}{a_3+} \frac{b_3}{a_4+} \cdots = a_1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\prod_{i=1}^{k} b_i}{q_k q_{k+1}}, \tag{4}$$

where $q_k = a_k q_{k-1} + b_{k-1} q_{k-2}$, $q_1 = 1$, $q_0 = 0$ (see Euler [1737, 1739b]).

Equality (4) allowed one to give a meaning to a large class of divergent series. For instance, in [1754–1755, 615] Euler observed that the solution of the differential equation

$$x^m \, \mathrm{d}x = x^{q+1} \, \mathrm{d}y + (p-m) x^q y \, \mathrm{d}x + y \, \mathrm{d}x$$

[6] The Bernoulli numbers B_r are defined by the relation $\frac{t}{e^t-1} = \sum_{r=0}^{\infty} \frac{B_r}{r!} t^r$.

[7] The first systematic investigation of continued fraction and infinite products was due to Euler (see Euler [1737, 1739b]; on this topic, see Ferraro [2000b, 85–90]).

was given by

$$y = \frac{x^m}{1+}\,\frac{px^q}{1+}\,\frac{qx^q}{1+}\,\frac{(p+q)x^q}{1+}\,\frac{2qx^q}{1+}\,\frac{(p+2q)x^q}{1+}\,\frac{3qx^q}{1+}\,\frac{(p+3q)x^q}{1+}\cdots$$

under the condition $y(0) = 0$. However, the solution could also given by the series

$$y = x^m - px^{m+q} + p(p+q)x^{m+2q} - p(p+q)(p+2q)x^{m+3q} + \cdots, \tag{5}$$

which in general has a radius of convergence equal to zero.

This type of power series (later, totally divergent series) was different from the type of power series characteristic of the formal-quantitative conception. An ordinary power series always had a positive radius of convergence and had a meaning that was independent of the formal transformation by which the series was derived. Since totally divergent series had a radius of convergence equal to 0, they *had no quantitative meaning by themselves*: their meaning was derived only by formal transformation (4), which linked them to a differential equation and a continued fraction or other expressions of quantities.

3. The debate on divergent series and Euler's systematization

The development of series theory between the 1720s and 1750s did not rely on convergence and dealt mainly with the combinatorial questions. In particular, the theory permitted the generalized use of divergent series. To simplify our reasoning, I shall employ the term "formal use" or "formal concept" of series to refer to a notion of series which was characterized by the use of formal principle (**IE**), but, unlike the formal-quantitative notion, did not give importance to convergence.

Some mathematicians rejected the idea that a divergent series (namely, a series that expressed no quantity) could be associated with a quantity. However, they were in the minority. I shall now try to understand the reason for this.

The first attempt at associating a divergent series with a quantity was due to Grandi, who derived the equality

$$1 - 1 + 1 - \cdots = 1/2 \tag{6}$$

in 1703. While Leibniz [1713] tried to justify (6) (see Panza [1992, 314–333] and Ferraro [2000c, 61–67]), some mathematicians disagreed with him. Indeed, in his [1715], Varignon observed that the equality

$$\frac{a}{m \pm n} = \sum_{i=0}^{\infty} (\mp 1)^i \frac{an^i}{m^{i+1}}$$

was true if $m > n(> 0)$. However, it was false if $(0 <)m < n$, and was simply $\infty = \infty$ if $m = n(> 0)$ and the denominator was the difference $m - n$. Finally, it was false if $m = n(> 0)$ and the denominator was the sum $m + n$.

Varignon's viewpoint was shared by other mathematicians. One of them was Nikolaus II Bernoulli, who criticized Euler's method for summing the series

$$\sum_n \frac{1}{n^{2k}}, \quad k = 1, 2, 3, \ldots$$

(see Ferraro [2000b, 90–92]). In 1742, he had written to Euler that there would be the possibility of error if the series were not convergent. On April 6 1743, Bernoulli stated:

> I cannot persuade myself that you (Euler) think that a divergent series... provides the exact value of a quantity which is expanded into the series. Indeed, e.g., $1/(1-x)$ is not $= 1 + x + xx + x^3 + \cdots + x^\infty$, but $= 1 + x + xx + x^3 + \cdots + x^\infty + x^{\infty+1}/(1-x)$. [Fuss, 1843, 2, 701–702]

Similar objections are found in d'Alembert, the most famous mathematician among those who criticized the use of divergent series during the 18th century. Although N. Bernoulli's and d'Alembert's conception might seem close to modern ideas about series, it was in fact rather problematic. To make this clear, let us consider d'Alembert's *Réflexions sur les suites et sur les racines imaginaires* [1768]. This paper is devoted to the approximation of the function $(1+x)^m$ by means of its series expansion

$$\sum_{n=0}^{\infty} \binom{m}{n} x^n.$$

D'Alembert did not determine the sum of this series but assumed the expansion of the function $(1+x)^m$ is $\sum_{n=0}^{\infty} \binom{m}{n} x^n$ (it is to be supposed that, according to him, this expansion was derived by the usual formal methods). He limited himself to observing that the series $\sum_{n=0}^{\infty} \binom{m}{n} x^n$ had to be decreasing in order for one to compute the values of $(1+x)^m$. Hence the ratio between the nth and $(n+1)$th terms of the series had to be greater than 1 (in absolute value) and, therefore,

$$|x| < \frac{n}{n-m-1} \quad \text{if } n > m+1.$$

Putting $n = \infty$, he concluded that, at least, the last terms of the series were decreasing (convergent) if $|x| < 1$. In contrast, the last terms were increasing (divergent) when $|x| > 1$ [1768, 173].

