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Developing into series and returning from series: A note on the foundations of eighteenth-century analysis

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Abstract

In this paper we investigate two problems concerning the theory of power series in 18th-century mathematics: the development of a given function into a power series and the inverse problem, the return from a given power series to the function of which this power series is the development. The way of conceiving and solving these problems closely depended on the notion of function and in particular on the conception of a series as the result of a formal transformation of a function. After describing the procedures considered acceptable by 18th-century mathematicians, we examine in detail the different strategies—both direct and inverse, that is, synthetic and analytical—they employed to solve these problems.

Sommario

In quest'articolo vengono analizzati due problemi relativi alla teoria delle serie di potenze nel secolo diciottesimo: lo sviluppo di una funzione in serie di potenze e il problema inverso, il regresso dalla serie alla funzione di cui tale serie è lo sviluppo. Il modo in cui questi problemi erano concepiti e risolti dipendeva dalla nozione di funzione e, in particolare, alla concezione di una serie come il risultato di una trasformazione formale di una funzione. Dopo aver caratterizzato le procedure di sviluppo considerate accettabili, vengono esaminate le differenti strategie—dirette e inverse, ovvero sintetiche e analitiche—usate per risolvere tali problemi.

Résumé

Dans cet article nous étudions deux problèmes concernant la théorie des séries entières au XVIII[ème] siècle : le développement d'une fonction donnée en une série entière et le problème inverse, le retour d'une certaine série entière à la fonction dont cette dernière est le développement. La manière dont ces problèmes étaient conçus et résolus tenait à la notion de fonction, et en particulier à la conception d'une série comme le résultat d'une transformation formelle d'une fonction. Après avoir présenté et discuté les différentes procédures de développement employées par ces mathématiciens, nous examinons avec plus de détail les différentes stratégies

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de solutions de ces problèmes, en distinguant entre procédures directes et procédures inverses, c'est-à-dire synthétiques et analytiques.

MSC: 01A50

Keywords: Function; Series; Convergence; Analysis/synthesis; Direct/inverse methods

1. Introduction

The 18th-century theory of series is the subject matter of several studies, which approach the topic in different ways. Some of them insist on the main results; they show how and when such results were reached, but seem to dismiss the early procedures as naive or meaningless and to recast them directly in terms of modern formalisms.[1] Others highlight how certain results can be interpreted in terms of modern special theories (nonstandard analysis or summability theory) and understand the results in this later context.[2] Finally, there are some writings which investigate the foundation and internal motivations of 18th-century theories.[3] Following this last approach, in the present paper we shall advance some historiographical theses which should serve as a possible key to the reading of 18th-century mathematical texts.

In the first part, we shall endeavor to establish the actual meaning of equalities[4] of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \tag{1}$$

in the 18th century. There exists, indeed, a radical difference between modern and 18th-century conceptions of series: even the fundamental terms, such as "function," "series" and "equality," have significantly different meanings.

In the second part, we shall consider the problem of developing a function into a series and suggest that the problem of summing a series was conceived merely as the inverse problem since it was viewed as the problem of the return from the series to the function. The relation between the problems of the sum and development was inverted with respect to today.

Finally, we shall investigate how these two problems were treated and try to classify different strategies for solving them.

In our inquiry, we shall attempt to identify those elements that seem to constitute evidence of a shared conception with respect to the foundation of analysis in the 18th century and therefore focus our attention on common elements in the works of the major mathematicians who dealt with series.[5] We shall not

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[1] See, for example, Dutka [1984–1985].

[2] See, for example, McKinzie and Tuckey [1997].

[3] See, for example, Fraser [1985] and [1989].

[4] In this paper, unless indicated otherwise, we shall use the term "equality" as a generic term to denote any expression in which there are two members connected by the symbol "=," independent of the specific meaning this symbol possesses in different cases.

[5] Fraser [1989] already tried "to identify as clearly as possible those elements that are common" in 18th-century analysis; according to him "these elements constitute evidence of a shared conception significantly different from the modern one"

discuss the differences between these mathematicians. Besides, we shall restrict ourselves to power series. Power series were not the only series considered in 18th-century analysis; however, they were largely dominant.[6]

2. Convergence and power series

It is well known that in the first half of the 18th century analysis gradually developed as a general theory of functions and was finally expounded as an organic theory by L. Euler in his *Introductio in analysin infinitorum* in 1748. The essential novelty of Euler's treatise consisted in the introduction of functions as autonomous objects and the construction of a comprehensive theory of these objects. However, according to 18th-century mathematicians, a function[7] was not an association between the elements of two given sets: it was a symbolic notation (which was termed "analytical expression," "formula," or "form") expressing a quantity in terms of another quantity.[8] It was not merely an expression, but the expression of a certain quantity, or else a function was a quantity as long as it was expressed, or could be in principle expressed, by a certain symbolic notation. Although mathematicians endeavored to enlarge the set of known functions, they always seemed to reason as if the set of functions was somehow fixed *a priori* by means of a genetic definition according to which a function had to derive from a finite number of elementary functions by applying a finite number of combination rules.[9] As long as it was conceived as an expression, a function was thought of as a finitary composition of two sorts of atomic symbols: the atomic symbols for constant or variable quantities (i.e., $a, b, \ldots; x, y, \ldots; 0, 1, \ldots$; etc.) and the atomic symbols for the elementary operations on these quantities. As there were a finite number of elementary operations (i.e., algebraic elementary operations, logarithm, exponential and trigonometric direct and inverse operations), a function was thus conceived as a composition of a finite number of elementary functions. It was conceived to be the expression of a quantity since these elementary functions

[1989, p. 318]. With reference to 18th-century analysis, Fraser mainly refers to Euler's and Lagrange's conceptions but seems to suppose that these conceptions were largely shared by the entire mathematical community during the period that began roughly in the 1740s and lasted till the first years of the 19th century. We agree with this opinion and would like to add some elements to Fraser's reconstruction, especially insisting on the earlier roots of such a "shared conception." Thus, we shall use the term "18th century" in a quite large sense, to refer to a period in the history of mathematics approximately starting from Newton's and Leibniz's research, and finishing with Lagrange's proposal to found the calculus on Taylor's expansions.

6 Apart from power series, from the end of the 17th century to about the 1740s, mathematicians used only series of the form $\sum_{n=1}^{\infty} a_n x^{\alpha_n}$, where α_n could be a negative integer or (in exceptional cases) a rational number. Only from the 1740s did other function series and in particular trigonometric series begin to be examined. The concepts and techniques originating in power series were applied to trigonometric series too; very interesting examples are in some of Euler's papers, such as [1773]. However, in certain cases, this application was rather problematic.

7 On the concept of a function, see Fraser [1989], Panza [1996], Ferraro [2000a]. Here we limit ourselves to a short summary.

8 By "quantity" 18th-century mathematicians meant what can be increased or decreased. The most convenient means of representing a quantity in modern mathematical terms is by means of real values (and we shall also use this representation, for the sake of simplicity). However, we should not imagine a quantity as an element of a given well-defined set such as $\mathbb{R}$, since this notion was lacking in 18th-century mathematics.

9 We shall use the term "function" even when we refer to authors, such as Newton or de Moivre, who never used this term. It seems to us that this terminological anachronism simplifies the exposition provided a "function" is understood in the terms outlined above.

were thought of as expressions of quantities and the rules of composition were conceived as conservative with respect to such a property of elementary functions.

This concept of a function implied that infinite series, as such, were "not themselves regarded as functions" [Fraser, 1989, p. 322]: they were instruments for facilitating the study of functions and for rendering them more intelligible (see Euler [1748, §.59]). During the 18th century, "infinite series were never introduced arbitrarily" (see Fraser [1989, p. 321]): they always arose in some definite way in a particular mathematical problem, process or procedure.

Power series were conceived of as quasi-polynomial entities (that is, mere infinitary extensions of polynomials). Even the symbolism was ambiguous and suggested this idea. Generally speaking, series were denoted by "$a + b + c + d + \&c.$" or "$a + bx + cxx + dx^3 + \&c.$," but the symbol "$\&c.$" was also used in some cases to denote a finite number of terms. The ambiguity of the notation depended on the fact that 18th-century mathematicians considered a series as being known when one could explicitly exhibit its first terms and knew the law for deriving the following ones. In many cases it did not matter whether or not, starting from a certain point, the terms were all equal to zero. For instance, the product of two series was not openly defined: it seemed obvious that

$$(a + bx + cxx + dx^3 + \&c.) \cdot (A + Bx + Cxx + Dx^3 + \&c.)$$

was equal to

$$\begin{aligned}
&aA \\
&+ (aB + Ab)x \\
&+ (aC + bB + cA)x^2 \\
&+ (aD + bC + cB + dA)x^3 \\
&+ \&c.,
\end{aligned}$$

independent of the meaning of "$\&c.$" in such expressions: the rule of ordinary multiplication between two polynomials was extended to infinite series without making a distinction between finite and infinite series.

This approach could lead us to think that series were considered as entirely formal objects, but the matter is different. To make this clear, let us consider two examples.

In *De vera proportione*, Leibniz (see Leibniz [1682, p. 44]; on Leibniz's theory of series see Ferraro [2000c]) argued that $\frac{\pi}{4}$ is equal to $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$ and justified it by observing that if we take the first term of this series, then $\frac{\pi}{4}$ is approximated with an error less than $\frac{1}{3}$, if we take the first two terms of this series, the error is less than $\frac{1}{5}$, etc. If the series is continued, the error becomes less than any given quantity and thus the whole series contains all approximations and expresses the exact value.

In his famous *Epistola posterior* to Leibniz of October, 24, 1676 (see [Newton C, II, pp. 110–161]), Newton considered several applications of the binomial expansion, which he wrote in the form

$$(P + PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m}{n}} Q + \frac{m(m-n)}{2n^2} P^{\frac{m}{n}} Q^2 + \&c. \tag{2}$$

In the case of the function $\sqrt[5]{c^5 + c^4x - x^5}$ he first put $P = c^5$ and $Q = \frac{c^4x - x^5}{c^5}$ and obtained

$$\sqrt[5]{c^5 + c^4x - x^5} = c + \frac{c^4x - x^5}{5c^4} - \frac{2c^8x^2 - 4c^4x^6 + 2x^{10}}{25c^9} + \&c.;$$

then he put $P = -x^5$ and $Q = -\frac{c^4x+c^5}{x^5}$ and obtained

$$\sqrt[5]{c^5 + c^4x - x^5} = -x + \frac{c^4x + c^5}{5x^4} + \frac{2c^8x^2 + 4c^9x + 2c^{10}}{25x^9} + \&c.$$

Finally, he observed that the first procedure is preferable when x is very small, the second when it is very large.

