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Alphonse Antonio de Sarasa and Logarithms

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The discovery of hyperbolic logarithms has been variously attributed to Gregory of St-Vincent and to Alphonse Antonio de Sarasa. In this paper we describe the relationship between St-Vincent and de Sarasa, the challenge from Mersenne which provoked de Sarasa's publication, de Sarasa's understanding of logarithms, and the propositions of de Sarasa where the connection between hyperbolic areas and logarithms was first claimed. Since de Sarasa's hyperbolae were not defined analytically, he did not insist on a particular base for his logarithms, nor did he choose a value for log 1 which would have implied $\log AB = \log A + \log B$, so we conclude that he did not focus on what Euler later called natural logarithms. An explanation of the different accounts of the origin of natural logarithms offered by twentieth century writers is tentatively proposed.

Die Entdeckung hyperbolischer Logarithmen ist verschiedentlich Gregorius a S.Vincentio und Alphonse Antonio de Sarasa zugeschrieben worden. In diesem Artikel beschreiben wir die Beziehung zwischen St.Vincentius und de Sarasa, die Herausforderung von Mersenne, die de Sarasas Veröffentlichung provozierte, de Sarasas Verständnis von Logarithmen und die Sätze de Sarasas in denen der Zusammenhang zwischen Hyperbelflächen und Logarithmen zum ersten Mal behauptet worden ist. Da de Sarasas Hyperbeln nicht analytisch definiert waren, forderte er keine Basis fuer seine Logarithmen. Er wählte auch keinen Wert fuer log 1, der $\log AB = \log A + \log B$ impliziert hätte. So können wir schliessen, dass er sich nicht auf das konzentrierte, was Euler später natürliche Logarithmen nannte. Eine Erklärung der unterschiedlichen Darstellungen über den Ursprung der natürlichen Logarithmen, die Autoren des 20. Jahrhunderts geben, wird ansatzweise versucht.

MSC 1991 subject classification: 01 A 45.
Key Words: logarithm; hyperbola; geometric progression; arithmetic progression; mean proportional.

INTRODUCTION

Historians' assessments of de Sarasa's contribution [de Sarasa 1649] to the development of logarithms have varied from that of Cantor [Cantor 1907, 896] who only gave de Sarasa credit for pointing to the work of St-Vincent, to that of Katz [Katz 1998, 492], who asserts that de Sarasa identified and claimed $A(ab) = A(a) + A(b)$ for hyperbolic areas. Kästner [Kästner 1799 III, 251–254] gave a brief but accurate account of de Sarasa but did not suggest that he was the discoverer of modern hyperbolic logarithms. In this paper the relevant work of de Sarasa is examined and its context described. This results in a reassessment of what de Sarasa meant by "logarithm" and a recognition that some caution should be exercised in describing de Sarasa's claims about hyperbolic areas.

Alphonse Antonio de Sarasa was born in Nieveport, in Flanders, in 1618, of Spanish parents, and died in 1667 in Brussels. He was a pupil and later colleague of his fellow Belgian Gregory of St-Vincent (1584–1667) and, like him, a member of the Jesuit

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order. In addition to the publication we will examine [de Sarasa 1649], de Sarasa published a book entitled *The Art of Always Rejoicing* (1663); sadly this was not about mathematics!

Gregory of St-Vincent had been admitted to the Jesuit order, as a novice, in Rome in 1605 where he was the mathematical protégé of Clavius (also a Jesuit) and an enthusiastic admirer of Galileo. St-Vincent stayed in Rome until the death of Clavius in 1612. St-Vincent then returned to Belgium and to a succession of appointments culminating in three years at Antwerp (1617–1620) and four years at Louvain (1621–1625). It seems to have been in this period [Looy 1984] that St-Vincent wrote much of his magnum opus *Opus Geometricum Quadraturae Circuli et Sectionum Coni* [St-Vincent 1647], which was not, however, published until 1647. (In [Hofmann 1942, 7] and [Dictionary of Scientific Biography 1975 XII, 74–75] there are remarkable accounts of St-Vincent's limited access to his own manuscripts.) In 1625, St-Vincent went to Rome seeking permission to publish his quadrature of the circle and returned in 1627. He was then called to Prague (1628–1631), to the household of the emperor (to whom Kepler was mathematician), before returning to Belgium and to a position in Ghent in 1632, where he stayed for the rest of his life [Bosmans 1901 XXI]. De Sarasa was admitted as a novice of the Jesuit order in Ghent in 1632, and later became a colleague of St-Vincent in the college there for seven years. After St-Vincent's death, he assisted in the publication of a further book of St-Vincent's mathematics [Looy 1984, 59]. He also held academic positions in Antwerp and Brussels [Sommervogel 1896 VII, 621–627].

In Book 6 (*de Hyperbola*), Propositions 125, 129, and 130, of his *Opus Geometricum Quadraturae Circuli et Sectionum Coni* [St-Vincent 1647], St-Vincent proved that if points were taken in geometric progression along one asymptote of a hyperbola, and lines drawn through these points parallel to the other asymptote, then the areas between the parallel lines, bounded at one end by the asymptote and at the other by the hyperbola, were equal. St-Vincent's proof focused on two adjacent strips and established their equality. He gave two proofs [Dhombres 1993], one echoing Archimedes' quadrature of the parabola, and another, which needs a little patching up, working directly with the strips. A full account of *de Hyperbola* is given by Bopp [Bopp 1907].