Then d'Alembert determined the bounds of errors. He observed that if $|x| < 1$, $\nu > 1 + m$, $S = \sum_{n=\nu-1}^{\infty} \binom{m}{n} x^n$, and $A = \left|\binom{m}{\nu-1}\right| x^{\nu-1}$, one had

$$S < \sum_{i=0}^{\infty} A|x|^i = \frac{A}{1-|x|}$$

and

$$S > \sum_{i=0}^{\infty} A\left(\frac{\nu-1-m}{\nu}\right)^i |x|^i = \frac{A}{1-|x|\frac{\nu-1-m}{\nu}}.$$

He stated that the sum of the series from A on lay between the bounds $A/(1-|x|)$ and $nA/(\nu-(\nu-1-m)|x|)$, which gave "a practicable enough way" of summing the series by approximation, and that the error was less than

$$\frac{A(m+1)|x|}{(1-|x|)(\nu-(\nu-1)|x|+m|x|)}$$

if one assumed S to be equal to a value between these bounds [d'Alembert, 1768, 177–178]. D'Alembert also discussed increasing the rate of convergence of the series and finally criticized the use of divergent series [1768, 181–183].

I stress that *d'Alembert accepted the principle of formal manipulation* upon which the early theory was based: this was precisely the principle that had led to asymptotic series, recurrent series, etc. In particular, d'Alembert did not consider series as autonomous objects but the result of transformations of given closed analytical expressions. Although it is true that d'Alembert used the inequality technique, it was only a tool for the numerical evaluation of a function. In no case was this technique used to prove the existence of a limit (see also Grabiner [1981, 63]). *D'Alembert had no knowledge of the ratio test*, if by this term we refer to a criterion to establish whether a given series has a finite sum or not. For him the condition $a_{n+1}/a_n < 1$ served to establish where the series approximated its known sum: the "ratio test" was not used to prove the existence of the sum but was only a procedure to determine the bounds of errors.

In conclusion, criticisms of the use of divergent series were weak because all mathematicians used formal procedures even if some of them, such as Euler, Lagrange, and Laplace, used them in a stronger form. Of course, Varignon, N. Bernoulli, and d'Alembert's ideas could be developed in a more modern sense, but this did not occur during the 18th century. People who criticized Euler and Lagrange were unable to reject formal methods entirely. When considered seriously, N. Bernoulli and d'Alembert's arguments involved a rethinking of the whole of analysis that went beyond the intention of the mathematicians of the time and the state of the art.

There was a second reason that Varignon, N. Bernoulli, and d'Alembert's standpoint was weak. It did not produce results of wide interest; by contrast, divergent series gave rise to a number of significant findings and were to prove fertile ground for further investigations in later decades. I would like to emphasize this point: *the formal point of view contributed to the development of mathematical knowledge*, whereas the approach of their opponents was substantially sterile during the middle of 18th century. I think that this was the heart of the question: the formal approach was triumphant because it was capable of producing new mathematics.

In the 1750s, the criticisms of his opponents led Euler to make the formal concept of series explicit and to go beyond the formal-quantitative approach. He sought to give a definition of the sum that generalized the old notion (which had never been explicitly formulated) and provided a basis for new findings.

Euler defined the sum of an infinite series to be the finite expression, the expansion of which generates the given series (see [1754–1755, 593–594]), namely, $f(x)$ is the sum of $\sum_{i=0}^{\infty} a_i x^i$ if the expansion of $f(x)$ is $\sum_{i=0}^{\infty} a_i x^i$.

I explicitly point out some consequences of Euler's notion:

(1) Every series was conceived to have its own generating function, which was identified with the series;
(2) The reciprocal substitution between a series and its generating function was always possible (see [1754–1755, 593–594]);

(3) A power series was not an autonomous object of study.[8] While the object function existed independently of its series expansion, a series was merely a transformed form. Consequently, a function could never be defined by a series.

This does not mean that the problem "given a series, find its sum" was not considered, and the investigation of certain series qua series was not deemed interesting. The crucial point was that a series was thought to be the expression of a function (or, more precisely, of an elementary function; see footnote 2), which was already known, at least in principle. Thus "to sum a series" meant that one had to find the elementary function from which the series derived. It is natural to ask: What happened if mathematicians did not know of a function from which the series could be derived?

This problem arose during the second part of the century, particularly in connection with the solution of differential equations. I will tackle this question in Sections 4.2 and 5 (for a more general discussion, with particular reference to the notion of a function, see Ferraro [2006, in press]).

Euler's concept of the sum of a series was applied to function series. As for numerical series, it is rather difficult to speak of the development of a number (even if Euler did so in [1755, 221–223]. However, 18th-century mathematicians usually considered a numerical series $\sum_{i=0}^{\infty} a_i$ as a particular case of a power series $\sum_{i=0}^{\infty} a_i x^i$ for $x = 1$ (or, more generally, they considered it as a particular case of a series $\sum_{i=0}^{\infty} b_i x^i$ for x equal to an appropriate c such that $b_i c^i = a^i$). In this way the search for the sum S of a series $\sum_{i=0}^{\infty} a_i$ was thought to be $f(1)$ (or $g(c)$), where $f(x)$ (or $g(x)$) was the function which generated $\sum_{i=0}^{\infty} a_i x^i$ (or $\sum_{i=0}^{\infty} b_i x^i$).[9]

It is worthwhile noting that in the formal-quantitative approach to the theory of power series it was obvious that to sum a series meant to invert the operation of development and that the sum of a power series was also a function to which the series converged, at least over an interval of values of x. According to Euler's definition, the fact that a series was the result of a formal transformation of a function was the only justification for stating that the function was the sum of the series. This definition made it possible to generalize the notion of series and could be applied to series that differed from ordinary power series (for instance, to totally divergent series). It was later applied to symbols of operations in the context of the calculus of operations.

I also observe explicitly that the more formal notion of series did not imply a rejection of the formal-quantitative approach, but only that the former was the necessary presupposition of the latter. Even though new types of series were admitted, mathematicians continued to use the same procedures to expand functions and always based these procedures upon principle (**IE**). The possibility of inventing new procedures that were not merely an infinitary extension of finite procedures was never considered.

4. Some developments of series theory during the second part of the 18th century

During the second part of the 18th century, the main lines of research on series increasingly came to stress the formal aspect. There were many remarkable results such as the Lagrange series, Laplace's

[8] On this see also Fraser [1989, 322].

