On the Area of a Semi-Circle

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Contents

A widespread view often encountered in historical studies is that ideas, or customs, practices, and beliefs, are the spontaneous reactions of the human mind to environing conditions. The notion is, of course, an assumption, often tacitly held. It is itself probably not spontaneous, but a part of the Greek rationalist ideology, of our Greek rationalist heritage.

In mathematics this view takes the following form: Mathematics arises from the activities of daily life; it is practical and has its origin in obvious practical applications. The Greeks invented proof, so it is generally held: before that mathematics was "empirical", whatever that means.

Opposed to this is the view that ideas are the products of certain special circumstances. A corollary is that any idea has a single (cultural) origin. This view is not a dogma, but a theory to be built up inductively. One technique is to examine a complex of ideas for parts having no inherent connection. If such features can be found, a cultural, and not merely logical, connection is indicated. At the same time, the accidental features may give a clue to the special circumstances of origin.

To illustrate: In 1877 M. CANTOR began a comparative study of Greek and Indian mathematics, and in particular studied G. THIBAUT's paper on the *Sulvasútras*, an Indian sacred work on altar constructions.[1] CANTOR notes (following THIBAUT) that for the Indians, the Theorem of PYTHAGORAS is not so much a theorem on triangles as a theorem on rectangles: "The cord stretched in the diagonal of an oblong", writes BAUDHAYANA, "produces both (areas) which the cords forming the longer and shorter side of an oblong produce separately". CANTOR compares this with the fact that HERON employs the Theorem of PYTHAGORAS to compute the diagonal of a rectangle before taking up the triangle. Moreover, the *Sulvasútras* give the theorem separately for a square and for an oblong; and HERON, in the place mentioned, gives two successive problems: one

[1] "Gräko-indische Studien," *Zeit. f. Math. u. Phys. (Hist.-lit. Abt.)*, vol. **22** (1877); "On the Sulvasutras," *J. Asiatic Soc. Bengal*, vol. 44, 1 (1875).

for the equal-sided rectangle, and one for the unequal-sided rectangle. CANTOR considers that these coincidences cannot possibly be accidental.

Even EUCLID defines *oblong*, though he never uses the term. As T. L. HEATH has observed, it is but a survival from earlier texts.[2]

The main point here is, of course, that the classification of rectangles into oblongs and squares is in no way inherent in the Theorem of PYTHAGORAS, and that this arbitrary element therefore indicates a historical connection.

In a previous work[3] we added the observation that the classification of rectangles into oblongs and squares is part of a widespread religious or theologic complex: thus the first principles of PYTHAGORAS are ten in number and consist of pairs of opposites, *e.g.* odd-even, male-female, *etc.*, and one of these pairs is square-oblong; the Indians also had this same duality—oblong bricks are human, square bricks divine; and we even found the duality as far away as Fiji, as "the Fijians who dwelt round the Koro Sea built oblong houses, but their temples were usually square". All this suggests that the Theorem of PYTHAGORAS was, in origin, part of a religious or theologic or, as we prefer to say, a ritual complex. The thesis that geometry has a ritual origin was exposed in our paper on *The Ritual Origin of Geometry*.

There are two theses, then: (1) that geometry had a single origin, and (2) that this origin was in ritual. The first of these, at least, is not new. Though not expanded upon, the first was definitely formulated by CANTOR: in his studies it occurred to CANTOR that perhaps in very ancient times ("roughly speaking, three or four thousand years ago") there already existed a not altogether insignificant mathematical knowledge common to the whole cultured areas of those times, which was further developed, here in one direction, there in another.[4] This is precisely our thesis. The problem then becomes to say in what this original knowledge consisted and what were the later developments; or, in other words, to put the contents of ancient mathematics into a chronological perspective.

There are two distinct traditions easily discernable in ancient geometry: one is computational or algebraic, the other is constructive or geometric. In the first, for example, the Theorem of PYTHAGORAS says that the diagonal of a rectangle is the square root of the sum of the squares of the sides; it is expressed with a computation in view. In the second, the theorem says that the square built on the diagonal is the sum of the squares on the sides; it is expressed with a construction in view. Or even simpler: the first says that the area of a triangle is one-half base times altitude—this is the sort of theorem that will not be found in EUCLID. EUCLID says, rather, that "if a parallelogram have the same base with a triangle and be in the same parallels, then the parallelogram is double the triangle (*Euclid*, I, 41)". The first tradition might be called the *Oriental tradition*, the other, the *Greek tradition*; or perhaps better, in order to avoid suggesting a place of origin, the *algebraic* and the *geometric*, respectively.

[2] *The Thirteen Books of Euclid's Elements*, vol. **1**, p. 428.

[3] "Ritual Origin of Geometry," *Archive for History of Exact Sciences*, vol. **1** (1962), p. 503.

[4] "Über die älteste indische Mathematik," *Archiv d. Math. u. Phys.*, vol. 8 (1904), p. 71.

It has usually been held, at least in recent times, that the algebraic tradition preceded the geometric. This view has been well exposed in B. L. VAN DER WAERDEN's work *Science Awakening*. His exposition takes into account what is known of the relative chronology of Babylonian and Greek mathematics, and also of the desire of the Greeks to overcome difficulties due to the existence of incommensurable quantities. But it totally fails to take into account the *Sulvasutras*—the work of THIBAUT and of CANTOR in this regard is not even mentioned. If the *Sulvasutras are* taken into acount, however, the opposite conclusion, we believe, will be reached. This view is argued in the paper of ours cited, and although the *Sulvasutras* have never been assigned a very early date, we do not hesitate to say that the geometry of the *Sulvasutras* was already old in Old-Babylonian times.

Our first main thesis is, then, that the elements of geometry as found in the ancient civilizations, in Greece, Babylonia, Egypt, India, and China, are a derivative of a system of ritual practices as disclosed in the Sulvasutras.

1. The Circle and Square in India

This is not the place to review the contents of the *Sulvasutras*, but we must recall that the main problem was to construct an altar (a plane figure) of given shape and area. The basic altar had an area of $7\frac{1}{2}$ square units *(Purushas)*. It was composed of a number of squares and rectangles, assembled into a form said to resemble (and which to some extent does resemble) a falcon. For its construction a knowledge of how to lay out a right angle is needed. In the *Sulvasutras* a right angle is constructed sometimes with but also sometimes without an application of the Theorem of PYTHAGORAS; and this theorem is not needed for the construction of the basic falcon-shaped altar. But the sacrificer was on a sacrificial ladder, his rank determined by, or determining, the area of the altar. The next highest rank was $8\frac{1}{2}$; and here the Theorem of PYTHAGORAS is actually and explicitly involved.

One of the shapes for an altar was a circle, and the problem of converting a square into a circle thus arises. We call this problem the *circulature of the square:* it is to be clearly distinguished from the problem of *squaring the circle*, which is to construct a square equal in area to a given circle. This latter problem is also treated in the *Sulvasutras*, but its solution (as we shall explain) is out of character with the rest of the work, and the squaring of the circle did not have, as far as we could tell, a sacred application.

In the *Sulvasutras* the circulature of the square is done as follows (see Fig. 1). In square $ABCD$, let M be the intersection of the diagonals. Draw the circle with M as center and MA as radius; and let ME be the radius of the circle perpendicular to the side AD and cutting AD in G. Let $GN = \frac{1}{3} GE$. Then MN is the radius of a circle having an area equal to the square $ABCD$.

For the reverse problem, that of squaring the circle, one is given the rule:

> "If you wish to turn a circle into a square, divide the diameter into 8 parts, and again one of these 8 parts into 29 parts; of these 29 parts remove 28, and moreover the sixth part (of the one part left) less the eighth part (of the sixth part)."

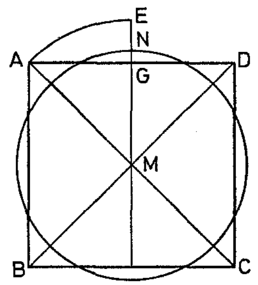

Fig. 1

The meaning is: side of required square $= \frac{7}{8} + \frac{1}{8 \cdot 29} - \frac{1}{8 \cdot 29 \cdot 6} + \frac{1}{8 \cdot 29 \cdot 6 \cdot 8}$ of the diameter of given circle.

One also finds the approximation: $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}$ (more precisely: the diagonal of a square $= 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}$ of a side).

In looking at the *Sulvasutras* as a whole, one notes that the squaring of the circle differs in character, in several respects, from the rest of the work. The work has definitely a geometric and not an arithmetic character. There is, to be sure, some arithmetic (not counting the squaring of the circle and the approximation to $\sqrt{2}$). For example, it is realized that the square of n units in length has area n^2; from this it is deduced that $\frac{1}{2}$ the side of a square produces $\frac{1}{4}$ the area of the square, and $\frac{1}{3}$, the ninth. It is also explicitly stated that $1\frac{1}{2}$ linear purushas produces $2\frac{1}{4}$ square purushas; and even the general rule $(a+b)^2 = a^2 + 2ab + b^2$ is set up. Fractions thus enter, but there is little arithmetic involved with them. There are some elaborate bird altars involving several types of bricks, but most of them have an integral number of sixteenths of a square purusha as area; and even the commentators, who are already in the algebraic tradition, make all their computations in terms of chaturthi-bricks ($= \frac{1}{16}$ square purusha). Thus most of the arithmetic is with integers, and there is nothing in the remainder to suggest that the ritualists (the earlier ones, that is) could work with the fractions mentioned in connection with the squaring of the circle.

As THIBAUT has pointed out, the squaring of the circle is "nothing but the reverse of the rule for turning a square into a circle"; that is, if $d =$ diameter, $s =$ side of an equal square, then the circulature of the square gives

$$\frac{d}{s} = \frac{2 + \sqrt{2}}{3}.$$

After replacing $\sqrt{2}$ by the rational approximation $1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}$, it is easy, by simple arithmetic, to find the reciprocal s/d. This gives

$$\frac{s}{d} = \frac{7}{8} + \frac{1}{8 \cdot 29} - \frac{1}{8 \cdot 29 \cdot 6} + \frac{1}{8 \cdot 29 \cdot 6 \cdot 8} - \frac{41}{8 \cdot 29 \cdot 6 \cdot 8 \cdot 1393},$$

which, neglecting the last term as explained by THIBAUT, is the expression in the *Sulvasutras*.

The circulature of the square involves no arithmetic. One may imagine an ancient ritualist starting from the square, observing that the inscribed circle is too small, the circumscribed circle too large, and guessing that one should take $GN = \frac{1}{3} GE$. (See Fig. 1.) The line of thought, though approximative, is geometric. We may suppose that this solution of the circulature of the square, having become fixed in tradition, became the starting point for squaring the circle. This reverse problem, though an easy exercise for us, may well have baffled the Vedic ritualists: How, given the circle of radius MN, is one to get hold of NG and thereby reverse the steps in the circulature of the square? Not being able to solve this problem geometrically, the ancients went over to an arithmetic solution. Here they needed a rational expression for $\sqrt{2}$; of course, they might have rationalized the denominator of

$$\frac{3}{2+\sqrt{2}}$$

i.e., brought

$$\frac{3}{2+\sqrt{2}}$$

to the form $\frac{3}{2}(2-\sqrt{2})$, but presumably they did not know enough algebra, either.

This leads to our second main thesis:

The first crisis in mathematics occurred because the ritualists could not reverse their (canonical) circulature of the square in a geometric way. The resulting efforts to find an arithmetic solution for the squaring of the circle gave rise to the algebraic tradition. Geometry was dislocated from its ritual base.

Though not of concern to us for the moment, we may mention a third thesis:

The problem of finding a rational expression for $\sqrt{2}$ arose from the attempt to square the circle. The discovery that there was no such expression gave rise to the second crisis in mathematics. This was resolved by a revival of the geometric tradition.

2. The Area of a Circle in Egypt

In our geometry paper we cited VAN DER WAERDEN's opinion that "... Egyptian geometry is ... merely applied arithmetic", and showed that if we accept his presentation of the evidence, especially that of Problem 10 of the Moscow mathematical papyrus (MMP), which goes "when you are told a basket (of $4\frac{1}{2}$) in diameter by $4\frac{1}{2}$ in depth, then tell me the area", then one must come to the opposite conclusion (*op. cit.*, p. 511). We introduced the notation $\frac{1}{4}\pi_1$ for the ratio of the area of a circle to the square on its diameter, and π_2 for the ratio of the circumference of a circle to its diameter. Of course, we know that $\pi_1 = \pi_2$, but the question is whether the ancients knew it. Now we showed that if one accepts VAN DER WAERDEN's interpretation (following T. E. PEET) of the cited problem, then one must (or should) conclude that the Egyptian knew that $\pi_1 = \pi_2$ (and the same goes also for O. NEUGEBAUER's interpretation, which VAN DER WAERDEN presents but rejects). Of course, when we say "the Egyptian knew that $\pi_1 = \pi_2$", this is only a shorthand for saying he understood certain essential relations between the area of a circle, its diameter, and its circumference, and not literally that he considered π_1 and π_2, much less their equality.

Now if the Egyptian (or a predecessor) realized that $\pi_1 = \pi_2$, then this realization must have come about by a geometrical analysis which, no matter how crude it was, was quite sophisticated. Thus Egyptian mathematics could not be "merely applied arithmetic".

As a *reductio ad absurdum,* we can find no fault with this line of argument; but we have been led to reject VAN DER WAERDEN's interpretations of the basket problem, as well as NEUGEBAUER's, and this prompts us to take up the point once more.

We shall return to the basket problem, but first must recall how the Egyptian computed the area of a circle of given diameter. Problem 50 of the Rhind Mathematical Papyrus (RMP) reads: Example of a round field of diameter 9 *khet.* What is its area? (Solution): Take away $\frac{1}{9}$ of the diameter; the remainder is 8. Multiply 8 times 8; it makes 64. Therefore it contains 64 *setat* of land. Do it thus: *etc.*

In modern shorthand we can write the procedure thus: $A = \left(d - \frac{d}{9}\right)^2$. Comparing this with our $A = \frac{\pi_1 d^2}{4}$, we may say the Egyptians took $\pi_1 = 4 \cdot \left(\frac{8}{9}\right)^2$.

E. T. BELL long ago wondered "what suggested the curious $(\frac{4}{3})^4$". [5] We believe that the main clue for an answer was given by K. VOGEL[6] when he called attention to the figure in RMP 48. Here is the figure (Fig. 2):

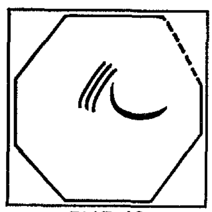

Fig. 2

VOGEL interprets this, correctly we believe, to represent a polygonal approximation to the inscribed circle; and he supposes this, also correctly we believe, to have been obtained by dividing each of the sides into 3 equal pieces and by joining the points of division.