This shows that, from the very origin of the theory of series, mathematicians were aware that certain series provide a close approximation of certain quantities, when a convenient number of their terms is considered, whereas this is not the case for other series. In this primordial sense, they were thus concerned with convergence. (We shall later argue that this was not inconsistent with considering series as quasi-polynomial entities in the previous sense.) Mathematicians of the 18th century also knew that the convergence of a power series $\sum_{i=0}^{\infty} a_i x^i$ depended on the value of its variable x. Of course, they did not possess the modern notion of interval of convergence, mainly because of the absence of any object such as the set $\mathbb{R}$ of real numbers.[10] Nevertheless, it seems to us that the term "interval of convergence" can be conveniently used, provided one takes into account that when referring to the interval over which the power series $\sum_{i=0}^{\infty} a_i x^i$ converges, we are not referring to a subset of $\mathbb{R}$, but merely to the fact that the series is convergent if the variable x varies from $-\delta$ to δ, where δ is an appropriate positive value. We shall also use the expression "non-null interval" to underline that the domain of variation of x does not reduce only to one value and, specifically, does not reduce to the single value $x = 0$.

There are difficulties with the term "convergence," too. Even if 18th-century mathematicians had generally no difficulty in distinguishing series that converge from others that did not converge, in the previous primordial sense,[11] they often used the terms "convergent" and "series" in an ambiguous way. Here we shall not classify and discuss the different meanings given to these terms in the 18th century. Later we shall use the term "convergent series" to refer to a series that satisfies the following condition:

($\mathbf{C}_0$) A power series $\sum_{i=0}^{\infty} a_i x^i$ is said to be convergent to $f(x)$ on a non-null interval I of the values of x if and only if, for any value α of x belonging to I, the sequence $\{\sum_{i=0}^{j} a_i \alpha^i\}_{j=0}^{\infty}$ approaches $f(\alpha)$ indefinitely when j increases and it is finally equal to $f(\alpha)$, when j is a infinite number.

At this juncture, some remarks are appropriate.

First, it is clear from the texts that 18th-century mathematicians considered this condition as salient and knew how to distinguish series depending on whether they satisfied ($\mathbf{C}_0$) or not. However it is certainly not

[10] Cf. the previous note (8).

[11] In certain cases, 18th-century mathematicians considered series that we today refer to as "divergent" to be equal to a finite quantity. The most famous example is that of series $1 - 1 + 1 - 1 + 1 - \&c.$ that Grandi and Leibniz (cf. Grandi [1703], Leibniz [1713] had taken as being equal to $\frac{1}{2}$ (for a discussion of such an example, cf. Panza [1992, Ch. III.1]; for other examples see Ferraro [2000b, 2002]). However, pretending that a series was equal to a finite quantity was not the same as asserting that it had a sum. We shall abundantly come back to this later.

a precise condition, and it is not possible to formulate it in more precise terms without adding elements which were essentially alien to 18th-century analysis.[12]

Second, a power series was considered as being the expression of a quantity (for any value of x belonging to I) if and only if it was considered to satisfy ($\mathbf{C}_0$). For instance, the series $\sum_{i=0}^{\infty}(-1)^i x^i$ expressed the ordinate of a hyperbola for certain values of x, as long as it was convergent to $\frac{1}{1+x}$ for these values of x.

Third, 18th-century mathematicians thought that even if the series $\sum_{i=0}^{\infty} a_i x^i$ converged to a function $f(x)$ only on a non-null interval of values of x, the relation $f(x) = \sum_{i=0}^{\infty} a_i x^i$ could be manipulated without regard to the interval of convergence. They did not limit the validity of this equality to the interval over which the series converged to the function.[13] For instance, though it was well known that the series $\sum_{i=0}^{\infty}(-1)^i x^i$ converges only for $|x| < 1$, the relation $\frac{1}{1+x} = \sum_{i=0}^{\infty}(-1)^i x^i$ was freely used in manipulations, without being restricted to $|x| < 1$. Thus, the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ stated a general result which concerned the formal nature of the function $f(x)$ and not the convergence of the series $\sum_{i=0}^{\infty} a_i x^i$.

3. The development of functions into series

At this juncture, a very natural question arises: What did the sign "$=$" mean in the equality (1)?

To answer such a question, we first consider the simpler case of the equality $f(x) = g(x)$ between two finite analytical expressions $f(x)$ and $g(x)$.

In the 18th century the equality $f(x) = g(x)$ meant that one of these expressions, say $g(x)$, resulted from a transformation of the other one. In Chapters 2 and 3 of the *Introductio*, Euler investigated the transformation of functions. According to him, "Functions are transmuted into other forms either by introducing another variable quantity instead of the one initially used, or retaining the same variable quantity."[14] For instance, the expression $2 - 3z + z^2$ becomes $(1 - z)(2 - z)$ by factoring and $\sqrt{a + bz}$ is transformed into bx by substituting $bx^2 - \frac{a}{b}$ for z.

[12] In particular the concept of limit had not been defined in mathematical terms and, moreover, there was ambiguity concerning when a limit had been reached. On the notions of "limit-achieving" and "limit-avoiding," see Grattan-Guinness [1969–1970].

[13] It is known that Euler dealt with power series such as

$$x^m - px^{m+q} + p(m+q)x^{m+2q} - p(m+q)(m+2q)x^{m+3q} + \cdots$$

or

$$1 - 2!x + 3!x^2 - \cdots,$$

which does not converge over any non-null interval (for instance, cf. Euler [1754–1755]). The investigation of these series originated in the attempt to solve certain differential equations or to calculate certain integrals by continued fractions. While a series convergent over a non-null interval was considered as the development of a certain function and was thus used to express or study quantities, totally divergent series were only considered as tools for relating integrals or differential equations with continued fractions, that is as formal links between different expressions of a quantity. On the relationship between continued fractions and divergent series in Euler, see Ferraro [2000b].

[14] See Euler [1748, I, p. 32]: "Functiones in alias formas transmutantur vel loco quantitas variabilis aliam introducendo vel eandem quantitatem variabilem retinendo."

This is also the case for the equality (1). The sign "=" interposed between a function and a series meant that the series was derived from the function by means of certain rules of transformation. Thus, the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ meant[15] that the power series $\sum_{i=0}^{\infty} a_i x^i$ was associated with the function $f(x)$ and that such an association depended on the fact that this power series resulted from operating on the expression $f(x)$ according to certain rules of transformation.[16] We shall express this fact by saying that the series $\sum_{i=0}^{\infty} a_i x^i$ is the development of $f(x)$.

As a consequence, the equality (1) was not logically symmetrical since the two expressions $f(x)$ and $\sum_{i=0}^{\infty} a_i x^i$ played different roles in such an equality. The first directly expressed a quantity and had a meaning *per se*; it was the proper object[17] of 18th-century analysis. The second was simply the result of a transformation of the given function $f(x)$. A series $\sum_{i=0}^{\infty} a_i x^i$ expressed a certain quantity only indirectly, since it was associated with the function $f(x)$ expressing this quantity. Therefore, the left-hand side, $f(x)$, of the equality (1) established the real object to be investigated, while the right-hand side, $\sum_{i=0}^{\infty} a_i x^i$, merely exhibited the result of a transformation useful to investigate the function in the left-hand side.

In speaking of certain rules of transformation, we mean a number of explicitly stated rules or a finite combination of them. Thus, a power series was associated with a given function and indicated as being equal to it if and only if it was derived from this function by means of the application of one of these rules or of a finite combination of them.

Eighteenth-century mathematicians presented the accepted procedures for the development of a function in different ways. In *De analysis*, composed in 1671 but only published in 1711,[18] Newton presented two procedures for expanding a given function in a power series. These procedures are generally known as Mercator's rules, since particular cases of them had already been used by N. Mercator in his *Logarithmotechnia*.[19] They consisted of the application of the arithmetical rules for dividing a number by another number, or for extracting a root of a given number to literal expressions (see below).

[15] It should be clear that we use the symbols "$f(x)$" and "$\sum_{i=0}^{\infty} a_i x^i$" in order to refer to any particular functions or power series. The relation expressed by (1) should be understood as a relation between a particular function and a particular power series, whatever these functions and series are.

[16] Let us note that, according to such a condition, the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ has a precise sense independent of its interpretation as an identity (one would have an identity if "$f(x)$" and "$\sum_{i=0}^{\infty} a_i x^i$" were considered as two different notations of the same object), or even as an equivalence (this case would occur if "$f(x)$" and "$\sum_{i=0}^{\infty} a_i x^i$" were considered as two names for distinct objects belonging to a common class of equivalence). The essential reason for this is that (1) does not concern primarily the object denoted by the symbols "$f(x)$" or "$\sum_{i=0}^{\infty} a_i x^i$", but these symbols themselves. It states that the finitary expression "$f(x)$"—i.e., not the quantity that this expression expresses, but this expression itself—has a certain relation (i.e., the relation of being transformable in) with the expression "$\sum_{i=0}^{\infty} a_i x^i$". Only once this relation between these two expressions had been stated, could one interpret (1) as being concerned with the quantity expressed by $f(x)$ (if the condition of convergence were satisfied).

[17] As a proper object of a certain (mathematical) theory, we mean an object whose conditions of identity do not depend on the conditions of identity of some other object, and thus constitutes the genuine matter of investigation of such a theory. Opposed to proper objects are conditional ones. They are conceived as forms of expressions of the proper objects and they are thus studied within this theory because of their power to express them. Paradigmatic examples of proper and conditional objects are respectively curves and expressions in Descartes's geometry, or, as we maintain here, functions and series in 18th-century analysis. For these notions, cf. Panza [1997b].

[18] See [Newton MP, II, pp. 206–247], for the original version and Newton [1711] for the published text. We refer here to [Newton MP, II, pp. 210–219].

[19] See Mercator [1668].