[9] There were only few cases in which a numerical series was used without referring to a function series. For instance, the series $\sum_{n=1}^{\infty} 1/n^2$, which, however, was often treated as a particular case of a power series (for this power series the considerations of Section 4.2 hold).

theory of the generating functions, and the calculus of operations. Further series were often used as solutions to differential equations, and trigonometric series became the subject of various types of research. In this section, first, I briefly mention the calculus of operations as an example of the formal approach, and second, I illustrate some problematic results concerning series solutions to differential equations and trigonometric series.

4.1. Series of operations

By 1772, Lagrange used series in which symbols did not denote quantities. The starting point was the Leibniz analogy, namely the observation that the powers of a binomial $(x+y)^n$ and differentials of a product $\mathrm{d}^n(xy)$ obey the same rule [1849–1863, 3,174–177]. In his *Sur une nouvelle espèce de calcul*, Lagrange [1772, 441–443] developed this analogy. Indeed, he noted that since $e^{h\xi}-1=\sum_{i=1}^{\infty}h^i\xi^i/i!$, one can write

$$e^{\frac{\mathrm{d}u}{\mathrm{d}x}\xi}-1=\sum_{i=1}^{\infty}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^i\frac{\xi^i}{i!}.$$

If we identify $\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^k$ with $\frac{\mathrm{d}^k u}{\mathrm{d}x^k}$ and put $\Delta u=u(x+\xi)-u(x)$, then we can write $\Delta u=e^{\frac{\mathrm{d}u}{\mathrm{d}x}\xi}-1$. Hence

$$\Delta^\lambda u=\left(e^{\frac{\mathrm{d}u}{\mathrm{d}x}\xi}-1\right)^\lambda.$$

Lagrange considered infinite series of symbols of operations and dealt with them on the basis of the analogy with power series. For example, in his [1792, 663–684], Lagrange investigated the differences $\Delta^m=\sum_{r=0}^{m}(-1)^r\binom{m}{r}T_{m-r}$, $m>1$, and $\Delta^0=T_0$ of a given sequence T_r.

He noted that $T_1=\Delta^0+\Delta^1$ and

$$\begin{aligned}T_n&=(T_1)^n=\left(\Delta^0+\Delta^1\right)^n\\&=\Delta^0+n\Delta^1+\frac{n(n-1)}{1\cdot 2}\Delta^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\Delta^3+\cdots,\end{aligned}$$

which is Newton's interpolation formula. Then he stated that $\Delta^0\Delta^1=\Delta^{0+1}=\Delta^1$ by applying the law of exponents and wrote

$$T_1=\Delta^0+\Delta^1=\Delta^0\left(1+\Delta^1\right).$$

Since $1+\Delta^1=e^{\log(1+\Delta^1)}$, he derived

$$T_n=(T_1)^n=\Delta^0 e^{n\log(1+\Delta^1)}.$$

Then Lagrange put

$$P_0=\Delta_0=T_0\quad\text{and}\quad P_r=\left[\log\left(1+\Delta^1\right)\right]^r,$$

expanded $e^{n\log(1+\Delta^1)}$, and obtained

$$T_n = P_0 + nP_1 + \frac{n^2}{2}(P_2)^2 + \frac{n^3}{6}(P_3)^3 + \cdots$$

and

$$1 + \Delta^1 = e^{P_1} = 1 + P_1 + \frac{1}{2}P_2 + \frac{1}{6}P_3 + \cdots.$$

The calculus of operations, which in a sense was the extreme result of the 18th-century theory of series, enjoyed a remarkable history, which continued during the 19th century (on this topic see Goldstine [1977] and Koppelman [1971]).

The calculus of operations was also related to the Laplacian theory of generating functions. Laplace formulated this theory in his [1779]. According to Laplace, if $u(t)$ is the sum of the series $\sum_{x=0}^{\infty} y_x t^x$ (in Euler's sense), then $u(t)$ is the generating function of the sequence y_x. Laplace developed a calculus of generating functions. For instance, he showed that if $u = u(t)$ is the generating function of y_x, then

$$u = u(t)t^n \quad \text{and} \quad \left(t^{-1} - 1\right)^n u(t)$$

are the generating functions of

$$y_{x-n} \quad \text{and} \quad \Delta^n y_x,$$

respectively (see [1779, 1–8]).

In [1779, 1812] Laplace used generating functions in a extremely powerful way: he showed how to derive a vast number of formal identities and provided several applications. For instance, in [1779, 9–10] he put $t^{-1} = 1 + \alpha t^{-r}$ and expanded t^{-i} as a function of α. He obtained

$$\frac{u}{t^i} = u + i\alpha + \frac{i(i+2r-1)}{2!}\alpha^2 + \frac{i(i+3r-1)(i-3r-2)}{3!}\alpha^3 + \frac{i(i+4r-1)(i-4r-2)(i-4r-3)}{4!}\alpha^4 + \cdots.$$

Hence, by applying the rules of calculus, he derived the interpolation formula

$$\begin{aligned} y_{x+i} = y_x + i\Delta y_{x-r} &+ \frac{i(i+2r-1)}{2!}\Delta^2 y_{x-2r} \\ &+ \frac{i(i+3r-1)(i-3r-2)}{3!}\Delta^3 y_{x-3r} \\ &+ \frac{i(i+4r-1)(i-4r-2)(i-4r-3)}{4!}\Delta^4 y_{x-4r} + \cdots. \end{aligned}$$

In a similar way Laplace [1799–1825, 4, 205–208] derived many interpolation formulas, for instance,

$$\int_0^n y_x \,\mathrm{d}x = \frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n - \frac{1}{2}[\Delta y_{n-1} - \Delta y_0]$$

$$- \frac{1}{24}\left[\Delta^2 y_{n-2} - \Delta^2 y_0\right] - \frac{19}{720}\left[\Delta^3 y_{n-3} - \Delta^3 y_0\right] + \cdots$$

(on this subject, see Panza [1992, 550–584]).