The horizontal and vertical lines joining the points of division (see Fig. 3)

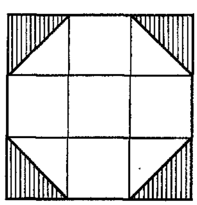

Fig. 3

[5] *Development of Mathematics,* p. 38.

[6] *Vorgriechische Mathematik* (1958), vol. 1, p. 66.

divide the big square into 9 equal little squares. The outer little triangles make $\frac{2}{9}$ of the big square, so the area of the octagon is $\frac{7}{9}\,d^2$.

With a minor qualification, RMP 48 is unique among the 87 problems of the papyrus in that there is no statement of the problem: the solution consists of a computation of 8×8 and 9×9. According to T. E. PEET, the problem "is clearly the comparison of the area of a square of side 9 khet with that of a circle of diameter 9 khet".[7] The solution says that a circle is to its circumscribed square as 64 is to 81.

The side of the square in RMP 48 is 9, so the octagon has area 63. The side of an equal square would be $\sqrt{63}$. VOGEL has suggested that this is approximated by $\sqrt{64}=8$, and that perhaps in this way one came to the formula $(\frac{8}{9}\,d)^2$. We think this contains some truth, but is too roughly said and does not correspond to the Egyptian's thinking. More recently R. J. GILLINGS & W. J. A. RIGG have taken up this point anew, and have come still closer to the truth.[8] We shall comment on their work also, but for the moment take for granted as substantially correct the reconstruction just outlined.

Let us return to VAN DER WAERDEN's thesis that Egyptian geometry is "merely applied arithmetic". He also writes (*loc. cit.*, p. 89): "At the start, in the first excitement of discovery, one is occupied with questions such as these: how do I calculate the area of a quadrangle, of a circle, ...?" If this were the case, how can one understand the Scribe's going over from the simple and direct program of computing $\frac{7}{9}\,d^2$ to that of computing $(\frac{8}{9}\,d)^2$, which involves an error he could presumably see? But if, as we suggest, there already existed the problem of squaring the circle, then one can. An older tradition *compelled* the Scribe (or a predecessor) to give the answer in the form of a square. Therefore he was quite willing (or, rather, obliged) to make yet another approximation (which, incidentally, gave him a better answer, but that was sheer luck).

To return to GILLINGS's & RIGG's explanation: they suggest that first a 9×9 square was drawn, and divided up into 81 little squares by lines parallel to its sides; the side (they say) was taken as 9 because 9 is exactly divisible by 3; each of the corners (to be excluded) has area $4\frac{1}{2}$; if the two top corners replace the top row of little squares, and the bottom corners the left column of little squares, (and if these are excluded from the 9×9 square) then the figure remaining would be an 8×8 square; it is true that in this way the upper left little square is removed only once instead of twice, but still the scribe could properly conclude that the area of the inscribed circle is very closely equal to a square of side 8.

[7] T. E. PEET, *The Rhind Mathematical Papyrus*, p. 88. PEET writes: "It is interesting in No. 48 to find the dimensions inserted throughout. It is still more so to notice that in the first line of all, '1 8 setat,' the unit is stated as *setat*. To our modern feeling this is wrong. The 8 in question is, strictly speaking, in units of long measure, *viz.* 8 *khet*, and it is not until we multiply it by another unit of long measure, viz. 8 *khet*, that it can logically be expressed in square units." PEET is quite wrong: if the details supplied by the Egyptian (for counting up the 64 setat) seem wrong to our "modern feeling" it is because this feeling has lost sight of the analysis upon which it is based. What *is* interesting is that traces of this analysis are to be seen in RMP 48.

[8] "The Area of a Circle in Ancient Egypt," *Australian J. Science*, vol. **32** (1969), p. 197.

First we shall indicate some difficulties in this explanation:

(a) The explanation is really a suggestion for a method of approximating the square root of 63. Of course, the scribe *could* have seen this, but no documentation is offered to suggest that he *did* see it. Nowhere in the rest of the Egyptian mathematical remains do we find that the square root was approximated in this way (or in any other way).

(b) There is the problem of where the the 9 comes from. It is true that 9 is exactly divisible by 3, but 3 is also exactly divisible by 3 and leads equally simply to $\frac{7}{9} d^2$. Taking the $d=9$ for granted is a form of begging the question. (Actually, we suppose that $d=9$ of RMP 50 comes because the scribe wants a d whose ninth will yield no arithmetical irrelevancies; but this does not get us very far, because then, of course, the question is: where did the $\frac{1}{9}$ come from? So the question of where the 9 comes from remains.)

(*A priori*, it may be the 8 (of the $\frac{8}{9}$) that needs explaining. This might be the case if we knew or suspected that the Scribe, or a predecessor, had the problem of the circulature of the square in mind. But we do not see how to get the 8 out of anything we know about the circulature of the square, and so suppose it really is the 9 that should be explained.)

(c) GILLINGS & RIGG note that the Scribe's answer is in the form of a square and that this form is obtained at the cost of an error of $\frac{1}{81}$, but they say nothing as to why the Scribe was willing to incur this error. (This difficulty has, however, already been met.)

In meeting these difficulties, let us assume for a moment that (b) has already been met, so in (a) it is a question of showing that the method for finding $\sqrt{63}$ is in accord with Egyptian thinking. Now in the *Sulvasutras* we find explicitly the problem of turning a rectangle (say a by b) into a square (see Fig. 4): with the

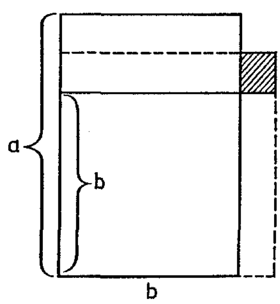

Fig. 4

shorter side (say b) one cuts off a square (yielding a square $b \times b$ and a rectangle $b \times (a-b)$); the remaining rectangle is divided into two rectangles (each $b \times (a-b)/2$); one of these is brought around to a side of the smaller square, and one is left with a square (of side $(a+b)/2$) minus a square of side $(a-b)/2$ at one of the corners; "one has been taught how to subtract the square", say the *Sulvasutras*.

If we apply this to a 9×7 rectangle, then we have rather closely the suggested method for finding the side of a square of area 63, except that the *Sulvasutras* do not neglect the "little" square: they had no need to, but arithmetically it usually *is* needed, in particular, in the case 9×7. (The Egyptian would, of course, have the relation $63=9\times 7$ in mind, especially as the 9 is already there.)

THIBAUT *(op. cit.)* has explained the approximation $1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$ for $\sqrt{2}$ in a similar way. He imagines the priests first to have looked for a square integer whose double was also a square. Trying the first few squares, they soon would have come to $12^2=144$ and $2\times 12^2=288$, which is only one short of $289=17^2$. Thus $\frac{17}{12}$ is an approximation to $\sqrt{2}$. $\left(\text{In Egyptian style, } \frac{17}{12}=1+\frac{1}{3}+\frac{1}{3\cdot 4}.\right)$ Now one considers a 17 by 17 $\left(\text{or rather } \frac{17}{12} \text{ by } \frac{17}{12}\right)$ square, composed of 289 equal little squares. One of these little squares has to be subtracted. One half of this little square is equal to a rectangle 17 by $\frac{1}{34}$ $\left(\text{respectively } \frac{17}{12} \text{ by } \frac{1}{12\cdot 34}\right)$. Thus one comes to the approximation $1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}$, where now a little $\frac{1}{34}$ by $\frac{1}{34}$ $\left(\text{or } \frac{1}{12\cdot 34} \text{ by } \frac{1}{12\cdot 34}\right)$ square has been neglected. Thus the reconstruction for $\sqrt{63}$ is in accord with THIBAUT's for $\sqrt{2}$; and we may be confident that it is in accord with the thinking disclosed in the *Sulvasutras*.

In 1877 CANTOR took the view, which he renounced in 1904, that the geometry of the *Sulvasutras* was a derivative of Alexandrian knowledge. In particular, he claimed that the approximation for $\sqrt{2}$ and the rational number for the squaring of the circle were Egyptian. In our geometry paper, we expressed ourselves ready to concede this, but took the view that the squaring of the circle (not the circulature of the square) was "interpolated", *i.e.*, added to the *Sulvasutras* after the composition of its characteristic portion. But we would not concede that it was interpolated at a late date. Now it is clear to us that the squaring of the circle in the *Sulvasutras* and its squaring in the *Rhind Mathematical Papyrus* belong to the same historical stratum.

At the time, although we observed that the Egyptian had the notion of square root, we also noted that all the square roots (known to us) came out even. We suspected that the problems were fixed so that this would happen and then conjectured that the Egyptian could approximate square roots (*op. cit.*, p. 514). If the above reconstruction is correct, this conjecture is now validated.

Let us now go to (b): where does the 9 come from? Starting from the inscribed octagon as described, we can imagine the Egyptian asking: How shall I compute the side of a square of equal area? He realizes that he must take away a fraction of the side of the big square, and even perhaps senses that this will be a small fraction, but what? We would call it x, but the Egyptian would call it 1—he knew the method of "false assumption"—and the side of the big square x, or better, $3x$, since it is already, divided into 3. Then he gets the equation $3x=x^2$ (*i.e.*, $1\cdot 3x=x^2$), whence $x=3$ and $3x=9$. So here we get the 9!

When one augments a square to a larger square, one vertex remaining fixed, the difference of the two squares is a figure called a *gnomon*. The gnomon makes itself rather obvious in considering a diagram showing that $(a+b)^2 = a^2 + 2ab + b^2$ (as in *Euclid* II, 4), or in the rule given for turning a rectangle into a square; and it also occurs in our reconstruction for computing square root. At any rate, the Greeks and Indians definitely knew the gnomon; but in our geometry paper (*op. cit.*, p. 511), we commented that we knew of no gnomon from Egypt. Now the Egyptians had a sign for square root. It is: .[9] Could this be the missing gnomon?[10]

Let us sum up in a fourth thesis:

The Egyptian procedure for finding the area of a circle is not merely a computation for area but is a true quadrature. It belongs to the same stratum as the squaring of the circle in the Sulvasutras and, like it, requires a technique for computing square roots approximately. The approximation $(\frac{8}{9})^2$ to $\pi_1/4$ can be resolved into two approximations, one of which is the approximation $\frac{8}{9}$ to the square root of $\frac{7}{9}$.

3. The Area of a Semi-Circle in Babylonia and China

In our geometry paper we insisted on the importance of comparative studies for the history of ancient mathematics, and we did compare the mathematics of Greece, India, Egypt, and Babylonia. We also emphasized the importance in this regard of the ancient Chinese mathematics, but except for a brief reference to the *Chou-Pei*, we refrained from discussing the Chinese mathematics. The reason for this restraint was that, from a general description of its contents, we deemed the *Chiu Chang Suan Shu (Nine Books on Arithmetic Technique)* the most relevant for our purpose; but of this ancient work, the fullest account accessible to us was that of Y. MIKAMI,[11] and he himself calls what he has a "summary". There was, indeed, a Russian translation by E. I. BEREZKINA (1957), unknown to us at the time. More recently (1968) K. VOGEL has brought out a German translation.[12]

Let us take as preliminary glance at the *Nine Books*. We confine ourselves mainly to the geometric parts.

Book I starts with *the area of a rectangular field*. Problem 1 reads: "Now one has a field; it is 15 steps wide and 16 steps long. The questions is: How large is the field?" The answer (= 1 Mou) is given; and a second problem of a similar kind is posed and the answer given. Then the general rule is stated. With a couple of minor exceptions,[13] this is the format used throughout the work: one or two problems of some type are posed, the answers given, and the general rule stated.

[9] See, *e.g.*, W. W. STRUVE, "Mathematischer Papyrus des Staatlichen Museums der Schönen Kunste in Moskau," *Quellen und Studien zur Geschichte der Mathematik* (Abt. A), vol. **1** (1930), Problem 6, Col. VIII, line 5, Table II, no. 6; *cf.* p. 125.

[10] Now we see that CANTOR had already suggested this (*op. cit.* (1904), p. 69). However, if the reconstruction for finding approximate square roots is correct, then the connection between *gnomon* and *square root* is not, as CANTOR feared it might be, accidental.

[11] "The Development of Mathematics in China and Japan," *Abh. z. Ges. d. Math. Wiss.*, vol. **30** (1912).

[12] K. VOGEL (Tr.), CHIU CHANG SUAN SHU, *Neun Bücher Arithmetischer Technik* (1968). VOGEL (*op. cit.*, p. 151) refers to *Istor.-matem. isaledovanija* **10**, 1957, pp. 423–584 for BEREZKINA's work.

[13] See VOGEL, *op. cit.*, p. 124, for the exceptions.

This is different, in the main, from the Babylonian procedure, where the problems are stated and *worked out*, but the general rule is not given (though there are a few instances of general statements).[14] Book I continues with arithmetical problems (addition, subtraction, *etc.* of fractions),[15] returning to geometry with Problem 25 which asks for *the area of a triangle*. Then comes *the trapezoid*. With Problem 31 we come to *the circle*: "Now one has a round field; the circumference is 30 steps, the diameter 10 steps. The question is: How large is the field. The answer says: 75 Pu." Clearly the ratio of the circumference to diameter is taken to be 3, though curiously no problem requires this knowledge, and throughout superfluous information is supplied. The value 3 is typically Babylonian, but the Babylonian scribe needs to know this in working his problems. The rule (in the *Nine Books*) for *the area of a circle* is to multiply one-half the circumference by one-half the diameter; three further rules are given, the third of which says to square the circumference and divide by 12—this is the Babylonian procedure. Then the *sector* is considered. Then come a couple of problems on the *segment*: given the chord ($=s$) and the "arrow" (= distance from midpoint of chord to midpoint of arc $=p$), to find the area (the second example is a semi-circle). The rule is: area $=(sp+p^2)/2$, and so appears to approximate the segment with a trapezoid of bases s and p and width p (the approximation is "correct" for a semicircle, *i.e.*, the formula is consistent with the other computations for the area of a semi-circle). The Babylonians have problems on the segment, but they are so far not understood,[16] and so, of course, cannot be said to be the same as the Chinese problems. Still, it appears that in the Old-Babylonian period, in Susa, the segment was assimilated to a bow; and in the *Mishnat ha-Middot* (I, 5), a Hebrew geometry compiled, according to S. GANDZ, about 150 A. D., the technical term "arrow" definitely occurs.[17] Moreover, HERON, who is often considered to be continuing the Babylonian tradition, has this same problem.

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[14] For some examples, see VAN DER WAERDEN, *Awakening Science*, p. 74. We take this occasion to suggest the following translation of of the first example (changes are in italics):

Length and width as much as area; let them be equal.
You in your procedure,
The *length* you take *again*.
From this you subtract 1.
You form the reciprocal.
With the *length* you have taken
You multiply and
The width it gives you.

