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Infinitesimals in the foundations of Newton's mechanics ☆

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Abstract

This paper discusses two concepts of "moment" (infinitesimal) used successively by Newton in his calculus and relates these two concepts to the two concepts of force that Newton presented in Law II and Def. VIII of the *Principia*, to which the approximations to the action of a centripetal force known as the polygonal and parabolic models are considered to be related. It is shown that in the context of the application of the calculus to mechanics, the transition in the use of these concepts of "moment" took place in 1684, between the writing of *De Motu* and its first revision.

Resumen

En este artículo se pretenden caracterizar dos concepciones de "momento" (un infinitesimal) empleadas sucesivamente por Newton en su cálculo. Estas dos concepciones se intentan relacionar con los conceptos de fuerza que Newton presentó en la Ley II y en la Def. VIII de los *Principia*, a los que se consideran vinculadas las aproximaciones a la actuación de la fuerza centrípeta conocidas como poligonal y parabólica. Se muestra que en la aplicación del cálculo a la mecánica la transición en el empleo de estos conceptos de "momento" pudo tener lugar en 1684, entre la redacción del *De Motu* y su primera revisión.

Keywords: Kepler's second law; Force; Galileo; Newton; Calculus; Infinitesimal

In their analysis of Newton's mathematical science, historians have identified a certain tension between the discrete and the continuous that Newton did not quite resolve. First, Newton declared that he had renounced the infinitesimal, although some specialists have found that he used infinitesimals in

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his method of first and ultimate ratios.[1] Second, Newton formulated two concepts of force in the *Principia*. In Law II he made force equal to the change in the quantity of motion [$F = \Delta(mv)$], while in Def. VIII he made it equal to the rate of this change [$F = \Delta(mv)/\Delta t$]. Third, with these two concepts of force he associated two models for the action of centripetal forces. He conceived the first, now called "polygonal," as a series of instantaneous impulses impressed on the body at equal time intervals. The path thus resulted in a polygonal line formed by rectilinear segments traversed with uniform motion. In the second, now called "parabolic," it was the force and not the velocity that he considered constant, and he employed infinitesimal elements of arc [Hankins, 1967; Westfall, 1971, 439, 1980, 417–418]. The equivalence between these approaches has not always been clear.

In the present article I cast some light on these questions, working from the claim that Newton successively adopted two different concepts of infinitesimal. I try to clarify some aspects of the mathematics used by Newton to solve mechanical problems.[2] It has been said that the analysis of motion elaborated in the Scientific Revolution was carried out by exploring the limits of preclassical mechanics [Damerow et al., 1992]. But we must not forget that the limits of classical geometry were also explored, and that the concepts of the new mechanics are expressed in the vocabulary of this geometry.

1. Intensities, quantities, and proportions

As is known, a geometrical procedure was developed in the Middle Ages for measuring qualities. It is known as the configuration of qualities, and it measures a quality through its *intensio*, its *extensio*, and its *duratio*. In dealing with the local motion of a body, each point of the body was assigned an intensity or degree of velocity. The intensity, together with the extension, was represented by a surface. The duration added a third dimension, and the resulting figure was a solid. Abstracting the extension of the body, the intensities of its motion (that is to say, that of its center of gravity) can be represented as a function of time. In Fig. 1 (although not used at the time, we have added the coordinate axes for greater clarity) v_1 and v_2 represent the degrees or intensities of velocity of two uniform motions in a time interval t. Today we take (as Newton also did) the areas $V_1 = v_1 t$ and $V_2 = v_2 t$ to represent the spaces traveled in this interval. For Galileo, these areas represented the "total or overall velocities" V_1 and V_2 or the "quantities" of velocity in the time t, since he considered the velocity as a characteristic magnitude of the motion as a whole [Souffrin, 1992]. In this respect, a degree of velocity is not velocity, because in an instant of time (a point if time is represented by a line) no space is traveled. Neither, for the same reason, is an instant time.

Galileo had no difficulty in comparing two uniform motions. Here the ratio between the intensities or degrees of velocity is as the ratio between the quantities of velocity, the aforementioned areas, measured in the same time or the same space. But the case of uniformly accelerated motion, shown in Fig. 2, was more difficult. In essence, the question was to find out if the same proportions found in comparing

[1] For example, Boyer [1949, 201], Lai [1975], and Bechler [1991, 249] maintain that Newton treated the infinitesimal as something continuous. Kitcher [1973, 44, n. 39] believes that he came to consider the infinitesimal as a variable. On the other hand, Arthur [1995] states that the infinitesimal was not present in Newton's fluxional scheme.

[2] The main idea is discussed in Sellés [1999]. Some related considerations are contained in Sellés [1998].