4.2. Series as solutions to differential equations

The use of series as solutions to differential equations went back to the very beginnings of the calculus. For instance, in *Supplementum geometriae practicae* [1693, 286], to determine the series solution to

$$\mathrm{d}y = \frac{a\,\mathrm{d}x}{a+x},$$

Leibniz wrote

$$a\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}y}{\mathrm{d}x} - a = 0$$

and set $y = Bx + Cx^2 + Dx^3 + \cdots$ (he implicitly assumed $y(0) = 0$). Putting this expression into the differential equation, he obtained

$$(aB - a) + (2aC + B)x + (3aD + 2C)x^2 + \cdots = 0.$$

Therefore $B = 1$, $C = -1/(2a)$, $D = 1/(3a)$, ... and

$$y = x - \frac{x^2}{2a} + \frac{x^3}{3a^2} - \frac{x^4}{4a^4} + \cdots.$$

In the 1692 version of his *De quadratura curvarum* [1967–1981, 7, 94–96], Newton exhibited a method of expanding the solution to a differential equation and believed that it was sufficient to expand any quantity (see Ferraro [2002]). In effect the application of series to the solution of differential equations was an integral part of the formal-quantitative approach to series. Mathematicians believed implicitly that any solution of differential equation could be expressed by a convergent series.

After 1750 Euler made ample use of the series solutions of differential equations and dealt with this method in his *Institutionum calculis integralis* lengthily. For example, he [1768–1770, II, 177–185] investigated the differential equation

$$x^2\left(a + bx^n\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\left(c + ex^n\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(f + gx^n\right)y = 0. \tag{7}$$

Euler tried a solution of the form

$$y=\sum_{j=0}^{\infty}A_j x^{\lambda+jn} \tag{8}$$

where $A_0\neq 0$. By replacing (8) into (7), he found

$$\beta_0 A_0+\sum_{j=1}^{\infty}(\alpha_{j-1}A_{j-1}+\beta_j A_j)x^{\lambda+jn}=0,$$

where

$$\begin{aligned}
\beta_0&=\lambda(\lambda-1)a+\lambda c+f,\\
\alpha_0&=\lambda(\lambda-1)b+\lambda e+g,\\
\alpha_j&=jn(jn+2\lambda-1)b+jne+\alpha_0\quad(\text{for } j>0),\\
\beta_j&=jn(jn+2\lambda-1)a+jnc\quad(\text{for } j>0).
\end{aligned}$$

Hence

$$\lambda(\lambda-1)a+\lambda c+f=0\qquad\text{and}\qquad \beta_j A_j=\alpha_{j-1}A_{j-1}\quad(\text{for } j>0).$$

If one chooses A_0 arbitrarily, the previous relations allow one to determine λ by solving the equation $\lambda(\lambda-1)a+\lambda c+f=0$ and A_j (for $j>0$) by recurrence.

Therefore, if $a\neq 0$, we can determine two values λ_1 and λ_2 of λ and so we have two series of the form (7), which furnish complete solutions to the differential equation. However, if $\lambda(\lambda-1)a+\lambda c+f=0$ has only one root or when the difference between the two values of λ is divisible by n (in this case the coefficients β_j become equal to infinity for one of the two roots), the general integral cannot be expressed as the sum of two series of the form (7). In this case Euler found that the general integral had the form

$$\log x\sum_{j=0}^{\infty}C_j x^{\lambda_1+jn}+\sum_{j=0}^{\infty}B_j x^{\lambda_2+jn}+\sum_{j=0}^{\infty}D_j x^{\lambda_1+jn},$$

where $\lambda_2\leqslant\lambda_1$ and the coefficients A_j, B_j, C_j depend on two arbitrary constants.

This example shows that the series solution to a differential equation could not always be transformed in a closed expression. Similar cases became increasingly numerous after 1760. I limit myself to the example

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}+\left(\alpha^2-\frac{\beta^2}{r^2}\right)=0$$

(α and β constants), which was derived by Euler while he was investigating the vibrations of a stretched membrane. In *De motu vibratorio tympanorum*, Euler assumed the existence of the power series solution to this equation and obtained

$$u(r)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(\beta+1)_k}\left(\frac{\alpha r}{2}\right)^{\beta+2k}$$

(see Euler [1764, 344–359]). Today this differential equation is called Bessel's equation and the solution $u(r)$ is the Bessel function $J_\beta(\alpha r)$.

Mathematicians were aware that differential equations could be solved by series that were not the expansions of elementary functions. This fact, however, did not convince them to modify their approach and, in particular, to abandon the use of formal methodology, which had its foundations in elementary functions.

Their formal approach prevented mathematicians from an appropriate use of series to deal with new functions. In effect, as Lagrange still stated in 1797, "an expression in series can always be regarded as the development of a finite expression" [Lagrange, 1797, 93]. Series were not considered as autonomous objects; they could not be used to define a function. When one was unable to integrate a function $f(x)$ or a differential equation $F(x, y, \mathrm{d}y/\mathrm{d}x, \ldots)=0$, the usual procedures of development[10] allowed one to determine a series $\sum a_n x^n$ that could be used to approximate the values of the integral. *This did not mean that the solution to a given differential equation was defined by the series*. In the same manner as $x^n/n!$ represented the quantity e^x and allowed one to investigate it but did not define it, a power series $\sum a_n x^n$ represented quantities of the types $\int f(x)\,\mathrm{d}x$ or $F(x, y, \mathrm{d}y/\mathrm{d}x, \ldots)=0$ and allowed one to investigate them but did not define them.

In effect, series solutions to differential equations played two roles. First, series were instruments that provided the approximate values of a quantity expressed by a differential equation. It was a commonplace that a solution by series was not an exact solution. According to Lagrange [1776, 301], the method of series was a method "for integrating by approximation the differential equations whose finite integral was impossible or, at the very least, extremely difficult." And in his [1780, 522–523], Euler regarded the representation of a quantity by a series as an approximate representation. Series did not provide the exact solution and did not express a quantity exactly.

Second, series could be instruments for expressing a link between different analytical expressions (in this role, convergence was not of importance and even totally divergent series could be used). For instance, in [1794], Euler showed that the solution to

$$x(1-x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left[\gamma-(\alpha+\beta+1)x\frac{\mathrm{d}y}{\mathrm{d}x}-\alpha\beta\gamma\right]=0 \tag{9}$$

was the hypergeometric series. By using this fact and the relation between the hypergeometric series and certain appropriate expansions of $\int_0^\pi(1+\alpha^2-2\alpha\cos\phi)^n\cos p\phi\,\mathrm{d}\phi$ and $\int_0^\pi(1+\alpha^2-$

[10] Usual procedures could be applied since integrals and differential equations were expressions of the type $f(x)\,\mathrm{d}x$ or $F(x, y, \mathrm{d}y/\mathrm{d}x, \ldots)=0$, where f and F were elementary functions.

