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Newton's Solution to the Equiangular Spiral Problem and a New Solution Using Only the Equiangular Property

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This paper is a study of Proposition IX of Book I of Newton's *Principia*, the problem of determining the centripetal force for an equiangular spiral. In Newton's main proof of this proposition there is an error concerning his reason for the figure SPRQT being "given in kind," and a very interesting technique of varying things in the neighborhood of a limit. This main proof utilized Newton's formula for the limit of SP^2QT^2/QR given in Corollary I to Proposition VI of the *Principia*. Newton also gave an alternate proof which utilized his formula for SY^2PV given in Corollary III to Proposition VI. The "given" of Proposition IX was "a spiral PQS, cutting all the radii SP, SQ, &c., in a given angle." Both the main proof and the alternate proof implicitly depend on the property of the equiangular spiral that the radius of curvature at any point is proportional to the pole distance SP. We here offer a new proof of Newton's proposition which does not depend on this implicit assumption. © 1992 Academic Press, Inc.

Cet article est une étude de la Proposition IX du Livre I des *Principia* de Newton, du problème de la détermination de la force centripète pour une spirale équiangle. Dans la preuve principale de Newton de cette proposition il y a une erreur concernant sa raison pour laquelle la figure SPRQT soit "donné en espèce," et aussi une technique très intéressante pour la variation des quantités dans le voisinage d'une limite. Cette preuve principale a utilisé la formule de Newton de la limite de SP^2QT^2/QR donnée dans Corollaire I de Proposition VI des *Principia*. Newton a donné aussi une preuve alternative qui utilise sa formule de SY^2PV donnée Corollaire III de Proposition VI. La donnée de la Proposition IX a été "une spirale PQS, coupant tous les rayons SP, SQ, &c., dans un angle donné." Et la preuve principale et la preuve alternative dépendent implicitement de la propriété de la spirale équiangle que le rayon de courbure daas un point quelconque est proportionnel à SP, la distance du pôle. Nous offrons ici une preuve nouvelle de la proposition de Newton qui ne dépend pas de cette supposition implicite. © 1992 Academic Press, Inc.

Dieser Artikel ist eine Studie über Lehrsatz IX aus dem ersten Buch von Newtons *Principia*, welcher von dem Problem, die Zentripetalkraft einer gleichwinkligen Spirale zu bestimmen handelt. In Newtons grundlegendem Beweis dieses Satzes steckt ein Fehler in seiner Begründung, daß die Figur SPRQT "given in kind" sei. Ausserdem verwendet er eine sehr interessante Technik, Gegenstände in der Umgebung eines Grenzwertes zu variieren. Dieser grundlegende Beweis verwendet Newtons Formel vom Grenzwert von SP^2QT^2/QR, welche im Korollar I zu Satz VI der *Principia* steht. Newton gab ebenfalls einen Alternativbeweis an, in dem er seine im Korollar III zu Satz VI stehende Formel für SY^2PV benutzt. Die Voraussetzung von Satz IX war "eine Spirale PQS, die alle Radien SP, SQ, etc. in einem gegebenen Winkel schneidet." Beide, der grundlegende Beweis und der Alternativbeweis, sind implizit von der Eigenschaft der gleichwinkligen Spirale, daß in jedem Punkt der

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Krümmungsradius proportional zum Polabstand SP ist, abhängig. Wir zeigen hier einen neuen Beweis von Newtons Satz, welcher diese implizite Voraussetzung nicht braucht.
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1. INTRODUCTION

In Proposition IX Newton presents two proofs that the centripetal force for the equiangular spiral goes inversely as the cube of the distance from the pole. The main proof is of great interest because there is an interesting error in it and an interesting technique of varying the "indefinitely small angle PSQ." His proof also depends on a certain infinitesimal figure being "given in kind." He gives the wrong reason for this. One can only speculate on what reason he might have given if this error had been brought to his attention, but a likely candidate depends on the implicit assumption that the radius of curvature of the spiral is proportional to the pole distance. The alternate proof offered by Newton also depended on the same implicit assumption about the radius of curvature for the spiral. After our analysis of Newton's proofs we offer a new proof which does not use Newton's implicit assumption about the radius of curvature. Thus our proof uses only the single "given" of Proposition IX, that the spiral is equiangular.