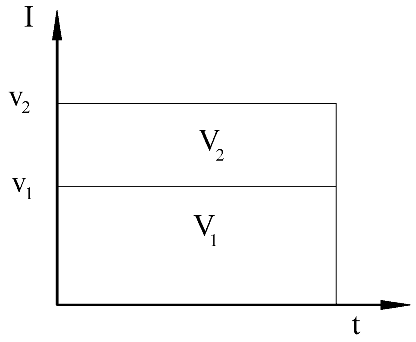

Fig. 1.

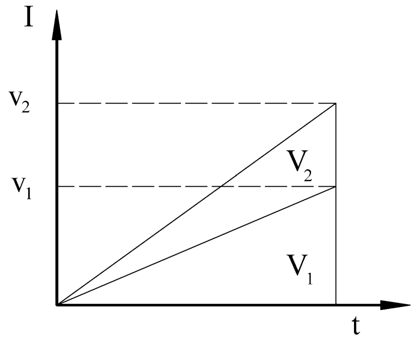

Fig. 2.

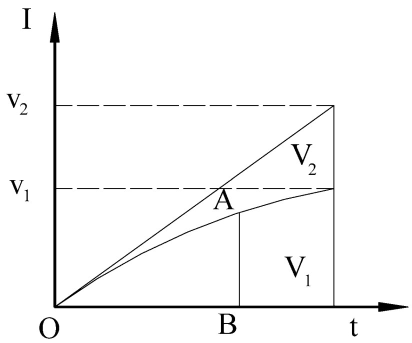

Fig. 3.

uniform motions remained valid when uniformly accelerated motions were compared. He found it so, whenever these motions were compared in the same time, counting from their beginning.[3]

The problem that Newton faced—always expressed in a very general way—was that of comparing a nonuniformly accelerated motion with a uniformly accelerated one (Fig. 3). And, as he states in his method of first and ultimate ratios, he found that the proportion could also be established, but only at the very beginning, or at the very end, of the motion (at the O point).[4]

There is a certain similarity in the formulations by Galileo and Newton of the kinematic problem and its geometrical representation, although their resolution methods were obviously different [Sellés, 2001b]. Therefore, we can expect to find in Newton's geometry the same concepts of intensity and quantity of a magnitude, the latter being measured throughout a certain extension, normally the duration. These concepts appear in a manuscript entitled "De gravitatione," which dates from around the time Newton was preparing the *Principia*.[5] Therein he presented a series of definitions (motion, force, *conatus*, *impetus*, *inertia*, pressure, and gravity), establishing that their quantities could be determined by taking

[3] The question raised here is dealt with in more detail in Sellés [2001b]. The brief interpretation presented here is based on the fragment "Dubito" [Galilei, 1964–1966, VIII, 386].

[4] Note that the same Fig. 3 continues to be valid if, instead of representing intensities of velocities, it represents spaces, and can be compared to the one accompanying Lemma IX, Book I, Section 1ª of the *Principia*.

[5] "De gravitatione et aequipondio fluidorum." In Hall and Hall [1962]. Recently a date of ca. 1684 has been agreed upon [Newton, 1999, 47]. My thanks to Professor Guicciardini, who informed me of this fact.

into account their intensity and extension, and that their "absolute quantity" was the product of intensity and extension. He also defined velocity as the intensity of the motion. These definitions underwent modifications in the *Principia*, but he kept the distinction between intensity and quantity.[6] Specifically, the "motive quantity of centripetal force" is defined as "the measure of this force that is proportional to the motion which it generates in a given time" [Newton, 1999, 407]. Later on, in the enunciation of Law II, he writes: "A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed." A little further on he specifies that this is accomplished "whether the force is impressed all at once or successively by degrees" [Newton, 1999, 416].

Erlichson [1991, 843] points out that the concept of force that plays the predominant role in the *Principia* is the quantity of the motive force of Def. VIII (a scalar quantity that increases with time). Geometrically represented according to the diagram of Fig. 3, this quantity will be an area whose base is the time interval under consideration. Its intensity (we agree to call it f_{II}) would be represented by a segment perpendicular to this base (represented as AB in Fig. 3). Because it is an intensity, it produces no change in the motion. On the other hand, the force acting "all at once" in Law II (we agree to call it f_{I}) does generate motion, even when it is also instantaneous. It thus produces an action commensurable to the quantity of motive force f_{II}, but not to its intensity or degree.[7] As mentioned previously, f_{I} would be associated with the polygonal model for the action of the centripetal force, and f_{II} would be associated with the parabolic one. Both forces and models can also be related to the two successive concepts of moment adopted by Newton. These are described in the next section.