$2\alpha\cos\phi)^{-n-1}\cos p\phi\,\mathrm{d}\phi$, Euler proved the equality of the integrals

$$\binom{n+p}{p}(1-\alpha^2)^{-n}\int_0^\pi (1+\alpha^2-2\alpha\cos\phi)^n\cos p\phi\,\mathrm{d}\phi$$

and

$$\binom{p-n-1}{p}(1-\alpha^2)^{n+1}\int_0^\pi (1+\alpha^2-2\alpha\cos\phi)^{-n-1}\cos p\phi\,\mathrm{d}\phi.$$

5. Trigonometric series

From the 1730s, trigonometric series appeared in some mathematical and physical investigations. Astronomy was the field where the need for such series was most strongly felt; indeed, they seemed well suited to describing periodic phenomena, clearly relevant to the subject matter of astronomy.

In his paper of 1749 on irregularities of the orbits of Saturn and Jupiter [1749b], Euler investigated the series expansion of $(1-g\cos\omega)^{-\mu}$. The interest in this formula originated from the fact that $(1-g\cos\omega)^{-3/2}$ was contained in certain differential equations describing the motion of these planets. The greatest difficulty in finding the solutions to these equations was the determination of a sufficiently fast expansion of $(1-g\cos\omega)^{-\mu}$. Indeed Euler observed that the expansion of this formula, "following ordinary rules," is

$$(1-g\cos\omega)^{-\mu}=1+\frac{\mu}{1}g\cos\omega+\frac{\mu(\mu+1)}{1\cdot 2}g^2\cos^2\omega+\cdots,$$

"but this series is not suitable for my purpose, in as much as it is not sufficiently convergent, since it contains powers of cos ω. As for the last disadvantage, one can remedy it by reducing the powers of the cosine of the angle ω, to the cosines of multiples of the angle" [1749b, 61]. Since

$$\begin{aligned}2\cos^2\omega&=\cos 2\omega+1,\\ 4\cos^3\omega&=\cos 3\omega+3\cos\omega,\\ 8\cos^4\omega&=\cos 4\omega+4\cos 2\omega+3,\\ &\vdots\end{aligned}$$

Euler stated that $(1-g\cos\omega)^{-\mu}$ must have an expansion of the type $\sum_{i=0}^{\infty}a_i\cos i\omega$. Then Euler determined the coefficients of this cosine series by giving a recursive formulas for a_i, a process involved a long sequence of calculations (see Golland and Golland [1993, 58–64]).

Soon afterward, in [1750–1751] Euler studied some functional equations, such as $y(x+1)=y(x)$, and found the solutions in the form of trigonometric series. Euler, however, did not conceive of this trigonometric series as the final result. He, in principle, imagined that such trigonometric series could be expressed as finite functions of $\sin\pi n$ and $\cos\pi n$.

Trigonometric series were also used in the problem of the vibrating string. In his [1747] d'Alembert described the motion of a stretched elastic string by equations equivalent to a partial differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

He solved this equation and found $y = f(t + x) + F(t - x)$, for $c = 1$, f and F being two arbitrary functions. D'Alembert thought that the solution to the problem had to be interpreted only by means of functions given by a single analytical expression, because the calculus was grounded in such functions (see Truesdell [1960]).

In contrast, Euler thought that f and F could be functions described piecewise or even functions without an analytical expression. These functions were termed discontinuous functions in opposition to "continuous functions," namely functions given by a single analytical expression. In the summary of *De usu functionum discontinarum in Analysi* Euler explained:

> The solutions that Geometers gave to the problem of the vibrating motion of strings include nothing but the assumption that the figure, which is given to the string at the beginning of the motion, is regular and can be represented by a certain equation. Instead they denied that the other case (if this figure is discontinuous or irregular) was of relevance for analysis or that the motion that originated from this configuration might be reasonably defined. [1765, 7]

Although Euler thought that other similar problems necessarily involved the use of discontinuous functions, he was unable to introduce them effectively in analysis (see Ferraro [2000a]).

This controversy is relevant to our purpose for two reasons. First, it showed the need to introduce quantities that were different from elementary functions in order to mathematize the study of natural phenomena; at the same time, it revealed the difficulties that 18th-century analysis (and in particular the theory of series) had in treating nonelementary functions. The treatment of these new quantities required a change in the conception of analysis, although this was not forthcoming. The result was that analysis could not fully develop its potential.

The second, and more specific, reason for which the controversy is of importance here is that, in his [1753], Daniel Bernoulli stated that all the initial positions could be represented by

$$y = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

and proposed the trigonometric series solution

$$y = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$$

for the differential equation. In his opinion, the trigonometric solution was general: he based this opinion on the assumption that all sonorous bodies potentially contain an infinity of sounds, whose vibrations can be superimposed on each other.

Instead in [1753, 236–237], Euler rejected Bernoulli's opinion by noting that "all the curves contained in that equation, although one increases the number of the terms to infinity, have certain characteristics which distinguish them from all other curves." Indeed, they were periodic and the sine series was also odd. As Grattan-Guiness and Ravetz [1972, 245] noted, periodicity was an insuperable difficulty.

In the 18th century, a function was considered a global entity: it was viewed as a whole and its behavior was a global matter, which could not be reduced to the sum of the behavior of the points of its domain. For example, consider the equality

$$\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots. \tag{10}$$

Today we think of it as an equality which is valid on a certain interval since the sum of the series is different from $\frac{1}{2}x$ when x does not belong to $(-\pi, \pi)$: the series gives rise to a function that is defined piecewise. However, 18th-century mathematicians did not consider equality (10) to be valid upon a certain interval, but to be valid everywhere (this was implied by the generality of algebra and the concept of function; see Ferraro [2000a]). Consequently, this relation could not be thought of as a quantitative relation. A trigonometric series was understood as a formally derived relation: it might have a quantitative meaning, though this was not necessarily the case. In any event, it was not defined quantitatively, namely as the limit of the partial sums.