(Because the result is the basis of all de Sarasa's claims, we outline one of St-Vincent's proofs that a geometric progression along the x-axis gives rise to equal hyperbolic areas below $y = 1/x$. With modern notation, St-Vincent proved that the area bounded by the x-axis, the lines $x = a$, $x = \sqrt{ab}$, and $y = 1/x$ was equal to the area bounded by the x-axis, the lines $x = \sqrt{ab}$, $x = b$ and $y = 1/x$. This result was then invoked for each adjacent pair of hyperbolic areas standing on the x-axis sectioned by the geometric progression. To prove this result St-Vincent noted that the trapezium with vertices $(a, 0)$, $(\sqrt{ab}, 0)$, $(\sqrt{ab}, 1/\sqrt{ab})$, $(a, 1/a)$ had the same area as the trapezium with vertices $(\sqrt{ab}, 0)$, $(b, 0)$, $(b, 1/b)$, $(\sqrt{ab}, 1/\sqrt{ab})$. So the corresponding hyperbolic areas were equal if and only if the convex hyperbolic segments on $(a, 1/a)(\sqrt{ab}, 1/\sqrt{ab})$ and $(\sqrt{ab}, 1/\sqrt{ab})(b, 1/b)$ were equal. St-Vincent proved the equality of these convex hyperbolic segments by exhaustion, inscribing triangles of maximum area in the segments in a manner reminiscent of Archimedes' quadrature of the parabola. The key geometrical properties for this proof were that the diameter of the hyperbola through $(\sqrt{ab}, 1/\sqrt{ab})$ bisected the line segment $(a, 1/a)(b, 1/b)$ and that the tangent at $(\sqrt{ab}, 1/\sqrt{ab})$ was parallel to the line $(a, 1/a)(b, 1/b)$. The exhaustion was effected by repeatedly inserting geometric means on the hyperbola for the areas still uncovered by the triangles. This outline summarizes *Opus Geometricum*, Book 6, propositions 102–109.)

Shortly after the publication of *Opus Geometricum* Descartes wrote a letter to Mersenne pointing out an error in St-Vincent's quadrature of the circle [Montucla 1796 II, 81]. Auzout and Roberval also criticized this quadrature to Mersenne [Bosmans 1901 XXI, 157]. In the following year, 1648, the year of Mersenne's death, Mersenne published a pamphlet (*Reflexiones Physico-mathematicae*), containing a brief review of *Opus Geometricum* quoted in [de Sarasa 1649], which was mildly derogatory of St-Vincent's efforts but did not identify any mistake. Mersenne challenged the supposed circle-squarer with what he regarded as an equally difficult problem. The temptation to suppose that Mersenne already knew that the answer lay in hyperbolic areas should be resisted. Mersenne had reason to suspect that the quadrature of the circle was unsolved. To then suggest as a comparably difficult problem one to which he already knew the answer would be either arrogant or deceitful. The testimony to his character as "A man of simple, innocent, pure heart, without guile ... A man whom all the arts and sciences to whose advance he tirelessly devoted himself, by investigating or by stimulating others ... " [Dictionary of Scientific Biography 1975 IX, 320] would suggest that the preamble to Mersenne's problem should be taken at face value. Mersenne posed the problem because he believed it was as difficult as the quadrature of the circle. From a nineteenth century perspective, both circle-squaring and finding a third logarithm, given two, require the construction of transcendental numbers, but there the affinity between the two problems ends. De Sarasa discussed the two problems in separate and independent sections of his publication.

MERSENNE'S CHALLENGE PROBLEM

Datis tribus quibuscumque magnitudinibus, rationalibus vel irrationalibus, datisque duarum ex illis Logarithmis, tertiae Logarithmum Geometrice invenire.	Given three arbitrary magnitudes, rational or irrational, and given the logarithms of two, to find the logarithm of the third geometrically.

(a) The Term "Logarithm"

Before looking at de Sarasa's response to this problem, we must acknowledge the difference between the term "logarithm" in this context and our modern, narrower, use. At least from the time of Euler, once the base had been chosen, the logarithms of all positive numbers were uniquely defined. Seemingly, only the logarithm of one number need be given to determine the rest uniquely. But to say this is to hide *our* assumptions that $\log 1 = 0$ and that logarithms are defined on the continuum. Knowing that $\log 1 = 0$ and knowing the base (the number whose logarithm is 1) has been enough for nearly 300 years to determine all other logarithms. To find all other logarithms, given the base, is the modern counterpart of Mersenne's challenge.

Before 1650, there was no consensus about what number had zero logarithm. For Napier's logarithms $\log 10^7 = 0$ [Napier 1614]; for Speidell's variant of Napier $\log 10^8 = 0$ [Speidell 1619]; [Cajori 1919, 152]; for quite different reasons Bürgi took $\log 10^8 = 0$ [Bürgi 1620]. Kepler took $\log 10^5 = 0$ [Kepler 1624]. Cavalieri took $\log 10^{-10} = 0$ [Cavalieri 1632]. Caramuel (1670) believed he had perfected the logarithms of Napier and Briggs and took

$\log 10 = 0$. Of these early logarithm constructors, only Briggs [Briggs 1617 and 1624] acting on the advice of Napier (after [Napier 1614]) took $\log 1 = 0$ [Coolidge 1990, 78]. (See [Naux 1971 II] for various early versions of logarithms.)

But without $\log 1 = 0$, the standard logarithmic property $\log AB = \log A + \log B$ fails. So what then did Mersenne mean by "logarithm"? Briggs' *Arithmetica logarithmica* [Briggs 1624] began with the statement that logarithms were numbers with constant differences matched with numbers in continued proportion (*Logarithmi sunt numeri qui proportionalibus adjuncti aequales servant differentias*). In other words, when the terms of a geometric progression were matched with the terms of an arithmetic progression, in sequence, the terms of the arithmetic progression were called the logarithms of the corresponding terms of the geometric progression. This description was the basis of all logarithm constructions, before 1649. It is repeatedly made clear in de Sarasa's pamphlet that this was exactly what he meant by logarithms. In his first chapter, Briggs [Briggs 1624, 1] illustrated his definition of logarithm with a table giving four versions of logarithms.

