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Newton's attempt to construct a unitary view of mathematics

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Abstract

In this paper Newton's persistent attempts to construct a unitary view of mathematics are examined. To reconcile the calculus of fluxions with Euclid's *Elements* or Apollonius's *Conics* appears, with the benefit of hindsight, an enterprise that cannot be accomplished simply by a widening of Greek mathematical thought. It requires a deep modification of the epistemological ground. Although Newton's attempts remained for the most part in manuscript form, it is hardly doubtful that Newton's ideas paved the way for the deep modifications that mathematics underwent in the succeeding centuries.

Riassunto

In questo articolo sono esaminati alcuni fra i tentativi, che hanno occupato l'intera vita di Newton, di costruire una visione unitaria della matematica. Riconciliare il calcolo delle flussioni con gli *Elementi* di Euclide o con le *Coniche* di Apollonio appare, con il senno di poi, un'impresa che non pu esssere condotta con una semplice estensione del pensiero matematico greco, senza una profonda modifica del suo fondamento epistemologico. Di conseguenza i tentativi di Newton, per la maggior parte, sono rimasti allo stato di manoscritti. Tuttavia difficile dubitare del fatto che le idee di Newton abbiano spianato la via alle profonde modificazioni subite dalla matematica nei secoli successivi.

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1. Introduction

When one considers Newton's writings composed after the *Principia*, and particularly his last writings, the necessity of making a comparison between Newton and Descartes is

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self-evident.[1] In these writings often, and sometimes in an obsessive way, Newton makes Descartes' work a target for his polemics, mixing within his harsh criticisms an increasing consideration for the mathematics of the ancients, a consideration that was at its height in his old age.

The modest consideration afforded by Descartes to classical mathematics is fairly well known. In a famous letter to Princess Elizabeth of Bohemia, Descartes asserts that

> I always observe, when seeking a geometrical question, that straight lines, which I make use of to find it, are parallel or intersect at right angles as much as possible; and I do not use other theorems, except that the sides of similar triangles are in the same proportion relative to each other, and that, in right-angled triangles, the square of the base equals the two squares of the sides.[2]

It is obvious that Descartes extracts *two single results* from the whole body of Euclid's *Elements*. He simply disregards the Euclidean *tour de force* that is required to obtain Proposition I.47 independent of the theory of proportion, as well as the subtleties of Book V. The same attitude appears in the *Regulae*[3] and in numberless letters. In the *Entretien avec Burman* he plainly states that mathematics has to be learned not from books but from its very practice.[4] In a well-known passage in the *Secondes Réponses* to the *Meditations*, Descartes opposes the value of analysis to that of synthesis.[5] This clearly implies a modest regard for the greater part of classical texts. Furthermore, a close look at the beginning of *La Géométrie* shows Descartes singling out some devices of the theory of proportion in order to introduce the correspondence between algebra and geometry, but again without reference to its origins.[6] Newton often severely criticizes this Cartesian attitude.[7] The

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[1] In composing this paper I have benefited greatly from [Guicciardini, 2009], where this subject is studied extensively. I have also used [Guicciardini, 1999].

[2] "J'observe tousiuors, en cherchant une question de Geometrie, que les lignes, dont je me sers pour la trouver soient paralleles, ou s'entrecouppent à angles droits, le plus qu'il est possible; & ie ne considere point d'autres Theoremes, sinon que les costez des triangles semblables ont semblable proportion entr'eux, & que, dans les triangles rectangles, le quarré de la base est égal aux deux quarrez des costez". See [Descartes, 1964–1974, Vol. 4, p. 38]. The letter is also edited in [Verbeek et al., 2003, pp. 155–158], where the precise date of 16 November 1643 is ascertained. The translation is mine.

[3] Where he declares his astonishment at the "insane exultations and sacrifices for trivial inventions" ("…insanae exultationes & sacrificia pro levibus inventis…"). See [Descartes, 1964–1974, Vol. 10, p. 276].

[4] …"this [mathematical science] has to be obtained not from books but from its very use …"("ea [scientia mathematica] non ex libris sed ex ipso uso hauriri debet") [Descartes, 1964–1974, Vol. 5, p. 176].

[5] See [Descartes, 1964–1974, Vol. 9–1, p. 122].

[6] The importance of the theory of proportion (as interpreted by Descartes) for the structuring of his masterpiece is well known. A whole chapter of the classic book by Vuillemin [1960] is devoted to this topic and a section of this chapter has the very revealing title "La Géométrie comme théorie des proportions." See also [Galuzzi, 1980].

[7] The similarity between the intellectual evolution of Leibniz and Newton with regard to classical geometry is examined in the classic text of Hofmann [1974]. Useful remarks are also given in [Taton, 1978]. An interesting comparison between the intellectual evolution of Leibniz and his famous disciples is given in [Roero, 1995].

following passage by Henry Pemberton, taken from the *Preface* of [Pemberton, 1728], is well known[8]:

> I have often heard him censure the handling geometrical subjects by algebraic calculations; [and] praise *Slusius*, *Barrow* and *Huygens* for not being influenced by the false taste, which then began to prevail. He used to commend the laudable attempt of *Hugo de Omerique* to restore the ancient analysis, and very much esteemed Apollonius' book *De sectione rationis* for giving us a clearer notion of that analysis than we had before.... Sir *Isaac Newton* has several times particularly recommended to me Huygens's style and manner. He thought him the most elegant of any mathematical writer of modern times, and the most just imitator of the ancients. Of their taste, and form of demonstrations Sir *Isaac* always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did ....

After this quotation, Whiteside adds:

> For our part we may be grateful he did not! If he had immersed himself in the styles and techniques of ancient geometry in his youth, he would surely never have gone on to make the magnificent advances in Cartesian geometry and calculus which were the highlight of his *annus mirabilis* of mathematical invention.[9]

Whiteside perceives a sort of opposition between the conquests of the young Newton and his growing interest in classical mathematics. He seems to suggest that the last writings of Newton reflect the attitude of a great scientist who, after the wonderful results of his youth, reconsiders his achievements mainly from a methodological or epistemological point of view, in the light of the ideas of the ancients, and feels the need to reorganize his ideas in order to give them the accomplished form of a treatise.

Whiteside's statement contains many elements of truth, but I think that it can be further deepened. Newton's desire to compose a systematic treatise is not ruled only by the intention of giving his previous contributions an elegant systematic form, a form that ideally extends the mathematics of the ancients; it is also linked to important structural elements. It is a deep reflection upon the concept of proof and it is the attempt to mold his proofs in the classical style that regulates Newton's last thoughts. I will try to show that Newton aims at constructing a unitary view of mathematics that would allow him to reconcile the calculus of fluxions with classical geometry, i.e., Euclid's *Elements* or Apollonius's *Conics*.

2. Descartes and Newton

Descartes, in his mathematical masterpiece, *La Géométrie*, gives us a *method* by which we can easily reduce all geometrical problems to a form in which "a knowledge of the lengths of certain straight lines is sufficient for their construction."[10] This assertion,

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[8] I reproduce the text given by Whiteside. See [Newton, 1967–1981, Vol. 7, p. 199].

[9] Ibid.

[10] "qu'il n'est besoin par aprés que de connoistre la longeur de quelques lignes droites pour les construire". See [Descartes, 1964–1974, Vol. 6, p. 369]. I have slightly modified the translation given in [Descartes, 1954, p. 2]. In the letter to Mersenne of March 1636, Descartes is more daring: "Finally, in *La Géométrie*, I aim at giving a general way to solve all the problems that have not yet been solved" ("Enfin, en la *Geometrie*, ie tache à donner une façon generale pour soudre tous les Problémes qui ne l'ont encore iamais esté.") See [Descartes, 1964–1974, Vol. 1, p. 340]. See also the introduction by De Buzon [De Buzon, 2009, pp. 29–30] in [Descartes, 2009]. This volume (the third of a new edition of Descartes' works) also provides the text of *La Géométrie*, enriched by a large apparatus of notes by A. Warusfel, and preceded by an interesting *Présentation* by the same author.

which is placed just at the beginning of the treatise, may appear rather emphatic, but it is well grounded in the contents of the text. To give just two examples: Descartes' magnificent solution of Pappus's problem paved the way to the theory of algebraic curves (and smoothed the way for calculus)[11] and Descartes' treatment of the problem of the square[12] which is set out in a theoretical context and gives some important criteria of solvability to the rising theory of equations, criteria that prefigure the concept of a field extension, one of the main tools of modern algebra.[13] The mathematical work of Descartes is of undeniable importance and Newton himself gained enormous benefit from it.

Descartes decided to give his mathematics the form dictated by his "method." He also decided to give a precise literary form to *La Géométrie*: "Imitating Montaigne, it [*La Géométrie*] is written in the first person and comprises a strongly autobiographical aspect."[14] In this text Descartes paints the portrait of himself as a "géomètre en honnête homme."[15] These factors make the task of a systematical organization of his results rather difficult. To do so would amount exactly to rewording the text in the style that Descartes abhorred: that of the mathematical practitioners such as Roberval.[16]

In Descartes' opinion, the mastery of algebra, once appropriately purified in order to produce "genuine" analysis, enables the proper tackling of all the most important mathematical problems. At the same time, Descartes often considered the solutions of *traditional* classical problems as some sort of "exercices de l'esprit" ("exercises of mind"). Of course a certain boastfulness is present when he declares himself to be lazy as far as they are concerned, or when he professes to have little interest in them. But it would not be inaccurate to say that Descartes considered the mathematical questions inherited from tradition mainly as necessary educational training for the mind so that it could grasp the real vital problems.[17]

[11] See [Vuillemin, 1960; Bos, 1981; Giusti, 1990; Freguglia, 1999; Gardies, 2004].
[12] This problem, which appears in the Third Book, is a classical νεῦσις (insertion) problem that requires a segment to be fitted between a given straight line and the side of a square (or its extension), the segment passing through a given point.
[13] Descartes' brilliant solution of the problem may be interpreted (with the benefit of modern hindsight) as a strategy that can be used to check if an irreducible polynomial of fourth degree may become reducible by the adjunction of a quadratic irrational. See [Galuzzi and Rovelli, 1997].
[14] "á l'imitation de Montaigne elle [*La Géométrie*] est écrite à la premiére personne et comporte un important aspect autobiographique". See [Descotes, 2005, p. 164].
[15] Ibid., p. 168.
[16] See also [Rabouin, 2009].
[17] I want to emphasize that the primary issue in this paper is not the intellectual evolution of Descartes, but the image that Newton could have had of him. After *La Géométrie*, and the bitter polemics that immediately followed it, Descartes' denial of any interest in the mathematics (of the practitioners) became more and more frequent, even if his participation in van Schooten's enterprise of the first Latin edition of *La Géométrie* is more than conjectural (see [Maronne, 2006]). The practitioners of mathematics according to Descartes were Roberval, Beaugrand, his protégé van Schooten, etc. (and Fermat at the beginning of the polemics about the method of tangents; although he soon had to change his mind when he learnt that Fermat was a "honnête homme" like himself). Princess Elizabeth, the daughter of a King, and Constantin Huygens, secretary to the Prince of Orange, cannot be considered practitioners.