[15] Problems 5 and 6, Book I, ask one to reduce $\frac{12}{18}$, $\frac{49}{91}$ to lowest terms. The EUCLIDEAN algorithm is used.

[16] See O. NEUGEBAUER & A. SACHS, *Mathematical Cuneiform Texts* (1945), pp. 57, 134, 135, 136.

[17] E. M. BRUINS & M. RUTTEN, *Textes Mathématiques de Susa*, in *Mémoires de la Mission Archéologique en Iran*, vol. **34** (1961). S. GANDZ, *The Mishnat ha-Middot, Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik*, Abt. A, vol. **2** (1932). The text from Susa appears to have "arrow" ("flèche") as a technical term; at least, so BRUINS & RUTTEN translate the term *pi-ir-ku* for the distance in question (*op. cit.*, pp. 25, 28). The Greeks have neither the term nor the conception; the Babylonians at least have the conception (GANDZ, *op. cit.*, p. 19, n. 33). AL-KHWARIZMI and BHASCARA have the term (GANDZ, *loc. cit.*)

His solution is $\frac{1}{2}(b+h)h+\frac{1}{14}(\frac{1}{2}b)^2$, where the correction term comes from the ARCHIMEDEAN value $\frac{22}{7}$ for π_2 (= circumference/diameter); he mentions that the "ancients" took $\frac{1}{2}(b+h)h$ and even conjectured that they did so because they took $\pi=3$.[18] Book I ends with a problem on the area of a ring (= circle minus concentric circle); the Babylonians also have ring problems.[19]

Books II and III are strictly arithmetical.

Book IV poses the problem: Given the area of a rectangular field and its width, what is its length? In the first 11 problems the width is $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ with n taking on successively the values $n=2, 3, \ldots, 12$. To us this looks like a *Sulvasutra* construction problem translated into arithmetic; the more usual view would, presumably, be that the *Sulvasutra* problem (or *Euclid* I, 44) is this arithmetic problem translated into a construction. Problem 12 asks for the side of a square of 55225 Pu (*Ans.* 235 steps). Thus square root comes in. The work is in the decimal system, not, as in Babylonia, in the 60-system. In problem 17, one is given the area of a circle, to find its circumference—again a problem in square root. Then comes cube root, and first to find the side of a cube of given volume. Book IV ends with the problem of finding the diameter d of a sphere of given volume v. The rule is: Take the cube root of $16\,v/9$; this amounts to saying that $v=9\,d^3/16$. This appears to be just another problem in cube root, but to us it is quite surprising. The answer is wrong, of course, but what we find surprising is that the problem was set up at all.[20] We have no corresponding problem from Old-Babylonia (or from Egypt of about the same time, i.e., of the Middle Kingdom).

Book V returns to geometry (although, as throughout, in computational form). Here volume is taken up. Problem 9, for example, computes the volume of a *cyclinder* ($v=c^2h/12$). Problem 10 gives the rule for a *truncated pyramid of square base*; problem 11 considers a *truncated cone*; problem 12, a *square based pyramid*; problem 13, a *circular cone* ($v=c^2h/36$); problem 14, a *prism*; problem 15, an oblong based *pyramid*; problem 16, a *tetrahedron*; problem 17, a *wedge* having two trapezoidal faces at right angles; problem 18, a special case of the next problem; problem 19, a *truncated pyramid-like body* having rectangular, but dissimilar, bottom and top.

Thus the *Nine Books* know the basic facts about pyramids. The Egyptians had a correct formula for the truncated pyramid, and we think the Old Babylonians did, too, though the evidence is not as clear as one might wish. HERON con-

[18] See T. L. HEATH, *A History of Greek Mathematics*, vol. 2, p. 330 in reference to HERON's *Metrica*, I, 30, 31.

[19] O. NEUGEBAUER, *Mathematische Keilschrift-texte (Erster Teil)*, *Quellen und Studien zur Geschichte der Math., Astro., und Physik*, Abt. A, vol. 3 (1935), pp. 153–177.

[20] On a great circle of the sphere build a circular cylinder tangent to the sphere. The sections of the sphere by planes through the axis of the cylinder are in one-to-one correspondence with the (square) sections of the circumscribing cylindrical can, each of the former being $\frac{3}{4}$ (*i.e.*, $\pi_1/4$) of the latter; from which it might have been concluded that the sphere is $\frac{3}{4}$ of the cylindrical can. This then gives $V=\frac{3}{4}[(\frac{3}{4}\,d^2)\,d]$. This makes the formula intelligible, and suggests that the Chinese (or their forefunners) made infinitesimal analyses.

siders a pyramid-like body as described in Book V, Problem 19, but the formula is different (HEATH, *op. cit.*, vol. 2, p. 332).[21]

Books VI and VII are essentially arithmetical. Several problems in Book VI deal with arithmetical progressions. The Egyptians and Old Babylonians have similar problems.[22]

Book VIII deals with simultaneous linear equations. The Old-Babylonians could also handle simultaneous linear equations.

Book IX is geometric, treating the right triangle, especially problems involving the Theorem of PYTHAGORAS; the Theorem of THALES, that an angle inscribed in a semi-circle is right, also comes in. In the course of this, a familiarity with PYTHAGOREAN number triples, i.e., integers a, b, c such that $c^2 = a^2 + b^2$, is disclosed. All this is familiar ground for the Old-Babylonians. Problem 15 asks for the side of the square inscribed in a 5 by 12 right triangle (*Ans.* $3\frac{9}{17}$, i.e., $5.12/(5+12)$). This was presumably worked either by similar triangles or by the Theorem of the Gnomon (*Euclid*, I 43). The Old-Babylonians worked with similar triangles.[23] Problem 16 asks for the diameter of the circle inscribed in an 8 by 15 right triangle (*Ans.* 6). We do not have this problem documented for the Old-Babylonians, but it would have been an easy exercise for them.

Thus we see that the *Nine Books* are on a high level, indeed.

As to the date of the *Nine Books*: the first notice of the work dates from 179 A. D., but the oldest manuscript is an edition from the middle of the third century, with commentary by LIU HIU, who says that the work was put together

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[21] According to the interpretation by NEUGEBAUER (MKT I, pp. 176, 187) of the Old-Babylonian text BM 85194, the volume of a truncated pyramid is given by $V = \left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right] h$. There is, indeed, a difficulty in this reconstruction: there is not enough space on the tablet for the computation of the $\frac{1}{3}\left(\frac{a-b}{2}\right)^2$. VAN DER WAERDEN (*op. cit.*, p. 75) suggests that the formula was $V = \left[\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2\right] h$, which is indeed wrong but agrees with the formula $V = \frac{1}{2}(a^2 + b^2) h$ of two other closely related texts. It is, however, difficult to imagine why anyone would want to go over from $\frac{1}{2}(a^2+b^2)$ to $\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2$; though one might want to go from $\frac{1}{3}(a^2+ab+b^2)$ to $\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2$ in order to use tables of squares. The presence of the formula $V = \frac{1}{3}(a^2+ab+b^2) h$ in Egypt *and* in China tends, we consider, to confirm its presence in Babylonia.

One may note that the formulae from Egypt and Babylonia are for the *truncated* pyramid and that, curiously, in the *Nine Books* (*V*, 10) the truncated pyramid is taken up before the pyramid. This suggests that, anciently, the truncated pyramid (and not, as in "scientific" times, the pyramid) was the starting point of the considerations. If, as is usually presumed, the pyramid was the starting point, one is led to wonder how the Egyptians got their formula from that for the full pyramid; but if the truncated pyramid was the starting point, then the necessity for such a reconstruction is removed.

[22] See PEET, *op. cit.*, problem 64, p. 107; and NEUGEBAUER & SACHS, *op. cit.*, pp. 100, 52.

[23] See, *e.g.*, MKT I, p. 177 and p. 259.

from older materials by CHANG T'SANG (fl. 165-142) in early years of the HAN dynasty (202 B. C. to 9 A. D.).

It is clear that from such notices we do not get the date of the contents of the *Nine Books*. It is not that we in the least doubt LIU HIU's word for it that CHANG T'SANG composed the work, but even LIU HIU says it was put together from "older materials", and it is the date (or dates) of the contents we would like to know.

For example, one might ask whether anything in the *Nine Books* is due to ARCHIMEDES (fl. 250 B. C.) or was influenced by his mathematics. Of course, had ARCHIMEDES lived after the HAN dynasty, we would have the answer, but the opposite is true: he flourished about 250 B. C. The trade route from Persia to China already existed in the second century B. C.;[24] so we have to concede that ARCHIMEDES' works could have gotten over to China in the early part of the HAN dynasty (and even without this route, we need not doubt the possibility). Thus from direct historical notices we cannot deny that ARCHIMEDES had an influence on the Chinese mathematics during the HAN dynasty.

On the other hand, if we look at the text itself, we see that there is nothing in it that is characteristically ARCHIMEDEAN: there is nothing in it, in the geometric part at any rate, which we cannot claim with good reason already to have been known by the Old-Babylonians. If one were to find on some newly recovered cuneiform tablet any geometric problem occurring in the *Nine Books* (except possibly the one on the sphere), no-one would be in the least surprised.

One can, perhaps, test this kind of textual criticism by a similar examination of the *Mishnat ha-Middot*, a Hebrew geometry composed about 150 A. D. by Rabbi NEHEMIAH (see footnote 17 above). This work is not, by far, on the level of the *Nine Books*, but contains some points of interest. In it we find $\pi_1 = \frac{22}{7}$ and $\pi_2 = \frac{22}{7}$ (briefly: we find $\pi = \frac{22}{7}$). Moreover the author goes out of the way to harmonize the $\frac{22}{7}$ with the 3 of I, *Kings*, 7, 23 and II, *Chronicles*, 4, 2. The value $\frac{22}{7}$ is, with as good grounds as we can hope for, ascribed to ARCHIMEDES: an ARCHIMEDEAN influence on the *Mishnat ha-Middot* is thus clear. Nor is this in the least surprising since 150 A. D. is after 250 B. C. and in 150 A. D. the largest Jewish community in the world was in Alexandria, the center of Greek mathematical study.

On the other hand, the *Nine Books* do not know the $\frac{22}{7}$, and always use the older 3. If ARCHIMEDES had influenced the work, surely this influence would have made itself evident in the problems on the circle.

We can say, then, with considerable assurance that the Nine Books, in particular that part of the work which relates to circles, is pre-Archimedean.

Let us come now to the main issue, namely, whether the Old Babylonians knew the basic relations between the area, diameter, and circumference of a circle.

In our geometry paper (*op. cit.*, p. 512), following VAN DER WAERDEN, we supposed the Egyptians knew that $\pi_1 = 4(\frac{8}{9})^2$ and that $\pi_2 = 4(\frac{8}{9})^2$ and concluded that they (or their forerunners) must have realized that $\pi_1 = \pi_2$, that this realization could only have come about by a geometric analysis, and hence that

[24] Lord RAGLAN, *How Come Civilization?*, pp. 63, 186.

Egyptian geometrical knowledge was not merely arithmetical. The main point was this last conclusion, which now we consider to have been established above; but now we are holding in doubt whether they knew that $\pi_2 = 4\left(\frac{8}{9}\right)^2$.

We continued: "The Babylonians used the formulas $A = \frac{C^2}{12}$ and $C = 3d$ from which it would appear that they knew $\pi_1 = \pi_2$, though here it is a bit more difficult to judge". We want to explicate this.

How did the Babylonian see that $A = \frac{1}{12} C^2$? Or is it possible that he wasn't supposed to see it? We mean: Perhaps the factor $\frac{1}{12}$ was found in much the same way that we find, say, the specific gravity of iron, namely, by an experiment. We have to allow this as a logical possibility, but there is not a shred of evidence that the Babylonians regarded geometry as an experimental science; pending the presentation of some such evidence, we may put this possibility aside. There remains, so far as we can see, only the possibility that $A = \frac{C^2}{12}$ is the transform of a relation standing closer to the intuition. Our guess is the formula $A = \frac{Cr}{2}$.

One often reads that the Babylonians took π as equal to 3. The 3 is considered to be a crude approximation to π, which, of course, it is (but whether that was the intention of the Babylonian is a different matter). Let us introduce π_1', π_2' as notation for approximations to π_1, π_2. Then from $\pi_1' = \pi_2'$ we cannot conclude $\pi_1 = \pi_2$. If π_1', π_2' are complicated expressions, even fractions, the conclusion $\pi_1 = \pi_2$ is plausible, but if π_1', π_2' are integers, the conclusion begins to lack plausibility: one has to face the possibility that π_1', π_2' are both but crude integral approximations independently arrived at. That is why we said that in the case of the Babylonians it was "more difficult to judge".

For example, one can imagine the circle to have been compared with the circumscribed square and estimated to be three-fourths of it (perhaps by averaging the inscribed and circumscribed square). In this way, the formula $A = \frac{3}{4} d^2$ could have resulted. A. P. JUSCHKEWITSCH has, indeed, envisioned such a comparison of the circle with the square on its diameter.[25] Actually, the formula $A = \frac{3}{4} d^2$ is nowhere documented in the Babylonian material at our disposal.[25a]

Such was the situation, as we understood it, at the time we wrote our geometry paper *(op. cit.)*. Now, however, a further, crucial, piece of information has come to our attention. In BM 85210 (Rs. I, 18) the area of a semi-circle is computed according to the prescription arc times diameter/4.[26] This confirms our guess

[25] *Geschichte der Mathematik in Mittelalter*, p. 57.

[25a] See, however, A. D. KILMER, "The Use of Akkadian *DKŠ* in Old Babylonian Geometry Texts," in studies presented to A. LEO OPPENHEIM, pp. 142–143 (= Böhl Collection, no. 1821), where the scribe clearly has in mind that the area between two concentric circles of radii R, r is given by $3(R-r)(R+r)$.

[26] MKT I, pp. 226, 232. In the mathematical text from Susa mentioned above in footnote 17, dating from the end of the first dynasty of Babylonia (*loc. cit.*, p. 5), there occurs a list of coefficients ("constantes fixes"); and one is told (p. 28) that "15 (*i.e.*, $\frac{1}{4}$) [is] the constant of the semi-circle." Following F. THUREAU-DANGIN's comment on BM 85290 (in *Textes Mathématiques Babyloniens*, p. 51), BRUINS & RUTTEN (*loc. cit.*, p. 31) take this to mean that the area is computed as $\frac{1}{4}$ arc times diameter. This indicates that the formula was standard.

that

$$A = \frac{C^2}{12} \text{ is the transform of } A = \frac{Cr}{2} \left(\text{and not of } A = \frac{\pi_1}{4} d^2\right).$$ [27]

Thus we need not doubt that the Babylonians knew that $\pi_1 = \pi_2$, or otherwise put, that area $= \frac{1}{2}$ circumference times radius, which is definitely testified for in the case of a semi-circle: if they had known it, how could they have expressed themselves better? Of course, they could have conveyed the information via a formula for a full circle also. But if they had, and then the result were combined with $C = 3\,d$, the issue would be obscured. It seems a plausible interpretation to say that the issue got lost in the formula $A = \frac{C^2}{12}$, but remained clear in the formula for the semi-circle.