In *De methodis*, composed in 1671 but only published in an English translation in 1736,[20] Newton presented another procedure, known as Newton's method of the parallelogram, to express by means of a series the solution of a given algebraic equation $P(x, y) = 0$.[21] The crucial idea of this procedure was the following: by substituting the indeterminate series $\sum_{k=0}^{\infty} b_k x^{\alpha_k}$ for y in $P(x, y)$ one obtains a new polynomial $Q(x)$, where all the coefficients of the powers x^{α_k} must be separately equal to zero. The method of the parallelogram was a method for determining the coefficients b_k ($k = 0, 1, \ldots$) in the series $\sum_{k=0}^{\infty} b_k x^{\alpha_k}$ under such a condition, supposing that the value of x is close to a certain given value (for example 0). In short, Newton reduced the given equation in such a way that the coefficients b_k could be determined step by step. He thus obtained a series convergent on the given interval. What is important here is not the specific nature of this method (it is well known), but the general principle on which it is founded. This principle, generally known as the principle of undetermined coefficients, states that a series $\sum_{k=0}^{\infty} b_k x^{\alpha_k}$ is equal to 0 for every x on a non-null interval (if and) only if all the coefficients b_k ($k = 0, 1, \ldots$) are separately equal to zero.

Generally speaking, we can classify the accepted procedures of development into two classes. The first class comprises

($\mathcal{P}_1$) The Mercator expansions of fractions and square roots of polynomials.
($\mathcal{P}_2$) The binomial expansion for any exponent.
($\mathcal{P}_3$) Any expansion following the method of undetermined coefficients.

Consider first the Mercator expansions. We have already observed that they arose from applying the usual rules of division and extraction of square root of numbers to literal expressions. Take, for instance, the fraction $\frac{a^2}{b+x}$ (see [Newton MP, II, pp. 212–214, and III, pp. 36–38]). By dividing a^2 by $b + x$, one obtains the quotient $\frac{a^2}{b}$ and the remainder $-\frac{a^2}{b}x$. By dividing such a remainder by $b + x$, one obtains the quotient $-\frac{a^2}{b^2}x$ and the remainder $\frac{a^2}{b^2}x^2$. By continuing *ad infinitum* one obtains the series $\frac{a^2}{b} - \frac{a^2}{b^2}x + \frac{a^2}{b^3}x^2 - \cdots$ and the equality

$$\frac{a^2}{b+x} = \frac{a^2}{b} - \frac{a^2}{b^2}x + \frac{a^2}{b^3}x^2 - \&c. \tag{3}$$

An analogous procedure can be applied in order to determine the terms of the development $\sum_{i=0}^{\infty} a_i$ of a square root, say $\sqrt{p+q}$ (see [Newton MP, II, pp. 214–216, and III, pp. 40–42]). One takes first $\sqrt{p}$ as being the first term a_0 and calculates the first remainder $R_0 = p + q - a_0^2 = q$. Then, one calculates the following terms by using the recursive rules

$$a_i = \frac{R_{i-1}}{2a_0} = \frac{R_{i-1}}{2\sqrt{p}}, \qquad R_i = R_{i-1} - \left[2a_0 a_i + \sum_{j=1}^{i} a_j a_{i-j+1}\right],$$

[20] See [Newton MP, III, pp. 3–372] for the original version and Newton [1736], for the published text. We refer here to [Newton MP, III, pp. 51–57].

[21] A similar procedure had been already presented in *De analysis*: see [Newton MP, II, pp. 218–233].

which give

$$\begin{array}{ll}
a_1 = \dfrac{q}{2\sqrt{p}}, & R_1 = q - \left[q + \dfrac{q^2}{4p}\right] = -\dfrac{q^2}{4p}, \\
a_2 = -\dfrac{q^2}{8p\sqrt{p}}, & R_2 = -\dfrac{q^2}{4p} + \left[\dfrac{q^2}{4p} + \dfrac{q^3}{8p^2}\right] = \dfrac{q^3}{8p^2}, \\
a_3 = \dfrac{q^3}{16p^2\sqrt{p}}, & R_3 = \dfrac{q^3}{8p^2} - \left[\dfrac{q^3}{8p^2} + \dfrac{5q^4}{64p^3}\right] = -\dfrac{5q^4}{64p^3}, \\
a_4 = -\dfrac{5q^4}{128p^3\sqrt{p}}, & R_4 = -\dfrac{5q^4}{64p^3} + \left[\dfrac{5q^4}{64p^3} + \dfrac{7q^5}{128p^4}\right] = \dfrac{7q^5}{128p^4}, \\
a_5 = \dfrac{7q^5}{256p^4\sqrt{p}}, & R_5 = \dfrac{7q^5}{128p^4} - \left[\dfrac{7q^5}{128p^4} + \dfrac{21q^6}{512p^5}\right] = -\dfrac{21q^6}{512p^5}, \\
\&c., & \&c.
\end{array}$$

and so:

$$\sqrt{p+q} = \sqrt{p} + \frac{q}{2\sqrt{p}} - \frac{q^2}{8p\sqrt{p}} + \frac{q^3}{16p^2\sqrt{p}} - \frac{5q^4}{128p^3\sqrt{p}} + \frac{7q^5}{256p^4\sqrt{p}} - \&c. \tag{4}$$

The equalities (3) and (4) could also be easily obtained by applying the binomial expansion (2), or better its simplified form,

$$(p+q)^r = p^r + rp^{r-1}q + \frac{r(r-1)}{2!}p^{r-2}q^2 + \frac{r(r-1)(r-2)}{3!}p^{r-3}q^3 + \&c., \tag{5}$$

where r is any rational exponent. When obtained by means of Mercator's procedures, the equalities (3) and (4) are, however, directly extracted from the given expressions $\frac{a^2}{b+x}$ or $\sqrt{p+q}$ by operating on such expressions, while, when obtained by means of binomial expansions, they result from a particularization of the general equalities (5) which has, in its turn, to be proved. Hence, as long as they were obtained by means of Mercator's procedures, the equalities (3) and (4) were viewed as particular confirmations of such a general equality, rather than as a consequence of it. After Newton, nobody really doubted the validity of (5) or was reluctant to apply it in order to get the development of particular functions. Nevertheless, much effort was devoted to providing this equality with a proof more satisfying than Newton's argument in support of it (which finally relied on an *a priori* assumption of the same extension of algebraic rules that (5) seem to guarantee), or to prove its generalization to irrational exponents. A simple way to do that would have been to derive (5) from "Taylor's theorem":

$$f(x+\xi) = f(x) + \frac{df}{dx}\xi + \frac{1}{2!}\frac{d^2f}{dx^2}\xi^2 + \frac{1}{3!}\frac{d^3f}{dx^3}\xi^3 + \&c. \tag{6}$$

However, this was not considered as acceptable since (5) and its particular consequences were thought to be independent of the differential calculus and the rules of differentiation of the elementary functions depended on (5).

Another way for obtaining many developments of particular functions—including the equalities (3) and (4)—by operating directly on these functions was to resort to the principle of undetermined coefficients. Unlike Mercator's procedures, this principle allowed one to determine a development in power series whose existence was previously supposed: one started from the hypothesis that the given function $f(x)$ could be developed in a power series and relied on such a principle in order to determine (or construct) this series. An example of such a procedure is found in Stirling's *Methodus differentialis*, where it is used in order to develop the function $\frac{1}{A+Bx+Cx^2}$ (see Stirling [1730, p. 2]). Stirling supposed that

$$\frac{1}{A+Bx+Cx^2}=\sum_{i=0}^{\infty}a_i x^i$$

and assumed that the consequent equality

$$\left[\sum_{i=0}^{\infty}a_i x^i\right]\left[A+Bx+Cx^2\right]-1=0$$

should hold for any x in a non-null interval. By multiplying and rearranging he derived

$$(Aa_0-1)+(Aa_1+Ba_0)x+(Aa_2+Ba_1+Ca_0)x^2+(Aa_3+Ba_2+Ca_1)x^3+\&c.=0.$$

Finally, by applying the principle of undetermined coefficients, he obtained the equations

$$\begin{aligned} Aa_0-1&=0,\\ Aa_1+Ba_0&=0,\\ Aa_2+Ba_1+Ca_0&=0,\\ Aa_3+Ba_2+Ca_1&=0,\\ &\&c., \end{aligned}$$

which allowed him to determine the coefficients:

$$a_0=\frac{1}{A};\quad a_1=-\frac{B}{A^2};\quad a_2=\frac{B^2-AC}{A^3};\quad a_3=\frac{2ABC-B^3}{A^4};\quad \&c.$$

The principle of undetermined coefficients is here employed to find a development of the given function as a power series. One supposes that this function has such a development and, by means of the principle, explicitly constructs it. This procedure may easily be justified as being a simple extension of algebraic rules, if one assumes that the undetermined series $\sum_{i=0}^{\infty}a_i x^i$ which is initially supposed to be equal to the given function converges to this function in a non-null interval of values of x, since, if it is so and the supposed equality $f(x)=\sum_{i=0}^{\infty}a_i x^i$ is transformed by algebraic manipulations in another equality $\sum_{i=0}^{\infty}b_i x^i=0$, then this latter equality should hold both for $x=0$ and for some x different from 0. It is thus enough to put first $x=0$ to obtain $b_0=0$, then to divide this equality by x and afterward to put again $x=0$ to obtain $b_1=0$, and so on. By operating in such a way, it is also possible, under the same condition, to prove that the development of $f(x)$ that is thus obtained is unique. A similar proof is found in the first volume of Euler's *Introductio* (see Euler [1748, I, pp. 230–231]; but see also Euler [1740, p. 471]. From the suppositions

$$f(x)=\sum_{i=0}^{\infty}a_i x^i,$$

$$f(x)=\sum_{i=0}^{\infty}b_i x^i,$$

it follows in fact that

$$\sum_{i=0}^{\infty}(a_i - b_i)x^i = 0$$

and then, successively:

$$a_0 = b_0; \quad a_1 = b_1; \quad a_2 = b_2; \quad a_3 = b_3; \quad \&c.$$

The procedures ($\mathcal{P}_1$), ($\mathcal{P}_2$), and ($\mathcal{P}_3$) are all concerned with an infinitary extension of algebraic rules and, therefore, we call them "*quasi*-algebraic" procedures. A second class of procedures consisted of:

($\mathcal{P}_4$) Any expansion deriving from simultaneous differentiation or integration both of a certain function $f(x)$ and a certain determinate power series $\sum_{i=0}^{\infty} a_i x^i$ already associated with $f(x)$, or of a function $f(x)$ and a certain undetermined power series $\sum_{i=0}^{\infty} a_i x^i$ assumed to be associated with $f(x)$—the operations on the power series being performed term by term.