The reader of Descartes' *Principia Philosophiæ*, a text in which Descartes claimed to have made use of all his mathematical ingenuity, can perceive very little of this mathematics.[18] At most a reader may suppose that the mathematics fostered a mental aptitude for rigorous exposition.[19]

Newton's strategy in the *Principia* is quite different.[20] Mathematical objects are directly compared to natural bodies and every effort is made to describe each situation in terms of explicit mathematical structures. The mathematical apparatus rules every page. This conception of mathematical objects somehow identified with natural bodies will become even more explicit in *De Quadratura Curvarum*.[21]

Simply stated, one may say that these two authors have different conceptions of mathematics and of its role in the description of the physical world. Descartes considered mathematics as an indispensable tool to educate the mind so it could penetrate the secrets of nature.[22] Newton looked at nature as a geometrically organized whole.

It is easy to understand that Descartes and Newton had different ideas in mind when composing a *treatise* intended to organize mathematical knowledge. Descartes could leave to van Schooten the task of composing a treatise that would describe the new "Cartesian

[18] The following well-known judgment by Brunschvicg is very characteristic: "The homogeneity of Cartesian physics struck his contemporaries; however, it is odd that this physics by a mathematician is hardly mathematical physics. Everything it contains can most likely be calculated; but one finds no effective calculations in the Principes, excepting those concerning the laws of collision, which laws are wrong anyway." ("L'homogénéité de la physique cartésienne a frappé les contemporains; et cependant, il est curieux que cette physique de mathématicien n'est guére une physique mathématique. Tout y est calculable sans doute; mais on ne trouve pas de calcul effectif dans les Principes, sauf en ce qui concerne les lois du choc qui, d'ailleurs, sont fausses") [Brunschvicg, 1937, pp. 44–45; Allard, 1963, p. 153].

[19] One could uphold that a "contingent" reason prevents Descartes' physics from being soundly grounded in his mathematics: Descartes does not yet have the necessary tools for an adequate description which, in principle, could be presented in mathematical terms. From Descartes' own perspective, it merely remained for him to complete the work. Clearly Newton did not share Descartes' point of view: even in the title of his masterpiece, the words 'mathematica' and 'principia' are tightly linked. See [Cunningham, 1991], and my review of this paper in *Mathematical Reviews*, MR1143668 (92m:01017). In addition, Book 2.9 of the *Principia* shows Newton's careful, if polemical, consideration of Descartes' *Principia Philosophiæ*.

[20] The main passages in which Newton criticizes Descartes in the course of his long career are collected, and carefully examined, in [Guicciardini, 2009, Chap. 4, 5].

[21] This important treatise, in its final form, is one of the two mathematical treatises appended to the "editio princeps" of Newton's *Opticks* [Newton, 1704].

[22] A well-known text in which Descartes explains his ideas about mathematics is the *Entretien avec Burman*, quoted above (see particularly [Descartes, 1964–1974, Vol. 5, pp. 176–177]). The letter to Mersenne of 27 May 1638 (ibid., Vol. 2, pp. 142–143) is also relevant. The celebrated passages of the *Discours*, where a certain amount of rhetoric is evident, are to be considered more cautiously, even if the passage of the *Entretien* quoted above is related to a statement of the Latin translation of the *Discours*. Precisely, "non aliam inde utilitatem espectarem [from mathematics] quam quod paulatim assuefacerem ingenium meum veritati agnoscendæ" ("I expected [from mathematics] no other utility but that it would accustom my mind to the recognition of truth"). See [Descartes, 1964–1974, Vol. 6, pp. 550–551]. Descartes's conception of mathematics and the relation of his mathematics to the whole of his philosophy is the subject of a vast literature, most notably [Bos, 2001; Sasaki, 2003; Serfati, 2002; Serfati, 2005; Jullien, 2006]. These texts also provide a wealth of references. The secondary literature exploring the different philosophical conceptions of Descartes and Newton is as abundant.

mathematics" and drawing the necessary epistemological consequences. In contrast, Newton all his life felt the need to expose his results in a classical fashion. Thus one could say that Descartes felt only the need to organize his results in a context which he believed to be the one of classical mathematics appropriately reformulated and purified by means of his new algebra, whereas Newton felt the need for a deep reflection upon the most suitable form for his demonstrations, seeking a mathematical style which would ideally integrate all his striking new results into the context of classical mathematics.[23]

What does it mean to *prove* something in the new world of geometrical objects produced by Newton in which fluxions and series live together (in his opinion) with the purely geometrical objects of classical mathematics? What is the connection between proofs that necessarily incorporate these new objects and proofs of the ancients? There is more at stake in the late thoughts of Newton than the belief in the *prisca sapientia* (ancient wisdom).

Many *new* geometrical objects derive from Descartes' *La Géométrie* as well. Consider, for example, the entirely new world of algebraic curves. However neither Descartes nor his critics of the time had a clear perception of this situation. In fact the (wrong) idea[24] that all algebraic curves are solutions of Pappus's problem made the concept of an algebraic curve (in Descartes' mind) the result of a "paradigmatic abstraction" of the very concept of curve received by the ancients but not, as we today perceive it, by setting polynomial equations in the foreground, the result of a "thematization".[25]

An example drawn from *De Analysi* allows us to introduce the methodological concerns that Newton addressed in his last years.

3. An example in *De Analysi*

This text, one of Newton's most famous,[26] begins with the following words:

[23] Considering the situation from a modern point of view, we see equally important revolutions accomplished by Cartesian mathematics (the enormous increase in the number of mathematical objects, for example, all the algebraic curves) and by Newtonian mathematics (the introduction/algebraization of infinitesimal tools). However, Descartes and Newton had different conceptions about mathematics. Descartes mainly thought of his mathematical work as a radical rectification of that of the ancients. He believed it was his duty to restore the methods of discovery of the ancients, which the latter had concealed out of a desire to maintain their status. In contrast, Newton mainly considered himself as the heir of a sound tradition heavily threatened by the modern abuse of algebra.

[24] A nice proof of the falseness of Descartes' assertion is given in [Rashed, 2005, pp. 47–48]. In [Galuzzi, 1990] Newton's proof of the fact that a "general" curve of high degree cannot be a solution of Pappus's problem is analyzed.

[25] I refer, for this opposition, to the classical study of Cavaillès [1976]. On the generation of new mathematical objects and their nature, see also [Giusti, 1999; Gardies, 2004].

[26] In the *Introduction* to *Part* 2 of the second volume of Newton's *Mathematical Papers* (see [Newton, 1967–1981, Vol. 2, pp. 163–171]), Whiteside gives a lively and deep account of the circumstances that led Newton to compose this celebrated book. The great English scholar emphasizes the fact that surely this tract was not "Newton's first attempt to display his doctrines to public view" (ibid, p. 165). On the contrary, its theme, "the employment of infinite series in elementary geometrical analysis, represented only a small portion of the wealth of his mathematical researches from 1664 onwards" (ibid, p. 165). Even a cursory reading of the first volume and of the first part of the second volume of the *Mathematical Papers* would convince a reader of the wealth of Newton's mathematical discoveries that precede *De Analysi*.

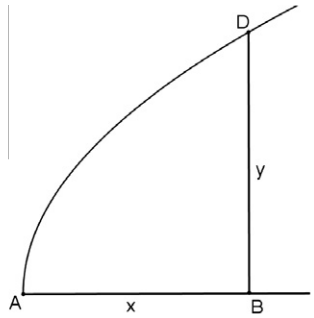

Fig. 1. The first rule in *De Analysi.*

> *The General Method*, which I have devised some considerable Times ago, for measuring the Quantity of Curves, by Means of Series, infinite in the Number of Terms, *is rather shortly explained, than accurately demonstrated in what follows.*[27]

Newton does not claim to give complete and rigorous proofs in *De Analysi.*

Let us consider the first Rule in particular. In Fig. 1, the base AB is denoted by x, the ordinate BD, perpendicular to the base, by y. $a, b, c, \ldots$ are given quantities, whereas m and n are integers. With the help of these conventions we have

Rule 1. *If* $ax^{m/n} = y$ *then*

$$\frac{an}{m+n} x^{\frac{m+n}{n}} = \text{Area}\, ABD. \qquad \square$$

The Rule is important not only in itself, but also for the notation it employs.[28] Newton proves this Rule at the end of his text,[29] but given my focus on the features of the proof, I give the proof immediately.

Observe that even in the context of the proof, Newton needs to begin with an example. By reference to Fig. 2, Newton supposes that, *at the starting instant*, $AB = x, BD = y$, Area $ABD = z$.

Then, he supposes that $B\beta = o, BH = v$, where v is such that

$$\text{Rectangle } B\beta KH = \text{Surface } B\beta\delta D.^{30}$$

Setting $A\beta = x + o$, it follows that Area $A\delta\beta = z + ov$. It is unnecessary to single out a particular curve for this premise. But Newton now picks the curve that subtends the area z such that

$$z^2 = \frac{4}{9} x^3. \tag{1}$$

It follows that

$$(z + ov)^2 = \frac{4}{9}(x + o)^3$$

[27] My emphasis. I use the English translation given by Whiteside in [Newton, 1964, Vol. 1, p. 3].
[28] Newton was responsible for considerable progress in the development of notation, in particular the complete generality in writing the power of a binomial is due to him. The latter point is dealt with in [Serfati, 2005], a very interesting text, even if at times rather daring in its theses.
[29] At the beginning he gives an example and claims that this example provides sufficient clarity.
[30] It was not until the nineteenth century that Newton's intuitive assumptions concerning this equality were called into question.