Let us review the argument in terms of the relevant formulae. We have definite documentation in the cuneiform texts for:

(1) $$C = 3\,d$$

and

(2) $$A = \frac{C^2}{12},$$

and we have seen reason to suppose that (2) is the transform of a formula standing closer to the intuition. Replacing one of the factors C in (2) by $3\,d$, we get

(3) $$A = \frac{C\,d}{4},$$

or

(3′) $$A = \frac{C}{2} \times \frac{d}{2};$$

and replacing both factors C by $3\,d$, we get

(4) $$A = \frac{3}{4} d^2.$$

Conversely, from (1) and (3) (or (3′)), we can get (2); and likewise from (1) and (4) we can get (2). Both (3) (or (3′)) and (4) stand closer to the intuition than (2), though to see (3) would require considerable sophistication. Thus (2) is presumably the transform of (3) (or (3′)) or of (4). Although we need not doubt that the Babylonian could see (4), if we stick strictly to the evidence, we should prefer the implication (3)$\Rightarrow$(2) to (4)$\Rightarrow$(2). Anyway, this is our choice.

Our considerations on Babylonia have depended, so far, on Babylonian material only. Now we may compare with China. As has been made evident, the *Nine Books* have a Babylonian look—in saying this we do not intend to say that Chinese mathematics is a derivative of the Babylonian, or vice versa, but merely that they have a common source. Now in the *Nine Books* we find explicitly what has to be reconstructed for Babylonia, namely, the formula $A = \frac{C}{2}$ times $\frac{d}{2}$.

[27] Although BM 85210 was known to us when we wrote our geometry paper *(op. cit.)*, the significance of the computation area $= \frac{1}{4}$ arc times diameter occurred to us upon reading A. J. E. M. SMEUR's work "On the value equivalent to π in ancient mathematical texts. A new interpretation," *Archive for History of Exact Sciences*, vol. 6 (1970), especially p. 264.

Could it be that they said it but didn't mean it? Could it be, for example, that the formula is but a mnemonic device? Since the Chinese even have a correct formula for a sector, we think this way out is vain.

As to the deduction $(3) \Rightarrow (2)$, this is no longer an issue, since (3) is explicitly present, but even the implication is perhaps attested in the *Nine Books*. Although the *Nine Books* proceed from the simpler to the more difficult, we cannot be sure that the development was intended to be logical. Still, we note that in the section dealing with the area of a circle, the formula $A = \frac{C}{2}$ times $\frac{d}{2}$ comes first; and one might be tempted, then, to see the others as derivative of this and $C = 3\, d$. This does not imply that $A = \frac{C^2}{12}$ was not the favorite formula; and there is some reason to think it was, since in the rules for a circular cylinder and circular cone the rule $A = \frac{C^2}{12}$ is followed. Thus if the Chinese had worked out their examples, and their rules had been lost, we probably would see exactly what we see in Babylonia. And what we see in Babylonia we explain in precisely this way: the rules are lost, but the favorite formula remains clear.

The comparison of the Babylonian and Chinese mathematics not only confirms our conclusion that the Babylonians knew that $\pi_1 = \pi_2$, but it also confirms our conclusion that the circle geometry of the *Nine Books* is pre-ARCHIMEDEAN—indeed, by some 1500 years at least.

There remains the question of where the relation $C = 3\, d$ came from. One speaks of measurements, crude to be sure, of a circumference, but we are skeptical that such measurements took place: it implies that geometry was considered to be an experimental science, and we have not a shred of evidence for such a view.[28] As far as we *know*, the evaluation of π has always been a conceptual matter, and may have been so from its first approximation as 3. The first thought that comes to our mind is that the Babylonians considered a regular hexagon inscribed in a circle: that the Babylonians had observed that such a hexagon is made up of 6 equilateral triangles of side equal to the radius may be documented from Babylonian material.[29] The circumference of this hexagon is three times the diameter; and if the Babylonian had considered the hexagon to approximate the circle, he may have taken $C = 3\, d$ equally well for the circle. Or the formula might have resulted in some other similar way. If it had come as we have imagined, surely the Babylonian realized that $\frac{C}{d}$ was not $= 3$ but somewhat larger. How much larger? Our view is that the Old-Babylonian didn't care! In the course of attempting to square the circle, someone saw the great, new, and correct theorem that $A = \frac{Cr}{2}$; but to compute A (or to construct a square equal to it in area), one had to know $\frac{C}{d}$, and this was just as baffling as the original problem. One gave up—gave up exact geometrical thought! Geometry became a background, an excuse, for arithmetical and algebraical problems. There are many absurdities in

[28] Moreover, no experiment, however refined, could tell us that circumference/diameter is constant for all circles, but could at most give us an approximation to this constant once it has been decided that there is such a thing.

[29] See NEUGEBAUER, MKT I, p. 141 in reference to BM 15285; or BRUINS & RUTTEN, *op. cit.*, p. 23 and Pl. II.

Babylonian geometry; for example, the volume of the frustrum of a pyramid is computed as one half the sum of the bases times the altitude; or the areas of the regular pentagon, hexagon, heptagon inscribed in a circle of given radius are computed, each time assuming that the perimeter of the polygon is $3d$, *i.e.*, the perimeter of the circle; or the area of a quadrilateral is taken to be the average of one pair of opposite sides times the average of the other.[30] We view this not as "practical" but as degenerate. The ratio was some number about 3: the Babylonian (or a forerunner) took it to be 3 and proceeded unconcernedly with his main interest, his calculations.

Once the 3 had been fixed in tradition, it would be difficult to change. (Rabbi NEHEMIAH's courageous attempt to change 3 to $\frac{22}{7}$ was soon frustrated.[31]) Still, the 3 is such a poor approximation to π, and Babylonian mathematics was at such a high level, that one cannot help but think they must have had a better approximation: NEUGEBAUER felt this way.[32] BRUINS has maintained that the Babylonians did have $3\frac{1}{8}$ as such an approximation (for π_1); and NEUGEBAUER has accepted this opinion (though previously he had rejected this interpretation for similar findings).[33] In any event, the new value, if such it is, appears to have had no impact on mathematics itself.

To sum up:

The Chinese and the Old-Babylonians (or a forerunner) had a correct notion of the relations holding between the area, diameter, and circumference of a circle. Though the arithmetical work was approximative (and necessarily so), the work was based on a realization of two basic relations: (1) *that the ratio of the circumference of a circle to its diameter is the same for all circles; and* (2) *that the area of a circle is one half its circumference times its radius.*[34]

4. The Area of a Semi-Circle in Egypt

The most contested problem of ancient Egypt is Problem 10 of the Moscow mathematical papyrus (MMP 10). In 1930, at its first publication in modern times, STRUVE astounded the world by declaring that the Egyptians of the Middle Kingdom knew the correct formula for the area of a hemisphere.[35] This was not an off-hand opinion: the mathematical reconstruction is quite un-

[30] See footnote 21 above; BRUINS & RUTTEN, *op. cit.*, p. 32; and NEUGEBAUER & SACHS, MCT, p. 47.

[31] VAN DER WAERDEN, *op. cit.*, p. 33 and GANDZ, *op. cit.*, pp. 8–9.

[32] *The Exact Sciences in Antiquity*, 2nd ed. (1969), p. 46.

[33] NEUGEBAUER, *op. cit.*, pp. 47, 52 and MCT, p. 59, n. 152 k. See also BRUINS & RUTTEN, *op. cit.*, p. 33. VITRUVIUS has $\pi_2 = 3\frac{1}{8}$. See *The Ten Books of Architecture* (tr. by M. H. MORGAN), p. 301.

[34] The reader may wish to compare our conclusion with SMEUR's (*op. cit.*, p. 264): "Thus we can be sure the Babylonians were not familiar with a formula like $A = \pi R^2$. We have to admit that separate prescriptions existed for the calculation of the circumference of a circle, the area of a whole circle and the area of the semicircle, and that the Babylonians, surely at least in the beginning, were not aware of any relation between the numbers 3, 5′ and 15′ [*i.e.*, 3, $\frac{5}{60} = \frac{1}{12}$, and $\frac{15}{60} = \frac{1}{4}$], and certainly not that those numbers were connected by one and the same factor or proportionality, our number π."

[35] *Op. cit.*, in footnote 9 above.

convincing, but the paleographic study appears to be most meticulous, and even all the subsequent objections appear to some extent to have been anticipated.

The reason for the astonishment is that it indicated an altogether higher level to ancient mathematics, especially ancient Egyptian mathematics, than one might have expected from what was already known. To be sure, MMP 14, which was free of paleographic difficulties and gave the correct formula for the volume of a truncated pyramid, showed a high level, indeed;[36] but even granting the truncated pyramid—the Great Egyptian Pyramid, as E. T. BELL called it—the hemisphere indicates a higher level still. This judgement is a matter of mathematical sensibility: the problem of finding the area of a sphere is hard.

This is not to minimize lesser achievements. On the contrary, our object here is merely to show that the Egyptians (or their forerunners) knew that $\pi_1 = \pi_2$, or, better, that they knew $A = \frac{C}{2} r$; or, still move explicitly, that they conceived of the circle as divided up into a large number of small (and, say, equal) sectors which were assimilated to triangles. It may be said that this is "obvious"; but our attitude is that nothing is obvious (unless one looks, and then the question is, why does one look?). The Indian ritualists were vitally interested in the circle, but it never once occurred to them to think of the length of the perimeter—which was eventually looked at, not because it is obvious, but because one was trying to square the circle, or find its area. Moreover, there is no *clear* evidence, in any of the papyri, that the Egyptians thought of it either; and there is no *unclear* evidence, except possibly in MMP 10. This, then, is our object: to show that in MMP 10 a circumference, or rather, a semi-circumference, was calculated.

In 1931, PEET took up MMP 10 again, and gave a different, or, rather, two different, interpretations: (1) that the object spoken about was a semi-circle, and (2) that it was a semi-cylinder.[37] NEUGEBAUER gave still another interpretation, that the object was a paraboloid-shaped basket and that the formula was an approximation.[38] Through this diversity of opinion the mathematician interested in history but knowing scarcely a hieroglyph may get a better perspective on the paleographic difficulties.

Curiously, the semi-circle, which will mainly concern us, has dropped out of NEUGEBAUER's book (and paper), and also out of VAN DER WAERDEN's.[39] As already said, PEET's semi-cylinder and NEUGEBAUER's paraboloidal segment already give us the desired conclusion $\pi_1 = \pi_2$. As for STRUVE's interpretation, presumably any analysis leading to the correct formula for the area of a hemi-

[36] MMP 14 had already been published in 1917 by B. T. TURAJEFF in *Ancient Egypt*, pp. 100–102.

[37] T. E. PEET, "A Problem in Egyptian Geometry," Journal of Egyptian Archeology, vol. 17 (1939), pp. 100–106.

[38] *Vorlesunq über Geschichte der Antiken Mathematischen Wissenschaften. Erster Band. Vorgriechische Mathematik*, 1934; "Die Geometrie der egyptischen mathematischen Texte," Quellen und Studien zur Geschichte der Mathematik, Abt. B., vol. **1** (1931), pp. 413–451.

[39] NEUGEBAUER does mention the semi-circle in connection with STRUVE's rendering (where it is properly excluded); see *Vorgriechisch Math.*, p. 131; and *op. cit.* (1931), p. 424 (see footnote 38 above).

sphere, even if very crude, would involve the knowledge that $\pi_1 = \pi_2$. That is why we are mainly concerned with the semi-circle. Of course, there is the possibility that the formula is a guess signifying nothing.

Let us start with STRUVE's translation. Square brackets [ ] enclose restorations of damaged or lost passages in the papyrus; round brackets ( ) enclose commentary. (Pointed brackets $\langle\ \rangle$ will, following PEET, enclose words or passages which never stood in the papyrus, but whose omission there is due, or presumably due, to an error on the part of the scribe.)

STRUVE's translation runs:

Kol. XVIII

1. Form der Berechnung eines Korbes (*nb·t*),
2. wenn man dir nennt einen Korb mit einer Mundung (*tp-r*)
3. zu $4\frac{1}{2}$ in Erhaltung. O
4. lasz du mich wissen seine (Ober)fläche. Berechne
5. du $\frac{1}{9}$ von 9, weil ja der Korb (*nb·t*)
6. die Hälfte eines E[ies][40] ist. Es entsteht 1.

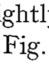
Kol. XIX

1. Berechne du den Rest als 8.
2. Berechne du $\frac{1}{9}$ von 8.
3. Es entsteht $\frac{2}{3}\ \frac{1}{6}\ \frac{1}{18}$. Berechne
4. du den Rest von dieser 8 nach
5. diesen $\frac{2}{3}\ \frac{1}{6}\ \frac{1}{18}$. Es entsteh[t] $7\frac{1}{9}$.

Kol. XX

1. Rechne du mit $7\frac{1}{9}$ $4\frac{1}{2}$mal.
2. Es entsteht 32. Siehe: es ist seine (Ober)fläche.
3. Du hast richtig gefunden.

PEET's translation into English of STRUVE's translation is: "Form of working out a basket. If they mention to you a basket with a mouth of $4\frac{1}{2}$ in preservation. Let me know its surface. Take a ninth of 9, since the basket is half an egg; result 1. Take the remainder, namely, 8. Take a ninth of 8; result $\frac{2}{3}+\frac{1}{6}+\frac{1}{18}$. Take the remainder of these 8 after (the subtraction) of this $\frac{2}{3}+\frac{1}{6}+\frac{1}{18}$; result $7\frac{1}{9}$. Reckon with $7\frac{1}{9}$ $4\frac{1}{2}$ times. Result 32. Behold, that is its surface. You have found rightly."

The first difficulty already occurs in Col. 18, line 1 with . (See Fig. 5.) It is agreed that this is a word and that the word is basket. But there is the possibility, already considered by STRUVE, that it is a technical term and, if so, could possibly mean semi-circle. That "basket" is, indeed, a technical term is definitely indicated in Col. 18, lines 5, 6, where we are told that a "basket" is "half of [ ]", half of something, but what the "something" is is for the most part destroyed. Now a basket itself is not, or would not be thought of as being, half of anything; rather it is the figure that "basket" denotes that is half of some other familiar figure. Thus that "basket" is a technical term is indicated; but let us for the moment accept STRUVE's opinion, in this regard not implausible, that "basket" means *hemisphere* (and the missing term *sphere*).

[40] The brackets here are supplied by us, those in Kol. XIX, line 5, by STRUVE.