Like ($\mathcal{P}_3$), these procedures also depend on the supposition that the series which is associated with the given function converges to it on a non-null interval. Moreover, the procedures ($\mathcal{P}_4$) also depend on an infinitary extension of the properties of linearity of differentiation and integration (today we know that they do not follow from simple convergence). In Chapter 2 of the second part of the *Institutiones calculi differentialis* (see Euler [1755, p. 235]), Euler justified this supposition for differentiation by asserting that from $f(x) = \sum_{i=0}^{\infty} a_i x^i$ it follows that

$$\begin{aligned} df(x) &= f(x+dx) - f(x) = \sum_{i=0}^{\infty} a_i(x+dx)^i - \sum_{i=0}^{\infty} a_i x^i \\ &= \sum_{i=0}^{\infty} a_i\left[(x+dx)^i - x^i\right] = \sum_{i=0}^{\infty} a_i i x^{i-1}\, dx. \end{aligned}$$

In Chapter 3 of the *Institutiones calculi integralis* (see Euler [1768–1770, Vol. 1, pp. 76–85]), he relies on an analogous rule for integration in order to state that the integral of a function whose development is $\sum_{i=0}^{\infty} a_i x^{m+in}$ is equal to $\sum_{i=0}^{\infty} \frac{a_i}{m+in+1} x^{m+in+1}$.

The first of these rules was used by Newton in a preliminary version of the *De quadratura curvarum*, in order to obtain the first version of Taylor's development of a function (cf. [Newton MP, VII, pp. 96–98]). If one puts

$$f(x) = \sum_{i=0}^{\infty} A_i(x-a)^i,$$

repeated term-by-term differentiation yields

$$\frac{df}{dx} = \sum_{i=0}^{\infty} A_i i(x-a)^{i-1},$$

$$\frac{d^2 f}{dx^2} = \sum_{i=0}^{\infty} A_i i(i-1)(x-a)^{i-2},$$

$$\frac{d^3 f}{dx^3} = \sum_{i=0}^{\infty} A_i i(i-1)(i-2)(x-a)^{i-3},$$

&c.

and then, by setting $x = a$:

$$A_1 = \left.\frac{df}{dx}\right|_{x=a},$$

$$A_2 = \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=a},$$

$$A_3 = \frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=a},$$

&c.

In the 18th century, various combinations of the procedures $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ were also used; namely, if the power series $\sum_{i=0}^{\infty} a_i x^i$, $\sum_{i=0}^{\infty} b_i x^i$, &c. were respectively associated with the functions $f(x)$, $g(x)$, &c., and a single function $F(x)$ was constructed by combining $f(x)$, $g(x)$, &c., then the power series constructed by combining the series $\sum_{i=0}^{\infty} a_i x^i$, $\sum_{i=0}^{\infty} b_i x^i$, &c. in the same manner was considered as being the development of $F(x)$.

In general, the accepted procedures for the development of a given function were reducible to the procedures $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ or a combination of them. This does not mean that these procedures were considered as the only elementary procedures[22] capable of providing power series developments of given functions. Mathematicians were open to the possibility of finding other procedures and other specific procedures were indeed applied in some particular cases.

Now, let us consider the question

($\mathbf{Q_1}$) Under what conditions was a particular power series $\sum_{i=0}^{\infty} a_i x^i$ associated with a certain function $f(x)$ in 18th-century analysis?

Of course, a particular power series $\sum_{i=0}^{\infty} a_i x^i$ was associated with a certain function $f(x)$ if it was the result of a transformation of this function. However, as long as a function was considered not merely as an expression but rather as the expression of a quantity, not all transformations could be accepted. Therefore, we cannot answer **($\mathbf{Q_1}$)** simply by listing a finite list of procedures of transformation like $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ (even though such an answer would be factually correct); indeed we have to complete this answer by examining another question:

($\mathbf{Q_2}$) Why was a certain procedure $\mathcal{P}$ transforming a function $f(x)$ into a series $\sum_{i=0}^{\infty} a_i x^i$ considered an acceptable rule of development[23] in 18th-century analysis?

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[22] As we have already observed, they are not completely independent of each other.

[23] By an acceptable rule of development, we mean a rule that, when applied to a given function, generates a power series, which can be considered as the development of the function.

We have previously observed that the equality $f(x) = g(x)$ between two functions meant that the expression $g(x)$ was derived by a transformation of the expression $f(x)$. However, the given expression $f(x)$ was taken into account insofar as it expressed a certain quantity: thus mathematicians thought that the result of the transformation had to express the same quantity, too.[24] The twofold nature of functions was thus transmitted to the equality $f(x) = g(x)$: on one side, this equality stated that $g(x)$ is the result of a certain transformation of $f(x)$; on the other side, it stated that $f(x)$ and $g(x)$ expressed the same quantity.[25]

Thus, the rules of transformation of a function into another one had to preserve the expressed quantity in order to be acceptable. This means that a rule of transformation $\mathcal{R}$ was considered as being acceptable only if it was ascertained or supposed that one of the following conditions was satisfied:

($\mathbf{C}_1$) for any function F, F and $\mathcal{R}(F)$ express the same quantity;[26]
($\mathbf{C}_2$) for any two functions F and G, if F and G express the same quantity, then $\mathcal{R}(F)$ and $\mathcal{R}(G)$ also express the same quantity.

If $\mathcal{R}$ was considered to satisfy ($\mathbf{C}_1$), then the equality $\mathcal{R}(F) = F$ was considered acceptable. If $\mathcal{R}$ was considered to satisfy ($\mathbf{C}_2$) and the equality $F = G$ was given, then the equality $\mathcal{R}(F) = \mathcal{R}(G)$ was considered acceptable.

It was precisely because the usual algebraic rules satisfied the condition ($\mathbf{C}_1$) that they were considered as acceptable rules of transformation; and it was precisely because the contemporary differentiation or integration of two finite functions $f(x)$ and $g(x)$ satisfied the condition ($\mathbf{C}_2$) that this rule was considered as an acceptable rule of transformation.

In order to extend this approach to the rules of transformation of a function into a series, a preliminary problem should be solved: under what condition could a power series be considered to be the expression of a quantity?

An initial answer to such a question could rely on the notion of convergence. We saw that a power series expressed a quantity if and only if it was convergent to this quantity and that if this quantity was analytically expressed by the function $f(x)$, then the series had to converge to $f(x)$. We could then answer ($\mathbf{Q_2}$) in the following way:

($\mathbf{A_2}$) In 18th-century analysis, a certain procedure $\mathcal{P}$ for transforming a function $f(x)$ into a series $\sum_{i=0}^{\infty} a_i x^i$ was acceptable if and only if the power series $\sum_{i=0}^{\infty} a_i x^i$ was convergent to $f(x)$ on a non-null interval I of the values of x.

By composing ($\mathbf{A_2}$) with ($\mathbf{C}_0$), we would obtain the following condition:

[24] See Euler [1748, I, p. 159]: "Si fuerit $y = \frac{1-zz}{1+zz}$ atque ponatur $z = \frac{1-x}{1+x}$, hoc valore loco z substituto erit $y = \frac{2x}{1+xx}$. Sumpto ergo pro x valore quocunque determinato ex eo reperientur valores determinati pro z et y sicque invenitur valor ipsius y respondens illi valori ipsius z, qui simul prodiit. Uti, si sit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$ et $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, si in $\frac{1-zz}{1+zz}$, cui expressioni y aequatur, ponatur $z = \frac{1}{3}$."

[25] See Euler [1748, I, p. 38]: "Omnis transformatio consistit in alio modo eandem functionem exprimendi, quemadmodum ex Algebra constat eandem quantitatem per plures diversas formas exprimi posse."

[26] Of course $\mathcal{R}(F)$ denotes the expression that is obtained by applying the rule $\mathcal{R}$ to F.

($\mathbf{C}_3$) In 18th-century analysis, a certain procedure $\mathcal{P}$ that transformed a function $f(x)$ into a series $\sum_{i=0}^{\infty} a_i x^i$ was an acceptable rule of development of such a function if and only if for any value α of x belonging to a certain non-null interval I, the sequence $\{\sum_{i=0}^{j} a_i \alpha^i\}_{j=0}^{\infty}$ approached $f(\alpha)$ indefinitely when j increases, and it was finally equal to $f(\alpha)$, when j was a infinite number.

Since ($\mathbf{C}_0$) is not a precise condition, ($\mathbf{C}_3$) is not a precise condition too or, at least, it does not provide a sufficiently clear criterion for deciding whether a certain procedure $\mathcal{P}$ is an acceptable rule of development. However, this did not mean that ($\mathbf{A}_2$) was not taken into account, but only that, in order to decide if a particular series was convergent, 18th-century mathematicians relied upon a criterion that did not depend on the intrinsic nature of the series but on the procedure of development generating the series.

Indeed a procedure of transformation was acceptable if and only if it was an infinitary extension of the accepted rules of transformation of finite expressions into finite expressions. Thus, what guaranteed the convergence of the development of a function $f(x)$ to this function on a non-null interval of values of x was the formal nature of the procedure of the development—the fact that it was an infinitary extension of finitary rules satisfying ($\mathbf{C}_1$) or ($\mathbf{C}_2$)—and not an analysis of the nature of the resulting series, in accordance with the definition ($\mathbf{C}_0$). It is precisely this reason that made the procedures ($\mathcal{P}_1$)–($\mathcal{P}_4$) and their combinations acceptable in the 18th century.

Therefore a satisfactory answer to ($\mathbf{Q_1}$) is the following:

($\mathbf{A_1}$) In 18th-century analysis, a certain power series $\sum_{i=0}^{\infty} a_i x^i$ was associated with a certain function $f(x)$ if it appeared as the result of the application to $f(x)$ of one of the accepted procedures ($\mathcal{P}_1$)–($\mathcal{P}_4$), of any finite combination of them, or of any other particular infinitary extension of the rules of transformation of finitary expressions satisfying one of the conditions ($\mathbf{C_1}$), ($\mathbf{C_2}$), and operating on a given, determinate or undetermined, power series term by term.