2. Indivisibles, infinitesimals, and infinites

We can distinguish between two concepts of the infinite. The first is denumerable, e.g., the set of natural numbers. The second can be related to what I shall call the "actual" infinite, e.g., the set of all the points composing a line, which as is known is a dense and uncountable set.[8]

The process of division of, for example, a straight line segment by a denumerable method would be unending. But, supposing it were to have an end, an entity called an infinitesimal would be obtained, having the paradoxical property of representing the end of the process and still being divisible (otherwise, the infinitesimals would bear no ratio to each other). Specifically this is the concept of the infinitesimal held by John Wallis [Jesseph, 1999, 173; Malet, 1997], and it was adopted by Newton when he started to work on the calculus. However, we may suppose that, given that the infinitesimal is still divisible and homogeneous with the magnitude from whence it arises (the infinitesimal of a line is a line too, and can be divided by marking a point on it), the division has not been carried to its "true" end. As Galileo already

6 This is clear in the definitions located at the beginning of the work: "quantity of matter" (Def. 1), "quantity of motion" (Def. 2), "absolute quantity of centripetal force" (Def. 6), "accelerative quantity of centripetal force" (Def. 7), and "motive quantity of centripetal force" (Def. 8). And he adds: "these quantities of forces, for the sake of brevity, may be called motive, accelerative, and absolute forces" [Newton, 1999, 403–407].

7 In today's terms, and recognizing that the cases are not the same, one could say that it is very similar to a delta of Dirac.

8 These two concepts are presented in Sellés [2001a], a paper written before Sellés [1998] and Sellés [1999], although it was published at a later date. For Galileo, these concepts are discussed in Sellés [2001b]. The distinction between the treatments by Galileo and Leibniz is studied in Knobloch [1999]. Regarding medieval concepts see Murdoch [1982].

pointed out, this end cannot be reached by a denumerable division process, but must be achieved by other means [Galilei, 1964–1966, VIII, 81].

As is known, Galileo considered the continuum to be composed of indivisibles. The given composition was geometrically heterodox, but did not lack predecessors among the medieval atomists. In this conception, the continuum, e.g., a line, will be composed of an actual infinity of points. In spite of the problems posed by its heterodoxy, Galileo was thus able to resolve the question of uniformly accelerated motion. It is from this conception that his statement affirming rest to be a state of infinite slowness is to be understood. If the infinite involved is the infinite called actual, and the time of a motion beginning from rest is represented by a line, then this state is the former extreme point of the line, and it can simultaneously be the last instant of rest and the first instant of motion without any contradiction, because an instant is not time, in the same way that "two o'clock" is the end of one o'clock and the beginning of two. The extreme point of a line can be reached by the motion, as a point in motion can survey all and every single point of a line segment without stopping at any end and, in a finite time, reach the true limit of the segment, its extreme point. This point is a truly indivisible entity, as it lacks extension. Galileo, however, did not use motion in his analysis of the continuum.

Newton (who was more orthodox than Galileo regarding the composition of the continuum) uses the concept of motion held by, among others, Thomas Hobbes and Isaac Barrow, according to which geometric figures should be genetically defined by means of a causal relation. The motion of a point generates a line, that of a line generates a surface, and that of a surface gives rise to a solid [Mancosu, 1996, 94]. As is known, this concept is found at the basis of Newton's fluxional calculus and in his later version of first and ultimate ratios.

Newton's rejection of the infinitesimal was a rejection of the conception according to which the infinitesimal would be eliminated at the end of a given process in the calculus, not because it was strictly null, as are indivisibles, but because of its infinite smallness. Although the error committed was less than any assignable magnitude, it still appeared to him unacceptable in geometrical rigor. Underlying this point of view is the idea that, even when curved lines, for example, are considered as being made up of infinitesimals, there is still a "true" curve, and these only constitute an approximation to it, however close this approximation may be.[9] Wallis constitutes an outstanding example of this way of thinking. In his treatise on the angle of contact (also called "angle of contingence" and "horn angle") that is formed by an arc and its tangent at a point, he tried to demonstrate that the given angle was smaller than a rectilinear angle, even when this angle was infinitesimal [Malet, 1997]. For Wallis curves are represented by polygonal lines composed of infinite rectilinear infinitesimal segments. But when studying the contact angle and the curvature, he considered that a "smooth" curve still exists and that it can be reached when these infinitesimal segments degenerate into points. The relation of these points to the segments was the same, in his opinion, as the one separating the contact angle from the rectilinear angle. The said contact angle belonged to a type of entity denominated "inceptive" or "inchoactive" of the magnitude, and which he characterized as follows:

> There are some things, which tho, as to some kind of Magnitude, they are nothing; yet are in the next possibility of being somewhat. They are *not* it, but *tantum not*; they are in the next possibility to it; and the Beginning of it: Tho'not as *primum quod sit* (as the Schools speak) yet as *ultimum quod non*. And may very

[9] This is one of the concepts discussed in the debates on the foundations of the calculus. See Mancosu [1989, 1996, 150–155].