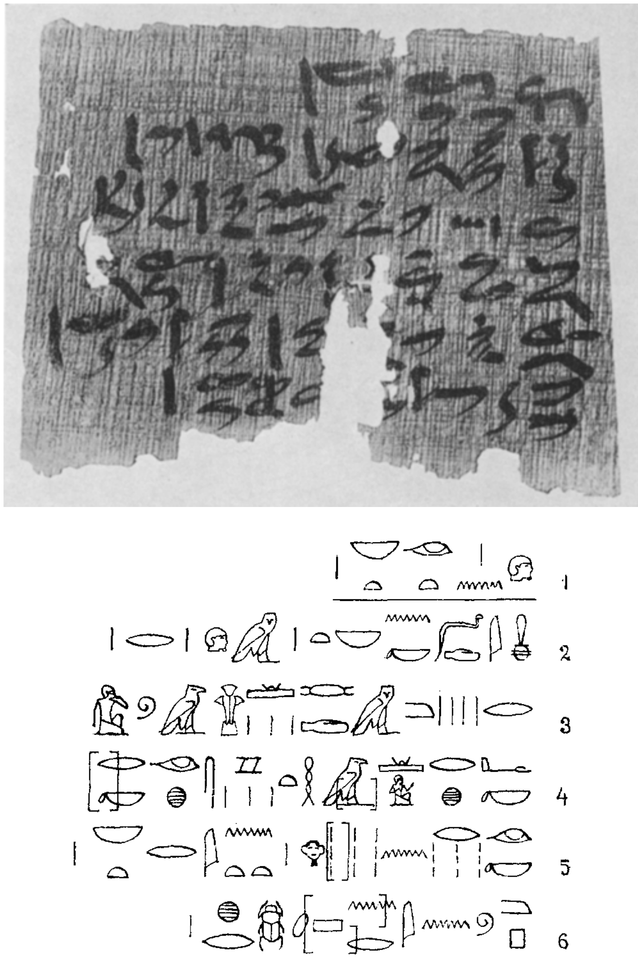


Fig. 5. Moscow Mathematical Papyrus, Problem 10

A second difficulty is, in PEET's words, "the 9 which unexpectedly turns up without explanation in line 5, where its sudden appearance is so disconcerting to Struve". Though the scribe almost never explains, the individual steps, no matter how simple, are usually given. Thus one expects a line: Take $4\frac{1}{2}$ two times. Result 9. This omission is what troubled STRUVE. He is hard-pressed to give another example, but points to MMP 23, where a like omission is claimed to occur: there numbers 10 and 5 make their appearance and it appears that the fact that 10 is twice 5, or that 10 divided by 5 is 2, is tacitly used. PEET has a different explanation of this problem, but the omission remains.[41] STRUVE also notes that in MMP 13 a whole complex of four operations is abbreviated; but in this case the problem is like an earlier one, MMP 9, where a detailed computation

[41] On MMP 23 see PEET's review of STRUVE's work in *J. Egyptian Archeology*, vol. **17** (1931), p. 158. *Cf.* STRUVE, *op. cit.*, p. 163.

is given. Moreover, STRUVE claims that the phrase "because a 'basket' is half of an X" shows that the Egyptians were also in possession of a formula for X—and in this we think STRUVE is surely right; MMP 10 is then like a missing problem for X (a sphere for STRUVE) and the idea is this: In computing the area of a sphere one would take 4 times the $4\frac{1}{2}$, *i.e.*, the diameter, but here one takes a half of 4 times the diameter since a hemisphere is half of a sphere. The compression, then, is somewhat more than the abbreviation of an operation. Still, the situation is most unusual: it is the one place in the papyri where a scribe cites a theorem; and the arithmetic is easy. So perhaps we can allow STRUVE the 9.

The main difficulty, however, are the words (lines 2, 3) which describe the basket:

$$nbt\ m\ tp\text{-}r\ r\ 4\tfrac{1}{2}\ m'\underline{d},$$

as PEET writes them, and which STRUVE translates as "einen Korb (*nbt*) mit einer Mundung (*tp-r*) zu (*r*) $4\frac{1}{2}$ in Erhaltung ($'\underline{d}$)". PEET (*cf.* footnote 37, above) has several criticisms of this translation that can be properly judged only by an Egyptologist—NEUGEBAUER agrees with them[42]—but there is one criticism that one can routinely check from a hieroglyphic transcription of the original (of MMP and the other mathematical papyri); and PEET himself calls this "the real rock on which Struve's rendering breaks up". Namely, that "*r* is never used in the mathematical papyri to introduce a dimension when only one dimension is given", though "it is used to introduce the second of two dimensions when two are given, and it then answers exactly to our 'by' in '6 feet by 3' ".[43] PEET considers this to be the clue to a correct interpretation of the passage, which he first restores as:

$$nbt\ \langle nt\ x\rangle\ m\ tp\text{-}r\ r\ 4\tfrac{1}{2}\ m\ '\underline{d},$$

"a basket (?) of x in mouth and $4\frac{1}{2}$ in $'\underline{d}$, where $'\underline{d}$, whatever it may mean, is the name of the second dimension given, just as *tp-r* is of the first." We are inclined to agree with PEET.

But even if we agree with STRUVE's translation, the question remains, as STRUVE realized, whether the Egyptians could have given a correct derivation of the formula: by "correct" we do not mean meeting our standards of rigor, but only meeting the Egyptians' standard. We reject STRUVE's suggestion for a derivation; and, moreover, cannot even imagine a derivation without the knowledge that $\pi_1 = \pi_2$. (STRUVE's derivation involves this.) If there were no derivation, the only remaining possibility is that the formula was a guess. Now *we* guess, so there is no reason to think the Egyptian could not have done likewise:

[42] *Op. cit.* (1931), p. 426, n. 54c.

[43] R. J. GILLINGS ("The Area of the Curved Surface of a Hemisphere," Australian J. of Science, vol. **30** (1967), p. 113) has rather freely translated PEET as saying that "the hieroglyph ○, read as '*r*' before the $4\overline{2}$ of line 3, is *always* used in the mathematical papyri as the equivalent of 'by', as in our modern '6 feet by 3 feet' ...". Taking exception, he adds that "it is also used in other senses, as (up to) RMP 40, 41, 42, 43, and 46; as (goes into) RMP 41, 44, and 45. It is also rendered (to), (for), (of), and (mouth)." All of these examples are irrelevant. The "*r*" of MMP 14, already discussed by STRUVE, where a truncated pyramid is said to be "6 in height, by 4 on the bottom side, by 2 on the top side," is also hardly a counter-example.

the question is whether the guess is based on an essentially correct understanding; if it encompasses only a little part of the truth, the guess signifies nothing.

Recently GILLINGS *(op. cit.)* has considered MMP 10 once more, and takes the view that the formula is a guess: he imagines the Egyptian to be looking at a hemisphere, a basket, and saying to himself that its area is greater than its opening—surely we can go along with that: If F is the area of a hemisphere and A the area of a great circle, then $F > A$. Then by an "inspired guess"—we would say: divinely inspired guess—the Egyptian judged that $F = 2A$.

Earlier, GILLINGS, in rejecting PEET's semi-circle says that "it removes in one fell swoop all the real difficulties of the problem and reduces the scientific and historical interest in it to almost nil." On the same basis, GILLINGS should have rejected his own considerations. But our disappointment over something having no value is hardly grounds for ascribing value to it, so we shall proceed in another way to show that the formula could hardly have been a guess for the area of a hemisphere.

Let us recall how the Egyptian computes the area A of a circle of diameter d. First he finds $\frac{8}{9}d$; but this he does not do in a single step: rather he first computes one ninth of d and subtracts the result from d—in a formula: $d - \frac{1}{9}d$. Then he squares the result: we may write $A = (\frac{8}{9}d)^2$, but this is an abbreviation for $A = (d - \frac{1}{9}d)^2$. The straightforward program for computing $2A$ is then

$$F = 2(d - \tfrac{1}{9}d)^2.$$

Instead, the Scribe proceeds according to the program:

$$F = [(2d - \tfrac{1}{9}2d) - \tfrac{1}{9}(2d - \tfrac{1}{9}2d)] \cdot d.$$

Note, first, that in the straightforward program, the doubling comes last, but in the actual program it comes first. What could have moved the Scribe to do this? Could it be that he took advantage of the fact that $d = 4\frac{1}{2}$ and that twice this is exactly divisible by 9? This could hardly be so. The Scribe was not working out problems on his precious papyrus: rather he was writing a text. It would, however, be in the spirit of textwriting for the Scribe to choose $d = 4\frac{1}{2}$ in order to simplify the first step. In other words, $d = 4\frac{1}{2}$ might be taken because doubling is the first step but doubling would not be the first step because $d = 4\frac{1}{2}$. Moreover, in the same spirit, he could have taken $d = 9$: the straightforward program $2(\frac{8}{9}d)^2$ would then proceed arithmetically as smoothly as possible and the idea that $F = 2A$ would have been conveyed in as clear a way as possible. The thesis that the formula was a guess, or even that the Scribe is conveying that $F = 2A$, thus leaves completely unexplained the transposition of the doubling operation. But let us write the straightforward program as $F = 2(\frac{8}{9}d)(\frac{8}{9}d)$ and let us assume, for no assignable reason, that this was rewritten as $(\frac{8}{9}2d)(\frac{8}{9}d)$, though the Scribe actually went over to $[\frac{8}{9}(\frac{8}{9}2d)]d$. What could have moved the Scribe to transpose the operation of multiplying by $\frac{8}{9}$ from the factor d to the factor $\frac{8}{9}2d$? That he can do so depends on the identity

$$[(2d - \tfrac{1}{9}2d) - \tfrac{1}{9}(2d - \tfrac{1}{9}2d)]\, d = (2d - \tfrac{1}{9}2d) \cdot (d - \tfrac{1}{9}d),$$

which can hardly be claimed to be obvious; but let us suppose the Scribe saw it. There still remains the question of why he applied it. Could it be that one program is computationally simpler than the other? This could hardly be so in general, but is it so even in the case at hand? The Scribe takes $\frac{1}{9}$ of 8. Result: $\frac{2}{3}+\frac{1}{6}+\frac{1}{18}$. He then subtracts this from 8. Result: $7+\frac{1}{9}$. The Scribe does not give the details, and the reader may wish to supply them in the spirit of Egyptian arithmetic. Though surely easy enough for an adept, they could well stump a novice.[44] On the other hand the calculation of $\frac{1}{9}$ of $4\frac{1}{2}$ in the program $(\frac{8}{9}\,2d)\cdot(\frac{8}{9}\,d)$ is simple enough. It would run:

$$\begin{array}{rcc} & 1 & 9 \\ & \frac{1}{2} & 4\frac{1}{2} \\ \hline \text{Answer:} & \frac{1}{2} & \end{array}$$

i.e., it takes one easy step Egyptian style to get the answer; after which one has to compute $4\frac{1}{2}-\frac{1}{2}$, obviously 4, and then multiply by 8 (the 8 of the $\frac{8}{9}\,2d$), to get 32. Therefore the alternate program chosen by the Scribe is much more complicated then the modified straightforward program. The shift of the $\frac{8}{9}$ operation also remains unexplained.

Thus *the hypothesis that the formula of MMP 10 is a guess for the area of a hemisphere, or, indeed, that it is a formula conveying that any area is twice the area of a circle, is excluded on arithmetical grounds alone.*

Let us proceed to the other interpretations of MMP 10, which we shall consider in historical order; so PEET's semi-circle is next. The first difficulty that PEET has to meet is that a basket is a 3-dimensional object, and a first presumption may well be that "basket" must refer to a 3-dimensional figure.

PEET himself is not disturbed by the suggestion that | 𓎟, "basket", could be a term for a semi-circle. Having paleographically disposed of STRUVE's "e[gg]", at least to his own satisfaction, he writes (*op. cit.*, p. 103):

"Let us now cut ourselves free from the assumption that the figure is a hemisphere and see where the data leads us .... The figure is written with the word *nbt*, a word means "basket", but which in this case, where we are dealing with geometry, must not necessarily be assumed to bear its literal meaning, though we should certainly expect it to represent some object of which the sign 𓎟 itself is not an unreasonable picture.

There appear to be two possibilities, according as we take the figure to be in two dimensions or in three. In the first case we have the semi-circle and in the second the semi-cylinder ...".

Although PEET was not disturbed over the 2-dimensional suggestion, apparently NEUGEBAUER was, for he dismissed the whole idea without a word.[45] It may be in order, then, to pursue this question a bit farther. First, in general terms, we need not expect a technical term to cover all the meaning of an older word from which it derives. An older word is put to a new use by, for example, narrowing down its meaning. To get away from generalities and at the same time provide some evidence, we can give an exactly parallel case from the *Nine Books*: there the word "basket" is used to denote an isosceles trapezoid, and, as

[44] For how the arithmetical details might have gone, see GILLINGS, *op. cit.*, p. 115.

[45] As already remarked, he does mention it in connection with STRUVE's rendering; see footnote 39.

VOGEL notes (*op. cit.* 1968, p. 30, n. 1,) the isosceles trapezoid is the section—or as one might say in draughtman's terminology: the "characteristic" view—of a basket having the form of a truncated cone. (We may remark that the Babylonians used "basket" to stand for the truncated cone itself.[46]) Thus in China we find precisely the phenomenon that some people find it so hard to contemplate.

PEET had restored lines 2, 3 as: "a basket(?) of x in mouth and $4\frac{1}{2}$ in *'ḏ*". That x is the diameter of a circle is clear enough, but no linguistic analysis can supply the value of x. This must come from the proposed interpretation itself. There are four cases: (a) for the sphere, $x=4\frac{1}{2}$; (b) for the semi-circle, $x=9$; (c) for the semi-cylinder, $x=4\frac{1}{2}$; (d) for the paraboloidal segment, $x=4\frac{1}{2}$.