This conception was rooted in the theory of power series. In [1773, 169] Euler stated this view in a resolute way. For instance, he noted that

$$\sum_{i=1}^{n} \cos ix = -\frac{1}{2} + \frac{\cos nx - \cos(n+1)x}{2(1-\cos x)}$$

and if one put $n = \infty$, then one had

$$\sum_{i=1}^{\infty} \cos ix = -\frac{1}{2}.$$

He took care to show that this result corresponded to his definition of the sum. Indeed, by considering $-1/2 = (\cos x - 1)/(2(1-\cos x))$ and expanding the last fraction, Euler derived $\sum_{i=1}^{\infty} \cos ix = -1/2$.

In the same way, when d'Alembert denied that $\sum_{i=1}^{\infty} \cos ix = -1/2$, Lagrange replied:

> I would pose the question whether every time one encounters an infinite geometric series in an algebraic formula, for example $1 + x + x^2 + x^3 + \cdots$, one can substitute $1/(1-x)$, though this quantity is really equal to the sum of proposed series only when one supposes the last term x^∞ to be zero. [1760–1761, 323]

This approach to trigonometric series prevented 18th-century mathematicians from realizing the potential of trigonometric series even though they derived many results that, in a sense, seem to anticipate Fourier's series. For instance, in his [1754] Lagrange derived the functional solution to equation

$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ by different approach. He used a discrete-masses model for the problem and obtained

$$y(x,t) = \left(\frac{2}{l}\int_0^l Y(x)\sum_{r=1}^{\infty}\sin\frac{r\pi x}{l}\,\mathrm{d}x\right)\sin\frac{r\pi x}{l}\cos\frac{r\pi ct}{l} + \left(\frac{2}{\pi c}\int_0^l V(x)\sum_{r=1}^{\infty}\sin\frac{r\pi x}{l}\,\mathrm{d}x\right)\sin\frac{r\pi x}{l}\cos\frac{r\pi ct}{l}, \tag{11}$$

where $Y(x)$ and $V(x)$ are the initial position and velocity functions of the string [1754, 100–101].

Here Lagrange seems to be very close to Fourier series. In reality, Lagrange was persuaded a priori of the impossibility of representing any function through trigonometric series (in his paper he also rejected Daniel Bernoulli's solution). Equation (11) "was for him only a step on the road to the Eulerian functional solution" [Grattan-Guinness and Ravetz, 1972, 248]. Lagrange used trigonometric series in a formal way according to typical 18th-century procedures and never considered them as autonomous objects, capable of representing a quantity by themselves.

Another particularly interesting result was obtained by Clairaut [1754, 544–564]. He returned to the problem that Euler had examined in [1749a] and sought the expansion of the function $f(x)$ in the form

$$a_0 + 2\sum_{k=1}^{\infty} a_k \cos kx.$$

In so doing he produced expressions for what later would be called the Fourier coefficients of the series. Indeed he wanted to interpolate the given function $f(x)$ for $x = 2\pi/n$ and obtained the interpolation formula

$$a_0 = \frac{1}{n}\sum_{h=0}^{n-1} f\left(\frac{2h\pi}{n}\right)$$

and

$$a_k = \frac{1}{n}\sum_{h=0}^{n-1} f\left(\frac{2h\pi}{n}\right)\cos\frac{2hk\pi}{n} \quad (k > 0).$$

For $n = \infty$, Clairaut obtained

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos kx\,\mathrm{d}x.$$

In 1777 Euler, in dealing with an astronomical problem, showed that the coefficients of the trigonometric series

$$f(x) = a_0 + 2\sum_{k=1}^{\infty} a_k \cos kx$$

could be obtained in a very quick way. He multiplied both sides of the last equality by $\cos mx$ and integrated the series term by term by observing that

$$\int_0^{2\pi} \cos mx \cos kx \, \mathrm{d}x = 0$$

for $m \neq k$ (see Euler [1793]).

These results did not change the common approach to trigonometric series. The expansion of a function into a trigonometric series was always recognized as being the result of applying a formal procedure.

Although the treatment of trigonometric series resembled that of power series, there were significant differences between the two cases. While the expansion of a function in a power series, which was convergent upon an interval, was considered to be guaranteed a priori, the expansion of a function into a trigonometric series was not guaranteed a priori by usual procedures and had to be justified, sometimes even by referring to the physical meaning of the trigonometric series.[11]

6. On the decline of formal methods

At the beginning of the 19th century the theory of series was re-founded upon new bases, which mainly consisted in the attempt to avoid formal manipulation and to give an exclusively quantitative interpretation to the equality (1).

The new approach to series appeared for the first time in a paper of Gauss published in 1813. However, Fourier had already looked at trigonometric series from a viewpoint which differed from Euler's and Lagrange's in his work on the propagation of heat submitted to the *Institut de France* in 1807 (published after a complicated series of events in [1822]). Cauchy gave the first systematic presentation of a theory based on an exclusively quantitative approach in his famous treatise of 1821.

A very important reason for the rejection of the formal concept of series was that, at the beginning of the 19th century, this concept no longer contributed to the growth of analysis. We saw that one of the reasons for the success of the formal conception had been its capacity to produce novel and interesting results. However, this capacity was exhausted by the end of the century. In this period, the circumscribed domain of elementary functions and their power series was not sufficient for the needs of astronomy, probability, physics, etc. These sciences required the mathematical investigations of new quantities and the introduction of new functions. The formal concept of series prevented scholars from dealing with

[11] Kline observed: "[Euler] did not accept the general fact that quite arbitrary functions could be so represented [by using trigonometric series]; the existence of such a representation, where he used it, was assured by other means" [1972, 517].

geometrical and physical quantities adequately and, in particular, prevented scholars from using series as autonomous objects that could be used to introduce and represent new quantities.