> well be called *Inchoactives* or *Inceptives*, of that somewhat to which they are in such possibility. [Wallis, 1684, 96]

As examples of this class of magnitudes, Wallis presented the point (inceptive of the length if considered with relation to motion), the line (inceptive of the surface) and the surface (of the solid). In motion, swiftness or celerity was inceptive to length and acceleration to celerity.[10] Here Wallis was advancing a new concept. From his standpoint, one was not dealing with an infinitesimal because it does not come from an infinite division of the associated magnitude, and conversely, it cannot constitute one by an infinite multiplication. Rather one was dealing with an indivisible that he somehow associates with a "generative capacity" of magnitude.

There appears to be no direct connection between these concepts of Wallis and those of Newton. However, the above quotation can be compared with Newton's definition of "moment" written around the same time, in one of the draft manuscripts of the *Principia*:

> The moments of quantities are their principles of generation or alteration, as time present of the past and future, present motion of past and future motion, centripetal or any other momentary force of impetus, as point of a line, a line of a surface, a surface of a solid, and a contacting angle of a straight angle. [Herivel, 1965, 312]

This definition did not appear directly in the *Principia*, but another similar one did. In the Scholium to Section 1ª of Book I, Newton insisted on the fact that these moments had no magnitude, although they could have a ratio:

> It can also be contended that if the ultimate ratios of vanishing quantities are given, their ultimate magnitudes will also be given; and thus every quantity will consist of indivisibles, contrary to what Euclid had proved concerning incommensurables in the tenth book of his *Elements*. But this objection is based on a false hypothesis. Those ultimate ratios with which quantities vanish are not actually ratios of ultimate quantities, but limits which the ratios of quantities decreasing without limit are continually approaching, and which they can approach so closely that their difference is less than any given quantity, but which they can never exceed and can never reach before the quantities are decreased indefinitely. [Newton, 1999, 442–443]

Once again, in Lemma II of the 2nd Section of Book II of the *Principia*, Newton warned against considering moments as finite particles:

> They must be understood to be the just-now nascent beginnings of finite magnitudes. For in this lemma the magnitude of moments is not regarded, but only the first proportion when nascent. It comes to the same thing if in place of moments there are used either the velocities of increments and decrements (which it is also possible to call motions, mutations, and fluxions of quantities) or any finite quantities proportional to these velocities. [Newton, 1999, 647]

In fact, Newton used both types of infinite in the *Principia*. Because, as De Gandt has already pointed out [1986, 202, 1995, 226], time enters in two ways: as the discrete time of successive operations per-

[10] As Malet points out [1997, 87–88], it seems as if Wallis was on the brink of defining differentiation.

formed by the mathematician (a countable process of subdivision), and as the continuous time of motion, providing the background for the increase and decrease of the magnitudes. This latter aspect gives the geometry of the *Principia* its kinematic character. The first type appears in Lemmas II to V of the 1st Section of Book I, which deals with the approximation of curvilinear figures by means of a series of parallelograms whose width diminishes *ad infinitum*. The second appears in Lemmas VI to XI, where the movement of a point over an arc AB is considered, thanks to which the point, moving from B, will coincide with A. In the first case, as we are dealing with a numerable infinite, the limit will never be reached, but Lemma I establishes that "Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which before the end of that time approach so close to equality that their difference is less than any given quantity, become ultimately equal" [Newton, 1999, 433]. And in Corollary 4 of Lemma III he takes pains to specify that these figures "are not rectilinear, but curvilinear limits of rectilinear figures" [Newton, 1999, 434]. Thus the lines are not polygons composed of an infinite number of infinitesimal segments: at the limit, an extrapolation in time, the "true" curve is reached. But in the second case, that of motion, the limit is reached in a finite time. In this latter case, and with reference to Fig. 3, it could be said that when the time diminishes, the ratio between the velocities of the uniformly and diformly accelerated motions (please remember that they are intensities) is only approximately equal to the ratio to be found in point O. However, the approximation increases as the time decreases, until at point O both become equals (and thus at this same starting point of motion the curve and its tangent will coincide, and will follow Galileo's law of the proportionality between the space and the time squared). The objection is, naturally, that at point O the velocities no longer exist. But Newton alleges that an "ultimate velocity" exists, at which the motion ceases (or commences). We must remember here that an instant, which is not time, can be the last instant of motion and the first instant of rest, simultaneously. But, as we have seen in the previous quotations, he remains undecided with respect to the existence of these quantities at the limit in which they "vanish." The quantities cannot exist, but their ratio still exists. Is this then the ratio that occurs between the "generating capacities" of these ultimate quantities, now indivisibles, and which are reached at the limit? Now, Newton does not explain it, and so the historian's interpretation also reaches its limit.

Must this second concept of moment be considered as an infinitesimal, although Newton did not conceive it as such? It depends on what we wish to understand by this term.[11]

3. The transition between the two concepts

At an initial stage, when Newton used what we could call the first concept of moment, he considered curves formed by an infinite number of rectilinear segments. In his first exposition of fluxional calculus, in November of 1665, when comparing the uniform motions of two bodies A and B, he wrote

> And though they move not uniformly yet are y[e] infinitely little lines w[ch] each moment they describe as their velocitys are w[ch] they have while they describe them. [Newton, 1967–1981, I, 385]

[11] Thus, De Gandt [1992] underlines the nature of the moment as an element "gros de la réalité à venir," and interprets it as an infinitesimal (also in De Gandt [1995, 224–225]).