Continuing his considerations for the semi-circle, PEET writes: "This construction has the very great advantage of bringing in as the first datum the figure 9 which seemed to occur so entirely without explanation in line 5." We agree. But no sooner had PEET written this down than he began to worry: "On the other hand it has one grave disadvantage, since it requires us to suppose that the Egyptian here gave two measurements, diameter and radius, of a semicircle, when one would have sufficed". Here we think PEET is uselessly worrying over nothing. The supplying of superfluous data is in itself not illogical. Moreover, we have seen that in the *Nine Books* there are several over-determined problems on the circle: the circumference and the diameter are both given. Besides, in the case of MMP 10, it is not clear that there is any superfluous data. If "basket" meant *circular segment*, then it is not known to be a semi-circle until we know that its "mouth" is twice its *'ḏ*. In the *Nine Books* we have precisely this situation: problems 35 and 36 of Book I speak of a "bow-shaped field"; in problem 35 the chord is 30, the arrow 15 (and so the field is a semi-circle), in problem 36 the chord is $78\frac{1}{2}$, the arrow $13\frac{7}{9}$. There is no separate term for a semicircle. The same appears to hold for Babylonia.[47] And it could have been just the same in Egypt. This does not conflict with the information that "the basket is half of a [ ]", which we learn only after the dimensions have been given.[48]

The cited passage from PEET continues: "For my own part I am not prepared to dismiss this possibility [*i.e.*, of superfluous data] out of hand. Egyptian mathematics was a very concrete and practical science, and a semi-circle was a plane figure which might for every-day purposes be regarded as having, like other plane figures, two measurements, length (*tp-r*) and breadth (*'ḏ*). Is it unthinkable that on the basis of this popular view of the figure there should exist a practical rule for finding the area of a semi-circle which proceeded not by halving the area of the complete circle, but by taking $\frac{8}{9}$ of $\frac{8}{9}$ of the diameter (length) and multiplying it by the radius (breadth)?"

Yes, we find this "unthinkable". A semi-circle is a circular segment, so what is still "thinkable" is that the semi-circle is worked out as a special case of a circular segment. But the Scribe says that the computation depends on the

[46] O. NEUGEBAUER & W. STRUVE, "Über die Geometrie des Kreises in Babylonien," *Quellen und Studien zur Geschichte der Math.*, Abt. B, vol. **1** (1929), pp. 86–88.

[47] See the remark on b_2 in MKT I, p. 230. The *Mishnat ha-Middot* uses the same term for semi-circle and for segment; *cf.* GANDZ, *op. cit.*, p. 13, n. 6.

[48] STRUVE uses the definite article ("der Korb") in line 5, PEET the indefinite.

formula for a circle. So one has to explain why he did not directly apply the formula $F=(\frac{8}{9}\,d)^2$, known from RMP 50; or one has to face the possibility that he was applying a different formula for the area of a circle.

Could it be that PEET was missing the simple observation that $\frac{8}{9}$ of $\frac{8}{9}$ of the diameter is (assuming $\pi_1=\pi_2$) one half the semi-circumference? This seems to be the case.

PEET continues: "Supposing for a moment that the figure really is a semi-circle—what is the force of the words of ll. 5–6: 'Take a ninth of 9, since a semi-circle is half a [circle]'? The phrase which begins with 'since' must explain either the figure 9, or the step as a whole. Now the 9 needs no explanation, being one of the data, and the words must therefore be taken as explaining why the procedure of taking a ninth (as a preliminary to taking $\frac{8}{9}$) associated with obtaining the area of a circle is here adopted."

But do they explain it? If we understand PEET correctly, he is suggesting that the passage is alluding to the formula $(\frac{8}{9}\,d)^2$ for the area of a circle, so that the passage has only an allusive and not an explanatory force. The transformation of the straightforward program $\frac{1}{2}(\frac{8}{9}\,d)^2$ to the program $[\frac{8}{9}(\frac{8}{9}\,d)]\,r$ remains unexplained.

To come to our explanation: our view, or hypothesis, is that the Scribe is, indeed, referring to the area of a circle, but not to the formula $F=(\frac{8}{9}\,d)^2$, rather to the formula $F=\frac{1}{2}$ circumference times radius, which as a program reads: $[\frac{8}{9}(\frac{8}{9}\,2d)]\cdot r$. In computing the area of a circle according to this formula, the first step is to multiply d by 2; and the Scribe is explaining that this preliminary multiplication is not necessary, "since a semicircle is half a circle".

Our hypothesis also explains the program actually adopted. It is true that this program is computationally more involved than taking $\frac{1}{2}(\frac{8}{9}\,d)^2$, but the Scribe is not, any more than we in our calculus courses, merely trying to show the student how to get the right answer, but is trying to convey the idea that the area, of a circle or a semicircle, can be obtained as $\frac{1}{2}$ arc times radius. It is even conceivable that the Scribe expects the student to know the answer, namely 64, to the familiar problem: "Find the area of a circle of diameter 9", so that the obvious answer 32 can then serve as a check on the new idea: surely this is the way we would proceed.

To repeat: we bring in our hypothesis to explain why the program $[\frac{8}{9}(\frac{8}{9}\,d)]\,r$ rather than the program $\frac{1}{2}(\frac{8}{9}\,d)^2$ was followed. There may be some other explanation, one not using our hypothesis, but what it could be, we cannot imagine.

Our explanation implies that the Egyptian knew the formula $C=\frac{8}{9}(\frac{8}{9}\,2d)$ for the semi-circumference of a circle. How would he have gotten this? He knows that $F=(\frac{8}{9}\,d)^2$ and that $F=C$ times $\frac{1}{2}\,d$. So he has to divide $(\frac{8}{9}\,d)(\frac{8}{9}\,d)$ by $\frac{1}{2}\,d$, whence he gets $C=\frac{8}{9}(\frac{8}{9}\,2d)$. This requires some algebra, to be sure; but he has the motive and he knew some algebra (for example, in MMP 6 he solves the problem: Given the area F of a rectangle and the ratio $b:l$ of the breadth to the length, to find the breadth and the length).

The formula $F=$ semi-circumference times radius does not flow from the formula $F=(\frac{8}{9}\,d)^2$: it requires new concepts and a new analysis of the area of a circle; the second is a direct squaring of the circle, the first is not. The two formulae, then, are different in style, and one is tempted to ascribe them to different traditions, or at least different periods. But traditions can cross, and in this case could have done so without conflict.

Although we think PEET lost the way at a couple of vital points, still he deserves great credit for suggesting that the "basket" of MMP 10 is a semi-circle. This was no off-hand polemical response to STRUVE's work of 1930. Already in 1923 PEET was in possession of photographs of the Moscow papyrus, sent to him on the reasonable understanding that he should not publish their contents. He thus had ample time to come to an independent conclusion and, according to his own statement, did so (*op. cit.*, p. 100).

We can now be brief with the remaining possibilities. For the semi-cylinder, as already noted, the diameter x equals $4\frac{1}{2}$. Here the difficulty is, again, that "the working ought to have begun with the multiplication of $4\frac{1}{2}$ by 2 to get 9". There is no way to meet this difficulty, except by shrugging it aside as not very significant.[49]

NEUGEBAUER, too, notes (*op. cit.*, 1931, p. 427) that the semi-cylindrical basket would have no (semi-circular) sides. This is surely worth noting, but can hardly be decisive. But we need not insist on these difficulties, since if we accept the semi-cylinder, we surely have the desired conclusion that the Egyptians had $\pi_1 = \pi_2$.

As to the paraboloidal segment, NEUGEBAUER (*loc. cit.*) considers that "basket" means *basket*, *i.e.*, it is not a geometrical term, but literally denotes a basket, and that the formula is a "crude" approximation to its area. Here the diameter x is $4\frac{1}{2}$ and there is another measurement $a = 4\frac{1}{2}$. To give force to lines 5–6, however, he is obliged to consider the "basket" purely as a mathematical figure. Besides, the 9 remains underived. Again we need not insist on the difficulties, as we would anyway get $\pi_1 = \pi_2$. Moreover, the infinitesimal analysis NEUGEBAUER offers is like the one we have in mind for the circle: indeed, he says that he got the idea after perusing COLEBROOKE's *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*, p. 88, where exactly the analysis we have in mind for a circle is given.

To set out our conclusions:

The area being computed in Problem 10 of the Moscow mathematical papyrus is that of a semi-circle. The work is based on the formula F = semi-circumference times radius, and only indirectly on the formula $F = (\frac{8}{9}\,d)^2$, for the area of a circle. Thus the Egyptians knew the basic relations between the area, diameter, and circumference of a circle.

5. The Area of a Semi-Circle in Greece

The father of Greek geometry is THALES, but the accomplishments which have been attributed to him have been evaluated in quite contrasting ways. According to HERODOTUS, THALES predicted a solar eclipse, nowadays presumed to be the eclipse of 585 B. C. XENOPHANES is said to have voiced his admiration of THALES for this prediction. VAN DER WAERDEN (*op. cit.*, p. 86) accepts the

[49] PEET adds rather lamely that STRUVE's interpretation suffers from the same difficulty.

attribution, and argues that therefore THALES, who according to all our sources was the first Greek astronomer, must have had some knowledge of Babylonian astronomy. NEUGEBAUER, on the other hand, thinks that the story about THALES prediction is no more reliable then the story that ANAXAGORAS predicted the fall of meteors (*Exact Sciences*, p. 142).

As to geometry, we have the same contrast in opinion. Our actual source is PROCLUS (4^{th} century A. D.), but PROCLUS derived his information from the history of EUDEMUS (4^{th} century B. C.), a work now unfortunately lost. VAN DER WAERDEN argues that EUDEMUS (and PROCLUS) are quite reliable. NEUGEBAUER, however, cannot see "a single reliable element in any of these stories which have become so dear to the histories of science".

VAN DER WAERDEN pictures THALES as getting wind of the Egyptian and Babylonian mathematics. "At the time of Thales", he writes, "the Egyptian and Babylonian mathematics had long been dead wisdom. The rules for computing could be deciphered and shown to THALES, but the train of thought which underlay them was no longer known. From the Babylonians he might hear that the area of a circle is $3r^2$, while the Egyptians asserted that it is $(\frac{8}{9}2r)^2$ ... ."

In this part, at any rate, NEUGEBAUER agrees, at least in a general way: although "there is nothing to do but to admit that we have no idea of the role which the traditional heroes of Greek science played", still "it is rather obvious that early Greek mathematics cannot have been very different from the Heronic Diophantine type", he writes (*op. cit.*, p. 148).

The images thus brought to mind are very plausible and easy to believe; yet it would be very difficult to offer anything quite definite as documentary evidence. Any formula of the "Heronic" type involves a unit, but the unit is notoriously absent in Greek classical geometry. PLATO in his dialogue on THEAETETUS mentions a square "one foot" on a side; and EUCLID X, Def. 3 speaks of an "assigned straight line" (*i.e.*, a line chosen as unit); but this is about as close as one can get to documentary evidence for a unit in classical Greek geometry.

However, there are grounds for believing that the unit was expunged from Greek mathematics! This would explain our lack of evidence for an early "Heronic" type of geometry in Greece. Still this is not evidence for its existence, but merely an argument for it. The history of Greek mathematics is not merely a descriptive matter, but is really a theory.

According to PROCLUS, following EUDEMUS, THALES was the first: (1) to prove that a circle is divided into two equal parts by its diameter, (2) to observe that the base angles of an isosceles triangle are equal (or, as had been more anciently said, "similar"), (3) to discover that when two lines meet, the opposite vertical angles are equal, and (4) to realize the truth of the congruence proposition Euclid I 26.

Point (1) is surely astounding! As HEATH noted, this is not proved even in EUCLID. Rather, in I, Def. 17, EUCLID merely asserts it.

One cannot help wondering what was the context for the proof of (1). If VAN DER WAERDEN is right, we may imagine that THALES was shown formulae for the circle and for the semi-circle. He would have been puzzled by the differences, and put things straight by proving (1). But we gladly concede that here we are on the ground of pure speculation.

On the other hand, as NEUGEBAUER suggests, propositions of a later age were attributed to heroes. Now with HIPPOCRATES OF CHIOS (430 B. C.) we definitely see the semi-circle enter classical geometry: he observed that the semi-circle on the hypotenuse of an isosceles right-angled triangle is the sum of the semi-circles on the sides. Since in his theory of lunules he was careful in one case to prove that an arc was greater than a semi-circle, and in another that it was less, point (1) conceivably arose in connection with this theory. Probably NEUGEBAUER would prefer this explanation (*cf. op. cit.*, p. 148).

6. The Area and Circumference of a Circle in Greece

The problem in this section will be to see how far back we can place a knowledge of $\pi_1 = \pi_2$ in Greece.

By "knowledge" we mean knowledge with reference to the local standard of rigor, and not with reference to our own. No one will dispute that the Babylonians "knew" that circumference/diameter is constant over all circles (though one will probably not get an easy agreement on how they knew it).

A corollary of this definition of knowledge is that something may be known at an earlier time and not known at a later, even if we confine ourselves to one culture and a generally advancing one at that; for the standard of rigor may go up without the technique being able to keep pace.

The main difficulty is that most of the pre-EUCLIDEAN mathematical works are lost. This is not due merely to the ravages of time, but is due in part to the nature of advancing knowledge: the excellence of EUCLID's *Elements* made previous compilations obsolete. We have only two pre-EUCLIDEAN mathematical works (of AUTOLYCUS) intact; for the rest we have a few fragments and passing references.

Still, a host of scholars have with great perspicuity reconstructed the pre-EUCLIDEAN history, back to HIPPOCRATES OF CHIOS (430 B. C.), the PYTHAGOREANS (of about 500 B. C.), and to some extent to THALES (585 B. C.). As an example, consider the history of the theory of proportionality. Greek mathematics knew three different definitions of proportionality, the second and third arising from inadequacies in their predecessors and giving rise to wide repercussions in the whole of mathematics. VAN DER WAERDEN *(op. cit.)* has given an excellent account of this development. It will be convenient for us to sketch this development briefly, as it will help to give a perspective on pre-EUCLIDEAN mathematics.

In the beginning, the concept of proportionality in geometry was the same as that which underlay the PYTHAGOREAN theory of numbers: four magnitudes are proportional if the first is the same part, or parts, or multiple of the second that the third is of the fourth. This is a perfectly good definition as far as the theory of (whole) numbers is concerned and retained its position in Book VII, Definition 20 of EUCLID's *Elements*, but it is not suitable for the comparison of line segments. If it were true that any two line segments had a common measure, then the definition would have been adequate, but the PYTHAGOREANS discovered, to some distress to themselves (so it is said), that even the diagonal of a square and its side do not have a common measure. This caused a crisis in geometry as the whole subject had to be built up anew.

To take a simple example, consider a rectangle $ABCD$ and a line parallel to one side, say AB, cutting the rectangle into areas U and V and a perpendicular side correspondingly into lengths u and v; and consider the theorem that U is to V as u is to v. The proof in the early part of the fifth century B. C. would have run: let w be a common measure of u and v, going m times into u and n times into v. Then the rectangle built on w and having the perpendicular side equal to AB would be a common measure for U and V, going m times into U and n times into V, from which the desired conclusion follows. But the definition had to be abandoned, and with it the proof. If the theorem was to be saved, a new definition was needed.

The third definition, the one due to EUDOXUS (370 B. C.), we know very well and see in action in *The Elements*, but of the second we manage to catch only a glimpse. This we get from a remarkable passage in the *Topica* of ARISTOTLE (158b):

"It appears also in mathematics that the difficulty in using a figure is sometimes due to a defect in definition, *e.g.*, in proving that the line which cuts the area parallel to one side (of a parallelogram) divides similarly both the line which it cuts and the area; whereas if the definition be given, the fact asserted becomes immediately clear; for the areas have the same antanairesis as have the sides: and this is the definition of 'the same ratio'."