2. NEWTON'S MAIN PROOF OF PROPOSITION IX

Figure 1 shows Newton's diagram for both of his proofs of Proposition IX. In this section we discuss only his main proof for which the line SY and the extension of SP to point V do not enter. In Fig. 1 shown, S is the attracting center and PQ .. is the equiangular spiral orbit. The equiangular spiral is characterized by a constant angle between the radius to a point and the tangent to the spiral at that point. In Newton's diagram PRY is the tangent to the spiral at point P. Q is a closely adjacent point on the spiral, QT is a perpendicular from Q on to SP, and QR is a parallel to SP drawn to the tangent PRY.

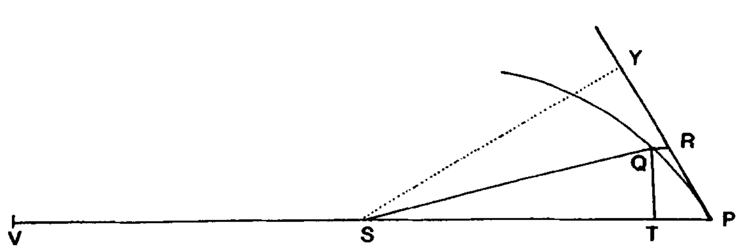

FIG. 1. Newton's diagram for Proposition IX.

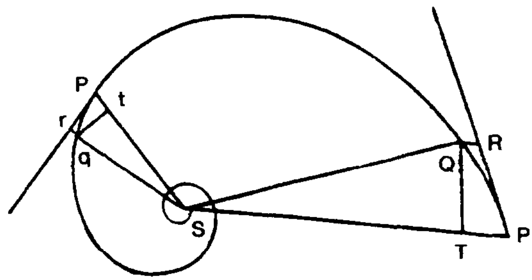

FIG. 2. Clarke's diagram for Proposition IX.

Newton begins his main proof with the sentence

> Suppose the indefinitely small angle PSQ to be given; because, then, all the angles are given, the figure SPRQT will be given in kind. [S1]

This sentence is filled with problems. Before considering these problems we note that by "given in kind" Newton clearly means "similar to the same figure constructed with the same angle pSq at any other point p on the spiral." Thus, "given in kind" has an implicit reference to another point p on the spiral (any other point) and makes Newton's proof equivalently a two-point proof. John Clarke used this two-point approach in his 1730 work entitled *A Demonstration of Some of the Principal Sections of Sir Isaac Newton's Principles of Natural Philosophy* [Clarke 1730, 158–164]. Figure 2 shows Clarke's diagram for his proof of Proposition IX. The figure Sprqt constructed at point p is similar to the figure SPRQT constructed at point P. It appears therefore that by S1 Newton is preparing us for a two-point proof; i.e., for the proof given by Clarke. However, in what follows S1 Newton never explicitly mentions any other point on the spiral.

Newton's reason for SPRQT being given in kind seems to be in error. This reason was that for SPRQT "all the angles are given." Now, the figure SPRQT is actually composed of the triangle STQ and the quadrilateral TPRQ. Although two triangles are similar if they have the same angles, two quadrilaterals are not necessarily similar if they have the same angles. Two quadrilaterals are necessarily similar if they have all corresponding sides in the same proportion. A square is not similar to a rectangle with unequal sides even though both figures have the same angles. So Newton's reason for "given in kind" was wrong. What he should have said instead of "all the angles are given" was "all the ratios of the sides are given."