In 1671, in a treatise on fluxional calculus,[12] he adopted the flow of time itself as a unit, instead of the uniform motion of one of the variables. He introduced the term "fluent" for the variables, and the term "fluxion" for their velocities, and started to avoid the term "moment," replacing it by "fluxion." Until then, an infinitesimal interval of time had been understood for "moment" [Newton, 1967–1981, I, 414]. However, in some additional pages that he wrote later his ideas changed. He established as a theorem that if the relation $A/B = C/D$ holds between four fluent quantities then $A \cdot fl(D) + D \cdot fl(A) = B \cdot fl(C) + C \cdot fl(D)$, where $fl(A)$ denotes the fluxion of A. He started the proof considering infinitely small moments, but he changed his mind, probably because he did not want to eliminate them at the end of the calculation due to their infinite smallness. Thus he considered that the last step took place in the ultimate moment of the defluxion, or in the first instant of the fluxion of the variables. According to Whiteside [Newton, 1967–1981, 3, 334, n. 16] and Westfall [1980, 231], this gave birth to the first-and-ultimate-ratios method. The matter was made clear in the preface to his "Geometria curvilinea" written ca. 1680:

> Those who have taken the measure of curvilinear figures have usually viewed them as made up of infinitely many infinitely small parts; I, in fact, shall consider them as generated by growing, arguing that they are greater, equal or less according as they grow more swiftly, equally swiftly or more slowly from their beginning. And this swiftness of growth I shall call the fluxion of a quantity. So when a line is described by the movement of a point, the speed of the point—that is, the swiftness of the line's generation—will be its fluxion. I should have believed that this is the natural source for measuring quantities generated by continuous flow according to a precise law, both on account of the clarity and brevity of the reasoning involved and because of the simplicity of the conclusions and the illustrations required. [Newton, 1967–1981, 4, 423; transl. of Whiteside]

Newton's two concepts of moment correspond to "analytical" and "synthetic" calculus methods, according to Newton's own terminology [Guicciardini, 1999, 28]. The appearance of the new method in the mechanical manuscripts can be dated to 1684, in the initial draft of *De Motu*. It is possible that a demonstration produced in 1679 or 1684 influenced this change, as the so-called "Kepler-motion papers"[13] are dated from around this time. In Proposition I, Newton proved Kepler's area law by applying the polygonal model, and in it he continued to consider the curve to be formed by infinitesimal segments. In Proposition II he considered a body moving in an ellipse under the attraction of a central force directed to one of the foci, and he determined the intensities of the force at the extremes of the long axis. In Proposition III he tackled the comparison of these intensities at any two points of the orbit. He drew a figure similar to Fig. 4, in which he considered points P and p, supposing that the segments PY and py are described in the same time interval. When these time intervals become infinitely small, the lines PQ, XI, and pq, xi become coincident and a proportion is established in which the intensities of the forces are inversely proportional to the squares of the distances from the focus.

With the new synthetic concept of calculus, this proof is no longer strictly true, as the aforementioned lines do not become coincident; they still remain separated by an infinitesimal distance. When Newton

[12] This exposition, although Newton did not give it a title, is known as "Tractatus de methodis serierum et fluxionum" [Newton, 1967–1981, 3, 32–328].

[13] The manuscript is titled "A demonstration that the Planets by their gravity toward the sun may move in Ellipses." A copy of this was found among Locke's papers, dated March 1689/1690. However, it is thought that it was possibly composed on the aforementioned dates, preceding *Principia*.

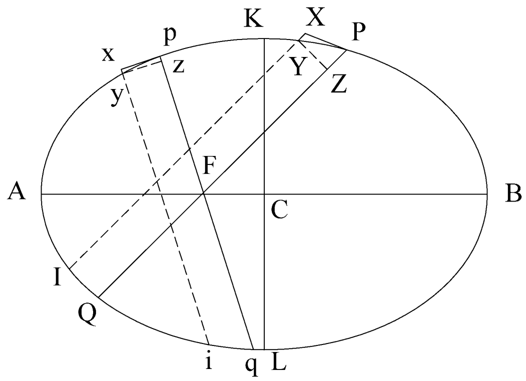

Fig. 4.