This should be a satisfactory formulation of a sufficient condition for the truth of (1), in 18th-century analysis. Certainly, this condition is not necessary. However, it is not necessary only in the following sense. A certain power series $\sum_{i=0}^{\infty} a_i x^i$ could be understood as convergent to $f(x)$ on a non-null interval I of values of x, according to ($\mathbf{C_0}$), even if it did not actually result from the application to $f(x)$ of one of the accepted procedures, and one did not actually know how to obtain it in such a way. But, in this case, an 18th-century mathematician would have assumed that this series could in principle be obtained in such a way.

In the conclusion of Section 2, we observed that the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ was conceived in the 18th century as concerned with the formal nature of the function $f(x)$ and not with the convergence of the series $\sum_{i=0}^{\infty} a_i x^i$ and that this equality was considered as valid independent of the value of x. This would seem to be contradicted by our last conclusion, namely that the validity of such an equality depends on the convergence of the series on a certain non-null interval. However, there is no contradiction. Simply, 18th-century mathematicians considered the equality $f(x) = \sum_{i=0}^{\infty} a_i x^i$ to be valid if and only if the series $\sum_{i=0}^{\infty} a_i x^i$ was considered as convergent to the function $f(x)$ on a non-null interval, but they did not think that the validity of such an equality had to be restricted to the values of x belonging to such an interval.

4. Direct and inverse problems in power series theory

At this juncture, it should be evident that the very heart of the 18th-century theory of series was constituted by the following pair of problems:

(**P.1.a**) To develop a given function into a power series;
(**P.1.b**) To return from a given power series to the function of which this power series is the development.

These problems should not be confounded with the following ones, with which modern real analysis is concerned:

(**P.2.a**) To look for a power series which converges to a given function of a real variable;
(**P.2.b**) To sum a given (convergent) power series of a real variable.

Clearly, both the pair (**P.1**) and the pair (**P.2**) consist of a direct and a inverse problem. By "inverse problem," we mean a problem that can only be formulated by referring to another problem, namely the direct one.

In modern analysis, problem (**P.2.b**) can be considered the direct one, since it is solved by summing a given series, namely by seeking the limit of the nth partial sums. Problem (**P.2.a**), by contrast, can be considered the inverse one. This is indeed the problem of seeking a power series the sum of which is the given function and is understood by referring to the operation of the sum and, therefore, to (**P.2.b**). For instance, one says that $\frac{1}{1+x}$ is the sum of $\sum_{i=0}^{\infty}(-1)^i x^i$ for $|x| < 1$, because $\lim_{n\to\infty} \sum_{i=0}^{\infty}(-1)^i x^i = \lim_{n\to\infty} \frac{1+(-1)^n x^{n+1}}{1+x} = \frac{1}{1+x}$ for $|x| < 1$, and that $\sum_{i=0}^{\infty}(-1)^i x^i$ is the development of the function $\frac{1}{1+x}$ for$|x| < 1$, because $\frac{1}{1+x}$ is the sum of $\sum_{i=0}^{\infty}(-1)^i x^i$.

Instead, in 18th-century analysis (**P.1.a**), the problem of developing a function into a series was the direct problem and (**P.1.b**) was the inverse one.[27] The path providing a solution to the problem (**P.1.a**) was a progressive path, since it progressed from the function, which was a proper object of 18th-century

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[27] Of course (**P.1.a**) and (**P.1.b**) were not the only pair of direct and inverse problems in 18th-century analysis. Generally speaking, given an operation O transforming an object α, belonging to a certain set **S** of objects, into an object $\beta = O(\alpha)$ belonging to a set **T**, we can look for a inverse operation O' such that $O'(\beta) = \alpha$. A problem arises when O, **S** and **T** are such that, for some β in **T**, there is no object α in **S**, such that $O'(\beta) = \alpha$. Today this problem is solved by defining a new set $\mathbf{S}^*$ (which can be thought of as an enlargement of **S**) whose objects are defined as the images of the objects of **T** under the operation O'. In this way, the problems of existence are settled *a priori*, by fixing the domain and range of the operations O and O' once and for all. In the 18th century, mathematicians viewed the matter differently. They did not define a set of objects $\mathbf{S}^*$ *a priori*, so that it is always possible to find an image of every object β of the set **T**, under the operation O'; instead, for every specific object β of **T**, they tried to construct a new object, somehow similar to the objects of **S**, so that one could arrive at such an object by applying the operation O' to β.

The difference between the modern and 18th-century approaches is crucial. Since an *a priori* definition is lacking, the object is constructed as the result $O'(\beta)$ of the application of the operation O' to the object β; for this reason, the nature of the object $O'(\beta)$ could only be understood by means implicit reference to objects that were already given outside the theory where the operations O and O' were initially defined. For example, the operation O and its inverse O' might have a geometric interpretation providing an explanation of the nature of new objects. This is the case for differentiation and integration. In his [1768–1770, I, p. 7], Euler defined the "integral" $\int g(x)\,dx$ of a function $g(x)$ as a function $f(x)$ such that $d[f(x)] = g(x)\,dx$. If for some $g(x)$, no known function $f(x)$ were such that $d[f(x)] = g(x)\,dx$, the symbol "$\int g(x)\,dx$" was used formally to mathematics, to the series, which was a particular expression associated with the function: in other words, it was a *synthetic* path. The path providing a solution to the problem (**P.1.b**) regressed from the series to the function: it led from a particular expression associated with an unknown object to this object (this is exactly what the verb "to return" indicates). It was a regressive path, that is, an *analytical* path.[28] For instance, given the function $\frac{1}{1+x}$, the direct problem was to develop such a function and the solution of this problem was given by the series $\sum_{i=0}^{\infty}(-1)^i x^i$. *Vice versa*, the inverse problem was to find the function whose development is given by the series $\sum_{i=0}^{\infty}(-1)^i x^i$. And thus the solution of this problem was given by the function $\frac{1}{1+x}$ just because $\sum_{i=0}^{\infty}(-1)^i x^i$ was considered as the development of $\frac{1}{1+x}$.

Eighteenth-century mathematicians used different terms to refer to the return from a power series to the original function (the function which this power series expresses). They, at times, used the term "regressus" (see, for example, [Leibniz GMS, III, p. 351] and de Moivre [1730, p. 123]); more often they preferred the term "sum". The sense in which this term was employed was made explicit by Euler: "As series in analysis arise from the expansion of fractions or irrational quantities or even of transcendentals, it will in turn be permissible in calculation to substitute in place of such a series that quantity out of whose development it is produced. For this reason [...] we employ this definition of sum, that is to say, the sum of a series is that quantity which generates the series" (see Euler [1754–1755, pp. 593–594]; translation in Barbeau and Leah [1976, p. 144]).

In order to use a clear and uniform language, we shall use the verb "to envelop" to refer to the passage from a given power series to the function of which the series is the development. Of course, by "envelopment" we shall denote the function that results from enveloping a series. Thus the problem (**P.1.b**) can be rephrased as follows:

(**P.1.b'**) To envelop a given power series into a function.

Using this terminology, we observe that when a mathematician of the 18th century spoke of summing a series, he meant enveloping it. Thus, for such mathematicians, the problem of summing a given power series was essentially different from our problem (**P.2.b**) and does not properly concern numerical series: there is no sense in speaking about the development or envelopment of a number. Numerical series cannot be enveloped but only summed. Eighteenth-century mathematicians had a perfect knowledge of the fact that certain series can be used to express numbers (in particular, irrational numbers); however, they usually considered a series like $\sum_{i=0}^{\infty} a_i$ as a particular case of the power series $\sum_{i=0}^{\infty} a_i x^i$ for the position $x = 1$, and thought that the most natural way to sum $\sum_{i=0}^{\infty} a_i$ was to determine the envelopment $f(x)$ of $\sum_{i=0}^{\infty} a_i x^i$ and then take $\sum_{i=0}^{\infty} a_i = f(1)$. This should make it clear that 18th-century analysis, unlike modern real analysis, was not a theory of real numbers. It was rather a theory of (continuous) quantities, insofar as they were expressed by means of a convenient expression.

To end this section, we make explicit a general condition concerning the problems (**P.1.a**) and (**P.1.b**), which was only implicit in the previous remarks. The generic symbol "$f(x)$," which indicates a function in the equality (1), is nothing but a written convention and cannot therefore support any formal procedure; the generic symbol "$\sum_{i=0}^{\infty} a_i x^i$," which indicates a power series in such an equality, is instead an explicit exhibition of a particular type of series and can support some formal procedures. Therefore, in order to go

denote an unknown function (but subject to certain general conditions) such that $d[\int g(x)dx] = g(x)\,dx$, and to which one could give a geometric meaning (area, length, ...).

[28] On the notion of analytic and synthetic as they are used here, cf. Panza [1997a].

from the first symbol to a complete determination of a particular object, one has to follow different steps which are part of a process of progressive determination that necessarily includes the determination of the particular form of the function. Only once these different steps are performed can a formal procedure be applied to $f(x)$. In order to proceed from the second symbol to a complete determination of a particular object, only one step has to be made, i.e., the determination of the coefficients occurring in it. And, even if this step is not performed, a formal procedure can be applied to the series. In other words: no formal procedure can be applied to a (completely) generic function, while certain formal procedures can be applied to (completely) generic power series.

5. Direct and inverse strategies to develop a function and envelop a series

We are now ready to consider different strategies to solve the problems (**P.1**). We shall distinguish four pairs of them, each pair being composed of a strategy to solve the problem (**P.1.a**) and a strategy to solve the problem (**P.1.b**). Since each of these problems is the inverse of the other, we could take each one of these strategies as the inverse of the other strategy belonging to the same pair. However, we shall also distinguish between the first pair, composed of two strategies aiming to solve the problem (**P.1.a**) and the problem (**P.1.b**) respectively, in a direct way, and the other three pairs, each of which is composed of two strategies aiming to solve the problem (**P.1.a**) and the problem (**P.1.b**) respectively, in an inverse way (that is, by looking for a solution of the inverse problem). Thus, we prefer to consider in general the two strategies belonging to the first pair as being direct strategies to solve the problems (**P.1.a**) and (**P.1.b**) respectively, and the six strategies belonging to the other three pairs as being inverse strategies to solve these same problems. This is because we shall denote, for short, the two strategies belonging to the first pair respectively as **DS.a** and **DS.b**, and the six strategies belonging to the three other pairs respectively as **iIS.a** and **iIS.b, tIS.a** and **tIS.b**, and **aIS.a** and **aIS.b**. The meaning of the small letters "**i**", "**t**" and "**a**" in these acronyms are the following: the letter "**i**" denotes immediate strategies; the letter "**t**" denotes transformative strategies; and the letter "**a**" denotes analogic strategies. We shall see what this means exactly in the different cases. For the time being, let us simply present the scheme, illustrating our classification in a compact way (see Table 1).