Proportional numbers	Alternative systems of logarithms			
1	1	5	5	35
2	2	6	8	32
4	3	7	11	29
8	4	8	14	26
16	5	9	17	23
32	6	10	20	20
64	7	11	23	17
128	8	12	26	14

These four versions of logarithms can be summarized algebraically by the four matchings: $2^n \leftrightarrow 1 + n$ in the first column, $2^n \leftrightarrow 5 + n$ in the second, $2^n \leftrightarrow 5 + 3n$ in the third, and $2^n \leftrightarrow 35 - 3n$ in the fourth. It was only in Chapter 2 that Briggs discussed and illustrated the case where $\log 1 = 0$. De Sarasa would surely have been familiar with Briggs' work through the completed edition published by Vlacq [Briggs 1628] in Gouda, Holland.

Here is a summary of some of the logarithmic systems of the 17th century:

	Geometric progression numbers	Arithmetic progression logarithms
Napier, 1614	$10^7(1 - 10^{-7})^n$	n
Briggs, 1617	10^n	n
Speidell, 1619	$10^7(1 - 10^{-7})^n$	$(10^8 - n)/10^2$
Bürgi, 1620	$10^8(1 + 10^{-4})^n$	$10n$
Kepler, 1624	$10^5(1 - 10^{-5})^n$	n
Cavalieri, 1632	10^n	$10 + n$
Caramuel, 1670	10^n	$10 - n$

It should be said that the notation in this table, with exponents, was not used in the early 17th century. The values of n were, in the first instance, zero and positive integers, and then, with interpolations, decimal fractions. We can summarize all such matchings of a geometric progression with an arithmetic progression with an algebraic form that respects the sequences; thus

Numbers in geometric progression					
Logarithms in arithmetic	a	$a+d$	$a+2d$	$\cdots$	$a+nd$
progression	A	AR	AR^2	$\cdots$	AR^n

Then products of *numbers* are matched by sums of *logarithms* in the sense that

$$AB = CD \Leftrightarrow \log A + \log B = \log C + \log D$$

$$\text{and also } A/B = C/D \Leftrightarrow \log A - \log B = \log C - \log D.$$

Here, A, B, C, and D are terms in the geometric progression. A modern justification of the first rule taking the numbers as AR^m, AR^n, AR^r and AR^s, runs

$$\begin{aligned}(AR^m)\cdot(AR^n) = (AR^r)\cdot(AR^s) &\Leftrightarrow A^2R^{m+n} = A^2R^{r+s}\\ &\Leftrightarrow m+n = r+s\\ &\Leftrightarrow 2a+(m+n)d = 2a+(r+s)d\\ &\Leftrightarrow (a+md)+(a+nd) = (a+rd)+(a+sd),\end{aligned}$$

assuming the numbers are positive, and providing, of course, that A, R, $d \neq 0$, and $R \neq 1$.

Napier claimed a special case of the first of these rules (when A, C, D, and B are consecutive terms in the particular geometric progression) [Struik 1969, 18, Proposition 38 from Napier 1889] and claimed the second in full generality [Struik 1969, 17, Proposition 36]. It is in the use of these rules that all early writers on logarithms saw their computational benefits. We should note that each of these rules is equivalent to $\log AB = \log A + \log B - \log 1$. In his second chapter, Briggs [Briggs 1624] claimed the great convenience of choosing $\log 1 = 0$, because of three consequences. Firstly, when the terms of a geometric progression starting with 1 were indexed (0, 1, 2, 3, ...), the logarithms were proportional to the indices. Second, the multiplication of numbers in a geometric progression starting from 1 was matched by the addition of logarithms. Third, the division of numbers in a geometric progression starting with 1 was matched by the subtraction of logarithms. We should be wary of saying that Briggs simply claimed $\log AB = \log A + \log B$, since his illustrations are always in the context of particular geometric progressions, and his interpolations are by repeated root extractions, that is, by the generation of increasingly dense geometric progressions.

When logarithms are characterized by a matching of the type $AR^n \leftrightarrow a + nd = \log AR^n$, there are exactly two degrees of freedom in setting up a system of logarithms. If the logarithms of A and AR are known (and they may be chosen arbitrarily provided only that they are different), the logarithms of the whole geometric progression may be determined. In this context, Mersenne's problem makes good sense.

(b) The Term "Geometrically"

Having clarified the context in which Mersenne used the term "logarithm," we may now focus on the other word in his problem which is not unambiguous, namely "geometrically."

In introducing his problem, Mersenne twice used the term "geometers" where today we would say "mathematicians." De Sarasa accepted the term "geometrical" in two senses: geometrical *constructions* and geometrical *rigour*. He used both expressions several times. In his final solution (Proposition 10) his logarithms were "geometrically assigned" with hyperbolic areas, and this was what de Sarasa derived from St-Vincent. But to seek "geometrical" (or Euclidean) *rigour* in relation to the determination of logarithms, is at best unexpected, and at worst, from the point of view of a constructor of tables, perverse. There was much approximation involved in the making of logarithmic tables. Napier's model of logarithms was a continuous one to which he approximated by multiplications by numbers of the form $1 - r/10^n$ and linear interpolations. Briggs started with the large common ratio, 10, and then repeatedly extracted square roots until he reached $\sqrt[2^{54}]{10}$, a number between 1 and $1 + 10^{-15}$. Briggs also used a binomial approximation for square roots [Whiteside 1961, 234]. Kepler, however, who had befriended Bürgi in Prague from 1603 [Klemm 1969, 142], had first seen Napier's tables in 1617 [Naux 1966 I, 130], and sought to construct a Euclidean edifice for them with postulates, axioms, common notions, and propositions. He published this account under the title *Chiliades logarithmorum* [Kepler 1624]. Excerpts from this publication were translated into English by Hutton [Hutton 1785] and parts of Hutton's translation have been reprinted in [Fauvel and Gray 1987, 9.E4]. Although the tables printed at the end of Kepler's book were like a scaled down version of 1000 of Napier's logarithms, Kepler's numerical illustrations indicated a multiplicity of ways of matching a geometric progression with an arithmetic progression. Kepler's rigour was repeatedly echoed in de Sarasa's text. Kepler, who had not seen Briggs' work, showed how repeated root extraction (which could be performed geometrically, and thus with Euclidean rigour) could refine a geometric progression to any required degree of denseness (Postulate 2). In Proposition 9, Kepler interpolated two mean proportionals between two cubes, three mean proportionals between two fourth powers, etc. (compare with de Sarasa, Proposition 5, below). Kepler's Proposition 8 bears such a close relation to de Sarasa's Proposition 6 that it must be quoted (in translation from [Hutton 1822, 51]) in full.