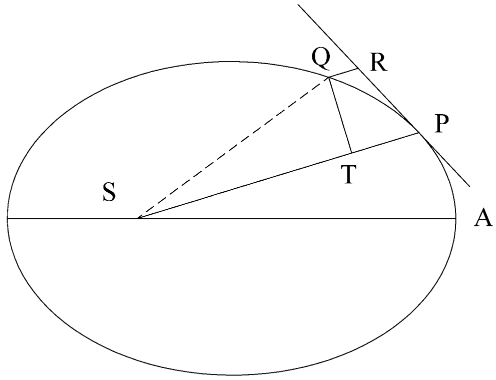

Fig. 5.

wrote the first draft of the *De Motu*, dated autumn 1684, he demonstrated Kepler's area law, still taking into account an infinitesimal moment of time [Newton, 1967–1981, VI, 34]. But in Theorem 3 he calculated the intensity of the centripetal force at a point on the trajectory, substituting the proof of the "Kepler-motion papers." The figure resembles Fig. 5. Here Newton does not compare the force intensities at two points on the trajectory; instead, he considers a single point P, the figure $QRPT$ as "indefinitely small," and obtains the value of the centripetal force in the limit, when P and Q coincide [Herivel, 1965, IX(1), 260; Newton, 1967–1981, VI, 40]. In this new proof, unlike the previous one, the said points no longer remain at an infinitesimal distance: they coincide exactly.

However, it could be said that there is no deviation at point P. But does an initial velocity exist with which the motion begins? These questions would take us back to the argument of the *Principia*.

4. The polygonal and the parabolic models

In the polygonal model, the action of the centripetal force is considered as a succession of impulses f_{I} impressed on the body at equal time intervals. When the number of impulses becomes infinite, the interval between them becomes infinitesimal. In this model, the force f_{I} is like the deviation it produces at each vertex. This deviation, produced by an instantaneous force, is described with uniform motion. The model conforms with the concept of a curve at a local level as a polygonal line of an infinite number of infinitesimal sides. In the parabolic model, force f_{II} is considered to be constant in an infinitesimal time interval. Here the deviation is described with uniformly accelerated motion. This agrees with the conception of the trajectory as composed of an infinite number of parabolic segments.

In the years 1665–1670 Newton applied both models to the study of circular motion [Herivel, 1965, Ms. IIa and IVa]. It seems that around this time he may already have been aware of the equivalence between them, given that in another manuscript he wrote:

> If a body move progressively in some crooked line and also circularly its center of motion shall have the same determination and velocity which the body hath. For this is trew when its motion is in a streight line but a crooked line may bee conceived to consist of an infinite number of streight lines. Or else in any point of the croked line the motion may be conceived to be on in the tangent. [Herivel, 1965, Ms. IId, 145]

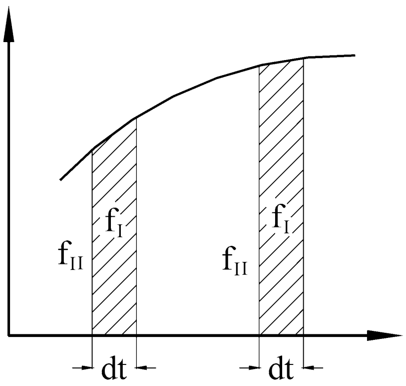

Fig. 6.

In Proposition III of the "Kepler-motion papers" (Fig. 4), the forces at the points P and p are like the displacements XY, xy described in the same time interval. Here Newton still applies the first concept of moment. When this time interval becomes infinitesimal, the displacements become infinitesimal, as do the forces f_{I} involved:

> Suppose now that the equal times in which the revolving body describes the lines PY and py become infinitely little, so that the attraction may become continual and the body by this attraction revolve in the perimeter of the Ellipsis: and the lines PQ, XI and also pq, xi becoming coincident and by consequence equal, the quantities $(PQ/XI) \cdot pF^{\mathrm{quad}}$ and $(pq/xi) \cdot PF^{\mathrm{quad}}$ will become pF^{quad} and PF^{quad}. [Herivel, 1965, 253–254]

Referring to this demonstration, Westfall [1971, 431] states that Newton began with a force f_{I} and ended with another force f_{II}. However, Erlichson [1991, 847–848] observes that even when individual motive forces f_{I} go to zero, in the limit where they are strictly zero their ratio still exists, which would be the ratio of the forces f_{II} in these points. The "continuous" curve is thus the limit of the polygonal. Herivel [1965, 256, n. 14] observes that this step to the limit would be a double one:

> The limiting process here envisaged is effectively a double one. It is necessary, in the first place, because the ellipse has been replaced (partially, at least) by an inscribed polygon; and in the second place, because the results already obtained are merely approximate. Only the second type of limiting process is necessary in Prop. 3 of the tract *De Motu* where the curve is not approximated to by an inscribed polygon.

Actually, this "double step" to the limit corresponds to the procedures of infinitesimal calculus that Newton had been using. The time interval becomes infinitesimal, and in a second step corresponding to the end of the calculation, the infinitesimal quantities are nullified. This nullification is produced "suddenly" and is equivalent to establishing that the ratio among the infinitesimal forces f_{I} before the nullification occurs (represented by the areas of base dt in Fig. 6; the curve represented is arbitrary) will be the same as the ratio between the intensities f_{II} after it has been produced. This then is an approximation in which the error committed will be less than any assignable magnitude (in short, infinitesimal).