During the period from 1770 to 1820 studies of new objects (gamma function, etc.) represented an important part of advanced mathematical research. Many results were achieved, mainly concerning certain integrals, although these results appeared as marginal additions to the analysis of the elementary functions and their power series. The potential of certain results was not appreciated. Mathematicians often succeeded only thanks to geometrical and physical considerations, but this was in contradiction to the declared independence of analysis from geometry, one of the cornerstones of Eulerian and Lagrangian mathematics.

In this context Fourier's treatise on the propagation of heat and Gauss's paper on the hypergeometric series were written.

In his treatise Fourier was obliged to develop and reinterpret the theory of trigonometric series to study heat diffusion mathematically. For instance, in the case of steady-state diffusion in a lamina, Fourier obtained the diffusion equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \quad (x\geqslant 0,\ -1\leqslant y\leqslant 1).$$

He solved this equation by separating variables and superposing simple states. He obtained

$$z=\sum_{k=1}^{\infty} a_k e^{-n_k x}\cos n_k y.$$

By considering the boundary condition $z=0$ (when $y=\pm 1$) and $z=1$ (for $x=0$) and replacing y by $2u/\pi$, he derived the trigonometric series $1=\sum_{k=1}^{\infty} a_k \cos 2(k-1)u$ (see Fourier [1807, 134–144]).

Fourier found the constants a_k by differentiating $1=\sum_{k=1}^{\infty} a_k \cos 2(k-1)u$ term by term infinitely many times. He put $u=0$ in all derived equations and obtained an infinite number of equations in the unknowns a_k. To solve this system Fourier considered the first seven equations in the first seven unknowns and found that

$$a_1=\frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot 13}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot 14}.$$

At this point Fourier stated that if one considered more equations, one would have found an expression of a_1 similar to the previous one. In the case of eight equations, he found

$$a_1=\frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot 13\cdot 15\cdot 15}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot 14\cdot 14\cdot 16}.$$

According to Fourier, if one considered all the infinite equations then one had

$$a_1=\frac{4}{\pi}=\frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot \cdots}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot \cdots}.$$

In the same way, he found the other coefficients [Fourier, 1807, 147–156]. This procedure is rather close to typical 18th-century procedures. However, Fourier changed the interpretation of the relation

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{12}$$

and this is the crucial point. He viewed this relation as a purely quantitative relation. Thus he [1807, 158] made clear that the equality

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}\cos(2k+1)u = \frac{\pi}{4}$$

did not hold when the variable u did not lie between $-\pi/2$ and $\pi/2$. Indeed, the function $\sum_{k=0}^{\infty}(-1)^k \times \frac{1}{2k+1}\cos(2k+1)u$ is equal to $\pi/4$ over $(-\pi/2, +\pi/2)$, it is 0 for $u = \pm\pi/2$, and it is equal to $-\pi/4$ over $(\pi/2, 3\pi/2)$. To make this clear he also provided a geometrical interpretation of the equation

$$y = \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}\cos(2k+1)u,$$

the curve having this equation being viewed as the limiting curve of the curves

$$y = \sum_{k=0}^{n}(-1)^k \frac{1}{2k+1}\cos(2k+1)u, \quad n = 1, 2, 3, \ldots$$

(see Grattan-Guinness and Ravetz [1972, 169–171] and Grattan-Guinness [1990, 594–601]).

Then he found the sum of $\sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}\cos(2k+1)u$ directly by showing that the nth sum (n even) of the series is $\frac{1}{2}\int_0^u \frac{\sin 2nx}{\cos x}\,dx$ and that it tended to $\pi/4$ as n went to infinity. Eventually he investigated the behavior of the integral $\frac{1}{2}\int_0^u \frac{\sin 2nx}{\cos x}\,dx$ to clarify that the sum of $\sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}\cos(2k+1)u$ is $\pi/4$ only for certain values of the variable (see Fourier [1807, 159–173]).

Fourier regarded the convergence of series as lying at the heart of the question. He thought it was easy to derive trigonometric series by different procedures:

> but the essential point is to distinguish the limits within which the value of the variable is to be taken. For instance the equation $\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots$, given by Euler, holds as long as the values of x lie between 0 and π or between 0 and $-\pi$. For all other values of x the right-hand side has a determined value very different from $\frac{1}{2}x$.
>
> ... It is by means of these observations that the contradictory consequences found by combining different series of sine and cosine are explained. [Fourier, 1807, 169]

In the following pages of his treatise, Fourier generalized the above results and showed that an odd arbitrary function $f(x)$ can be expanded into a sine series

$$\frac{1}{2}\pi f(x) = \sum_{k=1}^{\infty}\left(\int_0^{\pi} f(x)\sin ku \, \mathrm{d}u\right)\sin kx.$$

To derive this result he first used a complicated method based upon formal manipulation and the assumption that $f(x)$ could be expanded in Taylor series. Then he gave a quick method, namely the now standard method based upon the orthogonality of sine terms of the series [Fourier, 1807, 216–217].

By this method he also derived the general cosine series [1807, 223–224] and then obtained the coefficients of the full series (12) for an arbitrary function [1807, 260–262].

I think that the above description is sufficient to clarify the novelty of Fourier's approach. Although Fourier employed formal manipulations and geometrical arguments to support his theses (e.g., he considered the definite integral as the area under the curve), he used series based on the idea that the relation between series and its sum was only a quantitative relation. In so doing he succeeded in giving an analytical form to certain discontinuous functions and in enlarging the bounds of analysis. Fourier's treatise opened up a series of new problems which his followers were to pursue vigorously.

Another author who overcame the formal point of view in the first years of the nineteenth century was Gauss. In 1812 Gauss published a memoir entitled *Disquisitiones generales circa seriem infinitam* $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$ [1812] devoted to the investigation of the hypergeometric function and the closely related factorial and digamma functions.