Let us further pursue the meaning of "given in kind" by considering Fig. 3. In this figure the curve PQA is an equiangular spiral and the curve PQ′A′ is some other curve through P. PY is the tangent at P for both curves. The figures SPRQT and SPR′Q′T′ have equal angles but they are not similar figures. It is clear from considering Fig. 3 that what Newton wanted to call attention to for figure SPRQT

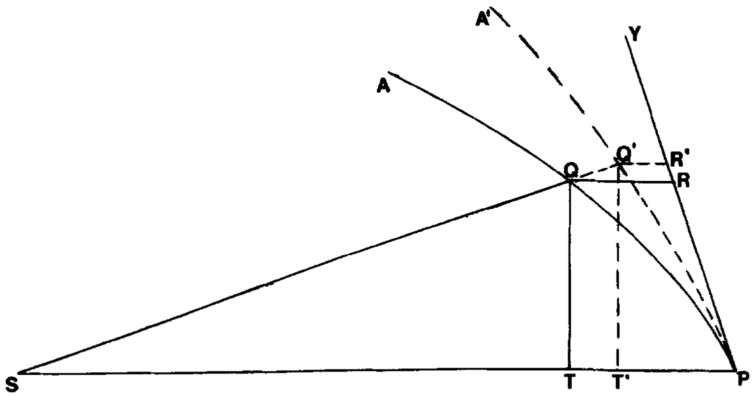

FIG. 3. Exploring the meaning of Newton's "figure SPRQT" is "given in kind."

was not the angles but the proportions of the sides of the figure. That which makes SPRQT similar to Sprqt at any other point on the spiral is that the proportions of the corresponding sides of the two figures are equal; i.e., that, for example,

$$\frac{\mathrm{QT}}{\mathrm{QR}} = \frac{\mathrm{qt}}{\mathrm{qt}}, \quad \text{or} \quad \frac{\mathrm{QT}}{\mathrm{SP}} = \frac{\mathrm{qt}}{\mathrm{Sp}}. \tag{1}$$

Another way of saying this is to say that once the angle PSQ is given, the ratios of all the sides of figure SPRQT are given.

Now, we ask, why is SPRQT "given in kind" for the equiangular spiral? If we look at SPRQT and SPR'Q'T' in Fig. 3 we see that the crucial thing is the curvature of the arc PQ. This is what differentiates the equiangular spiral arc PQ from the other arbitrary arc PQ'. And the unique thing about the spiral is that the radius of curvature at P is proportional to the pole distance SP. This is what clinches SPRQT as "given in kind." The angles for SPRQT and SPR'Q'T' were all the same but the curvatures at P were different.

Let us explore this further. Newton was surely aware that for any angle PSQ, finite or "indefinitely small," the figure SPRQT was "given in kind." Proving this for a finite angle PSQ is a bit of a task. However, proving it for an "indefinitely small" angle PSQ is somewhat immediate. Why so? Because for the indefinitely small PSQ one can assume that the curved section PQ is effectively approximated by the limiting circle of curvature in the infinitesimal neighborhood of point P. In this case PQ is now a very small arc of a circle and it is immediate that for a given "indefinitely small" angle PSQ all the sides of SPRQT are fixed in terms of ρ, the radius of curvature; i.e., that QR/ρ, QT/ρ, SQ/ρ, and PR/ρ are all given. So far, the curve can be any curve. But if now we have the equiangular spiral where ρ is proportional to SP (actually, SP $= \rho \sin \alpha$, see our Fig. 4) then QR/SP, QT/SP, SQ/SP, and PR/SP are all given; i.e., figure SPRQT is given in kind. So if Newton had been aware of his error in reasoning about why SPRQT was given in kind, then he would very likely have given the argument which we have just given as the correct reason for given in kind. And this argument assumes that the radius of curvature is proportional to SP. Thus, Newton's main proof assumes implicitly (if he had used our argument for given in kind, and, very likely, no matter what argument he used) that the radius of curvature is proportional to the pole distance.

We are now ready for Newton's next sentence. He says

> Therefore the ratio QT/QR is also given, and QT^2/QR is as QT, that is (because the figure is given in kind), as SP. [S2]

So, we have, for a fixed, indefinitely small angle PSQ, that

$$QT^2/QR \propto SP. \tag{2}$$

At this point Newton would like to pass to the limit as angle PSQ goes to zero, for it is this limit which enters into his measurement of the force at P. Here is a marvellously interesting situation. He is already in the immediate neighborhood of the limit; indeed he is arbitrarily close to the limit. SP is fixed and the limit of QT^2/QR exists from Newton's Lemma XI [Newton 1687, 36], (see Appendix on Lemma XI as applied to Proposition IX). Varying the indefinitely small angle PSQ keeps QT^2/QR the same because we are in the immediate neighborhood of the limit, so we can write

$$\lim_{PSQ \to 0} \frac{QT^2}{QR} \propto SP. \tag{3}$$

In Newton's words:

> But if the angle PSQ is any way changed, the right line QR, subtending the angle of contact QPR (by Lem. XI) will be changed in the ratio of PR^2 or QT^2. Therefore the ratio QT^2/QR remains the same as before, that is, as SP. [S3] and [S4]

The final step in the proof is done by Newton by referring to Corollaries I and V of Proposition VI. In these corollaries it is shown that the centripetal force goes inversely as $SP^2 \lim QT^2/QR$. Newton says

> And QT^2SP^2/QR is as SP^3, and therefore (by Cor. I and V, Prop. VI) the centripetal force is inversely as the cube of the distance SP. [S5]

Newton has

$$\begin{aligned} F_P &\propto \frac{1}{SP^2 \lim QT^2/QR} \propto \frac{1}{SP^2(SP)} \\ F_P &\propto \frac{1}{SP^3} \end{aligned} \tag{4}$$

and his main proof of Proposition IX is complete.

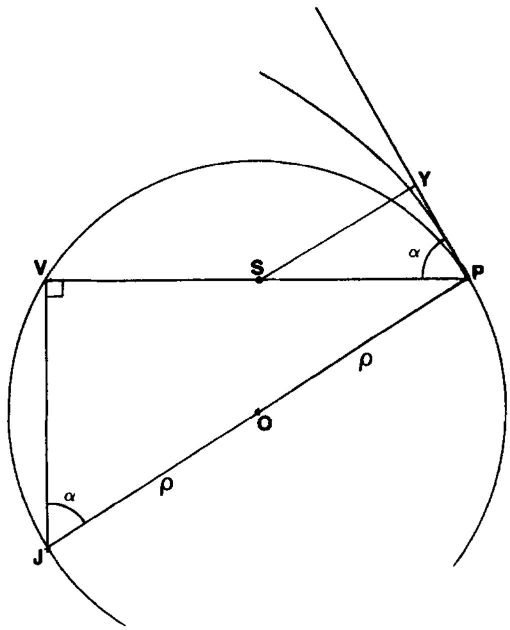

FIG. 4. Diagram showing that PV = $2\rho \sin \alpha$.

3. NEWTON'S ALTERNATIVE PROOF OF PROPOSITION IX

Newton also presented an alternative one-sentence proof of Proposition IX. This alternative proof utilized the SY^2PV measure of a force given by Newton in Corollary III to Proposition VI. Like the main proof it depended upon the proportionality between the pole distance SP and the radius of curvature. Referring to Fig. 1 Newton said

> The perpendicular SY let fall upon the tangent, and the chord PV of the circle concentrically cutting the spiral, are in given ratios to the height SP; and therefore SP^3 is as SY^2PV, that is (by Cor. III and V, Prop. VI) inversely as the centripetal force. [Newton 1687, 53] [S6]

From Fig. 1 we see that

$$SY = SP \sin \alpha \tag{5}$$

so that SY/SP is in the "given ratio" of $\sin \alpha$. Figure 4 shows Newton's equiangular spiral with the "circle concentrically cutting the spiral" at point P. O is the center of curvature and the radius of curvature at point P is OP, which we denote by ρ. From this diagram

$$PV = 2\rho \sin \alpha$$

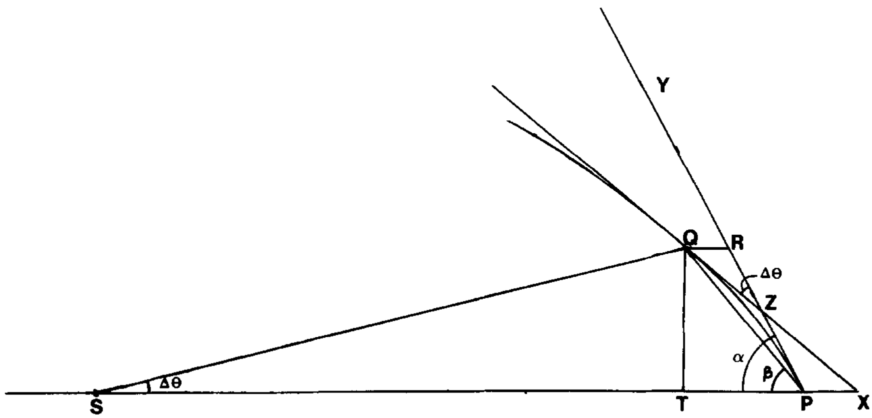

FIG. 5. Diagram for the new proof of Proposition IX.