Although he knew that this method led to correct results, Newton sought another more rigorous procedure. If a numerable division of time would only bring the points P and Y (or p and y) to an infinitesimal distance apart, the motion could carry them to a strict coincidence. Thus, in a second step to the limit, the infinitesimal areas f_{I} in Fig. 6 would continue decreasing in width until finally, when the points coincided, the areas disappeared as such, and were reduced to the intensities f_{II}. But please note that, with this procedure, comparison of the forces in two points of the trajectory is no longer required. The first step toward the limit converting the sides of the polygonal line into infinitesimals is not necessary either. One can start from an indefinitely small but finite distance between P and Y, and using the motion of Y toward P bring them to coincide without passing through the infinitesimal quantities whose existence is doubtful. This is the procedure of the *De Motu* proof referred to in our discussion of Fig. 5, which replaces the procedure explained in Fig. 4. The deviation RQ is still taken parallel to PS, as in the case of the impulse model; however, it is not because this model is being applied, but rather because it is the direction of said deviation in point P, the point where the aforementioned deviation "vanishes." And at this point—corresponding precisely to the point O of Fig. 3—the spaces are strictly as the squares of the times. No approximation is made.

Coming back to our two conceptions of "moment," it can be said that the infinitesimal quantities of motive force f_{I} fit with the first concept, whereas the intensities f_{II} fit with the second. As f_{II} is an intensity, its corresponding quantity can be expressed by the product between this force and the time interval in which it is measured. This quantity of f_{II} is expressed by f_{I}. With reference to Fig. 6, we can take the limit of f_{I} when the interval to time goes to zero (in this case, the rectilinear segments in the polygonal model go to zero too). This limit, when f_{I} vanishes, is f_{II}. The first concept of moment corresponds to the infinitesimal interval of time that occurs with the force f_{I} (or in the quantity of f_{II}), and the second concept corresponds to the instant of time associated with the intensity f_{II}. As in the case of the forces f_{I} and f_{II}, the two conceptions of moment are separated by one dimension.

Corresponding to the differences between the two moments, there are changes in the treatment of the polygonal and parabolic models. In the demonstration referred to in our discussion of Fig. 5, the force already is not supposed constant in an infinitesimal interval of time. Indeed an infinitesimal in the sense of the first concept of moment does not occur. In the same way, in the polygonal model it does not occur any more. The finite rectilinear segments between impulses go to zero without passing through an infinitesimal magnitude.

Thus, in the same way that the calculation of the centripetal force changed between the "Kepler-motion papers" and the first draft of *De Motu*, the proof of Kepler's area law changed between the initial draft of *De Motu* and its first revision.[14] In the initial draft the number of triangles was made infinite, converting them into infinitesimals, so that each individual time "moment" corresponds to only one triangle. In which case, he says, the centripetal force would act "without a break" [Herivel, 1965, 259; Newton, 1967–1981, VI, 34]. In the revision the area is no longer the sum of the infinitesimal triangles, but is the sum of the evanescent triangles, and the Proposition, as in the case of the *Principia*, is proved by Corollary 4 of Lemma III [Newton, 1967–1981, VI, 124]. In the *Principia* he added the observation that the ultimate perimeter of this figure will be a smooth line [Newton, 1999, 445].[15]

[14] Whiteside dates the first draft of the *De Motu* to the autumn of 1684, and the first revision to the winter or the beginning of spring 1684/1685.

[15] Some years ago D.T. Whiteside [1966, 1991; in Newton, 1967–1981, VI, 35–37, n. 19] and E. Aiton [1972, 103–104, 1989] pointed out a fault in Newton's proof of Kepler's area law in the *Principia*. If the deviation in the trajectory of a body produced by the centripetal force is considered as described by a uniformly accelerated motion, then the force is considered as constant during an infinitesimal interval of time. But if you take as starting point that the centripetal force acts as a succession

5. Conclusion

Given that Newton was not very explicit, it would appear that not much more can be said about this second concept of "moment." If the magnitude generated is represented by an area, as is the case for our first three figures, the corresponding "moment"—here the degree or the intensity of the velocity, which Newton calls simply "velocity"—is represented by a line, and it possesses a magnitude with its own dimension: two moments have a ratio. Thus it is possible to study its proportionality with the magnitude generated (the same will happen in the case of a solid). But, if the magnitude generated is represented by a line, then the moment is a point, and if one wishes to give it some type of dimension of its own, this must be done through some sort of device. This is what happens in algebra, whose domain is the real line. It is the classical problem of the infinitesimal (the "metaphysics of the calculus"), which nonstandard analysis resolved by granting the point a certain dimension, or considering a hyperdense continuum. In the age of Newton, however, it was thought that a line contained all the points there could be (given any two points of the line, another point could always be inserted between them).

