

NOTE

Oresme's Proof of the Density of Rotations of a Circle through an Irrational Angle

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One of the theorems of Nicole Oresme's (ca. 1320–1382) says that, for two points moving uniformly but incommensurably along a circle, "no sector of a circle is so small that two such mobiles could not conjunct in it at some future time, and could not have conjuncted in it at some time." A detailed study of his proof of this and related theorems shows that he was in the possession of all the arguments needed for the proof to be conclusive. © 1993 Academic Press, Inc.

Nicole Oresme (ca. 1320–1382) gibt folgenden Satz. Für zwei Punkte, die sich gleichförmig, aber inkommensurabel auf einem Kreis bewegen, gilt, daß "kein Sektor des Kreises so klein ist, daß nicht zwei solche bewegliche Punkte einmal in der Zukunft in Konjunktion stehen könnten, und einmal in Konjunktion hätten stehen können." Eine genaue Untersuchung von Oresmes Beweis dieses Satzes und einiger damit zusammenhängender Sätze zeigt, daß er alle Argumente zur Verfügung hatte, die man für einen schlüssigen Beweis braucht. © 1993 Academic Press, Inc.

Nicole Oresme (ca. 1320–1382) enuncia il teorema secondo il quale, dati due punti in movimento rotatorio uniforme ma incommensurabile, "non vi è alcun settore del cerchio tanto piccolo, che tali due mobili non possano congiungersi in futuro, e non avrebbero potuto farlo." Lo studio dettagliato della prova di questo teorema e di alcuni altri connessi dimostra che Oresme possedeva tutti gli argomenti necessari per una prova conclusiva. © 1993 Academic Press, Inc.

AMS 1991 subject classifications: 01A35, 01A55, 11-03, 11B05.

KEY WORDS: Oresme, density, Kronecker's theorem.

1. INTRODUCTION

In a two-part tract on the commensurability versus incommensurability of celestial motions, Nicole Oresme studied the properties of uniform circular motions [1]. In the second part, dealing with combinations of incommensurable motions, the following Proposition 4 appears. "No sector of a circle is so small that two such mobiles could not conjunct in it at some future time, and could not have conjuncted in it at some time [Grant 1971, 253]. If we assume the point-like bodies to rotate in the same sense, points of conjunction are those points of the circle which the bodies occupy at the same time. The arcs between two successive points of conjunction are equal in length, as are the angles [Grant 1971, 257]. Therefore Oresme is studying what are now called rotations of a circle through a

given angle. A sector obviously is an interval of positive length of the circle, and with this reading Oresme's proposition can be formulated as saying that rotations with an irrational angle yield a dense set of points. It is interesting to inquire how complete Oresme's suggested proof was, especially since his result is usually known as a special case of Kronecker's theorem of 1884 [2]. (Dirichlet remarked already in 1842 that this special case had been "known for a long time" [3].)

Oresme proves in the first part that two bodies in commensurable motion have a finite number of conjunction points so that their motion is periodic [Grant 1971, 199]. In the second part he proves that, in the case of incommensurability, there are infinitely many conjunction points. Any arc between two points of conjunction is cut by a further conjunction. At several places Oresme seems to take it for granted that one may infer his proposition from this. What is missing from the argument is that the length of all the arcs produced by the conjunction points diminishes without limit [4]. By following his (rather verbal) proof and ensuing remarks, we shall see that the omissions in Oresme's argumentation can be accounted for with reasonings very close to Oresme's own in all respects.

2. THE PROOF AND ITS ANALYSIS

Oresme denotes by d the first point of conjunction and by e the following one. These are separated by the arc de . The velocities of the two bodies are incommensurable, which is the same as saying the arc de is incommensurable to the circle. Referring to his previous Proposition 3, Oresme says "after an arc equal to de is applied a certain number of times, the circle is surpassed and this arc crosses beyond point d by cutting arc de in point g " [Grant 1971, 255]. Proposition 3 says that "whenever one of the two bodies is in the point where they are now, they will never be separated by a part commensurable to the circle." [Grant 1971, 253]. Its proof ends with the conclusion that no two conjunction points are separated by an angle commensurable to the whole circle [5]. This remark is the same as Proposition 2 [Grant 1971, 251]. Apparently Oresme thought it necessary to be explicit about not ending in point d but beyond it [6].