If we now assume, with Newton, that ρ is proportional to SP (actually $\rho \sin \alpha =$ SP) we have

$$PV \propto SP; \tag{6}$$

i.e., PV is in a "given ratio" to SP. If we now combine Eqs. (5) and (6) we have

$$SY^2PV \propto SP^3. \tag{7}$$

Since, by Corollary III to Proposition VI, the centripetal force is inversely as SY^2PV we find that

$$F \propto \frac{1}{SP^3}. \tag{8}$$

4. A NEW SINGLE POINT PROOF OF PROPOSITION IX WITHOUT NEWTON'S RADIUS OF CURVATURE ASSUMPTION

We have seen that both of Newton's proofs of Proposition IX depend on the assumption that the radius of curvature of the spiral is proportional to the pole distance SP. We next present a new proof of Proposition IX without this assumption. In this proof we use only the given that the radius SP of the spiral makes a constant angle α with the tangent PY for all points on the spiral.

Consider Fig. 5 which shows Newton's diagram with some added detail. The tangent QX to the spiral at point Q has been added. The tangent QX intersects the tangent PY at point Z. The constant angle between the tangent to the spiral and the radius is denoted by α. The very small angle QSP is denoted by $\Delta\theta$. Since the radius SP turns through an angle $\Delta\theta$ in going to SQ, the tangent to the spiral likewise turns through $\Delta\theta$ as the body moves from P to Q. Thus, angle QZR is equal to $\Delta\theta$.

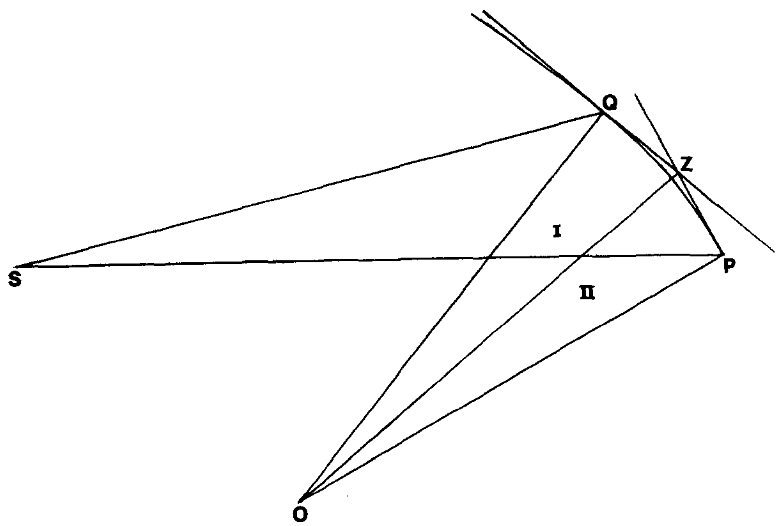

FIG. 6. Diagram to determine the limit of QP/QZ.

We first apply the law of sines to triangle QRZ. QR is parallel to SP, and since angle SPR $= \alpha$, angle QRZ $= \pi - \alpha$. The law of sines gives

$$\frac{\mathrm{QR}}{\sin \Delta\theta} = \frac{\mathrm{QZ}}{\sin \alpha}. \tag{9}$$

The law of sines applied to triangle SPQ gives

$$\frac{\sin \Delta\theta}{\mathrm{QP}} = \frac{\sin \beta}{\mathrm{SQ}} \tag{10}$$

where we have denoted angle SPQ by β. Eliminating $\sin \Delta\theta$ between Eqs. (9) and (10) and solving for QR gives

$$\mathrm{QR} = \frac{\mathrm{QP}(\sin \beta)\,\mathrm{QZ}}{\mathrm{SQ}\sin \alpha}. \tag{11}$$