> Of any quantities placed in the order of their magnitudes, if the intermediates lying between any two terms be not among the mean proportionals which can be interposed between the said two terms then such intermediates do not divide the proportion of those two terms into commensurable proportions.

After Proposition 11, Kepler gave an example of a triple (8, 13, 18) which could not belong to any geometric progression whatever. In his Proposition 6, de Sarasa gives a geometrical example of such a triple. De Sarasa did not cite Kepler explicitly but the similarity of viewpoint is unmistakable.

The repeated taking of square roots and insertion of geometric means leads from a given geometric progression to an arbitrarily dense geometric progression, and St-Vincent's study of the hyperbola used just this as a limiting process to compare two hyperbolic areas. Logarithms had not (before 1649) been defined by a limiting process, but by geometric progressions. De Sarasa understood the satisfactory solution of Mersenne's problem to depend on the existence of a geometric progression containing three arbitrary numbers.

(See the quotation from the Scholion after Proposition 9, below.) Because the insertion of mean proportionals allows any given geometric progression to be embedded in a denser one, it was not obvious, at first sight, whether a geometric progression could always be found which contained three given positive numbers, among the host of progressions containing any two of them.

DE SARASA'S RESPONSE

On 8 November 1651 Christiaan Huygens (who exposed the fallacy in St-Vincent's circle-squaring in [Huygens 1651]) wrote a letter to Saint-Vincent, complimenting him on the fine defence [of *Opus Geometricum*] which had been put up by *his colleague* de Sarasa [Bosmans 1901 XXI, 157]. And in one of St-Vincent's extant manuscripts [St-Vincent, Ms.5782] St-Vincent himself affirmed that he had been the moving spirit behind the various defences that had been offered for *Opus Geometricum*, including that of de Sarasa [Bosmans 1901 XXI, 170] and [Looy 1984, 70]. In his preface [de Sarasa 1649], de Sarasa says that Mersenne's review of *Opus Geometricum* was brought to his attention by "a friend," presumably St-Vincent. It seems that St-Vincent was deeply involved in de Sarasa's publication, but we will nonetheless refer to its contents as de Sarasa's except where de Sarasa quotes St-Vincent explicitly.

De Sarasa's reply to Mersenne's review was in two parts. The first part was a discussion of logarithms. The second part was a defence of St-Vincent's quadrature of the circle. We will leave aside part two and concentrate on the discussion of logarithms. To get the main thrust of his argument we give the penultimate paragraph of the preamble to his 10 propositions.

> From all this it will be clear that if quantities A and C are given, and their logarithms, and in the same way a third quantity L is given, which cannot be in any series [*of continued proportion*] containing the quantities A and C, however much that series is extended or divided or multiplied (which may be shown to be possible); in this case, it is not possible to find the logarithm of the quantity L, and therefore the problem has been badly formulated. But apart from this limitation we may find what is required in the problem, and to reduce the problem to a geometrical construction we apply what may be seen to be legitimate possibilities. [de Sarasa 1649, 6]

De Sarasa [de Sarasa 1649] gathered results from *Opus Geometricum* [St-Vincent 1647] (de Sarasa's Propositions 1–3, and a Corollary to Prop.1) and then, before giving his main argument, offered a Scholion in which he discussed the nature of logarithms and illustrated their relationship with hyperbolic areas. Then he investigated the embedding of geometric progressions in denser ones (Proposition 4–5). In Propositions 6–9 he studied the consequences of having two incommensurable/commensurable hyperbolic areas. And in Proposition 10 he solved Mersenne's problem for three numbers in a geometric progression, and after the demonstration stated explicitly that hyperbolic areas were like logarithms. Logarithms were not mentioned in Propositions 1–9, nor in their demonstrations, but after reading the Scholion following Proposition 3, one cannot but look at hyperbolic areas through logarithmic spectacles!

We now examine de Sarasa's propositions. In modern terms, de Sarasa's first proposition claims that points (a, br^5), (ar, br^4), (ar^2, br^3), (ar^3, br^2), (ar^4, br), (ar^5, b), ... lie on a rectangular hyperbola with the coordinate axes as asymptotes. If the abscissae of points of a rectangular hyperbola are in geometric progression then the ordinates are also in a geometric progression with the same common ratio, the one increasing the other decreasing.

De Sarasa's arguments were habitually in terms of ordinates; modern arguments normally work with abscissae. This proposition makes the transposition unproblematic. De Sarasa reproduced the same diagram three times; for Proposition 1, for Proposition 2, and for the Scholion following Proposition 3 (see below).

PROPOSITION 1. *Given any series of line segments AB, AC, AD, AE, AF, AG, etc., which continue in the same ratio as AB to AC; line segments are erected at right angles to the line AG at the points G, F, E, D, C, B that we call GN, FM, EL, DK, CI, BH, which are in the same ratio as the line segments AB, AC, AD, AE, AF, AG. I say that the points H, I, K, L, M, N, & c. are on a hyperbola with perpendicular asymptotes AV and AG.*

This proposition was followed by a Corollary affirming that the feet of the perpendiculars from the points H, I, K, L, M, N, etc. onto the asymptote AV also gave a geometric progression along that asymptote. Proposition 1 was quoted "ex.298" *de Hyperbola*, which I cannot trace. It also figures in *de Hyperbola* at Proposition 46. This property of the hyperbola is not original to St-Vincent and goes back at least to Apollonius [Heath 1981, 149], who was a major inspiration for St-Vincent [Looy 1984, 63].