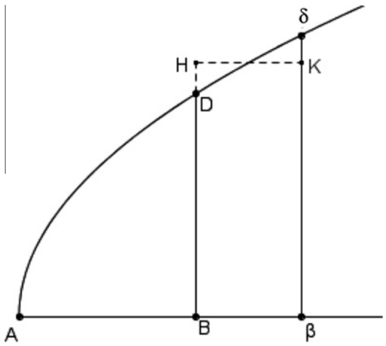

Fig. 2. The "derivative" in *De Analysi.*

and consequently[31]

$$\cancel{z^2} + 2zov + \boxed{o^2v^2} = \frac{4}{9}\left(\cancel{x^3} + 3x^2o + \boxed{3xo^2 + o^3}\right).$$

Newton states that:

> Now if we suppose $B\beta$ to be diminished infinitely and to vanish, or to be nothing, v and y in that Case will be equal...[32]

It follows that

$$\frac{4}{9} \times 3x^2 = 2 \times \frac{2}{3}x^{3/2}y$$

and

$$y = \sqrt{x}.$$

Newton's general proof proceeds along the same lines. He begins by letting

$$\frac{na}{m+n}x^{\frac{m+n}{n}} = z; \tag{2}$$

then, since he is confining himself to using only integer exponents, he states that $\frac{na}{m+n} = c, m+n = p$. By raising to the nth power, he changes (2) into

$$c^n x^p = z^n. \tag{3}$$

The substitutions $x \to x + o, z \to z + ov$ made by Newton (or $z \to z + oy$: he hastily affirms that it is the same thing) and the use of the binomial rule give

$$c^n(x^p + pox^{p-1} + \cdots) = z^n + noyz^{n-1} + \cdots. \tag{4}$$

It follows that

$$c^n px^{p-1} = nyz^{n-1}. \tag{5}$$

A new substitution gives

$$ax^{\frac{m}{n}} = y. \tag{6}$$

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[31] I have crossed-out the quantities that can be eliminated in view of (1), and I have boxed the quantities that Newton neglects. Of course I use this notation only for the benefit of a modern reader.

[32] See [Newton, 1964, Vol. 1, p. 23].

Newton draws the conclusion that, reciprocally, if $ax^{\frac{m}{n}} = y$, then also

$$\frac{n}{m+n} ax^{\frac{m+n}{n}} = z.$$

An essential tool for his proof is the binomial rule. But can we say that Newton possesses a real proof of the binomial expansion, a proof that may be compared for its rigor with the ones of the ancients? Or is it better to say that he has at most a demonstrative analogy?[33]

Besides, in order to consider a situation more general than the one given by a curve with equation $y = x^{\alpha}$, Newton systematically uses series expansions, obtained by using term-by-term integration (Rule 2). In fact, after some elementary examples, he provides Rule 3:

> But If the Value of y, or of any of it's Terms be more compounded than the foregoing, it must be reduced into more simple Terms; by performing the Operations in Letters, after the same Manner as Arithmeticians divide in Decimal Numbers, extract the Square Root, or resolve affected Equations;[34]

It is difficult to deny that the magnificent results contained in this text press for sound demonstrations, especially when these results aim at extending the knowledge of very well-known mathematical objects (such as conic sections), which were endowed with a solid rigorous tradition. In the final part of *De Analysi*, Newton tackles the problem of the *convergence* of the series obtained. With the help of X.1 of the *Elements*, he proves that the geometrical series $\sum_{n=0}^{\infty} x^n$ converges for $|x| \leqslant \frac{1}{2}$ and that the result can be extended to every infinite series $\sum_{n=0}^{\infty} a_n x^n$ such that $|a_n| < 1$. But considerations on convergence[35] are rather exceptional in Newton's works.

In the light of these observations, the *Geometria Curvilinea*[36]—the important text, probably written about 1680, in which Newton deals with the possibility of giving a direct geometrical approach (ideally extending Euclid's *Elements*) to the calculus of fluxions—constitutes an important step towards the new style employed in the *Principia*.[37] In Section 6.3, I will give an example of the direct geometric method of treating fluxions similar to the one of the *Geometria Curvilinea*.

The examination of the basic elements of an important proposition within the *Principia*, which I study in Section 5, shows (by the plain heterogeneity of its components) that the ideal of a unitary demonstrative structure that Newton was to pursue in his last writings has a real necessity. It is not only the fruit of his falling in love with the mathematics of the ancients.[38] But I want first to deal with an example that can be seen as a natural continuation of *De Analysi*.

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[33] See [Whiteside, 1961; Knobloch, 1991; Panza, 2005, pp. 170–181]. Of course in this example the expansion is limited to the case of integer exponents. But in other texts Newton is compelled to utilize the rule in all its generality.
[34] See [Newton, 1964, Vol. 1, p. 6].
[35] Clearly Newton ignores the fact that Grégoire de Saint-Vincent had just proved the convergence of the geometrical series for $|x| < 1$. See the interesting remark by Whiteside in [Newton, 1967–1981, Vol. 2, Note 146, pp. 246–247].
[36] This text is edited, accompanied by a rich commentary by Whiteside, in [Newton, 1967–1981, Vol. 4, pp. 409–521]. See also [Guicciardini, 2009, Chap. 9; Galuzzi, 1995] on Newton's synthetic calculus of fluxions.
[37] See [Galuzzi, 1995].
[38] Of course I do not want to argue that the whole scientific career of Newton is permeated by a complete rationality. In his old age he certainly came to have some odd attitudes related to the *prisca sapientia* and to the superiority of the mathematics of the ancients.

4. The solution of differential equations

One of the most spectacular results of the "new analysis" is given by the solution, by means of series, of differential equations. I will take an example from *De methodis serierum et fluxionum*, the treatise which was composed just after *De Analysi* and which stayed in manuscript state till 1736.[39] Newton in this text denotes the fluxions of $x, y, z, \ldots$ by $m, n, p, \ldots$. In the first part of the text he explains how to "prepare the equation," that is, how to write it in the form

$$\frac{n}{m} = f(x, y).$$

Among the equations he then considers is

$$\frac{n}{m} = 1 - 3x + x^2 + (1 + x)y, \tag{7}$$

for which he supposes that $y(0) = 0$. The solution is explained with the help of the following table:

	$+1$	$-3x$	$+x^2$				
$+y$		$+x$	$-x^2$	$+\frac{1}{3}x^3$	$-\frac{1}{6}x^4$	$+\frac{1}{30}x^5$	$+\cdots$
$+xy$			$+x^2$	$-x^3$	$+\frac{1}{3}x^4$	$-\frac{1}{6}x^5$	$+\cdots$
Summa	1	$-2x$	$+x^2$	$-\frac{2}{3}x^3$	$+\frac{1}{6}x^4$	$-\frac{4}{10}x^5$	$+\cdots$
$y\ =$	x	$-x^2$	$+\frac{1}{3}x^3$	$-\frac{1}{6}x^4$	$+\frac{1}{30}x^5$	$-\frac{1}{41}x^6$	$+\cdots$

A modern reader can easily grasp the algorithm: since $y(0) = 0$, we have from (7) that $\frac{n}{m}$, evaluated in $x = 0$, takes the value 1, which gives, for the beginning of the solution, $y = x + \cdots$. In the third column of the table we have the term of first degree of the right-hand side of (7) (in the first line), the term given by y (in the second line), and there are no terms in the third line, because the product xy begins with $x^2 + \cdots$. The fourth line gives the sum of the terms, which is $-2x$ (*Summa*), and the following term of the solution is reached by the integration of what is found in the *Summa* (fifth line). Hence we have $y = x - x^2 + \cdots$. We have obtained the term of degree 2, which allows us to go on to the fourth column, etc.[40]

Thus Newton is in possession of a technique that allows him to develop into a series the solution of a differential equation for which it is possible to give an explicit solution (in modern terms) only if one accepts that the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$$

belongs to the class of known functions.

---

[39] The epistemological changes that arose in the transition from *De Analysi* to *De Methodis* are analyzed in [Panza, 2005; Panza, forthcoming], a text that on this point, reworks and extends the previous text.

[40] See [Newton, 1967–1981, pp. 98–100]. It is of some interest to see also the translation by Buffon in [Newton, 1740, pp. 34–35]. The procedure given by Newton is not very different from the one given by him in *De Analysi* in order to find the roots of numerical equations. A modern reader may find it easier to assume that a solution is given in the form $y = a_1x + a_2x^2 + a_3x^3 + \cdots$ and to make a direct substitution. It amounts to the same thing.

While it is possible both to liken the method of quadratures to the method of exhaustion and to look at the projection "per umbras" ("by shadows") as a generalization of the method by which conic sections are constructed using the help of a circle, it is not possible to find room in classical mathematics for the technique used by Newton to solve differential equations.

Besides, if one takes seriously Newton's frequent claims on the primacy of synthesis over analysis and if one continues by asking oneself what sort of conversion could change this procedure (the solution by series of a differential equation) into a kind of synthesis, an answer comes naturally to mind: the "calcul des limites" of Cauchy.[41] And although the road that leads to Cauchy's treatment of differential equations still has far to be traveled,[42] it was Newton's determination to organize his results into a form as classical as possible that paved the way for the completion of the journey. Newton may have encountered insurmountable difficulties in gathering together all his results for the planned treatise of geometry but it was his attempt which spurred on the scholars who succeeded him.[43]

5. An example in the *Principia*

The First Section of the First Book of the *Principia* constitutes, according to Newton, an adequate foundational base for the arguments of infinitesimal nature successively used. I will not deal with this topic here. The literature on this subject is plentiful.[44] In this Section I will show how arguments of infinitesimal nature are entangled with classical elements (or elements claimed by Newton as such).

Proposition XI of the Third Section of Book I offers an excellent example: the problem, which Newton magnificently solves, consists in finding the measure of the centripetal force acting upon a body that revolves into an ellipse, assuming that this force tends to one focus (the direct problem of central forces). In the Appendix I have summed up the whole proof. Here I limit myself to giving a list of the principal elements on which it is built, in order to exhibit their remarkable epistemological differences.[45]

- The calculation of the centripetal force,[46] having its center in the point S of a body that describes a given trajectory γ is usually given by Newton in the following way: one has to take the *limit* of the quantity $\frac{QR}{SP^2 \times QT^2}$ as Q tends to P (Fig. 3). What is given by this process is a quantity proportional to the force.[47] This calculation, made by taking the limit

[41] The method appears in several *Comptes Rendus* from 1839 to 1842. An important contribution is [Cauchy, 1842]. This method seems the most natural to "complete" Newton's strategy.

[42] See the *Introduction* by Gilain in [Cauchy, 1981]. Lacroix, in his *Traité*, still more or less sticks Newton's way by proposing his "méthodes pour résoudre par approximation les équations différentielles du premier ordre". See [Lacroix, 1797–1798, Vol. 2, pp. 284–288].

[43] Even though this great treatise, as well as other important works, was not published, Newton's ideas spread throughout the milieu of English scholars via manuscript circulation. See [Guicciardini, 1989].

[44] See, for example, [Pourciau, 2001].