In any case, with the second concept of moment, the problem was how to measure an intensity that was a mere potentiality. Colin Maclaurin, in his *Treatise of Fluxions* (1742), gave the classic answer:

> The velocity with which a quantity flows at any term of the time while it is supposed to be generated, is called its *Fluxion*, which is therefore always measured by the increment or decrement that would be generated in a given time by this motion, if it was continued uniformly from that term without any acceleration or retardation. [Maclaurin, 1742, 57]

In the case of velocity, the fluxion of space, Maclaurin's measurement could be represented graphically by our Fig. 1. If, instead of the velocity, we consider f_{II}, that is, the fluxion of the velocity, this would be measured by the velocity that f_{II} would generate in a given time, assuming that f_{II} stayed constant; this would be a motion uniformly accelerated. It could be represented graphically by Fig. 2 at each point O of the trajectory.[16] The alternative would be to suppose f_{II} constant in a $\mathrm{d}t$, which would give rise to the same uniformly accelerated movement in the infinitesimal time interval (the parabolic model). The former does not suppose an approximation, and the latter does not suppose one either, if the $\mathrm{d}t$ is considered to be *inside* the instant t.

This takes us back to Galileo's problem, represented by our Fig. 2. It seems that there are no data enabling a direct relationship to be established between Galileo's "moment" and that of Newton. However, it is very interesting to note that, in Galileo's hands, it has a meaning very similar to that of Newton's concept. Galileo used the term "moment" in two contexts, weight and velocity. In both cases, the meaning

of instantaneous impulses, this implies that the velocity is supposed constant, and then the rectilinear segments traveled at uniform velocity should be second order infinitesimals. Such infinitesimals would approach a parabolic segment which would be a first order infinitesimal, which in turn would approach the true trajectory. Thus Newton when using the impulse model would only have carried out a valid proof for infinitesimal trajectories. The foregoing exposition supports Erlichson's assertion [1992] that from the polygonal model you do not need to go through the parabolic approximation in order to reach, in the limit, the curve. (The parabolic approximation supposes that the areas f_{I} of our Fig. 6 are approximated by rectangles.) Recently B. Pourciau [2004] has published a demonstration of the mathematical equivalence between the limit of a polygonal impulse motion and a tangent deflection one.

16 Maclaurin calls the intensity "power" and makes it proportional to the acceleration. He reserves the term "force" for the quantity of this intensity in a finite time, which is the same as the velocity acquired in this time [Hankins, 1967, 58].

was that of a *tendency*. The first of these moments was the tendency of the body to go down, or a modification of this. The second has traditionally been identified with the degree of velocity, as this usually appears in expressions such as "the degree or moment of velocity." In fact it should be understood as the tendency of the velocity to increase or decrease, that is to say, as the acceleration.[17] Both moments are, then, intensities related to motion. When studying the descent of a body along an inclined plane, Galileo finally used a dynamic principle, according to which the moments of descent along the vertical side and the inclined side would be the same as their overall velocities, measured in the same time interval. This case is represented in Fig. 2, where the said velocities would be as the degrees of velocity reached at the end of this time along the vertical and the inclined sides. In turn, these degrees will be equal to the corresponding moments of velocity accumulated during this time. In other words, the forces would be as the accelerations.[18] However, other measurements exist. If, with Newton, we call velocity the degree of velocity, then the quantities of the moments compared in this time interval (the areas below the lines in Fig. 2), are as the squares of the velocities reached at the end of this time (that is to say equal to what we call nowadays kinetic energies). This relates the moment and its quantity, respectively, to what Leibniz called *vis mortua* (dead force) and *vis viva* (living force). From this point of view, we can say that the "*vis viva* controversy" that developed in the first half of the 18th century revolved around the question of the intensity of the force and its quantity.

We see therefore that concepts of mechanics and concepts linked to the mathematical representation of motion were indissolubly intermingled in the gestation of classical mechanics, which also was the stage of gestation of mathematical analysis. Thus when studying one we must not forget the other.

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[17] These questions are somewhat involved, and I am currently preparing a paper about them. Justification of these statements is not possible here. But, please note, in Fig. 2, that if a uniformly accelerated motion starting from rest is considered, then the line representing the degree of velocity also represents the (actually) infinite additions "non quante" undergone by this degree of velocity from the beginning of the movement until the instant of time considered.

[18] In the *Principia*, Newton stated that Galileo had analyzed accelerated motion using Newton's first two laws. There is no evidence that Newton had direct knowledge of Galileo's *Discorsi*, and even less of the annex written by his disciple Viviani, which appeared in the second edition in which this principle was presented. But proof to the contrary does not exist either [Herivel, 1965, 36–37]. On the other hand, other later authors, such as Maupertuis, interpreted what we know today as the second law of Newton in terms of Galileo's law [Hankins, 1967].

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