From triangle TPQ we have

$$\mathrm{QT} = \mathrm{QP}\sin \beta, \tag{12}$$

so that combining Eqs. (11) and (12) we find

$$\frac{\mathrm{QT}^2}{\mathrm{QR}} = \frac{\mathrm{QP}(\sin \beta)\,\mathrm{SQ}\sin \alpha}{\mathrm{QZ}}. \tag{13}$$

We need the limit of $\mathrm{QT}^2/\mathrm{QR}$ as $\Delta\theta \to 0$. In this limit $\beta \to \alpha$ and SQ $\to$ SP. Hence,

$$\lim_{\Delta\theta \to 0} \frac{\mathrm{QT}^2}{\mathrm{QR}} = \mathrm{SP}\sin^2\alpha \lim_{\Delta\theta \to 0} \frac{\mathrm{QP}}{\mathrm{QZ}}. \tag{14}$$

To determine the limit of QP/QZ we use Fig. 6. At point Q a perpendicular to tangent QZ is constructed, and at point P a similar perpendicular to tangent PZ is constructed. These two meet at point O. In the limit as $\Delta\theta$ goes to zero the point Q approaches point P as a limit, the distance OQ approaches the distance OP, and the two right triangles OQZ and OPZ (show as triangles I and II in the figure) approach congruence. Hence, in that limit the ratio QZ/ZP approaches equality. In that same limit the distance QZ + ZP approaches the distance QP, so we have

$$\lim_{\Delta\theta\to 0}\frac{\mathrm{QP}}{\mathrm{QZ}} = 2. \tag{15}$$

We can now substitute Eq. (15) into Eq. (14) to find

$$\lim_{\Delta\theta\to 0}\frac{\mathrm{QT}^2}{\mathrm{QR}} = 2\,\mathrm{SP}\sin^2\alpha. \tag{16}$$

Since the centripetal force at P goes inversely as the limit of $\mathrm{SP}^2\mathrm{QT}^2/\mathrm{QR}$, we have

$$\begin{aligned} \mathrm{F} &\propto \frac{1}{\lim\limits_{\Delta\theta\to 0}\mathrm{SP}^2\mathrm{QT}^2/\mathrm{QR}} \\ \mathrm{F} &\propto \frac{1}{2\,\mathrm{SP}^3\sin^2\alpha}. \end{aligned} \tag{17}$$

The angle α is a constant, so

$$\mathrm{F} \propto \frac{1}{\mathrm{SP}^3}; \tag{18}$$

i.e., the centripetal force at any point P on the spiral goes inversely as the cube of the distance from the force center S. This completes the new proof of Newton's Proposition IX.

5. THE PROPORTIONALITY CONSTANT BETWEEN THE CENTRAL FORCE AND THE LIMIT GIVEN IN COROLLARY I TO NEWTON'S PROPOSITION VI

Newton worked with proportionalities in his calculations of centripetal forces. It is of some interest to derive the proportionality constant between Newton's limit and the centripetal force. The distance QR in Fig. 1 is very closely given by

$$\mathrm{QR} = \tfrac{1}{2}a(\Delta t)^2,$$

where Δt is the very short time interval for the body to go from P to Q. The distance QT is given very closely by the transverse speed $v_\perp$ multiplied by Δt; i.e.,

$$\mathrm{QT} = v_\perp\,(\Delta t).$$

In the limit as $\Delta t \to 0$ these expressions for QR and QT will be exact. Thus,

$$\lim_{\Delta t\to 0}\frac{\mathrm{QR}}{(\mathrm{QT})^2} = \frac{1}{2}\frac{a}{\mathrm{v}_\perp^2}. \tag{19}$$

Now writing r for SP we have

$$\lim_{\Delta t \to 0} \frac{\mathrm{QR}}{\mathrm{SP}^2\mathrm{QT}^2} = \frac{a}{(2r^2v_\perp^2)}.$$

The constant angular momentum l of the body of mass m is given by

$$l = \mathrm{mrv},$$

so that, by substitution,

$$\lim_{\Delta t \to 0} \frac{\mathrm{QR}}{\mathrm{SP}^2\mathrm{QT}^2} = \frac{m^2a}{(2l^2)}.$$

Thus, we find

$$\mathrm{F} = ma = \frac{2l^2}{m} \lim_{\Delta t \to 0} \frac{\mathrm{QR}}{\mathrm{SP}^2\mathrm{QT}^2}. \tag{20}$$

We see that the constant relating the force and Newton's limit is twice the square of the angular momentum divided by the mass of the body.