Let us first imagine that a particular function $f(x)$ is given. An initial obvious way for solving the problem (**P.1.a**) with respect to this function is:

(**DS.a**) To apply directly to $f(x)$ one of the procedures $(\mathcal{P}_1)$–$(\mathcal{P}_4)$, an appropriate combination of them, or any other particular accepted procedure of development.

Table 1

Strategies	To solve **P.1.a**	To solve **P.1.b**
Direct	**DS.a**	**DS.b**
Inverse		
Immediate	**iIS.a**	**iIS.b**
Transformative	**tIS.a**	**tIS.b**
Analogic	**aIS.a**	**aIS.b**

Let us now imagine, instead, that a particular power series $\sum_{i=0}^{\infty} A_i x^i$ is given. An obvious strategy to solve the problem (**P.1.b**) with respect to this series is:

(**DS.b**) To operate directly on the given series $\sum_{i=0}^{\infty} A_i x^i$ and transform it, by suitable manipulations and/or substitutions, into a finitary expression $f(x)$, which is assumed to be the sum of the series.

A subsequent application of strategy (**DS.a**) to $f(x)$ can then confirm that this function is precisely the solution to the problem (**P.1.b**) with respect to $\sum_{i=0}^{\infty} A_i x^i$.

Two examples of (**DS.a**) are Euler's and Lagrange's developments of the exponential function $y = a^x$, respectively, in Euler [1748, 1, pp. 123, 124] and in Lagrange [1797, pp. 18–20] and [1813, pp. 31–33; Œuvr., 9, pp. 45–48]. Let us consider the second of these examples. Lagrange started from the identity

$$a^x = \left[\left(1 + (a-1)\right)^n\right]^{\frac{x}{n}} \tag{7}$$

and, by applying the binomial expansion, he obtained

$$a^x = \left[1 + n(a-1) + \frac{n(n-1)}{2!}(a-1)^2 + \frac{n(n-1)(n-2)}{3!}(a-1)^3 + \&c.\right]^{\frac{x}{n}}.$$

By rearranging this equality, it is possible to put it in the form

$$a^x = \left[1 + H_1 n + H_2 n^2 + H_3 n^3 + \&c.\right]^{\frac{x}{n}}, \tag{8}$$

where the first coefficient H_1 is the series

$$(a-1) - \frac{1}{2!}(a-1)^2 + \frac{2}{3!}(a-1)^3 + \&c.$$

By applying the binomial expansion to (8) Lagrange obtained, then:

$$\begin{aligned} a^x = {} & 1 + x\left(H_1 + H_2 n + H_3 n^2 + \&c.\right) \\ & + \frac{x(x-n)}{2!}\left(H_1 + H_2 n + H_3 n^2 + \&c.\right)^2 \\ & + \frac{x(x-n)(x-2n)}{3!}\left(H_1 + H_2 n + H_3 n^2 + \&c.\right)^3 + \&c. \end{aligned}$$

This equality clearly holds for any n, whatever the values of the variable x and of the constant a. This is because n is, as Lagrange remarks, "entirely arbitrary." Thus the second member of such an equality has to be independent of n, which is possible only if all its terms where n occurs cancel each other. Thus,

$$a^x = 1 + Ax + \frac{A^2}{2!}x^2 + \frac{A^3}{3!}x^3 + \&c.,$$

where

$$A = (a-1) - \frac{1}{2!}(a-1)^2 + \frac{2}{3!}(a-1)^3 + \&c.$$

As a first example of (**DS.b**) let us consider the sum of the geometric series $\sum_{i=0}^{\infty} x^{\alpha+i\beta}$. In his [1732–1733, pp. 44, 45], Euler set

$$S=\sum_{i=0}^{m} x^{\alpha+i\beta}$$

and obtained

$$S-x^{\alpha}=\sum_{i=1}^{m} x^{\alpha+i\beta}. \tag{9}$$

By adding $x^{\alpha+(m+1)\beta}$ to both the sides of (9) and dividing them by x^{β}, he obtained

$$\frac{S-x^{\alpha}+x^{\alpha+(m+1)\beta}}{x^{\beta}}=\sum_{i=0}^{m} x^{\alpha+i\beta}=S$$

and then

$$S=\sum_{i=0}^{m} x^{\alpha+i\beta}=\frac{x^{\alpha}-x^{\alpha+(m+1)\beta}}{1-x^{\beta}}. \tag{10}$$

Taking $m=\infty$ in (10) and assuming that $|x|<1$, Euler concluded that

$$\sum_{i=0}^{\infty} x^{\alpha+i\beta}=\frac{x^{\alpha}}{1-x^{\beta}}. \tag{11}$$

As another example, consider the series $\sum_{i=1}^{\infty}(2i-1)\frac{x^i}{i!}$. In his [1732–1733, pp. 70, 71], Euler first set

$$\sum_{i=1}^{m}(2i-1)\frac{x^i}{i!}=S(x,m)$$

and, by integrating term by term (and assuming that the constant of integration is null), derived from it the equality

$$\frac{1}{2}x^{\frac{1}{2}}\left[\int x^{-\frac{3}{2}}\big[S(x,m)\big]\,dx\right]=\sum_{i=1}^{m}\frac{x^i}{i!}. \tag{12}$$

He differentiated (12) and obtained

$$\frac{\int x^{-\frac{3}{2}}[S(x,m)]\,dx}{4x^{\frac{1}{2}}}+\frac{S(x,m)}{2x}=1+\sum_{i=1}^{m}\frac{x^i}{i!}-\frac{x^m}{m!}.$$

By comparison with (12) he derived

$$(1-2x)\int x^{-\frac{3}{2}}\big[S(x,m)\big]\,dx=4x^{\frac{1}{2}}-\frac{2S(x,m)}{x^{\frac{1}{2}}}-\frac{4x^{m+\frac{1}{2}}}{m!}.$$

Assuming that $m=\infty$, this equality reduces to

$$\int x^{-\frac{3}{2}}\big[S(x)\big]\,dx=\frac{4x-2S(x)}{(1-2x)x^{\frac{1}{2}}}, \tag{13}$$

where $S(x) = S(x, \infty)$. By differentiating (13) and considering $S(x)$ as an independent variable S, Euler had

$$\frac{S dx}{x\sqrt{x}} = \frac{2x\,dx + 4x^2\,dx + S\,dx - 6Sx\,dx - 2x\,dS + 4x^2\,dS}{(1-2x)^2 x\sqrt{x}}.$$

Hence

$$dx + 2x\,dx - S\,dx - 2xS\,dx - dS + 2x\,dS = 0$$

and

$$dS + \frac{S(1+2x)\,dx}{1-2x} = \frac{(1+2x)\,dx}{1-2x}.$$

Euler then multiplied this equality by $\frac{e^{-x}}{1-2x}$ and noted that the left-hand side becomes equal to the differential of the function $\frac{e^{-x}S}{1-2x}$ of two variables x and S. Thus he obtained

$$\frac{e^{-x}S}{1-2x} = \int \frac{e^{-x}(1+2x)}{(1-2x)^2}\,dx = \frac{e^{-x}}{1-2x} - 1$$

(the constant -1 being determined under the condition $S(0) = 0$) and finally

$$S = \sum_{i=1}^{\infty}(2i-1)\frac{x^i}{i!} = 1 - e^x(1-2x). \tag{14}$$

A third example is taken from the *Institutiones calculi differentialis* [Euler, 1755, 2, pp. 217, 218]. Supposing that a power series $\sum_{i=1}^{\infty} A_i x^i$ is given, Euler transformed it by the substitution

$$x = \frac{y}{1+y}.$$

Since, for any integer i, we have

$$x^i = \left(\frac{y}{1+y}\right)^i = \sum_{k=0}^{\infty}\binom{-i}{k} y^{i+k},$$

he obtained

$$\sum_{i=1}^{\infty} A_i x^i = \sum_{i=1}^{\infty} A_i \left[\sum_{k=0}^{\infty}\binom{-i}{k} y^{i+k}\right] = \sum_{i=1}^{\infty}\left[\sum_{k=0}^{i-1}\binom{k-i}{k} A_{i-k}\right] y^i$$
$$= \sum_{i=1}^{\infty}\left[\sum_{k=0}^{i-1}\binom{k-i}{k} A_{i-k}\right]\left(\frac{x}{1-x}\right)^i,$$

and, since

$$\sum_{k=0}^{i-1}\binom{k-i}{k} A_{i-k} = \Delta^{i-1} A_1,$$

he derived

$$\sum_{i=1}^{\infty} A_i x^i = \sum_{i=1}^{\infty} \Delta^{i-1} A_1 \left(\frac{x}{1-x}\right)^i. \tag{15}$$

If the differences $\Delta^j A_1$ are equal to zero for large enough j, then $\sum_{i=1}^{\infty} \Delta^{i-1} A_1 (\frac{x}{1-x})^i$ reduces to a finite expression which Euler assumed to be the envelopment of the given series $\sum_{i=1}^{\infty} A_i x^i$. For instance, by applying (15) to the series $\sum_{i=1}^{\infty} i^2 x^i$, one has

$$\begin{aligned} \sum_{i=1}^{\infty} i^2 x^i &= \sum_{i=1}^{\infty} \Delta^{i-1} \left[\left(i^2\right)_{i=1} \right] \left(\frac{x}{1-x}\right)^i \\ &= \frac{x}{1-x} + 3\left(\frac{x}{1-x}\right)^2 + 2\left(\frac{x}{1-x}\right)^3 \\ &= \frac{x+x^2}{(1-x)^2}, \end{aligned}$$

because

$$\begin{aligned} \Delta^0\left[\left(i^2\right)_{i=1}\right] &= 1, \\ \Delta^1\left[\left(i^2\right)_{i=1}\right] &= 3, \\ \Delta^2\left[\left(i^2\right)_{i=1}\right] &= 2, \end{aligned}$$

and

$$\Delta^r\left[\left(i^2\right)_{i=1}\right] = 0,$$

for any $r > 2$.