Oresme continued: "The arc lying between the second and third points of conjunction will be cut in exactly the same manner, and so on in succession until the whole circle will be so divided that no part greater than arc dg will remain undivided." [Grant 1971, 255]. Here we encounter a problem. The argument assumes, without stating it, that $dg > ge$ [7]. In this case it is true after *two* full circles are completed that there is no part greater than dg left, as in Figure 1. In the contrary case of $dg < ge$, this would not be the case. But if we iterate one full circle from g onward, a point j_1 is reached, as in Figure 2. Since $dg = gj_1$, $gj_1 < ge$ so that j_1 divides the arc ge . One more iteration gives a point j_2 cutting j_1e if $gj_1 < j_1e$, so that after a certain number n of iterations, the point j_n passes e and the remaining part is smaller than dg . In other words, there is no part greater than dg as claimed by Oresme.

Another repetition of Oresme's procedure gives a point k in the arc dg , "until finally the circle will be divided in such a way that no part of it will be greater

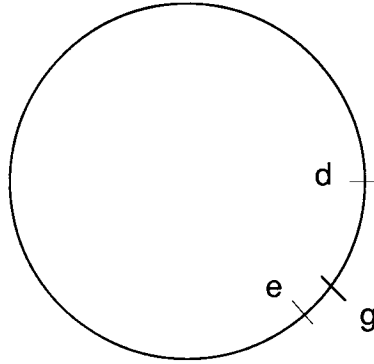


FIGURE 1

than arc dk " [Grant 1971, 255] [8]. Oresme says "this process can be carried into infinity by always dividing the circle into smaller parts *ad infinitum*. Thus, no part of the circle will remain but that it could not, at some time, be imagined as divisible in this way" [Grant 1971, 255]. Oresme compares this to the division of the diagonal of a square by its side. Taking the remaining part, or excess of the side, and applying it, another part is left. Iteration produces a set of points on the diagonal cutting any two previous points reached. Here Oresme assumes that the remaining part "diminishes to infinity" so that a dense set on the diagonal is produced. A similar diminution is assumed when he concludes the proof of his proposition, stating that "no sector of the circle will be so small but that at some time in the future the mobiles could not conjunct in some point of it, and this is what we have proposed" [Grant 1971, 255-257].

There is a problematic point in Oresme's proof, and another in his proposition.

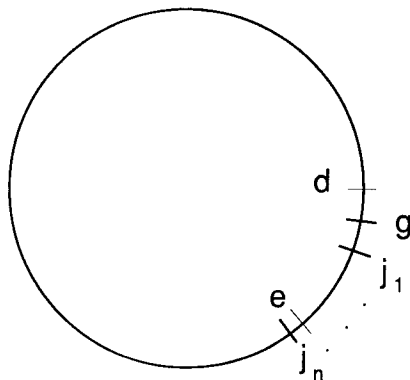


FIGURE 2

He does not prove that the arcs “diminish to infinity,” only that they are cut smaller. Sometimes he takes it as obvious that if there is an infinity of different conjunction points, they must be dense [Grant 1971, 255, 257]. In the proposition, he says any *sector* will have a conjunction point. But in the proof he only concludes this for *parts* of the circle. A part is an arc between two conjunction points that have already been reached by the process. Oresme is fully aware that given a conjunction point, there will be infinitely many distinct points in the past as well as in the future; further, there is also space in the circle for the commensurable points not reached by the process at all [Grant 1971, 257–259]. Therefore not all sectors are parts of the circle in Oresme’s sense.

How should one supply the missing limiting argument in Oresme? Is there an elementary proof of Proposition 4, and how closely would it follow arguments used by Oresme? Oresme only proves that the greatest arc is cut smaller, but does not say anything about the proportion—for example, by giving it upper and lower bounds distinct from 1 and 0. An idea that immediately comes to mind is the procedure suggested above for handling the case of $dg < ge$. There, the arc ge was divided in a finite number of steps into parts not greater than dg . Since dg is smaller than half of the original arc de , after n steps all the parts reached are smaller than half of de . By repeating the process *ad infinitum*, it can now be seen as true that the greatest arc “diminishes to infinity.” This would also at once justify the step from parts to sectors, for any sector would contain a part.

How faithful is this argument? It turns out that the very next result, Oresme’s Proposition 5, provides the answer. This proposition says that objects in incommensurable motion “will conjunct infinitely close to any given point of conjunction, and have already conjuncted infinitely near to it” [Grant 1971, 259]. The proof is very brief. Let d be a point of conjunction, and c another point close to it. By Proposition 4, there is a conjunction between c and d . “And if another point, say f , were assigned halfway between, it is again obvious—by the same—that the points will conjunct between d and f . In this manner their conjunctions will approximate infinitely close” [Grant 1971, 259–261]. The method of proof used here is the one we suggested above as a simple argument for the proof of Proposition 4. One cannot argue directly from Proposition 5 to Proposition 4, for the former is based on the latter, specifically, on the crucial step from “smaller parts *ad infinitum*” to an arbitrarily small part. That is exactly the step in need of justification. But if we only apply the method, the proof it gives for Proposition 4 is our refined version of the original. Therefore, we may conclude that Oresme’s proof for the density of rotations of a circle by an irrational angle, while not entirely conclusive, can be completed by elementary arguments used by himself in the very same context.