In modern terms De Sarasa's second proposition seems to say that

$$n \cdot (\text{hyperbolic area between } x = a \text{ and } x = ar)$$
$$= (\text{the hyperbolic area between } x = ar \text{ and } x = ar^{n+1}).$$

But if you follow his proof, you realize he is claiming more than that; in fact, for $r > 1$, that

$$n \cdot (\text{hyperbolic area between } x = a \text{ and } x = ar)$$
$$< (\text{the hyperbolic area between } x = ar \text{ and } x = as) \Leftrightarrow r^{n+1} < s.$$

See Fig. 1.

PROPOSITION 2. *With the same figure as before, let AV and AG again be the asymptotes of the hyperbola HIKN: and let us consider segments BH, CI, GN parallel to the asymptote AV, bounding the hyperbolic*

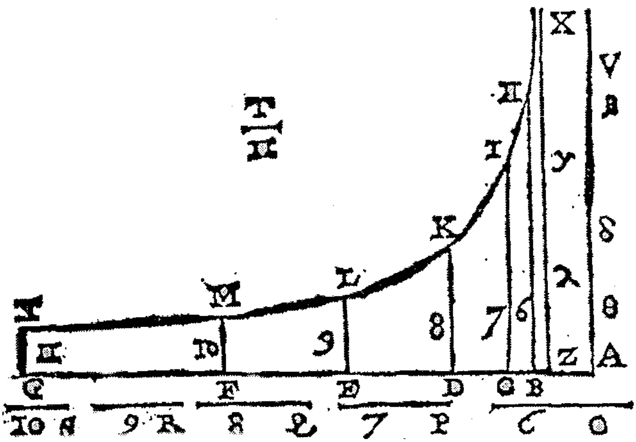

FIG. 1. Figure for Propositions 1, 2, and 3 and Scholion.

> *segments HBCI and ICGN. I say that the number of times the ratio HB : IC multiplied* [*by itself is within*] *the ratio IC : NG, is the number of times the area HBCI* (*which for convenience we will denote by HC*) *is contained in the area IG*; *and conversely.*

For the proof, de Sarasa cited Propositions 125 and 129 from *Opus Geometricum*, Book 7 [*sic*]. For the proof of Proposition 125, where the two areas are commensurable, St-Vincent's demonstration is of the type

$$(\text{area between } x = a \text{ and } x = ar^n) = n \cdot (\text{area between } x = b \text{ and } x = br),$$

for $n = 4$; for the proof of Proposition 129, where the two areas are incommensurable, St-Vincent's demonstration is of the type

$$n \cdot (\text{area between } x = b \text{ and } x = br) < (\text{area between } x = a \text{ and } x = as) \Leftrightarrow r^n < s,$$

though the stated proportions are of ordinates, not abscissae.

In modern terms, de Sarasa's third proposition states that the hyperbolic area between $x = a$ and $x = ar$ equals that between $x = ar$ and $x = ar^2$, and that between $x = ar^2$ and $x = ar^3$, etc., and conversely that if the hyperbolic area between $x = a$ and $x = b$ equals that between $x = b$ and $x = c$ and that between $x = c$ and $x = d$, etc., then a, b, c, d, etc., are in geometric progression, though it should be noted that in de Sarasa's presentation the stated proportions are of ordinates, not abscissae.

> PROPOSITION 3. *Make the same assumptions.*
> *If the lines HB, IC, KD, LE, MF, NG, etc. or, which reduces to the same thing, if the lines AB, AC, AD, AE, AF, AG, etc., are in continued proportion: I say that all the hyperbolic areas HC, CK, KE, EM, MG, etc., are equal. And if the hyperbolic areas between the lines HB, IC, KD, etc., parallel to the asymptote AV are determined and are equal, I say that all the lines HB, IC, KD, LE, etc., and likewise all the lines AB, AC, AD, AE, etc., are in continued proportion.*

For the proof of Proposition 3, de Sarasa cited Proposition 130 from *Opus Geometricum*, Book 7. In fact Proposition 3 incorporated both Proposition 109 (first sentence) and its converse, Proposition 130 (second sentence).

(Although de Sarasa cited Book 7, *de Hyperbola* is in fact Book 6 of *Opus Geometricum.* Hughes has suggested that de Sarasa was working from a manuscript, not the printed text [Hughes 1995, 171].)

Propositions 1, 2, and 3 were taken from *Opus Geometricum*. Neither in St-Vincent's original nor as cited by de Sarasa did they mention logarithms. They do, however, form the basis of Cantor's affirmation [Cantor 1892 II, 654; and 1907 II, 896; Bopp 1907, 264] that the relationship between logarithms and the hyperbola was found by St-Vincent in all but name. De Sarasa would have agreed with Cantor on this point for in the last paragraph of the preamble to his 10 propositions he wrote:

> In order that we may deal finally with this question with geometrical rigour, we will repeat here the most important teaching from Part 4 [*De Segmentis hyperbolicis convexis et concavis*] of Book [6] *de Hyperbola*, from *Opus Geometricum* of Gregory of St-Vincent; the foundations of the teaching embracing logarithms are contained there.

After Proposition 3, de Sarasa added a Scholion, a discussion, in which he expounded the "nature of logarithms" and their relationship to hyperbolic areas and set the scene for what was to follow. The full text of this Scholion is given below. De Sarasa used the same diagram as for Propositions 1, 2, and 3. For a particular decreasing geometric progression O, P, Q, R, S, T, illustrated by line segments, and with these terms having declared logarithms of 6, 7, 8, 9, 10, and 11, de Sarasa constructed a hyperbola with six ordinate lengths equal to O, P, Q, R, S, T respectively. (The segment of length T and the "11" have overflowed from the bottom of the figure into the middle!) The given logarithms related to hyperbolic areas in the light of Proposition 3 in an obvious way. He then used the hyperbola to construct two further logarithmic systems for the given geometric series:

Original series	Ordinates	1st logarithm	2nd logarithm	3rd logarithm
	XZ	(5)		(0)
O	HB	6	$XG = 6MG$	MG
P	IC	7	$HG = 5MG$	$LG = 2MG$
Q	KD	8	$IG = 4MG$	$KG = 3MG$
R	LE	9	$KG = 3MG$	$IG = 4MG$
S	MF	10	$LG = 2MG$	$HG = 5MG$
T	NG	11	MG	$XG = 6MG$

In each case a geometric progression was matched with an arithmetic progression. This is what the "nature of logarithms" required. The magnitude with logarithm 0 was different in each case. For the first logarithm it was a magnitude greater than XZ (in fact O^7/P^6). For the second logarithm $\log T^2/S = 0$, and for the third logarithm, the ordinate XZ, equal in length to O^2/P, has zero logarithm. De Sarasa exhibited just the kind of flexibility which is required to address Mersenne's problem. The Scholion finished with a poorly described line-ratio version of the third logarithm. For two of these logarithms (first and third), the logarithm increases as the number decreases, as with Napier's logarithms. One hundred years later Euler was to study logarithms to different bases (or common ratios) with the common assumption that $\log 1 = 0$. De Sarasa, quite distinctively, by working with a given hyperbola showed what it meant to have logarithms with the same base (or common ratio) but with different A such that $\log A = 0$.

Scholion following proposition 3. But, you say, I do not want these digressions. Yet I will lead you to logarithms, however distant this may seem from our purpose. Briefly then I explain how to understand the teaching on logarithms.

Assume again the same figure. Let there be some sequence of magnitudes O, P, Q, $[R,]$ S, T in continuous ratio whose logarithms are 6, 7, 8, 9, 10, etc. These numbers always exceed one another by the same amount as demanded by the nature of logarithms. Assuming again a certain hyperbola *HIN* with asymptotes *AV* and *AG*, erect the lines *HB*, *IC*, *KD*, $L[E]$, *MF*, *NG*, etc., parallel to the asymptote *AV* and equal to the lines O, P, Q, R, S, T respectively, all of which is easily done by the first corollary in this book.

Now by Proposition 3 in this book, all the areas *HC*, *CK*, *KE*, *EM*, *MG* are mutually equal. Hence if the ratio *IC* to *HB* is continued and becomes proportional to the same *XZ* and by the first corollary in this book it belongs to the same hyperbola, then the area *XB* will be equal to the areas *HC*, *CK*, etc. Whence the total hyperbolic area *XG* exceeds the hyperbolic area *HG* by the same amount as the hyperbolic area *HG* exceeds the area *IG*. Again: the hyperbolic area *IG* exceeds the area *KG*, by the same amount, and so on with the others. Wherefore, in place of the numbers 6, 7, 8, 9, 10, 11, etc., which are the logarithms of the magnitudes O, P, Q, R, S, T, we can adopt the hyperbolic quantities *XG*, *HG*, *IG*, *KG*, *LG*, *MG*, or better *MG*, *LG*, *KG*, *IG*, *HG*, *XG*; or if you prefer not to mention the hyperbola, the quantities (and hence the ratios) exceed one another by a no less equal amount as the logarithmic numbers which had been assumed. For this reason you see that the nature of logarithms with its continuation and excess of terms is adapted exactly to the hyperbola, so that in place of the numbers, you may take the parts of the hyperbola or the given ratio of the lines. (English translation taken with slight emendation from [Hughes 1995])

In Proposition 4, de Sarasa began his main investigation as to the circumstances under which a geometric progression *not* containing a certain number might be embedded in a new geometric progression which did contain this number.

PROPOSITION 4. *Let there be given a series of line segments* A, B, C, D, E, *etc., continued* [and decreasing] *according to the ratio of A to B. Moreover, let an arbitrary F be given. It is necessary to show whether F can be exhibited in a series with ratios* $A : B : C$, *if it is produced on either side, according to the ratio* $B : A$ [if F is] *greater* [than B] *or according to the ratio* $B : C$ [if F is] *less.* [*Textual modifications, in square brackets, in the light of de Sarasa's preamble*]

In Proposition 5, de Sarasa pointed out that a geometric progression with first term a and common ratio r might be embedded in a geometric progression with first term a and common ratio $\sqrt{r}$, or common ratio $\sqrt[3]{r}$, $\sqrt[5]{r}$, or $\sqrt[100]{r}$ ("etc.," in de Sarasa's text, means in our terms $\sqrt[n]{r}$). This form of interpolation was more precise than that used by Napier, who had a continuous (and so noncomputational) model to give plausibility to his linear interpolations. It incorporated Briggs' method of taking repeated square roots to interpolate in a geometric progression and Kepler's $n - 1$ interpolations between a^n and b^n. It did not involve changing the hyperbola under discussion.

PROPOSITION 5. *Again, let there be given a series in continued proportion* A, B, C, D, *etc. I say that an infinity of series may be constructed of which the lines* A, B, C, D, *etc. are parts, namely those for which the ratio* $A : B$ *is the double, the triple, the quintuple, or the hundredfold, etc., of the ratio which the first term has to the second term of the other constructed series.*

Appealing to Proposition 2, de Sarasa established in Proposition 6 that if two adjacent hyperbolic areas were incommensurable then the lengths of their three bounding ordinates were not terms in any single geometric progression. Propositions 7 and 8 dealt with the contrary situation. De Sarasa reproduced the same diagram for Propositions 6 and 7 (Fig. 2).

PROPOSITION 6. *Let AB and AC be the asymptotes of the hyperbola DFH, and let three line segments DE, FG, and HC be parallel to the asymptote AB, and let them bound the hyperbolic areas DG and GH. Let the ratio of these areas be as the side of a square to its diameter, so that the areas are*

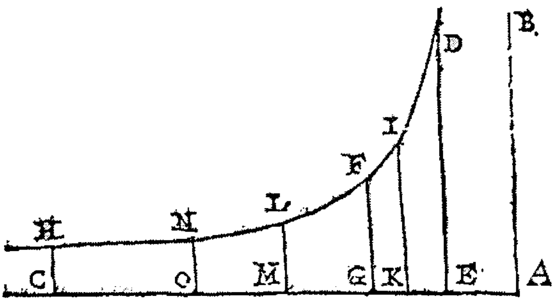

FIG. 2. Figure for Propositions 6 and 7.

incommensurable. I say that the line segment HC does not belong to any series whatever in which the line segments DE and FG are found.