[45] A deep analysis of this Proposition is given also in [Brackenridge, 1995, pp. 102–118]. See also my review in *Mathematical Reviews*, MR1658137 (99j:01006).

[46] See Prop. VI, Theor. V, in Newton (1687, pp. 44–45), and for the French translation of the third edition [Newton, 1759, pp. 39–40]. See also [Nauenberg, 2003].

[47] Newton expresses his results by utilizing (or rather by pretending to utilize) the tools of the classical theory of proportion, as given in the Fifth Book of the Euclid's *Elements*. I permit myself some simplification.

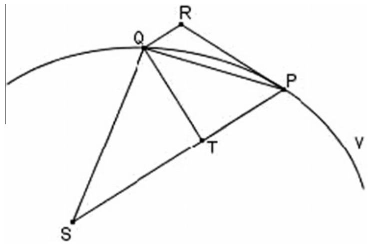

Fig. 3. The measure of the force.

of a curvilinear area that, in the final instant, is considered as the area of a triangle, scarcely finds its place in classical mathematics.

- Newton makes use of the "equations" of conic sections in classical terms. In the case of the ellipse (Fig. 4),[48] the one considered in this proposition, we have

$$QV^2 = \text{const.} \cdot GV \cdot PV,$$

where PG is a diameter and QV is an ordinate relative to this diameter.
- To complete the proof Newton has to demonstrate a new lemma of his own. Precisely, he proves that $PE = AC$, where the direction of EC is that of the tangent at P, and S, H are the foci (Fig. 5). The proof of this lemma retains, at least partially, classical vestiges.
- Last, Newton uses another lemma[49] that he had proved in his youth when reading the *Elementa Curvarum Linearum* by Jan De Witt, a text where conics are considered in a way very different from that of Apollonius.[50] At the moment of writing the *De Motu*, the text which is enlarged into the *Principia*, he did not remember his proof and hastily wrote "Constat ex Conicis" ("it is established from the *Conics*"), a statement that is also repeated in the *Principia*.

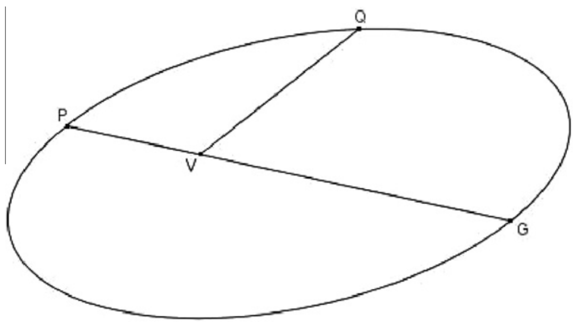

Fig. 4. The "equation" of the ellipse.

[48] See Prop. XI, Prob. VI in [Newton, 1687, pp. 50–51]. See also the solution added in the third edition [Newton, 1972, p. 120; Newton, 1759, pp. 45–46].
[49] See Lemma XII, in [Newton, 1687, p. 47] or in [Newton, 1759, p. 43].
[50] The text of De Witt figures in [Descartes, 1659–1661]. Actually, the lemma is Proposition VII.31 of Apollonius' *Conics*, but it is unlikely that Newton knew the translations available at that time. A proof is given by Newton himself in the margin of p. 220 of the text of Jan De Witt. See [Galuzzi, 1990, pp. 396–397].

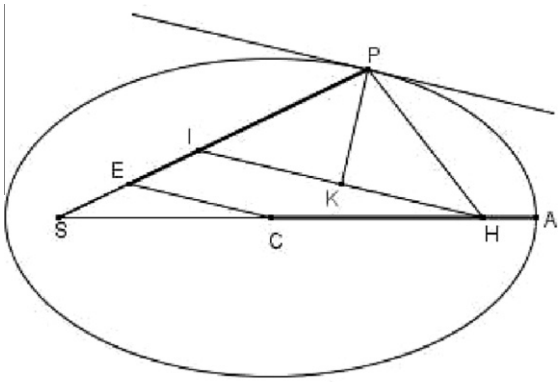

Fig. 5. Newton's Lemma about conics.

The lemma is the following: "All the parallelograms circumscribed about any conjugate diameters of a given ellipse or hyperbola are equal among themselves" (Fig. 6).[51]

This list, or more exactly the proof summarized in the Appendix, shows the difficulty of definitely dividing Newton's proofs into classical and modern. It becomes evident that Newton has interiorized, starting from the experience of the *Geometria Curvilinea* (in my opinion), some sort of "infinitesimal geometry" that at first sight makes the *Principia* appear to be a text written in a style not too distant from the classical texts.

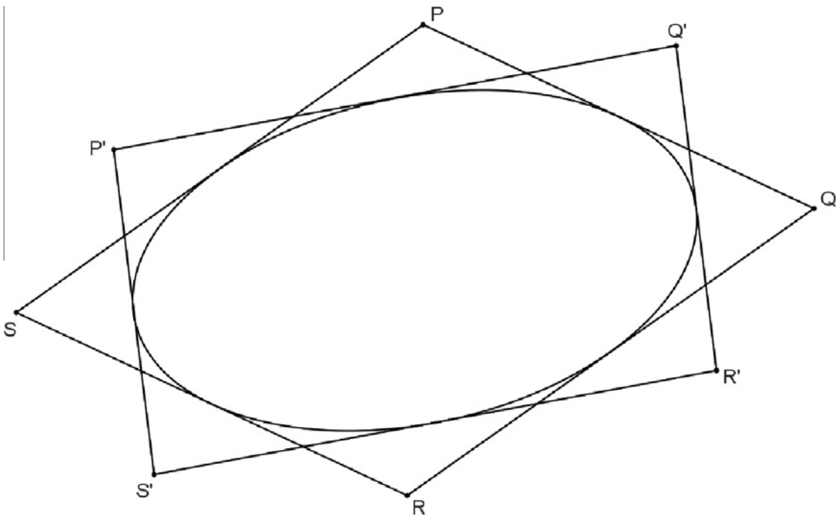

Fig. 6. Diagram illustrating Proposition VII.31 of Apollonius's *Conics*: parallelograms circumscribed about conjugate diameters have the same area.

[51] "Parallelogramma omnia circa datam Ellipsin descripta esse inter se aequalia. Idem intellige de Parallelogrammis in Hyperbola circum diametros ejus descriptis." See [Newton, 1687, p. 47]. This English translation by Motte, revised by Cajori, is given in [Newton, 1962, p. 53]. Newton's proof in the margin of De Witt's text is analyzed in [Galuzzi, 1990].

6. The attempts at writing a great treatise

In the 1680s, Newton resolved to write a great treatise in which all his mathematical conquests would harmoniously coexist with the mathematics of the ancients. This treatise in its final form[52] contains many more of Newton's own findings—such as the classification of cubics, the method of fluxions, the method for quadrature—than the analysis or the synthesis of the ancients. In contrast, in the texts that Whiteside collected as preparatory material, classical mathematics (somewhat idealized) has a more important presence. In the following sections, I will give an idea of Newton's treatment of the problem of porisms. The latter treatise has remained in manuscript form. But it was by utilizing its materials that Newton composed *Enumeratio linearum tertii ordini*[53] and *De Quadratura Curvarum*, the two texts collocated as appendixes to the *Opticks* of 1704. In these texts the pretended link with the ancients is more a sort of epistemological manifesto than a real source of mathematical inspiration. And the thesis that the new analysis of the 19th century inherited a large number of ideas from these texts is not totally unfounded.[54]

6.1. The porisms

The analysis of the first porism in the text that Whiteside has classified as "*Geometry: the first Book I*" is enough to characterize Newton's strategy on the subject (see Fig. 7). The text that Newton gives (following Pappus) is the following:

> If from the two given points A, B straight lines AZ, BZ be inclined to the straight line CZ given in position, let one, AZ cut off from the straight line EX given in position a segment EX, terminated at the given point E in it, having a given proportion to the other CZ, given in position, then the other, BZ, will cut off the segment EY having a given proportion to the same CZ. Of course EX and CZ will be parallel—understanding that the points E, C, A, B lie in a straight line.[55]

Newton does not explain what he is doing, but as a matter of fact he had long before elaborated a precise strategy.[56] This strategy can be explained in the following way. Let $EX = x$

[52] This final form is conjectured by Whiteside. He observes, by giving a justification for the title of Section 3 of [Newton, 1967–1981, Vol. 7] (that is, "*The final Geometriæ libri duo*"), that "Or so it appears if we trace aright the internal sequence of the corpus of surviving autography drafts of Newton's projected treatise on *Geometry*." Ibid., Note 1. See also [Guicciardini, 2009, Chap. 14].

[53] I do not deal in this paper with this really remarkable contribution of Newton's. A well informed review is given in [Guicciardini, 2009, Chap. 6].

[54] With respect to studying the development of the ideal of rigor in the 18th century, it would be interesting to investigate the influence of the numerous references to Newton contained in the work of d'Alembert and Lagrange. Regarding the spread of Newton's ideas in the British context, I refer again to [Guicciardini, 1989].

[55] "Si a duobus datis punctis A, B ad rectam lineam positione datam CZ rectæ lineæ inflectantur, abscindat autem una AZ a recta linea positione data EX ad datum in ipsa punctum E segmentum EX ad alteram positione datam CZ proportionem habens datam, abscindet et altera BZ segmentum EY ad eandem CZ proportionem habens datam. Quippe parallelæ erunt EX, CZ puta si puncta E, C, A, B jacent in directum". I have used here and afterwards Whiteside's translations from Latin. See [Newton, 1967–1981, Vol. 7, pp. 310–313].

[56] See for instance the text collected by Whiteside, "Solutio problematis veterum de loco solido" ("Solution of the Ancients' problem of the solid locus") [Newton, 1967–1981, Vol. 4, pp. 282–321].