But what is the antanairesis? As VAN DER WAERDEN explains (*op. cit.*, p. 176):

"The lexicon derives *ἀνταναιρεσις*, deduction, from the verb *ἀντ-αν-αιρειν*, subtract, literally, 'balancing against each other', which is used especially for sums of money, for instance in drawing up the balance sheet. The commentator Alexander of Aphrodisias adds here that by antanairesis, Aristotle means the same as by anthyphairesis. This brings us some help, because in Euclid VII 2 and X 2, 3 the verb *ἀνθυφαινειν* means 'to take away in turn' the smaller of two numbers or line segments from the larger one, for the determination of the greatest common divisor."

The idea is this: if two line segments AB, CD have a common measure d, and if, say, $AB > CD$, then AB, CD, and their difference EF will also have d as common measure. Now we can repeat the argument for CD and EF; and in this way, continuing, we can find the greatest common measure. If AB, CD are whole numbers, the same idea leads to their greatest common divisor, say d. This is the famous EUCLIDEAN algorithm for finding the greatest common divisor (also known to the Chinese in the *Nine Books*). If $AB = md$ and $CD = nd$, then, thinking of d as a big unit, one sees that the algorithm for m to n is parallel to that for AB to CD. Thus one sees that, for integers, AB, CD, UV, WX will be proportional if and only if AB, CD have the same antanairesis as UV, WX. Now keeping this part of the notion of proportionality and generalizing, we get the second definition, the one referred to by ARISTOTLE.

The trouble with this definition is that proving the theorem of the interchange of means gives difficulty: the theorem says that if a, b, c, d are four quantities of the same type and if a is to b as c is to d, then a is to c as b is to d. Now if a, b, c, d are line segments one can prove that a is to b as c is to d if and only if the rectangle on the means (b and c) is equal to the rectangle on the extremes (a and d), from which the theorem on the interchange of means follows.[50] If a, b, c, d are not line segments, one would (presumably) first have to find line segments a', b', c', d' proportional to them, apply the theorem to a', b', c', d', and then return to a, b, c, d. This is a reconstruction, of course, but that it is close to the truth we can see from another passage of ARISTOTLE (*Anal. Post.* 15):

"Formerly," says ARISTOTLE, "this proposition was proved separately for numbers, for line segments, for solids, and for periods of time. But after the introduction of the general concept which includes numbers as well as lines, solids, and periods of time" (namely, the general concept of magnitude), "the proposition could be proved in general."

Here ARISTOTLE is referring to EUDOXUS' theory of magnitude. We need not recall EUDOXUS' definition of "equal ratio," given in V, Def. 5. of *The Elements*. What is important to have in mind is that two quantities have a ratio if they are "capable, when multiplied, of exceeding one another;" and that the ratio for new types of figures is investigated by approximating them by figures whose properties are already known. Thus, in proving that circles are to each other as the squares on their diameters (*Elements*, XII, 2), one approximates the circles by regular polygons; in proving that pyramids of the same height and having triangular bases are to each other as the bases (XII, 5), one approximates the pyramids by sums of prisms.

In EUDOXUS' method, one works with inequalities; from the famous fragment on lunules, one knows that HIPPOCRATES already worked with inequalities.

In this same fragment one learns that HIPPOCRATES "considered as the foundation and as the first of the propositions which serve his purpose, that similar segments of circles are in the same ratio as the squares of their bases. He demonstrated this by showing first that the squares of the diameters have the same ratio as the circles."

According to VAN DER WAERDEN (*op. cit.*, p. 132) "it is still an open question whether Hippocrates actually proved this rigorously". He does not say why one should be skeptical, though he adds that the proof in EUCLID XII, 2 comes from EUDOXUS. Perhaps the difficulty is to imagine what the proof of HIPPOCRATES could have been. Let A_1, A_2 be two circles and d_1^2, d_2^2, the squares on their diameters. Then by an antanairesis, one could show that $A_1 : d_1^2 = A_2 : d_2^2$; and then by an interchange of means that $A_1 : A_2 = d_1^2 : d_2^2$. Though the antanairesis of two areas is conceptually harder to visualize than an antanairesis of two line segments, it would not have been beyond HIPPOCRATES' range. Still, we do not make the suggestion with great confidence, as the idea is not in the Greek style.

(The cited text continues: "For the ratio of the circles is that of similar segments, since similar segments are segments which form the same part of the circle." In symbols: if Σ_1, Σ_2 are similar segments of circles A_1, A_2 on chords c_1, c_2, then $A_1 : A_2 = \Sigma_1 : \Sigma_2$ since $\Sigma_1 : A_1 = \Sigma_2 : A_2$. We may note in passing that

[50] For more details, see VAN DER WAERDEN, *op. cit.*, pp. 177–178. There is a difficulty in proving: *P. If in a proportion the consequents are equal, then the antecedents are equal as well.* O. BECKER, "Eudoxus-Studies I," *Quellen und Studien*, Abt. B., vol. **2** (1933), p. 320, has shown how to meet this difficulty in a pre-EUDOXIAN way. (On p. 320, line 16, *instead of* $r_1' = r_0 - z_1 D \leqq r_0 - D$ *read* $r_1' = b - z_1(r_0 + D) = r_1 - z_1 D \leqq r_1 - D$.)

The Elements contains no *proposition* like $\Sigma_1 : A_1 = \Sigma_2 : A_2$, whether for circular segments or any other kind of areas. Thus one will not find a proposition stating that a triangle is to a square on a side as any similar triangle to the square on the corresponding side, though in VI, 19 one finds that two similar triangles are to each other as the squares on corresponding sides. The difference is trifling in view of the Theorem on the Interchange of Means, but must have some historical significance.)

Having seen how meagre are the materials for penetrating the history of pre-EUCLIDEAN mathematics, let us see what can be said for the area and circumference of a circle.

The first explicit statement on the circumference occurs in ARCHIMEDES' *Measurement of a Circle.*[51] Proposition 1 says that the area of a circle is that of a right triangle one of whose sides is the radius and the other the circumference; or, as we would say: $A = \frac{Cr}{2}$. This, together with EUCLID XII, 2, gives us the basic fact ($\pi_1 = \pi_2$) about circles.

ARCHIMEDES' proof is a typical double *reductio ad absurdum* argument.[52] In the course of it he uses that the perimeter of a (regular) inscribed polygon is less, and the perimeter of a (regular) circumscribed polygon is greater, than the perimeter of the circle. The reason is not given, but the assertion on the circumscribed polygon is Proposition 1 of *On the Sphere and Cylinder* I, a presumably earlier but at any rate allied work.

The next documentary evidence concerns the curve called *quadratrix.* This curve can be described, following PAPPUS (320 A. D.), as follows (see Fig. 6):

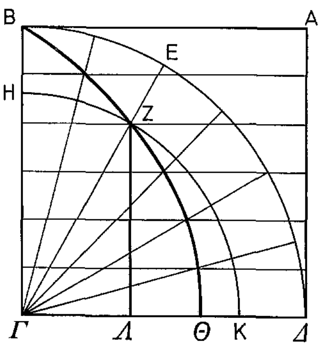


Fig. 6

Describe a circular arc $BE\Delta$ about Γ in a square $AB\Gamma\Delta$. Let the straight line ΓB rotate uniformly about Γ so that B describes the arc $BE\Delta$, and let the line BA move uniformly towards $\Gamma\Delta$, remaining parallel to $\Gamma\Delta$. Let both uniform motions take place in the same time, so that both ΓB and BA will coincide with

[51] See T. L. HEATH (Tr.), *The Works of Archimedes.*

[52] For brevity we refer to this technique as EUDOXIAN, but we do not intend to say it was original with EUDOXUS.

$\Gamma\Delta$ at the same moment. These two moving lines intersect in a point which moves along with them and which describes a curve $BZ\Theta$ [the so-called *quadratrix*]. If ΓZE is one definite position of the rotating line and Z the point of intersection with the line which moves parallel to itself, then, according to the definition, $B\Gamma$ will be to the perpendicular $Z\Lambda$ as the entire arc $B\Delta$ is to the arc $E\Delta$.

From the point of view of a draughtsman, there is no difficulty in constructing, *i.e.*, drawing, a quadratrix.

According to PROCLUS, HIPPIAS OF ELLIS (420 B. C.) had investigated, and presumably invented, the quadratrix. There is no difficulty in believing this, as the mathematics for the construction of the curve is quite simple.[53] PROCLUS says that HIPPIAS "derived the symptom" of the quadratrix. By the *symptom* of a curve, the ancients meant the condition which a point has to satisfy to lie on the curve, roughly then, the equation of the curve.

If we take $\Gamma\Delta$, ΓB of PAPPUS' description as x, y axes of a coordinate system, and place $\Gamma B = a$, then the quadratrix has the equation: $y = x \tan \frac{\pi}{2a} y$; here $\pi = \pi_2$. HIPPIAS would not have written the symptom in this way, but we need not doubt that he could have done it in some equivalent way.

According to PAPPUS, DINOSTRATUS (350 B. C.), the brother of MENAECHMUS, used the quadratrix to square the circle; and he gives the mathematical details for doing this. If one were to ask where (Θ) the quadratrix meets the x-axis, then one would be lead to study $y/\tan \frac{\pi}{2a} y$ as y goes to zero. Since $\theta/\tan\theta$ goes to 1 as θ goes to zero, one sees that $y/\tan \frac{\pi}{2a} y$ goes to $\frac{2}{\pi} a$ as y goes to zero. Thus $\Gamma\Theta = \frac{2}{\pi} a$. In effect, PAPPUS establishes this.

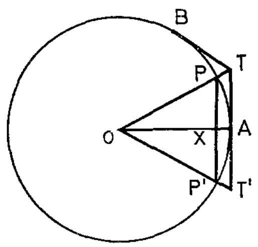


Fig. 7

PAPPUS's argument is a double *reductio ad absurdum* argument: if the asserted proportion arc $\Delta EB : B\Gamma = B\Gamma : \Gamma\Theta$ does not hold, then the fourth proportional ΓK is either greater or less than $\Gamma\Theta$. If $\Gamma K > \Gamma\Theta$, then it is established that $Z\Lambda = \text{arc}\, ZK$, which is declared to be absurd. In Fig. 7, this amounts to a situation in which $XP = \text{arc}\, AP$. Similarly, for the assumption $\Gamma K < \Gamma\theta$, PAPPUS obtains the conclusion arc $AP = AT$, which is also declared to be absurd.

[53] The quadratrix obviously brings angular measure directly into relation with linear measure. Hence it has been presumed that HIPPIAS invented the quadratrix in order to trisect the angle. This is not a universal opinion, however.

HEATH observes:

"The ... proof is presumably due to Dinostratus (if not to Hippias himself), and, as Dinostratus was a brother of Menaechmus, a pupil of Eudoxus, and therefore flourished about 350 B. C., that is to say, some time before Euclid, it is worthwhile to note certain propositions which are assumed known. These are, in addition to the theorem of Euclid VI, 33, the following: (1) the circumferences of circles are as their respective radii; (2) any arc of a circle is greater than the chord subtending it; (3) any arc of a circle less than a quadrant is less than a portion of the tangent at one extremity of the arc cut off by the radius passing through the other extremity. (2) and (3) are of course equivalent to the facts that, if α is the circular measure of an angle less than a right angle, $\sin \alpha < \alpha < \tan \alpha$."

There are two opposite views that one can take of this: (1) that PAPPUS is entirely reliable, from which one plausibly concludes that DINOSTRATUS knew the basic facts concerning a circle, or (2) that no one before ARCHIMEDES knew that $\pi_1 = \pi_2$, and hence that PAPPUS is unreliable. Of course, it is possible that DINOSTRATUS squared the circle, as PAPPUS says, but with a different proof.

A. J. E. M. SMEUR has adopted the second view (*op. cit.*, p. 258). He emphasizes that we have only three direct references to the quadratrix from ancient times: that of PAPPUS, which mentions DINOSTRATUS but not HIPPIAS; that of PROCLUS (450 A. D.), which mentions HIPPIAS and NICOMEDES (240 B. C.) but not DINOSTRATUS; and that of IAMBLICHUS (4th century, A. D.), which mentions NICOMEDES but neither HIPPIAS nor DINOSTRATUS. We may, perhaps, also include SPORUS (3rd century, A. D.), whose objections to the quadrature were mentioned by PAPPUS, and who was obviously PAPPUS's source. It is well to be thus reminded of the scantiness (and lateness) of the testimony.

Let us examine other parts of SMEUR's argument. After noting, following HEATH, that the quadrature presupposes that the ratio of circumference to radius is constant, he adds (*op. cit.*, p. 257):

"This supposition is a fundamental one .... As we have mentioned before, it is just this important relation that is missing in Euclid's *Elements*".

Now it is true that this theorem ($c/d =$ constant) is not in EUCLID, but if we argue as SMEUR does, we should conclude that the constancy of c/d was not known before ARCHIMEDES (or, at least, not known in Greece). Now even the Old-Babylonians knew that c/d is constant (SMEUR admits this), so the only way out from an absurdity is to hold that Old-Babylonian circle geometry made no impression on pre-Archimedian Greek geometry.

Moreover, EUCLID is not our only source. According to HEATH (*Greek Math.*, vol. 1, p. 344), "the *Mechanica* included in Aristotle's writings is not indeed Aristotle's own work, but it is very close to it in date, as we may conclude from its terminology ... ." In the *Mechanica* we read:

"Since the greater radius is moved more quickly than the less by an equal weight, and there are three elements in the lever, the fulcrum ... and two weights, that which moves and that which is moved, therefore the ratio of the weight moved to the moving weight is the inverse ratio of their distances from the fulcrum."[54]

Here we are clearly told that the weights are inversely proportional to the distances from the fulcrum; and, though less clearly, that this is because the distances traversed by the weights as the beam rotates about the fulcrum are inversely proportional to the weights. Eliminating the weights, we get that circumferences of circles are to each other as their radii.

That the reasoning supplied is strictly ARISTOTLE's is testified to by the following passages from ARISTOTLE's *De caelo* and the Physics (*cf.* HEATH, *Greek Math.*, vol. 1, p. 345)

"A smaller and lighter weight will be given more movement if the force acting on it is the same .... The speed of the lesser body will be to that of the greater as the greater body is to the lesser."

"If A be the mover, B the thing moved, C the length through which it is moved, D the time taken, then

$$A \text{ will move } \tfrac{1}{2}B \text{ over the distance } 2C \text{ in the time } D,$$

and

$$A \text{ will move } \tfrac{1}{2}B \text{ over the distance } C \text{ in the time } \tfrac{1}{2}D;$$

thus proportion is maintained."