An alternative expression for the constant $2l^2/m$ in terms of the areal speed $\dot{A}$ can be obtained by noting that

$$2\dot{A} = \frac{l}{m}$$

so that

$$\frac{2l^2}{m} = 2(2\dot{A})^2m.$$

Equation (20) can then be written as

$$F = ma = 2(2\dot{A})^2m \lim_{\Delta t \to 0} \frac{\mathrm{QR}}{\mathrm{SP}^2\mathrm{QT}^2}. \tag{21}$$

Another alternative is to solve for the accelerative force, F/m, and use $h = l/m = 2\dot{A}$. This yields

$$\frac{F}{m} = \text{acc. force} = a = 2h^2 \lim_{\Delta t \to 0} \frac{\mathrm{QR}}{\mathrm{SP}^2\mathrm{QT}^2}. \tag{22}$$

APPENDIX. LEMMA XI AND THE LIMIT OF $\mathrm{QT}^2/\mathrm{QR}$

Figure 7 shows the equiangular spiral of Fig. 1 and Newton's diagram for Lemma XI arranged to illustrate his Case 2. From that case we have

$$\lim_{\mathrm{QP} \to 0} \frac{\mathrm{QP}^2}{\mathrm{Q'P}^2} = \frac{\mathrm{QR}}{\mathrm{Q'R'}}. \tag{A1}$$

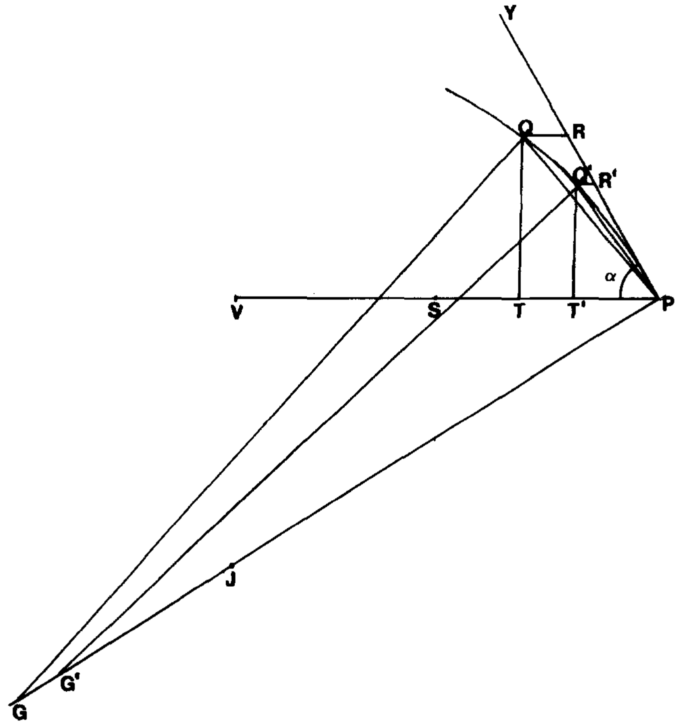

FIG. 7. The equiangular spiral and the Lemma XI diagram.

From Cor. I to Lemma XI

$$\lim_{QP\to 0} \frac{PR^2}{PR'^2} = \frac{QR}{Q'R'}. \tag{A2}$$

Since

$$QT = PR \sin \alpha$$

$$Q'T' = PR' \sin \alpha$$

$$\lim_{QP\to 0} \frac{QT^2}{Q'T'^2} = \frac{QR}{Q'R'} \tag{A3}$$

or

$$\lim_{QP\to 0} \frac{QT^2}{QR} = \frac{Q'T'^2}{Q'R'} \tag{A4}$$

Thus Lemma XI can be used to establish the limit of QT^2/QR.

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