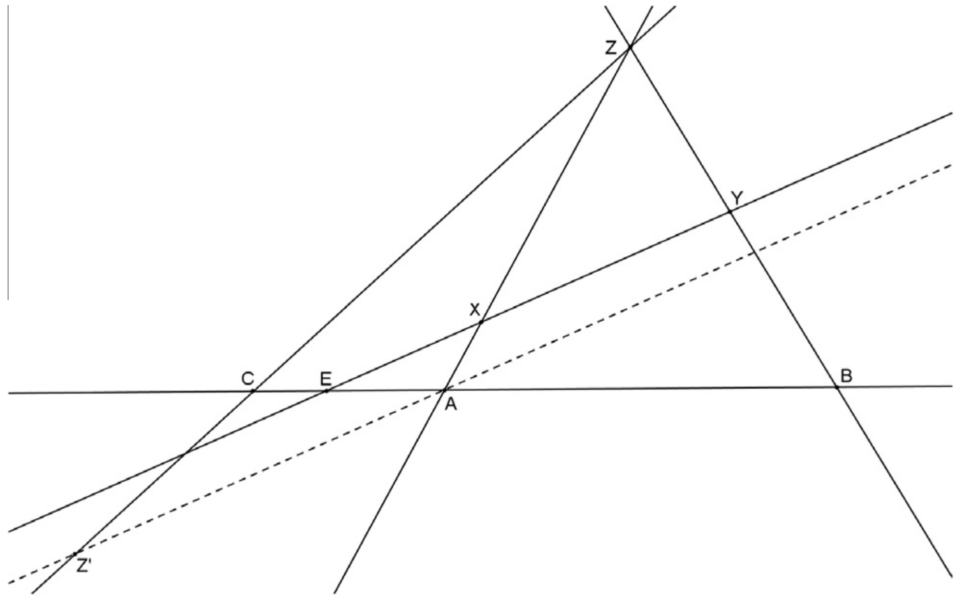

Fig. 7. Newton and porisms.

and $EY = y$. The relation between these quantities is given "per simplicem geometriam" ("by plain geometry"), which means that there is a bilinear relation of the form

$$\alpha xy + \beta x + \gamma y + \delta = 0. \tag{8}$$

Since the ratios EX/CZ and EY/CZ are given, the ratio $EX/EY = x/y$ is given as well. It follows that $\alpha = \delta = 0$. The quantities x and y must vanish at the same time, which implies that E, A, B, C are collinear. Also, we have $x = \infty \Longleftrightarrow y = \infty$, which implies that the straight line EX is parallel to CZ.[57]

The issue here is not only that of classical geometry. The geometrical situation gives birth to an *algebraic formulation*, an equation, that contains the variables x and y. Newton easily understood that, since there is a bijective correspondence between the points X and Y, this equation has to be of the form (8). Starting from this algebraic formulation, one may perceive a specific configuration in the geometrical situation given at the beginning.[58] The role of algebra will become even more evident in the following section.

6.2. Algebra behind geometry

The beginning of the First Book of the final edition of the *Geometria* surely amazes a reader who, after having read the preliminary materials, expects a text deeply indebted to the ancients. The book begins as follows (see Fig. 8):

> Problems according to the number of solutions which they admit are distinguishable into grades. ... If for instance, a straight line AB is to be extended to D so that the point D shall be at a given distance away from some point C which is given up above, the prob-

[57] In Fig. 7 the dotted line AZ' is drawn, which gives $EX = \infty$. It is clear that we do not at the same time have $EY = \infty$, because CZ is not parallel to EX.

[58] This strategy is employed in [Newton, 1967–1981, Vol. 4, pp. 306–313].

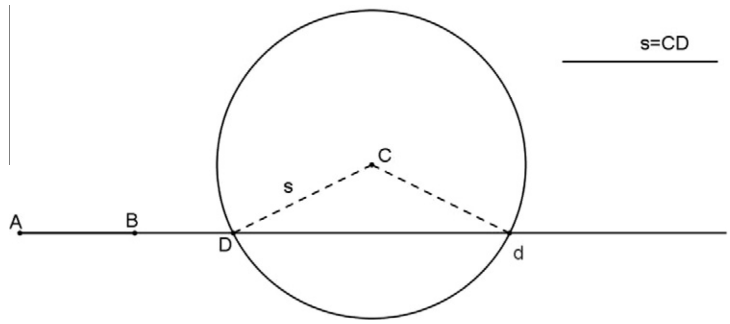

Fig. 8. Two real solutions.

> lem will be solved if with center C and that given distance as radius there be described a circle and the straight line be produced till it shall meet its circumference: from the double meet D and d there will prove to be a double solution, one by means of the line AD and the other by the line Ad—which shows that the problem is of second grade.[59]

The number of solutions seems to be linked to a geometrical configuration. But after having introduced the positive and negative quantities (*directæ* and *retrorsæ*) in the usual fashion (for a modern reader), Newton observes:

> The quantities by means of which we answer a question may sometimes also prove to be impossible: as in the present case the quantities BD and Bd when the interval CD is assigned too small for the circle to be able to cut the extended straight line. And whenever two or may be four or more solutions are impossible (for the number of impossibles is always even) the grade of a problem will be reckoned not from the number of reals alone but from the total number, that is of all which in any case whatever of the problem generally proposed can come to be real.[60]

The degree of a geometrical problem is thus revealed by means of its algebraic formulation. It is not enough, however, to consider only problems such that their number of solutions, if generally posed, corresponds to the degree of the equation. One must go beyond that. If the equation comes to have a certain degree, and it is impossible to modify the geometrical data in order to have the right number of solutions, it is necessary to produce a slight modification of the problem itself in order to restore the correspondence.

The example given by Newton is very interesting. The problem of inserting two mean proportionals x, y between two given quantities a, b obviously leads to the equation (for the quantity x)

---

[59] "Problemata pro numero solutionum quas admittunt distingui possunt in gradus ... Ut si data recta AB producenda est ad D ita ut punctum D dato intervallo distet a puncto aliquo C quod in sublimi datur: solvetur Problema si centro C intervallo isto dato describatur circulus et producatur recta illa usque dum circonferentia circuli hujus occurrat: et duplici occursu in D et in d fiet duplex solutio, una per lineam AD altera per lineam Ad quod ostendit Problema secundi gradus esse." See [Newton, 1967–1981, Vol. 7, pp. 402–404].

[60] "Hæ quantitates autem per quas respondemus quæstioni aliquando etiam impossibiles evadunt; ut in hoc casu quantitates BD et Bd ubi intervallum CD minus assignatur quam ut circulus rectam productam secare possit. Et quando duæ vel forte quatuor aut plures sunt impossibiles (nam numerus impossibilium semper est par) gradus Problematis non æstimabitur ex numero solarum realium sed ex numero omnium, id est omnium qui in quocunque casu Problematis generaliter propositi reales evadere possunt." See [Newton, 1967–1981, Vol. 7, p. 404].

$$x^3 = a^2 b. \tag{9}$$

This equation has a single real root for every possible choice of a and b. But it is possible to modify (a little!) the problem by the introduction of two other quantities c, d to arrive at the proportion

$$a : x = (x + c) : y = (y + d) : b. \tag{10}$$

We get the equation

$$x^3 + cx^2 + adx - a^2 b = 0, \tag{11}$$

which reduces to (9) for $c = d = 0$ and which can have three real roots if

$$-a^2 \left(27 a^2 b^2 + 18 abcd - c^2 d^2 - 4 bc^3 + 4 ad^3\right) > 0.$$

Since the Eq. (9) may be considered as a particular case of the Eq. (11), Newton draws the conclusion that even the problem of inserting two mean proportionals between two given quantities is a third-degree problem.[61] Before the awakening of algebra, the problems which today are considered of third degree were not handled in a unitary way. They were tackled either by means of the insertion (νεῦσις) or by the intersection of conic sections or by the help of particular curves such as the conchoid.[62] After the spread of algebra in the modern period, the different possible cases given by a third-degree equation were framed into an unitary strategy (in particular by Descartes).

Even though the solution of the equations of third and fourth degree considered in the Third Book of Descartes' *La Géométrie* (a text familiar to Newton) is given by a single apparatus (that is, the intersection of a circle and a parabola), this unitary approach does not interfere with the geometrical data. The examination of the *casus irreducibilis* by the tools of Cartesian algebra sheds some light on the differences between the *geometrical* problems of angle trisection and cube duplication.[63]

Newton's proposal of seeing all third-degree problems as particularizations of problems that "in general" have three real solutions gives to algebra a supremacy over geometry that is truly astonishing. However, since a third-degree equation has three real roots only if we add a condition on the coefficients, in what sense can Newton's proposal be considered "general"?

6.3. The fluxions in the Geometria

6.3.1. Fluxions and geometrical objects

The direct consideration of fluxions of *geometrical* objects does not have in the *Geometria* the same importance as it had in the *Geometria Curvilinea*. It is preceded by a more traditional approach to the calculus of fluxions (similar to the one of *De methodis*, for example). An interesting example[64] that reminds us of the *Geometria Curvilinea* is given

[61] The perfect mastery of all the subtleties related to the third-degree equation and its real or complex roots is evident in Newton's studies on cubics. See [Guicciardini, 2009, Chap. 6].
[62] See Books III and IV of [Pappus, 1982].
[63] See [Bos, 2001].
[64] Newton begins it with the statement that "In figuris hæc est methodus," by which he means that his method is mainly based on the consideration of figures. See [Newton, 1967–1981, Vol. 7, p. 495].

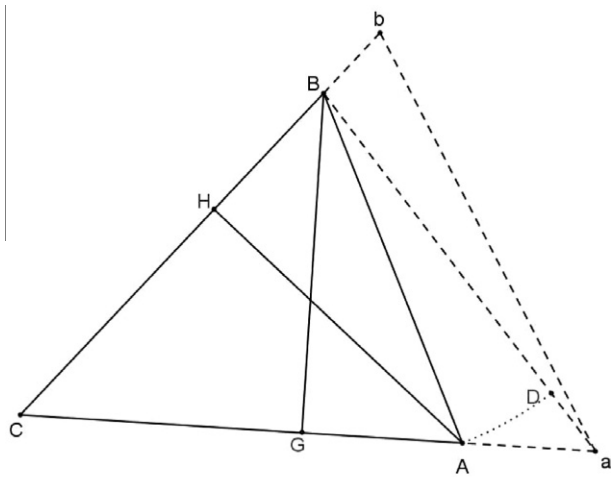

Fig. 9. Fluxions and geometrical objects.

by a triangle ABC that has angle C given (see Fig. 9).[65] I limit myself to illustrating how Newton calculates the fluxion of the side AB in terms of the fluxions of CA and CB.

Newton supposes that BC is given. He then assumes that CA becomes "fluendo" (flowing) Ca and that in the same time BA becomes Ba. Upon Ba Newton takes $BD = BA$ and the "partes genitæ" (the generated parts, i.e., the instantaneous increases) will then be Aa and Da. Let the height BG be drawn. We have $BG^2 + GA^2 = AB^2$. Consequently, following Newton's notation, i.e., denoting the product of quantities by a "prime" and writing the fluxions in lower case letters, we get

$$2GA'\,ga = 2AB'\,ab$$

(since BG is given).