PROPOSITION 7. *As before let AB and AC be the asymptotes of the hyperbola DFH; and let the lines DE, FG, and HC, parallel to the asymptote AB, cut off two commensurable areas DG and GH. I say that the line segments DE and FG are in a series of some ratio, in which the line segment HC can be found.*

PROPOSITION 8. *The hyperbola DFH with asymptotes AB and BC is given. The line segments DE, FG, HC are taken parallel to one of the asymptotes and contain the commensurable areas DG and GH. It is necessary to exhibit the greatest common measure of these areas.*

Proposition 8 was followed by two corollaries. The first said that the finding of a greatest common measure for two areas enabled this to be done for other areas. The second said that if there was no common measure then the areas were incommensurable. In Propositions 6, 7, and 8, de Sarasa determined precisely when there existed a geometric progression containing three given ordinate lengths; that is, when and only when the hyperbolic areas bounded by ordinates of these magnitudes were commensurable.

In Proposition 9 de Sarasa applied his Propositions 7 and 8 to the problem which he had raised in Proposition 4. Since the hyperbolic areas bounded by ordinates in geometric progression were equal in area, it sufficed to select any two of the ordinates in the progression (say A and B) and to declare whether the areas bounded by the three ordinates A, B, and F were commensurable or not.

PROPOSITION 9. *Given the progression series A, B, C, D, and an arbitrary magnitude F, which does not lie in the ratio series A, B, C, D, it is required to determine whether F is to be found in any series of which A, B, C, and D are part.*

In the Scholion following Proposition 9 he added:

Atque hinc patet ulterius non recte Problema a Mersenno fuisse propositum, *Datis tribus magnitudinibus, datisque duarum Logarithmis, tertiae Logarithmum Geometrice invenire*; planeque contra naturam Logarithmorum id peti, quod absolute semper exhiberi non potest.

And hence, it is furthermore evident that the Problem of Mersenne is not properly formulated, *given three magnitudes, and given the logarithms of two, to find the logarithm of the third geometrically*; that which is sought is plainly contrary to the nature of logarithms, and cannot always be absolutely exhibited.

For de Sarasa the "nature of logarithms" was discrete, and an answer to the challenge posed depended on the existence of a geometric progression containing all three numbers. From Proposition 6, this may not exist.

Having determined when a solution to Mersenne's problem was not possible, de Sarasa provided a solution in the case where, in his terms, a solution existed. This was the first point in his argument at which the hyperbola was not essential. An algebraic solution was readily to hand. If I, J, and K, are positive integers and $\log AR^I = i = a + Id$, $\log AR^J = j = a + Jd$, and $\log AR^K = k = a + Kd$, then $\frac{k-j}{j-i} = \frac{K-J}{J-I}$, which gives k in terms of i, j, I, J, and K. This solution might have been exhibited with similar triangles, but he had used the rectangular hyperbola at every stage so far, so he retained this convenient illustration with which to calculate the answer.

PROPOSITION 10. *Given three magnitudes A, B, and C, which can be shown in one and the same geometric progression, and given the logarithms of two of the three magnitudes, say those of A and B, to determine the logarithm of the third, C, geometrically.*

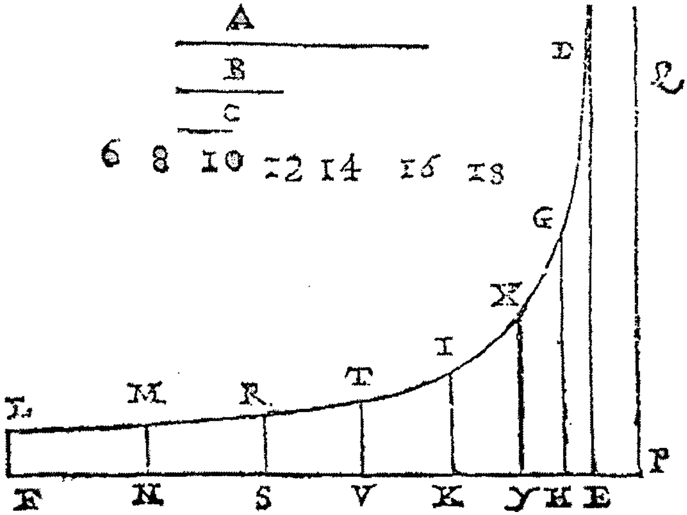

FIG. 3. Figure for Propositions 10.

De Sarasa's construction reflected the algebra described above. In Fig. 3, $GH = A$, $IK = B$, and $LF = C$. Since A, B, C were terms in some geometric progression, the areas GK and KL were commensurable, and NL was taken as their common measure. Now KL was four times the area of NL, and GK was twice the area of NL. The proportions of these areas were as the differences of the respective logarithms.

It was in the illustration of this result that the distinctive claim of de Sarasa was made:

Unde hae superficies supplere possunt locum logarithmorum datorum.	*Whence these areas can fill the place of the given logarithms*

or, more loosely, *hyperbolic areas are like logarithms*.

De Sarasa then gave two numerical illustrations, the first slightly garbled, the second spelt out in great detail. The detailed one takes $\log A = 6$, $\log B = 10$ and then deduces that $\log C = 18$. The garbled one seems to take $\log MN = 6$ and $\log TV = 10$ and to propose the computation of $\log DE$ (in fact 18, again).

DE SARASA AND NATURAL LOGARITHMS

Mersenne's question was an entirely general question about logarithms and assumed that the base of the logarithms (the common ratio of the geometric progression) was open and that the number whose logarithm is 0 was also open. De Sarasa's answer and his hyperbolic illustration of logarithms retained the full generality of Mersenne's question. Even Napier's logarithms, for example, are equal to hyperbolic areas if the appropriate hyperbola is chosen. The flexibility of hyperbolic curves to illustrate different systems of logarithms was pointed out in [Hutton 1822, 85].