In this paper Gauss listed some frequently used functions that could be obtained from the hypergeometric series by giving particular values to the parameters α, β, γ and then explained the goal of his research in the following way:

> the previous functions [the functions which he had listed] are algebraic, logarithmic and circular transcendent. In no case, however, do we undertake our general inquiry because of those, but rather to promote the theory of higher transcendental functions, our series containing a very large number of them. [1812, 128]

To achieve his aim, Gauss began by defining[12] the hypergeometric function $F(\alpha, \beta, \gamma, x)$ to be the limit of partial sums of the series

$$1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots. \tag{13}$$

Therefore Gauss did not identify the hypergeometric series with the function whose development generated the series. Rather he viewed the hypergeometric series as an autonomous object which could be used to define and introduce a new function. According to him:

[12] In effect he gave two different definitions (the first is found in [1812], the second in a sequel entitled *Determinatio seriei nostrae per aequationem differentialem secondi ordinis*, which was found in his Nachlass and published in Gauss's *Werke* [1863–1929, 3, 207–229]).

> It is clear that since our function is defined to be the sum of the series, the inquiry as to its nature is restricted to the case in which the series actually converges. Therefore it is inappropriate to ask about the value of the series for x greater than unity. [1812, 126]

In other words, a series can be used to define a function only if one accepts a quantitative point of view.

For this reason Gauss [1812, 126] investigated the conditions under which the series (13) could actually be considered as a function of x. He, first, observed that γ cannot be either 0 or a negative integer (to avoid infinitely large terms) and that when α and β are either negative integers or zero the series is the expansion of a rational function. Then he determined the convergence of the series by the ratio test. The ratio of the coefficients of x^m and x^{m+1} is

$$\frac{(m+1)(\gamma+m)}{(\alpha+m)(\beta+m)} = \left(1+\frac{\gamma+1}{m}+\frac{\gamma}{m^2}\right) : \left(1+\frac{\alpha+\beta}{m}+\frac{\alpha\beta}{m^2}\right). \tag{14}$$

Equation (14) becomes ever closer to unity as m increases; therefore, if x is a real number such that $|x| < 1$, the series is convergent, at least after a certain term, and has a finite sum. The same occurs if x has the form $a + b\sqrt{-1}$, provided $|x|^2 = a^2 + b^2 < 1$. Instead, if x is a real or complex number such that $|x| > 1$, then the series is increasing (if not initially, at least after a certain term): consequently, one cannot refer to the sum of the series. In Section 3 of [1812], he also investigated the more complicated case $|x| = 1$ and showed that the series converges if and only if $\gamma - \alpha - \beta > 0$ by formulating a specific convergence test:

> Given a sequence M_k, $k = 0, 1, 2, \ldots$ such that the ratios M_{k+1}/M_k are of the type
>
> $$\frac{M_{k+1}}{M_k} = \frac{P_\lambda(m+k)}{p_\lambda(m+k)}, \quad k = 0, 1, 2, \ldots,$$
>
> where m is an positive integer, $P_\lambda(t) = t^\lambda + A_1 t^{\lambda-1} + A_2 t^{\lambda-2} + \cdots + A_\lambda$ and $p_\lambda(t) = t^\lambda + a_1 t^{\lambda-1} + a_2 t^{\lambda-2} + \cdots + a_\lambda$, the series $\sum M_k$ converges if $A_1 - a_1 < -1$. [1812, 139–143]

Using a quantitative notion of series, Gauss succeeded in introducing new functions into analysis and in dealing with them in an appropriate way.

Fourier seems to have been interested mainly in the application of his results and did not hesitate to resort to geometrical interpretations. Although Gauss's work was certainly motivated by applied considerations, he attempted to organize the results he obtained into a purely analytical theory (see Ferraro [2006, in press]).

In the years which followed the rejection of divergent series and formal manipulation and the adoption of quantitative approach were at the basis of Cauchy's work.

7. Conclusion

In this paper I have attempted to highlight some aspects of the history of series theory during the 18th century. My theses can be summarized as follows.

At the beginnings of the calculus, mathematicians used series in order to represent geometrical quantities and solve geometrical problems. They thought that series had a quantitative meaning (namely, that they represented quantities) if, and only if, they were convergent. Although mathematicians had an intuitive idea of convergence, a distinction between finite and infinite sums was lacking and this gave rise to formal manipulations, namely the use of procedures that were the infinitary extension of finite procedures. Mathematicians thought that the quantitative and the formal could coexist and that formal manipulation was a tool for deriving convergent series. By the 1720s, this way of conceiving the relationship between the quantitative and the formal yielded several results that could not be reduced to the original concept. Mathematicians introduced recurrent series and emphasized the law of formation of coefficients, independent of the convergence of series. The attempt to increase the speed of convergence of series subsequently led to the emergence of asymptotic series, which showed the possibility of using divergent series to obtain appropriate approximations. Furthermore, the investigation of continued fractions and infinite products and certain applications of series increasingly stressed the formal aspects.

The need to validate these results gave rise to the Eulerian systematization. A series was thought to be the result of a formal transformation of an analytical quantity expressed in closed form. This transformation gave a meaning to the series, even when the latter was not convergent. However, mathematicians were not free to invent transformations by a free creative act. They limited themselves to using transformations that were used in the original theory or, at least, that were compatible with it. This seemed to guarantee that the new more formal conception was a generalization of the formal-quantitative conception. The latter did not imply any incompatibility with the former but was valid for series different from power series convergent over an interval. The formal-quantitative concept remained the essential basis from which all the parts of the series theory were subsequently generated.

The formal approach was predominant during the second part of the 18th century for two main reasons. First, mathematicians who were critical of it were not able to eliminate the formal aspects and found a really new theory: they also used the formal methodology that had led to asymptotic series and to the combinatorial use of series in a very natural way. Second, the formal concept of series contributed to the growth of mathematics. It led to many new discoveries and even to a new branch of analysis: the calculus of operations.

However, the formal approach became unsuited to most advanced mathematical research towards the end of the 18th century and the beginning of the 19th century. Applied mathematics stimulated research and encouraged the introduction of new functions into analysis, but formal methodology was unable to treat quantities which were not elementary quantities and series which were not power series. The need to use trigonometric series in the analytical investigation of heat led Fourier to reject the formal concept of series and to embrace an entirely quantitative notion of series. The need to introduce hypergeometric and gamma functions into analysis and to place them within an adequate analytical theory forced Gauss to highlight the quantitative meaning of the sum of series and to reject formal manipulations.

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