The fluxions of CA and GA are the same and consequently

$$\frac{GA'\,ca}{AB} = ab. \tag{12}$$

By following a similar procedure and by drawing the height AH onto the side BC, Newton determines the fluxion of AB as

$$\frac{GA'\,ca + HB'cb}{AB}. \tag{13}$$

Let us now put the situation in modern terms.

Let $CA = x, BC = y, \widehat{ACB} = \theta$. It follows that

$$AB^2 = x^2 + y^2 - 2xy\cos\theta$$

and the fluxion of AB is given by

$$\dot{AB} = \frac{(x - y\cos\theta)\dot{x} + (y - x\cos\theta)\dot{y}}{AB}.$$

As in the *Geometria Curvilinea*, it is obvious that the direct consideration of simple geometrical objects is not reflected by similar simplicity in calculus; i.e., the concept of simplicity is

[65] See [Newton, 1967–1981, Vol. 7, pp. 495–497].

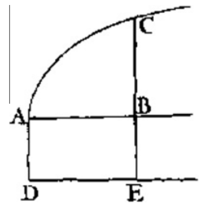

Fig. 10. Ratio of fluxions in *De Quadratura Curvarum*. Diagram excerpted from (Newton, 1704, p. 211).

not the same in the two domains. It was a matter of which Newton was well aware but for which he did not have a solution.[66]

6.3.2. The materials for De Quadratura Curvarum

Many propositions of the Second Book of the *Geometria* were used by Newton in the final version of *De quadratura*.[67] Before stopping to analyze three distinctive propositions, I briefly sum up the conceptual apparatus that supports most of Newton's demonstrations concerning the quadrature of curves.

Let ABC be the figure of which we seek the area, and let BC be a perpendicular ordinate whose abscissa is AB (see Fig. 10). Let us extend CB to E, so that $BE = 1$, and let us complete the parallelogram $ABED$: the fluxions of the areas ABC, $ABED$ will be as BC to BE.[68] If one supposes that the area ABC equals v and $z = AB$, one has $ABDE = z \times 1$ and the ratio of the fluxions is given by $\dot{v}/\dot{z}$ (I do not think that Newton's idea is misinterpreted by putting $\dot{z} = 1$).

I examine now three propositions that rely on this apparatus.

> *Prop. III*, *Theor. I.* If for the abscissa AB and area AE, that is $AB \times 1$ there be indiscriminately written z and for
>
> $$e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} + \&c$$
>
> there be written R, let, however the area of the curve be $z^{\theta}R^{\lambda}$ then the ordinate BC will be equal to
>
> $$\left\{\theta e + (\theta + \lambda\eta)fz^{\eta} + (\theta + 2\lambda\eta)gz^{2\eta} + (\theta + 3\lambda\eta)gz^{3\eta} + \&c\right\}z^{\theta-1}R^{\lambda-1}.$$ [69]

Let us take the fluxion of $v = z^{\theta}R^{\lambda}$, that is,

$$\dot{v} = \theta\dot{z}z^{\theta-1}R^{\lambda} + \lambda z^{\theta}\dot{R}R^{\lambda-1} = \left\{\theta\dot{z}R + \lambda z\dot{R}\right\}z^{\theta-1}R^{\lambda-1}.$$

[66] The subsequent development of calculus in the 18th and 19th centuries and its sharp separation from classical geometry clearly show the difficulty of what Newton was attempting.

[67] See Note 1 of Whiteside: [Newton, 1967–1981, Vol. 7, p. 508]. A very interesting analysis of this text, especially from the foundational point of view, is given in [Guicciardini, 2009]. The development of the "Fundamental Theorem" of calculus in Newton's work is examined in [Giacardi, 1995].

[68] See [Newton, 1967–1981, Vol. 2, p. 232; Vol. 3, p. 78; Vol. 7, p. 516; Newton, 1704, p. 175].

[69] See [Newton, 1967–1981, Vol. 7, p. 519].

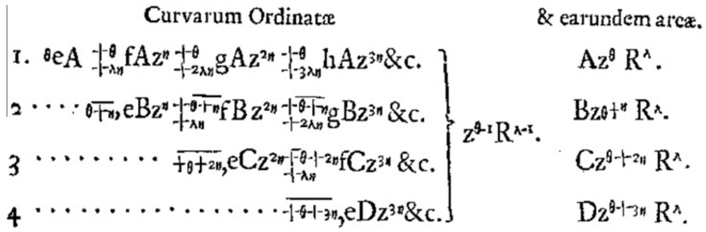

Fig. 11. Newton's table (Newton, 1704, p. 178).

Since

$$\dot{R} = \eta f z^{\eta-1}\dot{z} + 2\eta g z^{2\eta-1}\dot{z} + \cdots,$$

the calculation of $\dot{v}/\dot{z}$ gives Newton's result. □

The manipulation of the series in this case is very simple. The case of $z^\theta R^\lambda S^\mu$, considered in Prop. IV, Theor. II, is more complicated,[70] but Newton leaves it to the reader. This latter case also contains the case of the product RS $(\theta = 0, \lambda = \mu = 1)$ and the case of the quotient R/S $(\theta = 0, \lambda = 1, \mu = -1)$. However, Newton does not bother to explain the simplest rules of calculus (an attitude very different from that of Leibniz). For Newton, everything must come from a skilful use of series expansions.

Prop. V, *Theor. III* exhibits Newton's great ingenuity in manipulating series and in predicting the formal structure of the recursive formulas.[71] This time he wants to calculate the area subtended by a curve whose ordinate is given by $z^{\theta-1}R^{\lambda-1}S$; that is,

$$z^{\theta-1}\left(e + fz^\eta + gz^{2\eta} + hz^{3\eta} + \cdots\right)^{\lambda-1}\left(a + bz^\eta + cz^{2\eta} + dz^{3\eta} + \cdots\right). \tag{14}$$

He knows that, in general, the solution may be given by a series expansion having the form

$$z^\theta R^\lambda (A + Bz^\eta + Cz^{2\eta} + Dz^{3\eta} + \cdots). \tag{15}$$

Since every term of (15) represents an area, Newton can employ *Prop. III*. The ordinate corresponding to an area having the form $c_k z^{\theta+k\eta}R^\lambda$ may be given in the form $c_k z^{\theta-1}R^{\lambda-1} \cdot T_k$, where T_k is a power series in z^η with first term $z^{k\eta}$. The recursive structure of the calculations is explained with the help of the table in Fig. 11.

It is possible to calculate the coefficient A and recursively B, C, etc. by comparing this table with (14). For the benefit of the modern reader I give Newton's result with the help of modern notation. Let

$$R = a_0 + a_1 z^\eta + a_2 z^{2\eta} + a_3 z^{3\eta} + \cdots, \tag{16}$$

$$S = b_0 + b_1 z^\eta + b_2 z^{2\eta} + b_3 z^{3\eta} + \cdots, \tag{17}$$

and set $r_k = \theta/\eta + k\lambda$. If we write T in the form

$$T = A_0 + A_1 z^\eta + A_2 z^{2\eta} + A_3 z^{3\eta} + \cdots \tag{18}$$

the formulas given by Newton become

$$A_0 = \frac{b_0/\eta}{a_0 r_0}, \tag{19}$$

$$A_n = \frac{b_n/\eta - \sum_{k=1}^{n}(r_k + n - k)a_k A_{n-k}}{(r_0 + n)a_0}. \tag{20}$$

[70] See [Newton, 1967–1981, Vol. 7, p. 518].
[71] See [Newton, 1967–1981, Vol. 7, pp. 520–521; Newton, 1704, pp. 177–178].

Accordingly, if one obtains a sequence of terms A_k by starting from the writing of the ordinate,[72] the area of the curve will be given "in terminis finitis" (in finite terms).

The examples chosen by Newton clearly show the power of his method. To consider but one: suppose that we have

$$\frac{3k - lz^2}{z^2\sqrt{kz - lz^3 + mz^4}} = z^{-5/2}(k - lz^2 + mz^3)^{-1/2}(3k - lz^2). \tag{21}$$

The application of the procedure gives

$$z^{-5/2+1}(k - lz^2 + mz^3 - 2)^{-1/2+1}(-2 + 0 + 0 + \cdots) = -2\sqrt{\frac{k - lz^2 + mz^3}{z^3}}. \tag{22}$$

It is rather difficult to suppose that Newton proceeded from (21) to (22). It is easier to imagine the contrary. Newton's virtuosity in calculations may have led him to lay out well-organized tables, but it also contained elements of weakness, due to the absence of explicit algebraic rules.[73]

Of course Newton, from the period of his *annus mirabilis*, knew the fluxions of "elementary functions." In *De Analysi*, for example, he states that the fluxion of $\sin x$ is $\cos x$. The conceptual structure is (to put it into a very schematic form) the following:[74]

$$\begin{array}{ll}
\sin x \xrightarrow{\text{expansion into a series}} & x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots \\
& \Big\downarrow \text{ term by term differentiation.} \\
\cos x \xleftarrow{\text{the expansion is recognized}} & 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots .
\end{array}$$

Newton has perfect control of this type of situation and is able to make very elaborate tables of quadratures. However, when we are confronted with a simple combination of functions such as $\sin^3 x \cdot \cos^2 x$ and the need to obtain the fluxion or fluent that generated it, we will find using series expansions a much more sophisticated option than the straightforward employment of Leibniz's algebraic rules.

7. Conclusion

In his very interesting book, Guicciardini observes that the "method of series and fluxions" which

> [...] appears, with the benefit of hindsight, to be Newton's greatest achievement was perceived as just one among many alternative approaches to problem solving by its inventor.[75]

The attempt to place porisms side by side with series, the invectives against the abuse of algebra, the organic construction of curves, and finally all the materials that Newton tried to gather in his unpublished treatise on geometry plainly show that the method of fluxions

[72] Which may be written in different ways. For example, $(1+z)\sqrt{1+z}$ may be considered as $R \cdot (1+z)$, by setting $R = \sqrt{1+z}$, or, again, as $R \cdot (1 + 1/2z + \cdots)$, by setting $R = 1 + z$.

[73] The tables of integrals which Newton produced, starting in his prime years, are carefully examined in [Panza, 2005].

[74] See [Newton, 1967–1981, Vol. pp. 236–238]. Of course, in his long career, Newton elaborated a rich set of procedures that allowed him to master all the rules and subtleties of calculus. However, he always emphasized the preeminence of the series approach.

[75] See [Guicciardini, 2009, p. 9].

and series was not Newton's only interest. However, among these interests, the one that posed the greatest foundational worries was the opposition between the "new analysis," i.e., the algorithm in which infinitesimals and infinite series occurred, and the "common analysis," i.e., finite Cartesian algebra. The latter could sometimes be substituted by an ingenuous use of classical tools, but it was scarcely possible to do the same for the former.