Montucla [Montucla 1968 II, 82] chided de Sarasa for not adopting a continuous notion of logarithms and thereby providing a general solution to Mersenne's problem. Napier had considered a continuous model in his discussion of the invention of logarithms, and de Sarasa's argument depended on the continuity of his hyperbolas. Had de Sarasa had a continuous view of logarithms, then from $\log A = a$, $\log B = b$, and $\log C = c$, he could have deduced $\frac{c-b}{b-a} = \frac{H(B,C)}{H(A,B)}$, where $H(B, C)$ denotes the hyperbolic area bounded by ordinates with abscissae B and C. But the lack of a quadrature for the hyperbola would be a reason for not accepting the ratio of arbitrary hyperbolic areas as having geometrical rigour according to the statement of the original problem.

To get from de Sarasa's notion of logarithm to natural or hyperbolic logarithms, so named by Euler [Euler 1988 I, 97] (which logarithms a number of 20th century writers have attributed to St-Vincent or to de Sarasa, for example Toeplitz [Toeplitz 1963, 55–57], Edwards [Edwards 1979, 156], Dhombres *et al.* [Dhombres *et al.* 1987, 188], and Katz [Katz 1998, 492]), six steps must be taken.

1. The hyperbolic areas must be seen as related to the abscissae of their vertices rather than their bounding ordinates. This is the only one of the six steps which would have been a trivial adjustment for de Sarasa.
2. The hyperbolic *illustration* must become the basis of a construction or a *definition*.
3. It is necessary to adopt $\log 1 = 0$. This pinpoints the place from which areas are measured and guarantees that $\log AB = \log A + \log B$.
4. A convention for measuring areas, left to right or right to left, must be adopted under which areas are signed and logarithms are increasing. (Of the five examples of logarithms which de Sarasa gives, three are decreasing, two are increasing, and all can be illustrated with the same hyperbola.)
5. The hyperbola must be taken as $y = 1/x$, which determines the base of the logarithms. There is no way of telling from de Sarasa's text whether his hyperbola was $y = 1/x$, $y = 2/x$, or even $y = 1/(\ln R)x$, the last of which, with modern conventions, gives logarithms to the base R. (De Sarasa's logarithms retain the generality of Briggs' definition: $AR^x \leftrightarrow a + xd = \int_{AR^{-a/d}}^{AR^x} \frac{d}{\ln R}\frac{dt}{t}$.)
6. The logarithm must be presumed to be defined on the continuum.

There seems to be no doubt that all these steps were soon taken by de Sarasa's contemporaries, though even in 1691, Leibniz, a careful reader of St-Vincent, defined logarithms discretely, just as Briggs or de Sarasa would have done, giving $\log 1 = 0$ as one possibility amongst others [Leibniz 1993, 94–95].

Perhaps Mengoli (1650) was the first to identify the natural logarithms of positive integers. His identification was by an arithmetical limit. For a positive integer a he defined two functions:

$L(a, n) = 1/n + 1/(n+1) + \cdots + 1/(na-1)$, which he called the hyperlogarithm of a,

and

$l(a, n) = 1/(n+1) + 1/(n+2) + \cdots + 1/na$, which he called the hypologarithm of a.

L decreases as n increases. l increases as n increases. $L(a, n) > l(a, n)$ and $L(a, n) - l(a, n)$ tends to 0 as n increases, so a limit $L(a)$, the logarithm of a, is defined as the limit of one or other of the functions. $L(ab, n) = L(a, n) + L(b, an)$ and the modern logarithmic property follows [Naux 1971, 45]. An extension of this definition to rational numbers $a > 1$ is described in [Whiteside 1961, 224] as found in *Geometria speciosa* [Mengoli 1659].

It was in his notebooks of 1667 that Nicolaus Mercator, while investigating $1/(1 + x)$ (for $x = 0.1$ and 0.21) for his second treatment of logarithms, incorporated de Sarasa's awareness of the relation between logarithms and hyperbolic areas. This treatment was published in *Logarithmotechnica* [Mercator 1668; Cajori 1919, 188].

Felix Klein [Klein 1945, 85 and 268] proposed that history be used to help shape the teaching of mathematics and also proposed that $\int_a^b \frac{dx}{x} = \int_{ca}^{cb} \frac{dx}{x}$ be used as the basis of defining the logarithm in school [Klein 1911, 350; 1945, 156]. The two suggestions were not linked in his lectures. Although de Sarasa did not mention this property of logarithms (which is in fact valid in the form $\log b - \log a = \log cb - \log ca$ for all logarithmic systems, ancient and modern), results equivalent to it were developed in Propositions 105, 112, 113, 114, 115, 116, and 117 of Part 4 of Book 6 of *Opus Geometricum* [St-Vincent 1647], mostly under the guise of establishing equal segments of a hyperbola at the two ends of a pair of parallel chords, and without reference to geometric progressions. Possibly pursuing Cantor's attribution of natural logarithms to St-Vincent in all but name [Cantor 1907 II, 896], Toeplitz (in his programme to motivate calculus by posing the historical problems which generated the subject) claimed these propositions as the historical origin of natural logarithms [Toeplitz 1949, 53–55; 1963, 55–57], appending his own proof. (The editor of [Toeplitz 1963], enhanced Toeplitz's proof with a simple footnote.)

Propositions 111 and 112 of *de Hyperbola*, taken together, give Toeplitz's claim from St-Vincent, which Toeplitz then applied to obtain natural logarithms. Toeplitz's claim and its application have been followed by others, who have transferred some of the credit to de Sarasa, [Edwards 1979, 155] and [Katz 1998, 491–492], for example. This rational reconstruction of history has been fed more by the student's need to focus on the relation $\log AB = \log A + \log B$ than by de Sarasa's concern with geometric progressions.

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