Shortly afterwards, Euler [1755, 2, pp. 240–242] considered a series $\sum_{i=0}^{\infty} A_i x^i$, such that $A_i = u_i v_i$, where the envelopment of $\sum_{i=0}^{\infty} v_i x^i$ is a known function $f(x)$, and $\{u_i\}_{i=0}^{\infty}$ is a suitable sequence. To envelop such a series, he put it in the form

$$\sum_{i=0}^{\infty} A_i x^i = \sum_{i=0}^{\infty} C_i \frac{x^i}{i!} \frac{d^i f(x)}{dx^i},$$

where the coefficients C_i had to be determined. To determine these coefficients, Euler remarked that from the assumed equality

$$f(x) = \sum_{i=0}^{\infty} v_i x^i,$$

the other equality

$$C_i \frac{x^i}{i!} \frac{d^i f(x)}{dx^i} = \sum_{j=0}^{\infty} C_i \binom{j}{i} v_j x^j$$

follows. Thus

$$\sum_{i=0}^{\infty} u_i v_i x^i = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} C_i \binom{j}{i} v_j x^j\right) = \sum_{j=0}^{\infty} v_j x^j \left(\sum_{i=0}^{j} C_i \binom{j}{i}\right),$$

and then, by the method of undetermined coefficients,

$$u_j = \sum_{i=0}^{j} C_i \binom{j}{i};$$

that is

$$C_i = \Delta^i u_0,$$

and therefore

$$\sum_{i=0}^{\infty} A_i x^i = \sum_{i=0}^{\infty} u_i v_i x^i = \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i f(x)}{dx^i} \Delta^i u_0.$$

Once again, if the differences $\Delta^i u_0$ are equal to zero for large enough i, then $\sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i f(x)}{dx^i} \Delta^i u_0$ reduces to a finite expression, which Euler assumed to be the envelopment of the given series $\sum_{i=0}^{\infty} A_i x^i$.

As an example, Euler considered the series $\sum_{i=0}^{\infty} \frac{(i+1)^2+1}{i!} x^i$. Since one knows that $\sum_{i=0}^{\infty} \frac{1}{i!} x^i = e^x$, one can set $\sum_{i=0}^{\infty} \frac{(i+1)^2+1}{i!} x^i = \sum_{i=0}^{\infty} u_i v_i x^i$, where $(i+1)^2 + 1 = u_i$ and $\frac{1}{i!} = v_i$. One would thus obtain

$$\begin{aligned}\sum_{i=0}^{\infty} \frac{(i+1)^2+1}{i!} x^i &= \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i (e^x)}{dx^i} \Delta^i \left[\left((i+1)^2+1\right)_{i=0}\right] \\ &= e^x \sum_{i=0}^{\infty} \frac{x^i}{i!} \Delta^i \left[\left((i+1)^2+1\right)_{i=0}\right] \\ &= e^x \left(2 + 3x + x^2\right),\end{aligned}$$

since

$$\begin{aligned}\Delta^0\left[\left((i+1)^2+1\right)_{i=0}\right] &= 2, \\ \Delta^1\left[\left((i+1)^2+1\right)_{i=0}\right] &= 3, \\ \Delta^2\left[\left((i+1)^2+1\right)_{i=0}\right] &= 2,\end{aligned}$$

and

$$\Delta^r\left[\left((i+1)^2+1\right)_{i=0}\right] = 0$$

for any $r > 2$.

These examples show that the strategy (**DS.b**) could have different forms. The previous examples rely on the following versions of it:

(**DS.b.1**) To construct the sequence $\{\sum_{i=0}^{j} A_i x^i\}_{j=0}^{\infty}$ of the partial sums of the given series $\sum_{i=0}^{\infty} A_i x^i$, and to search for a (recursive or direct) rule of formation of the terms of this sequence giving the expression of its generic term $\sum_{i=0}^{m} A_i x^i$; if this expression reduces to another finitary expression $f(x)$ for the position $m = \infty$ this latter expression can be supposed to be the envelopment of the given series.

(DS.b.2) Imagine that, using some known developments, it is possible to transform a given power series $\sum_{i=0}^{\infty} A_i x^i$ into another series $\sum_{i=0}^{\infty} B_i x^i$, the terms of which are equal to 0 when i is greater than an appropriate m; then the series $\sum_{i=0}^{\infty} B_i x^i$ is reduced to a finitary expression which is supposed to be the envelopment of the given series.

Both (**DS.a**) and (**DS.b**) are direct strategies for solving the problems (**P.1.a**) and (**P.1.b**), respectively: indeed they lead us to the desired result by manipulating the given object. In the case of (**DS.a**), by manipulating a known function, one derives a series which is its development; in the case of (**DS.b**), by manipulating a given series, one derives a function which is its envelopment. To use a classic expression, we can say that these strategies are synthetic. This is because, by performing them, one operates on a known object to find the object considered to be unknown in the formulation of the problem.

Direct strategies are very natural, but they are not the only possible ones, and, as a matter of fact, they are not the only ones that were followed in 18th-century analysis. In fact, although the problems (**P.1.a**) and (**P.1.b**) were conceived as essentially distinct from each other, it is clear that the solution to one of them also provides the solution of the other, supposing that in this latter problem what is considered as given is that which is sought in the former and vice versa. As an example, consider the equality (14). It has been obtained following a direct strategy and states that $1 - e^x(1 - 2x)$ is the envelopment of the given series

$$x + 3\frac{x^2}{2!} + 5\frac{x^3}{3!} + 7\frac{x^4}{4!} + \&c. \tag{16}$$

It is clear that, once this equality has been stated, and the function $1 - e^x(1 - 2x)$ is assumed to be given, one can easily conclude that the series (16) is its development. This is a simple example of the following inverse strategy to solve the problem (**P.1.a**):

(iIS.a) If a particular function $f(x)$ is given and it is possible to recognize it as a known envelopment of a known series $\sum_{i=0}^{\infty} A_i x^i$, then it can be immediately concluded that $\sum_{i=0}^{\infty} A_i x^i$ is the development of $f(x)$.[29]

A similar strategy can be followed in order to solve the problem (**P.1.b**). Let us imagine, for example, that the series

$$2 + x - 6x^2 - 3x^3 + 18x^4 + 9x^5 - 54x^6 - 27x^7 + \&c.$$

is given and one recognizes it as the development of the function $\frac{x+2}{1+3x^2}$. One can thus immediately conclude that $\frac{x+2}{1+3x^2}$ is the envelopment of this series.

This is the strategy Newton used in a sketch of a treatise on quadratures and binomial developments composed in the summer of 1665, in order to express the area of the hyperbola of equation $y = \frac{a^2}{(b+x)^2}$ by

[29] An interesting example of this procedure can be found in Euler [1730–1731a, 1730–1731b, 1732–1733]. We prefer not to present it here because it involves some difficulties concerning the nature of integrals which are beyond of the scope of our paper.

means of a finitary expression (cf. [Newton MP, I, p. 129]. By using the development of y into a power series, he first found that this area could be expressed by the power series

$$\frac{a^2}{b^2}x - \frac{a^2}{b^3}x^2 + \frac{a^2}{b^4}x^3 - \&c.$$

Then he compared this series with the development of $\frac{a^2}{b+x}$, that is,

$$\frac{a^2}{b} - \frac{a^2}{b^2}x + \frac{a^2}{b^3}x^2 - \&c.,$$

and concluded that the area he was looking is was equal to

$$\frac{a^2}{b} - \frac{a^2}{b+x}.$$

Generally speaking, such a strategy is the following:

(iIS.b) If a particular series $\sum_{i=0}^{\infty} A_i x^i$ is given and it is possible to recognize it as a known development of a known function, then, it can be immediately concluded that $f(x)$ is the envelopment of $\sum_{i=0}^{\infty} A_i x^i$.

To illustrate how this strategy works, let us consider the series

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c. = \sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\binom{j-k}{k}\right]x^j.$$

According to the binomial expansion for positive integers exponents, it follows that

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c. = \sum_{i=0}^{\infty}\left(x + x^2\right)^i.$$

But, by setting $x + x^2 = y$ and applying the previous procedure to sum a geometric series, we have

$$\sum_{i=0}^{n}\left(x + x^2\right)^i = \sum_{i=0}^{n} y^i = \frac{1-y^n}{1-y},$$

and thus

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c. = \sum_{i=0}^{\infty}\left(x + x^2\right)^i = \frac{1}{1-y} = \frac{1}{1-x-x^2},$$

and $\frac{1}{1-x-x^2}$ is thus the envelopment of the given series $1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c.$, this result being obtained by observing that this latter series is the sum of an infinite number of finite developments.

The strategies (**iIS.a**) and (**iIS.b**) are two inverse strategies for solving (**P.1.a**) and (**P.1.b**), but they are also immediate, since they immediately refer to an already given solution to the opposite problem. Both cases allow one to obtain the desired result by using an already known result which has been derived by operating on the object that is considered as unknown in the given problem. Using a more classic language, we can say that they are analytic procedures.

It is interesting to note that these procedures are analytic though they do not solve the problem which is proposed by manipulating a generic function or a generic series, but by handling a function or a series guessed at in some way, i.e., by following a synthetic path. This shows that an analytic procedure, consisting in working on an unknown object K which is to be found as if it were given, can sometimes be performed by following a synthetic path: one works on a determinate object K and verifies that the given object can be derived from K. This is not the same as operating on a undetermined object X which is precisely what has to be found. In this latter case the path also is analytic.[30]

Let us now imagine that a particular function $f(x)$ is given and that it is possible to recognize it as the result of the application of a certain operation (also applicable to power series, term by term) to a known envelopment $g(x)$ of a known series $\sum_{i=0}^{\infty} A_i x^i$. This provides another strategy to solve the problem (**P.1.a**):

(**tIS.a**) The operation, which leads from a function $g(x)$ to another function $f(x)$, can be applied to the development $\sum_{i=0}^{\infty} A_i x^i$ of $g(x)$; this produces a new series $\sum_{i=0}^{\infty} B_i x^i$, which is the development of the function $f(x)$.

The successive application of one of the strategies (**DS.a**) or (**DS.b**) to $f(x)$ or to $\sum_{i=0}^{\infty} B_i x^i$, respectively, can successively confirm such a result.