The reasons behind Newton's unsuccessful attempt to compose a great treatise are rooted in the profound conceptual difficulties I have described. But I believe this lack of success cannot be interpreted simply as a loss. Newton's numerous attempts paved the way to successive important developments of mathematics. A hint at the evolution of some mathematical contents of the *Principia* may clarify this point. Proposition XI, whose contents I sketched in Section 5 and which are exposed at greater length in the Appendix, is followed in the *Principia* by the examination of the analogous cases of the hyperbola and the parabola (Propositions XII and XIII). They constitute as a whole a brilliant solution of the direct problem of central forces. Newton added something more in Corollary 1:

> From the three last Propositions it follows, that if any body P goes from the place P with any velocity in the direction of any right line PR and at the same time is urged by the action of a centripetal force that is inversely proportional to the square of the distance of the places from the center, the body will move in one of the conic sections, having its focus in the center of force; and conversely.[76]

The claimed solution to the inverse problem of central forces sketched in this corollary, for a force that varies reciprocally as the square of the distance, ended up appearing somehow insufficient to Newton himself who, after a paper of John Keill on the same subject (1708), "In a letter dated 11th October 1709 ... gave instructions to Roger Cotes ... to complete Corollary 1..."[77] This is the completion given by Newton in the letter, which appeared in the second edition of the *Principia*:

> For the focus, the point of contact, and the position of the tangent being given, a conic section may be described which at that point shall have a given curvature. But the curvature is given from the centripetal force [and velocity of the body] being given; and two orbits, touching one the other, cannot be described by the same [centripetal] force [and the same velocity].[78]

[76] "Ex tribus novissimis Propositionibus consequens est, quod si corpus quodvis P, secundum lineam quamvis rectam PR, quacunq; cum velocitate exeat de loco P & vi centripeta quæ sit reciproce proportionalis quadrato distantiæ a centro, simul agitur; movebitur hoc corpus in aliqua sctionum Conucarum umbilicum habente in centro virium; & contra." [See Newton, 1687, p. 55]. The English translation provided is given in [Newton, 1962, p. 61].

[77] See [Guicciardini, 1995, p. 543].

[78] "Nam datis umbilico & puncto contactus, & positione tangentis, describi potest sectio conica, quaæ curvaturam datam ad punctum illud habebit. Datur autem curvatura ex data vi centripeta, [& velocitate corporis]: & Orbes duo se mutuo tangentes eadem vi [centripeta eademque velocitate] describi non possunt." See [Newton, 1959–1977, Vol. 5, p. 5]. Once again the English translation is given in [Newton, 1962, p. 61]. Between brackets I have inserted the modifications introduced in the third edition: see [Newton, 1972, Vol. 1, p. 125]. See also [Guicciardini, 1995, p. 543, Note 15], where the differences between the editions of the *Principia* are described and commented. In Note 5, added to clarify the meaning of this letter, the editors of Newton's *Correspondence* boldly affirm that "This is Newton's first direct proof of the inverse problem of central motion when the force is reciprocally as the square of the distance; the force may be deduced from the curve, since the curve must coincide with that generated from a known force" (ibid., p. 6).

This sober statement constitutes all that is necessary, in Newton's opinion, to give a complete solution of the inverse problem of central forces.

As is fairly well known, Johann Bernoulli and the Leibnizians had a very different opinion. The real issue, they maintained, was not the simple need for clarification. In 1710 Johann Bernoulli severely questioned Newton's claim that Corollary 1, as given in the first edition of the *Principia*, might give a solution to the inverse problem of central forces. He declared that the solution claimed by Newton was simply a *non sequitur*.[79] Even the additions in the later editions of the *Principia* (1713, 1726) could not prevent the rising of an acrimonious controversy between the leading scholars of the day, some of whom upheld the soundness of Newton's addition while others judged it insufficient. This polemic still enjoys a certain favor and probably will survive. Indeed, the question at stake deals with what it is fair to suppose about Newton's assumptions.[80]

A rough outline of the situation nowadays may be given in the following terms: the scholars who feel up to reading Newton's text, to some extent at least, in terms of differential equations are inclined to underestimate the soundness of Newton's addition to Corollary 1, while those who pay more attention to the proper geometrical context are more tolerant (considering the existence and uniqueness assumption about trajectories for appropriate initial data as "natural").

It seems to me that the real problem is given by the status of conic sections in Newton's *Principia*. Is it possible to consider them both, on the one hand, as particular *trajectories* and, on the other, as the *classical curves* described by Apollonius? Undoubtedly, Newton, in Section 5 of Book I (where the classical Pappus problem is tackled) deals with conics in terms that Apollonius could have understood. However, when the *curvature* needs to be considered, as in Newton's Corollary 1, no classical source can help and conics are simply trajectories.

Even a giant such as Newton was unable to integrate the conceptual tools of the new analysis used in the *Principia* into the framework of the classical world he so much admired in his old age. However, the multiform nature of conic sections as considered by Newton, just to continue with this example, which springs out of the *Principia* and of other Newtonian texts—sections of the cone, simple algebraic curves of second order, trajectories—makes his mathematics a richer subject of investigation, in spite of (or maybe because of) the difficulty of a unitary approach.

Of course these special considerations may be extended to the whole of Newton's unfinished treatise.

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[79] See [Guicciardini, 1995, pp. 553–554].

[80] See, for example, [Aiton, 1988; Weinstock, 1992] and my reviews of both papers in *Mathematical Reviews*: MR1065876 (91i:01020), MR1150882 (99j:01006). A look at the demonstration Newton gave to Locke shows that even in Newton's time and for his friends (of the moment) there was some difficulty: see [Hall and Boas Hall, 1962, pp. 293–301]. A few years ago, when Feynman claimed not to be able to expound a solution to the inverse problem by using the geometrical tools of conics as employed by Newton, he was of course joking (see [Goodstein and Goodstein, 1997, p. 146]). However, Feynman's choice of a different style is not a joke.

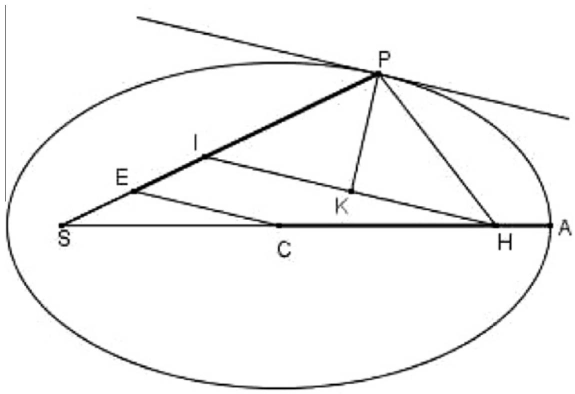

Fig. 12. A lemma about conics.

Appendix. Proposition XI of Section III of the First Book of the *Principia*

Let us suppose that a body describes a conic section (I consider only the case of the ellipse, that is of Proposition XI) being acted on by a central force directed towards a focus of the ellipse.[81] Newton wants to prove that this force is inversely proportional to the square of the distance from the focus. To prove that we need the lemma quoted above in Section 4.

Let S and H be the foci of the ellipse and let C be its center (see Fig. 12). Let us draw the straight line HI parallel to the tangent in P and also let CE be parallel to the tangent. Let PK be perpendicular to the tangent. We want to prove that

$$PE = AC.$$

Actually, since clearly $ES = EI$, we have

$$\begin{aligned} PS + PI &= PI + EI + ES + PI \\ &= 2(PI + EI) = 2PE. \end{aligned}$$

By the property of the tangent to an ellipse, PK is the bisector of the angle $\widehat{SPH}$ and, consequently, $PI = PH$. It follows that

$$PS + PI = PS + PH = 2PE.$$

We also have $PS + PH = 2AC$, and it follows that $2EP = 2AC$ and the lemma is demonstrated.

Now consider Fig. 13, where QR is parallel to PS and PF is perpendicular to the diameter CD, which in turn is parallel to the tangent in P. QT is perpendicular to PS.

In what follows, the boxes are used to emphasize the assumptions that go beyond classical geometry.

We have the *equation* of the ellipse:

$$GV \times PV = QV^2 \frac{PC^2}{CD^2}.$$

The triangles PUV and PEC are similar and from the equality $QR = PU$, we have

[81] See [Newton, 1687, pp. 50–51; Newton, 1759, pp. 45–46].

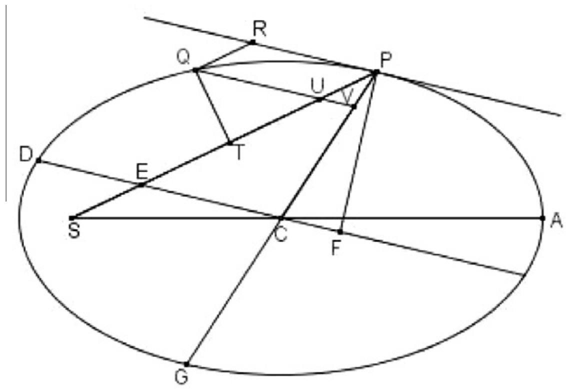

Fig. 13. The Proposition XI. Adapted from (Newton, 1687, p. 50).

$$QR = \frac{PE}{PC} PV. \tag{23}$$

Since PF is perpendicular to the tangent, the triangles QUT, PEF are similar as well. Besides, $\boxed{QU \approx QV}$. It follows that

$$QU : QT = PE : PF \Rightarrow QV : QT = PE : PF,$$

and therefore

$$QT = QV \frac{PF}{PE}. \tag{24}$$

Equalities (23) and (24) give

$$\frac{QR}{QT^2} = \frac{PE^3}{PF^2 \, GV \, \frac{CD^2}{PC}}.$$

$\boxed{Q \to P}$ and consequently we have $\boxed{GV \to 2PC}$. It follows that the ultimate ratio of QR and QT^2 is given by

$$\frac{QR}{QT^2} = \frac{1}{2} \frac{PE^3}{PF^2 \times CD^2}.$$

We know that $PF \times CD = \text{const.}$ ("constat ex *Conicis*") and that $PE = AC$, so that

$$\frac{1}{SP^2} \times \frac{QR}{QT^2} = \frac{1}{SP^2} \times \frac{1}{2} \frac{PE^3}{PF^2 \times CD^2} = \frac{1}{SP^2} \times \text{const.},$$

which means that the force is proportional to SP^{-2}.

References

Aiton, E.J., 1988. The solution of the inverse-problem of central forces in Newton's *Principia*. Archives Internationales d'Histoire des Sciences 38, 271–276.