3. GEOMETRIC DIOPHANTINE APPROXIMATIONS

Oresme approached the density theorem by geometric means. This was necessary since he had no number concept available for formulating the property. A comparison with [Dirichlet 1842, 635] is instructive in this respect. Dirichlet indi-

cated that the following result from the theory of continued fractions had been known “for a long time”: if α is irrational, there is always an infinity of integers x and y depending on each other (“zusammengehörig”), such that $x - \alpha y$ is smaller than $1/y$. Since y must take an infinity of separate values, the decimal part of multiples of α has arbitrarily small values. A few steps give the connection to Oresme’s theorem. We can obtain from an irrational α an incommensurable motion by taking α as the arc length between two conjunctions and by taking the circumference of the circle as rational, say, to simplify matters, of unit length with $\alpha < 1$. With these conventions, the result Dirichlet refers to says that there is an infinity of numbers n such that the decimal part of αn is less than $1/n$ [9]. In terms of rotations, a point is reached whose arc length is $1/n$ units from the starting point. Iteration gives a point $2/n$ distant, and so on, yielding a dense set on the circle.

Oresme’s tract contains a beautiful description of a dense trajectory for the center point B of the sun. Assuming the incommensurability of the (apparent) daily and yearly rotations of the sun, the combined motion gives a spiral-like line of motion of B between the two tropics. “In accordance with what has been imagined here, the whole celestial space between the two tropics is traversed by B , leaving behind a web- or net-like figure expanded through the whole of this space. The structure of this figure was already infinitely dense [*in infinitum inspissata*] through the course of an infinite past time, and yet, nonetheless, it will be made continually more dense, since it produces a new spiral every day [Grant 1971, 277]. In reading Oresme it at times appears as if he thought the incommensurable conjunctions would be equally distributed in all directions. Such equidistribution results were first proven by Bohl, Sierpinski, and Weyl in 1909–1910 [10]. In Oresme’s work, only vague hints in this direction can be found, as when he says that “by means of the greatest inequality, which departs from every equality, the most just and established order is preserved” [Grant 1971, 257].

ACKNOWLEDGMENT

I thank Glenn Shafer for useful suggestions concerning the presentation.

NOTES

1. The text, written around 1360, is found in [Grant 1971]. This volume also explains the more general significance of Oresme’s work.

2. Kronecker’s result can be found in his [1884].

3. See Dirichlet [1842, 93]. The same result can be formulated as the statement that the combination of two incommensurable harmonic oscillations describes a dense motion. It can be given as a trajectory on a torus, known since 1857 as a Lissajous figure. This special case of Kronecker’s density theorem seems to have been well known. It appears, for example, in Ludwig Boltzmann’s paper on gas theory, as a motivation for ergodic motions of gas molecules; see [Boltzmann 1871].

4. This goes without comment in [Grant 1971, 45; Grant 1961, 445], as well as in [von Plato 1981].

5. There is a slip or misprint in the translation here, rendering the Latin *commensurabilem* into its contrary; cf [Grant 1971, 252, line 64].

6. Why he refers to Proposition 3 instead of 2 is perhaps because its proof concludes with the remark Oresme needs, immediately preceding Proposition 4, as he notes. Grant refers to Oresme’s

Proposition 1 [Grant 1971, 44, note 64]. This proposition states that in the case of incommensurable motion, two mobiles now in conjunction will never conjunct at that point at other times [Grant 1971, 249].

7. This is noted by Grant, who says it is implicitly assumed [Grant 1971, 44]. See also [Grant 1961, 445]. The notation for inequality and equality of arcs is just a shorthand. Oresme had a geometric notion of irrationals as incommensurable quantities. He would not consider them numbers.

8. After what we have presented above, this should be put: no part remains that is greater than the smaller of dk or ke .

9. By putting for x the integer next greater to αn . A similar case, referred to in Oresme's proof of Proposition 4, would be given by the square with a diagonal of unit length.

10. See [Hlawka & Binder 1986] for the history of these results.

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