A similar strategy for solving (**P.1.b**) is then obvious. Let us imagine then that a particular series $\sum_{i=0}^{\infty} A_i x^i$ is given and that it is possible to recognize it as the result of the application of a certain operation (also applicable to finitary analytic forms) to a known development $\sum_{i=0}^{\infty} B_i x^i$ of a known function $f(x)$; then:

(**tIS.b**) The operation, which leads from $\sum_{i=0}^{\infty} B_i x^i$ to $\sum_{i=0}^{\infty} A_i x^i$, can be applied to $f(x)$; this produces a new function $g(x)$, which is the envelopment of the series $\sum_{i=0}^{\infty} A_i x^i$.

The successive application of one of the strategies (**DS.a**) or (**DS.b**) to $g(x)$ or to $\sum_{i=0}^{\infty} A_i x^i$, respectively, can successively confirm such a result.

(**tIS.a**) and (**tIS.b**) are two inverse strategies for solving (**P.1.a**) and (**P.2.b**). Indeed, in this case as well, one arrives at the result by operating on an object that is not given in the proposed problem. However, this object is not the unknown object of the problem but an object that is connected to the unknown object by means of a certain transformation. We can say that these strategies are not only inverse (i.e., analytic) but also transformative.

As an example of the strategy (**tIS.a**), let us consider Euler's development of the function $y = \log(1 + x + x^2 + x^3)$ in his [1768–1770, 1, p. 83]. Euler started from the equality

$$y = \log\bigl(1 + x + x^2 + x^3\bigr) = \log \frac{1 - x^4}{1 - x} = \log\bigl(1 - x^4\bigr) - \log(1 - x)$$

[30] This remark should justify Euler's use of the term "synthetic" to characterize the procedure of development he adopted in his [1732–1733]. Here Euler termed as "synthetic" a procedure he described as consisting in wondering "what the series could be whose sums are expressed" by certain formulas (cf. Euler [1732–1733, p. 42]. By asserting that this is a synthetic procedure, Euler seems to insist on the logical nature of the path rather that on the logical nature of the argument. He underlines in effect that, even if his procedure is regressive, it does not consist in operating on an unknown object since it concerns knowledge that is already available. A scheme based upon this point of view is presented in Ferraro [1998].

and observed that

$$\frac{d}{dx}(y) = \frac{d}{dx}\big(\log\big(1-x^4\big) - \log(1-x)\big) = -\frac{4x^3}{1-x^4} + \frac{1}{1-x}$$

and, according to (11),

$$\frac{x^3}{1-x^4} = \sum_{i=0}^{\infty} x^{3+4i}; \qquad \frac{1}{1-x} = \sum_{i=0}^{\infty} x^i.$$

He concluded from here that integrating term by term the series

$$\sum_{i=0}^{\infty} x^i - \sum_{i=0}^{\infty} 4x^{3+4i}$$

(and supposing that the constant of integration is null), one should have the development of $\log(1+x+x^2+x^3)$ which was sought,

$$\log\big(1+x+x^2+x^3\big) = \sum_{i=0}^{\infty} \frac{x^{i+1}}{i+1} - \sum_{i=0}^{\infty} \frac{x^{4+4i}}{i+1} = \sum_{i=0}^{\infty} (-3)^{([\frac{i+1}{4}]-[\frac{i}{4}])} \frac{x^{i+1}}{i+1},$$

where the symbol $[q]$ denotes the integral part of the number q.

As an example of (**tIS.b**) consider the series $1-2x+3x^2-4x^3+\&c$. It is easy to recognize that this series can be obtained by differentiating the power series $\sum_{i=0}^{\infty}(-1)^{i+1}x^i$ term by term and dividing the result by the differential dx. Since $\sum_{i=0}^{\infty}(-1)^{i+1}x^i$ is the known development of $-\frac{1}{1+x}$, it is then sufficient to calculate the differential ratio of the last function in order to derive the sum of the given series:

$$1-2x+3x^2-4x^3+\&c. = \frac{1}{(1+x)^2}.$$

This is the procedure by which Euler found in his [1761, pp. 71, 72] the sum of the series

$$1-2^n x+3^n x^2-4^n x^3+\cdots$$

for $n = 2, 3, \ldots, 6$, which are thus

$$\frac{1-x}{(1+x)^3},$$
$$\frac{1-4x+x^2}{(1+x)^4},$$
$$\frac{1-11x+11x^2-x^3}{(1+x)^5},$$
$$\frac{1-26x+66x^2-26x^3+x^4}{(1+x)^6},$$
$$\frac{1-57x+302x^2-302x^3+57x^4+x^5}{(1+x)^7},$$

respectively.

Let us finally suppose that a particular function $f(x)$ is given and that it is in a certain respect similar to a known envelopment $g(x)$ of a known series $\sum_{i=0}^{\infty} A_i x^i$. One could try to transform this latter series into a new series $\sum_{i=0}^{\infty} B_i x^i$ as similar to it as $f(x)$ is similar to $g(x)$. If this is possible, one could guess that $\sum_{i=0}^{\infty} B_i x^i$ is the development of the given function $f(x)$, and then try to verify this conjecture in same way, for instance by applying the strategy (**DS.b**) to $\sum_{i=0}^{\infty} B_i x^i$. This is a further strategy to solve the problem (**P.1.a**):

(**aIS.a**) If a particular function $f(x)$ is given and it is in a certain respect similar to a known envelopment $g(x)$ of a known series $\sum_{i=0}^{\infty} A_i x^i$, and it is possible to transform $\sum_{i=0}^{\infty} A_i x^i$ into a new series $\sum_{i=0}^{\infty} B_i x^i$, which is as similar to $\sum_{i=0}^{\infty} A_i x^i$ as $f(x)$ is similar to $g(x)$, one could guess that $\sum_{i=0}^{\infty} B_i x^i$ is the development of $f(x)$, and try to verify this conjecture in some way.

The inverse strategy to solve (**P.1.b**) is obvious, in this case as well:

(**aIS.b**) If a particular series $\sum_{i=0}^{\infty} A_i x^i$ is given and it is in a certain respect similar to a known development $\sum_{i=0}^{\infty} B_i x^i$ of a known function $f(x)$, and it is possible to transform $f(x)$ into a new function $g(x)$ as similar to $f(x)$ as $\sum_{i=0}^{\infty} A_i x^i$ is similar to $\sum_{i=0}^{\infty} B_i x^i$, one could guess that $g(x)$ is the envelopment of $\sum_{i=0}^{\infty} A_i x^i$, and try to verify this conjecture in some way.

To perform this latter verification, one can apply, for instance, the strategy (**DS.a**) to $g(x)$.

(**aIS.a**) and (**aIS.b**) are also inverse and indirect strategies for solving (**P.1.a**) and (**P.2.b**), respectively. Indeed, in these cases we reach the result by operating on an object that is neither given in the proposed problem nor is the unknown object of this problem. But, in this case, the object we operate on is connected to the unknown object not by means of a certain transformation, but by a sort of analogy. We can therefore say that not only are these strategies inverse (i.e., analytic) and indirect, but also analogic, rather than deductive.

An example of an analogic procedure is the transition from the discrete to the continuous, known as "Wallis's interpolation." Following Newton, Jakob Bernoulli ([1689–1704, 2, pp. 957–958] and [1713, p. 294]) considered the development of $(\frac{l}{m-n})^r$ for any natural integer r and derived from it the development of $(\frac{l}{m-n})^\alpha$ for any real number α. In general, such a procedure is performed by examining the known developments of a sequence of functions $F_n(x)$ according to the scheme

$$\begin{aligned}
&F_0(x) = A_{0,0} + A_{1,0}x + A_{2,0}x^2 + A_{3,0}x^3 + A_{4,0}x^4 + A_{5,0}x^5 + \cdots,\\
&F_1(x) = A_{0,1} + A_{1,1}x + A_{2,1}x^2 + A_{3,1}x^3 + A_{4,1}x^4 + A_{5,1}x^5 + \cdots,\\
&F_2(x) = A_{0,2} + A_{1,2}x + A_{2,2}x^2 + A_{3,2}x^3 + A_{4,2}x^4 + A_{5,2}x^5 + \cdots,\\
&\qquad \&c.
\end{aligned}$$

If the law of coefficients $A_{n,m}$ is known, one has

$$F_\alpha(x) = A_{0,\alpha} + A_{1,\alpha}x + A_{2,\alpha}x^2 + A_{3,\alpha}x^3 + A_{4,\alpha}x^4 + A_{5,\alpha}x^5 + \&c.,$$

where α is any real number. If the function is given, this procedure is an example of (**aIS.a**); instead, if the series is given, it is an example of (**aIS.b**).

We have thus presented the four pairs of strategies to solve the problems (**P.1.a**) and (**P.1.b**) we announced at the beginning of this section. Of course, we do not claim that this exhausts all the possible analytic or synthetic procedures used by 18th-century mathematicians to solve these problems. For instance, we can hypothesize that, if a relation of the form (1) between a certain function and a determinate power series is already given, it is then possible to move on from this relation to determining the development of other functions or the envelopments of other series. For instance, in his [1730], de Moivre considered the development

$$\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \&c.$$

and observed that

$$\frac{1}{1-x-x^2} = \frac{1-x^2}{1-3x^2+x^4} + \frac{x}{1-3x^2+x^4},$$

$\frac{1-x^2}{1-3x^2+x^4}$ and $\frac{x}{1-3x^2+x^4}$ being, respectively, an even function and odd function. From here he concluded that

$$\frac{1-x^2}{1-3x^2+x^4} = 1 + 2x^2 + 5x^4 + \&c.$$
$$\frac{x}{1-3x^2+x^4} = x + 3x^3 + 8x^5 + \&c.$$

Given the distinctions we have introduced, it would seem moreover that further even subtler distinctions can be identified. For example, the strategy (**DS.a**) can be performed in different ways according to the procedure chosen among the procedures $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and their possible combinations. Another possible distinction has already been mentioned briefly: the distinction between the procedures that follow a synthetic path, like (**iIS.a**) and (**iIS.b**), and the procedures that follow an analytic path. An analogous distinction could be made between the procedures of development of a function $f(x)$ which merely operate upon this given function, and the procedures which operate on this function and upon the generic form of a power series, for example by following the method of undetermined coefficients. According to such a method, the coefficients of the development participate in the procedure before being determined and we could say, also in this case, that such an analytical procedure follows a synthetic path.

Of course, other different classifications can be made, based upon different pairs than the pairs analytic/synthetic or direct/indirect. However, it is not our aim to investigate this possibility here.

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