Allard, J.-L., 1963. Le mathématisme de Descartes. Éditions de l'Université d'Ottawa, Ottawa.

Bos, H.J.M., 1981. On the representation of curves in Descartes' *Géométrie*. Archive for History of Exact Sciences 24, 295–338.

Bos, H.J.M., 2001. Redefining Geometrical Exactness. Descartes' Transformation of the Early Modern Concept of Construction. Springer, New York.

Brackenridge, J.B., 1995. The Key to Newton's Dynamics. University of California Press, Berkeley, Los Angeles, London.
Brunschvicg, L., 1937. René Descartes. Rieder, Paris.
Buzon, F. de, 2009. La première publication de Descartes. In: Descartes, R., Œuvres complètes. III. Discours de la Méthode et Essais, J.-M. Beyssade and D. Kamboucher (Dir.). Gallimard, Paris, pp. 15–41.
Cauchy, A.L., 1842. Memoire sur l'emploi du nouveau calcul, appelé calcul des limites, dans l'intégration d'un système d'équations différentielles. Comptes Rendus de l'Académie des Sciences, 15. In: Œuvres, vol. 7, pp. 5–17.
Cauchy, A.L., 1981. Équations différentielles ordinaires. Éditions Études Vivantes; Johnson Reprint Corporation, Ville Saint-Laurent, QC; New York. Preface by J. Dieudonné, Introduction by C. Gilain.
Cavaillès, J., 1976. Sur la logique et la théorie de la science, third ed. Vrin, Paris.
Cunningham, A., 1991. How the *Principia* got its name; or, taking natural philosophy seriously. History of Science 29 (4), 377–392.
Descartes, R., 1659–1661. Geometria à Renato Descartes anno 1637 Gallice edita; postea autem Unà cum Notis Florimondi De Beaune [...]. Operâ atque Studio Francisci à Schooten [...] Apud Ludovicum & Danielem Elzevirios, Amstelædami.
Descartes, R., 1954. The Geometry of René Descartes with a Facsimile of the first Edition, translated from the French and Latin by D.E. Smith and M.L. Latham. Dover, New York.
Descartes, R., 1964–1974. Œuvres. Adam, C., Tannery, P. (Eds.), new presentation by J. Beaude, P. Costabel, A. Gabbey and B. Rochot, 11 vols. Vrin, Paris.
Descartes, R., 2009. Œuvres complètes. III. Discours de la Méthode et Essais. J.-M. Beyssade and D. Kambouchner (Eds.). Gallimard, Paris.
Descotes, D., 2005. Aspects littéraires de la *Géométrie* de Descartes. Archives internationales d'histoire des sciences 55, 163–191.
Freguglia, P., 1999. La geometria fra tradizione e innovazione. Bollati Boringhieri, Torino.
Galuzzi, M., 1980. Il problema delle tangenti nella *Géométrie* di Descartes. Archive for History of Exact Sciences 22, 37–51.
Galuzzi, M., 1990. I *Marginalia* di Newton alla seconda edizione latina della *Geometria* di Descartes e i problemi ad essi collegati. In: Belgioioso, G., Cimino, G., Costabel, P., Papuli, G. (Eds.), Descartes: il metodo e i Saggi. Atti del Convegno per il 350° anniversario della pubblicazione del *Discours de la métode* e degli *Essais*. Istituto della Enciclopedia Italiana, Roma, pp. 387–417.
Galuzzi, M., 1995. L'influenza della geometria nell'evoluzione del pensiero di Newton. In: Panza, M., Roero, C.S. (Eds.), Geometria, flussioni, differenziali. Tradizione e innovazione nella matematica del Seicento. La città del sole, Napoli, pp. 271–288.
Galuzzi, M., Rovelli, D., 1997. Storia della geometria e didattica: qualche osservazione. In: Quaderno 19/2 del Ministero della Pubblica Istruzione. *L'insegnamento della geometria*. Liceo scientifico statale *A. Vallisneri*, Lucca, pp. 70–110.
Gardies, J.-L., 2004. Du mode d'existence des objets de la mathématique. Vrin, Paris.
Giacardi, L., 1995. Newton, Leibniz e il *Teorema fondamentale* del Calcolo integrale. Aspetti geometrici e Aspetti algoritmici. In: Panza, M., Roero, C.S. (Eds.), Geometria, flussioni, differenziali. Tradizione e innovazione nella matematica del Seicento. La città del sole, Napoli, pp. 289–328.
Giusti, E., 1990. La *Géométrie* di Descartes fra numeri e grandezze. In: Belgioioso, G., Cimino, G., Costabel, P., Papuli, G. (Eds.), Descartes: il metodo e i Saggi. Atti del Convegno per il 350° anniversario della pubblicazione del *Discours de la métode* e degli *Essais*. Istituto della Enciclopedia Italiana, Roma, pp. 419–439.
Giusti, E., 1999. Ipotesi sulla natura degli oggetti matematici. Bollati Boringhieri, Torino.
Goodstein, D.L., Goodstein, J.R., 1997. Il moto de pianeti intorno al sole. Una lezione inedita di Richard Feynman (Italian translation by P. Cavallo) Zanichelli, Bologna.

Guicciardini, N., 1989. The Development of Newtonian Calculus in Britain, 1700–1800. Cambridge University Press, Cambridge.
Guicciardini, N., 1995. Johann Bernoulli, John Keill and the inverse problem of central forces. Annals of Science 52, 537–575.
Guicciardini, N., 1999. Reading the Principia. Cambridge University Press, Cambridge.
Guicciardini, N., 2009. Isaac Newton on Mathematical Certainty and Method. MIT Press, Cambridge, MA.
Hall, A.R., Boas Hall, M., 1962. Unpublished Scientific Papers of Isaac Newton. Cambridge University Press, Cambridge.
Hofmann, J.E., 1974. Leibniz in Paris, 1672–1676. Cambridge University Press, Cambridge.
Jullien, V., 2006. Philosophie naturelle et géométrie au *XVII*^{e} siècle. Honoré Champion Éditeur, Paris.
Knobloch, E., 1991. L'analogie et la pensée mathématique. In: Rashed, R. (Ed.), Mathématiques et philosophie de l'antiquité à l'âge classique. Hommage à Jules Vuillemin. CNRS, Paris, pp. 217–237.
Lacroix, S.F., 1797–1798. Traité du calcul différentiel et du calcul intégral. J.B.M. Duprat, Paris.
Maronne, S., 2006. Sur une lettre de Descartes à Schooten qu'on dit de 1639. Revue d'histoire des mathématiques 12, 199–248.
Newton, I., 1687. Philosophiae Naturalis Principia Mathematica. Iussu Societatis regiae ac Typis Josephi Streater, Londini. Anastatic reprint, Culture et Civilisation, Bruxelles, 1965.
Nauenberg, M., 2003. Kepler's area law in the Principia: filling in some details in Newton's proof of Proposition 1. Historia Mathematica 30 (4), 441–456.
Newton, I., 1704. Opticks or a Treatise of the Reflexions Inflexions and Colours of Light also Two Treatises of the Species and Magnitudes of Curvilinear Figures. S. Smith and B. Walford, London.
Newton, I., 1740. La méthode des fluxions et des suites infinies (Trans. Buffon) Chez Debure l'aîné, Paris. Anastatic reprint, Blanchard, Paris, 1994.
Newton, I., 1759. Principes mathématiques de la philosophie naturelle. Chez Desaint & Saillant, Paris. Second Edition. Preface by Voltaire. Translation by the Marquise du Châtelet. Anastatic reprint, Dunod, Paris, 2005.
Newton, I., 1959–1977. The Correspondence of Isaac Newton (Turnbull, H.W., Scott, J.F., Hall, A.R., Tilling, L. Eds.). Cambridge University Press, Cambridge.
Newton, I., 1962. Principia. Motte's Translation Revised by Cajori. University of California Press, Berkeley, Los Angeles, London.
Newton, I., 1964. The Mathematical Works of Isaac Newton (Whiteside, D.T. Ed.). vol. 1. Johnson Reprint Corporation, New York, London.
Newton, I., 1967–1981. The Mathematical Papers of Isaac Newton (Whiteside, D.T. Ed.). Cambridge University Press, Cambridge.
Newton, I., 1972. Philosophiae naturalis Principia Mathematica, vol. I. Cambridge at the University Press, Cambridge.
Panza, M., forthcoming. From velocities to fluxions. In: Janiak, A., Schliesser, E., (Eds.), Interpreting Newton. Cambridge University Press, Cambridge.
Panza, M., 2005. Newton et les origines de l'analyse: 1664–1666. Blanchard, Paris.
Pappus, 1982. La Collection mathématique, avec une Introduction et des notes par Paul Ver Eecke. Nouveau tirage, Librairie scientifique et technique Albert Blanchard, Paris.
Pemberton, H., 1728. A View of Sir Isaac Newton's Philosophy. S. Palmer, London.
Pourciau, B., 2001. Newton and the notion of limit. Historia Mathematica 28, 18–30.
Rabouin, D., 2009. Mathesis Universalis. L'idée de "mathématique universelle". PUF, Paris.
Rashed, R., 2005. Les premières classifications des courbes. Physis 42 (2), 1–64.
Roero, C.S., 1995. Sul retaggio della tradizione geometrica nel calcolo infinitesimale leibniziano. In: Panza, M., Roero, C.S. (Eds.), Geometria, flussioni, differenziali. Tradizione e innovazione nella matematica del Seicento. La città del sole, Napoli, pp. 353–395.

Sasaki, C., 2003. Descartes's Mathematical Thought. Kluwer Academic Publishers, Dordrecht, Boston, London.
Serfati, M. (Ed.), 2002. De la méthode. Presses Universitaires Franc-Comtoises, Besançon.
Serfati, M., 2005. La révolution symbolique. La constitution de l'écriture symbolique mathématique. Petra, Paris.
Taton, R., 1978. L'initiation de Leibniz à la géométrie (1672–1676). Studia Leibnitiana Suppl., XVII, pp. 103–129. In: Taton, R., Études d'histoire des sciences. Brepols, Turnhout, pp. 159–185.
Verbeek, T., Bos, E.-J., van der Ven., J., 2003. The correspondence of René Descartes, 1643. Zeno, Utrecht.
Vuillemin, J., 1960. Mathématiques et métaphysique chez Descartes. PUF, Paris.
Weinstock, R., 1992. Newton's *Principia* and inverse-square orbits: the flaw reexamined. Historia Mathematica 19, 60–70.
Whiteside, D.T., 1961. Newton's discovery of the general binomial theorem. The Mathematical Gazette 16, 175–180.