Still more directly, in the *De caelo*, ARISTOTLE, in speaking of the speeds of the circles of the stars, says (*cf. Greek Math., loc. cit.*):

"it is not at all strange, nay it is inevitable, that the speeds of the circles should be in the proportion of their sizes."

The "size" of a circle could be measured either by the circumference or by the radius; but, if by the circumference, then the assertion is a tautology, so by "size" ARISTOTLE presumably means radius.

Thus we may be certain that ARISTOTLE's school had the "missing proposition", and nearly certain that ARISTOTLE himself had it.

So much for HEATH's first point (1); as for his point (3), we note that in the *De Caelo* ARISTOTLE mentions the following proposition (*cf.*, *Greek Math.*, vol. 1, p. 340):

"Of all closed lines starting from a point, returning to it again, and including a given area, the circumference of a circle is the shortest."

At the very least, this shows that the circumference of a circle was studied in pre-Euclidean times. Comparison between the circumference of a given circle and the arclengths of other figures must have been made; otherwise the proposition makes no sense. Moreover, merely allowing that the circle with a smaller radius has a smaller circumference, the basic relation $\theta < \tan\theta$, or arc $AP < AT$ in Fig. 7, is an immediate corollary.[55]

[54] See IVOR THOMAS, *Greek Mathematical Works*, vol. **1**, p. 431. The Greeks had no word for radius (HEATH, *Thirteen Books*, vol. **1**, p. 199).

[55] ZENODORUS, shortly after ARCHIMEDES, worked on theorems like ARISTOTLE's. ZENODORUS cites Prop. 1 of ARCHIMEDES' *Measurement*.

How is it that these elementary things known to ARISTOTLE did not make their way into EUCLID's *Elements*? Our answer is that EUCLID was not satisfied as to their rigor.

Before entering into a mathematical critique of the relation $\pi_1 = \pi_2$, let us still look at the reference to pre-EUCLIDEAN circle squaring.

It is said that ANAXAGORAS (450 B. C.) occupied himself with the problem while in prison. The comic poet ARISTOPHANES (410 B. C.) makes a poetic joke about it in "The Birds".

Of a more mathematical nature, we have (besides the fragment on lunules) references by ARISTOTLE to ANTIPHON (a contemporary of SOCRATES) and BRYSON (who came a generation later than ANTIPHON). According to ANTIPHON, one may inscribe a regular polygon (say a triangle) in a circle, then bisect the arcs to obtain a polygon with double the number of sides, *etc.* ultimately obtaining a polygon coinciding with the circle; since the polygons can be squared, ANTIPHON argued, so can the circle. BRYSON has a variant of this in that he considers not only the inscribed polygons but also the circumscribed ones. ARISTOTLE sneeringly dismisses the arguments of both men (*cf. Greek Math.*, vol. 1, pp. 221, 223). This does not seem quite just, as ANTIPHON and BRYSON appear to be groping with the notion of limit.

Coming now to a mathematical critique of the quadrature via the quadratrix, we have already noted that the basic point consists in the inequalities $\sin\theta < \theta < \tan\theta$, *i.e.*, in Fig. 7 with $OA = 1$, $AOP = \theta$, $XP < \text{arc}\, AP < AT$. Now this essential point comes up in our calculus courses when we wish to find the derivative of $\sin\theta$. It is necessary to know that $\frac{\theta}{\sin\theta}$ goes to 1 as θ goes to zero; this follows from $\sin\theta < \theta < \tan\theta$ (or even from $\sin\theta\cos\theta < \theta < \tan\theta$), by dividing by $\sin\theta$ to obtain $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$ $\left(\text{respectively, } \cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}\right)$, and observing that $\cos\theta$, and therefore also $\frac{\theta}{\sin\theta}$ goes to 1 as θ goes to zero. But how does one obtain the basic relation $\sin\theta < \theta < \tan\theta$ (or, at least, $\sin\theta\cos\theta < \theta < \tan\theta$)? This is often explained as follows. In Fig. 7, area triangle $OXP <$ area sector $OAP <$ area triangle OAT, whence, taking $OA = 1$, $\theta =$ angle AOP, one gets $\frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$. (Observing that $PP' = 2\sin\theta$ and arc $PP' = 2\theta$, and appealing to the notion that a straight line is the shortest distance between two points, one might deduce $\sin\theta < \theta$.)[56] In this explanation it is assumed that θ is arclength and the formula area of sector $= \frac{1}{2}$ arc times radius is used.

How does one see that area of sector $= \frac{1}{2}$ arc times radius? If pressed for a reference, one could probably not do much better (not counting modern sophisticated improvements) than to refer to ARCHIMEDES, *Measurement of the Circle*, Prop. 1. Now Prop. 1 is based on Prop. 1 of *On the Sphere and Cylinder* I, which in turn is based on the relation $\sin\theta < \theta < \tan\theta$; so the above proof begs the question!

A way out of the above difficulty is simply to define the arc length by the formula arc $= 2$ area of sector/radius! One then has immediately that $\theta < \tan\theta$

[56] Text books properly avoid this argument; however, one finds the argument that, in Figure 7, $\frac{1}{2}\sin\theta =$ area triangle $AOP <$ sector $AOP = \frac{1}{2}\theta$.

(from area sector $OAP <$ area triangle OAT). Similarly one gets $\sin\theta\cos\theta < \theta$; since $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta < 2\cdot\frac{1}{2}\theta$, one gets $\sin\theta < \theta$.

Simple as this is, the Greek mathematician could not have proceeded in this way, as he *never defines* length, area, volume (though he might have found some other logically equivalent way out).[57]

Now getting back to history, we have seen that the *Nine Books* knows that the area of a sector equals $\frac{1}{2}$ arc times radius; and that the Babylonians and the Egyptians at least had it in the case of a semi-circle. It is, then, at least possible that the pre-EUCLIDEAN Greeks knew it, that it got into the stream of Greek mathematics as other things of the Ancient East did. It would appear to be plausible, then, that DINOSTRATUS argued just as our calculus texts do in establishing that $\theta < \tan\theta$. However, when EUCLID—figuratively speaking—came to write this up, he saw that the question was being begged. Or, otherwise put, he could not see how to make the notion of arclength, even for a circle, tractable.

There would have been good reasons for such a difficulty: the problem of comparing areas (in the plane) is essentially simpler than that of comparing lengths. Besides the notion of invariance under congruence, EUCLID knows no principles of comparison except such as are comprised in "common notions": the whole is greater than the part; if equals be added to equals, the wholes are equal; if equals be subtracted from equals, the remainders are equal, *et alia*; there is also the "Axiom of ARCHIMEDES", already known to EUDOXUS. The case of arclength, even for circles, requires some further principle, since there is no way to make a line segment coincide with a circular arc. There is no way, for example, to see from EUCLID that the circumference of a circle having radius 1 mile is greater than a segment 1 inch long. Nor is there a way of seeing that a segment a mile long is greater than the circumference of a circle 1 inch in diameter. As to the first of these, it would follow if we assume that a straight line is the shortest distance between two points. One might think that EUCLID could have put this down as an axiom; but, despite what has been oft repeated, *Euclid*, Book I, does not have the notion of axiomatic geometry.[58] So, obvious as this obvious truth is, EUCLID probably would not have allowed it. He himself *proves* that the sum of two sides of a triangle is greater than the third (*Book* I, 20). But even if he had allowed it, he still would have been stuck with the second difficulty mentioned, for which it is not obvious how to frame an axiom.

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[57] In fact, it is clear that all quadratrices are similar, so that not only does B, in Figure 6, determine Θ, but, conversely, Θ determines B. Now let c be the circle of center Γ and radius $\Gamma\Theta$ and let T be a point (in the first quadrant) on the tangent to c at Θ; and let c cut ΓB in Φ. Let the ray ΓT meet c in P. Let Q be on the ray ΓT and such that area triangle $\Theta\Gamma Q =$ area sector $\Theta\Gamma P$. Then Q varies on a quadratrix q, since the ordinate of Q varies directly with the area of sector $\Theta\Gamma P$. Moreover area triangle $\Theta\Gamma P <$ area sector $\Theta\Gamma P =$ area triangle $\Theta\Gamma Q <$ area triangle $\Theta\Gamma T$, whence Q is between P and T. As T goes to Θ also P goes to Θ, and hence so does Q. Hence the quadratrix q goes through Θ, and hence also through B. As P approaches Φ, Q approaches B. Hence area sector $\Theta\Gamma\Phi =$ area triangle $\Theta\Gamma B$, Q.E.D. (*Cf.* T. DANTZIG, *The Bequest of the Greeks*, p. 138.)

[58] See A. SEIDENBERG, "Peg and Cord in Ancient Greek Geometry," *Scripta Mathematica*, vol. **24** (1959); and "Pasch," in the *Dictionary of Scientific Biography* (to appear).

Axioms, or "assumptions", for overcoming these difficulties were put down by ARCHIMEDES in a preface to his work *On the Sphere and Cylinder* I. The first of these says that: "Of all lines which have the same extremities the straight line is the least." The second requires a prior definition of "concave in the same direction", which involves a notion like ours of convexity. The second assumption reads: "Of other lines in a plane and having the same extremities, [any two] such are unequal whenever both are concave in the same direction and one of them is either wholly included between the other and the straight line which has the same extremities with it, or is partly included by, and is partly common with, the other; and that [line] which is included is the lesser [of the two]."

The only place the first assumption is used is to prove that the perimeter of a polygon inscribed in a circle is less than that of the circle; this is done in the preface. The only place the second assumption is used is to prove that the perimeter of a polygon circumscribing a circle is greater than that of the circle. This is done in Prop. I and essentially comes to proving $\theta < \tan\theta$. With reference to Figure 7, the sum $TA + TB$ of the tangents is greater than the arc AB, by Assumption 2, whence $\theta < \tan\theta$.

Just as a matter of mathematics, this is on the face of it surely unsatisfactory. One wishes to prove $TA + TB > \text{arc}\, AB$ and introduces a much more general assumption, from which the inequality follows as a special case. Obviously, there is some background to the assumptions that we cannot see.

ARCHIMEDES realized that to produce EUDOXIAN arguments on arclength one needed bounds from below and above. That there was no way out except by assumptions is a great insight by ARCHIMEDES (or a predecessor). Still, what allows him to make the assumptions? Just as a matter of straightforward mathematics, he had first to check his assumptions for (convex) polygonal paths. The first is immediate; and the second is also easy. In fact, the essential idea for a proof is contained in *Euclid*, I 21, which says that if triangle ADC is contained in triangle ABC, then $AB + BC > AD + DC$ (and also that angle $ADC >$ angle ABC). The proposition (I 21) is not, we believe, anywhere used. To us, this suggests that the notions of convex paths and their lengths goes back to EUCLID, though with no satisfactory resolution at that time of the problems they give rise to.

A modern mathematician, having checked that the assumptions hold for the largest class of convex arcs for which he has information (namely, the class of convex polygonal paths), and realizing that he has no information (on the larger class of all convex arcs) contradicting his assumptions, might seek a consistency proof for the assumptions. This, however, is too far from the point of view of the ancients to be worth pursuing here. Rather, they would want to be assured that the assumptions are *true*.

Perhaps it is worth speculating on how ARCHIMEDES (or a predecessor) convinced himself of the truth of the second assumption. We would suggest the following. Let APB, $AP'B$ be two arcs of the kind mentioned in Assumption 2, with $AP'B$ inside $APBA$. Now let $AC'D'E' \ldots B$ be a polygonal path inscribed in the arc $AP'B$. Let the rays AC', AD', ... extended meet APB in $C, D, \ldots$. Then the length of the polygonal path $ACD \ldots B$ is greater than (or at least equal to) the length of the polygonal path $AC'D' \ldots B$. Hence the length of arc APB is greater than any polygonal path inscribed in arc $AP'B$. Now officially ARCHIMEDES cannot express the idea that arc $AP'B$ can be approximated in length arbitrarily closely by polygonal paths. But unofficially he can feel convinced about it. It then follows that arc $APB >$ arc $AP'B$.

Whether this speculation hits the truth is not important. It serves, however, to emphasize that ARCHIMEDES' assumptions must have had some background which made them acceptable to his contemporaries. That there was some such familiar background is also suggested by ARCHIMEDES' prefatory letter to DOSITHEUS. In it ARCHIMEDES lays claim in no uncertain terms to some of his results, but he makes no special claim on the Assumptions; yet in the *Quadrature of the Parabola* he goes out of his way to credit "earlier geometers" with the so-called Axiom of ARCHIMEDES.

To complete this account, we may still sketch a modern treatment for the length of a convex arc. First, one may define an arc as the image under a continuous mapping of a segment $A'B'$; if A, B are the images of A', B', then the arc is said to be from A to B. We now confine the discussion to convex arcs. Let C', D', ... be a finite sequence of points on the $A'B'$, in the stated order from A' to B', and consider the images C, D, ... and the polygonal path $ACD \dots B$, which is convex. Since the mapping defining arc AB is given by continuous functions, the arc will be bounded, that is, one can enclose it in a square S. The length of the polygonal path is less than the length of S. Now considering all such polygonal paths $ACD \dots B$, let l be the least number equal to or greater than their lengths; that there is such a number is a basic property of the field of real numbers. We then *define* the length of the arc to be l. With this definition, it is an easy matter to prove the two assumptions of ARCHIMEDES.

So, the basic idea is to define the length of the arc as the least number equal to or greater than the lengths of the inscribed polygonal paths. Quite aside from the logical difficulties involved in defining the field of real numbers, this procedure is beyond the ken of the ancients.[59]

As SMEUR notes, following HEATH, the quadratrix argument first rectifies the circle; to square it, one still needs to know Prop. 1 of ARCHIMEDES' *Measurement of the Circle* (*i.e.*, that the area of the circle is that of a triangle with circumference as base and radius as altitude).[60] But this difficulty is, in the presence of EUDOXIAN techniques, on the same level as the difficulties already met: had EUCLID been able to establish to his own satisfaction that $\sin\theta < \theta < \tan\theta$, something ARISTOTLE surely knew, then there would have been nothing in the way of his including Prop. 1 of the *Measurement* in his *Elements*.

To sum up:

The formula for the area of a sector (area $= \frac{1}{2}$ arc times radius) *was inherited by the Greeks and was known to Dinostratus. However, Euclid could not establish it to his own satisfaction. Basing himself on this formula, Dinostratus squared the circle using the quadratrix of Hippias.*

[59] For a construction of the plane starting from the real field and including a discussion of arclength, see G. HOCHSCHILD, *A Second Introduction to Analytic Geometry*. The *area* of a circle, however, is not taken up!

[60] In fact, PAPPUS himself notes this and cites the mentioned proposition of ARCHIMEDES. See I. THOMAS, *op. cit.*, vol. **1**, p. 347.

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(Received September 6, 1972)