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The emergence of open sets, closed sets, and limit points in analysis and topology

Gregory H. Moore

Department of Mathematics, McMaster University, Hamilton, Ontario L8S 4K1, Canada

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Abstract

General topology has its roots in real and complex analysis, which made important uses of the interrelated concepts of open set, of closed set, and of a limit point of a set. This article examines how those three concepts emerged and evolved during the late 19th and early 20th centuries, thanks especially to Weierstrass, Cantor, and Lebesgue. Particular attention is paid to the different forms of the Bolzano–Weierstrass Theorem found in the latter's unpublished lectures. An abortive early, unpublished introduction of open sets by Dedekind is examined, as well as how Peano and Jordan almost introduced that concept. At the same time we study the interplay of those three concepts (together with those of the closure of a set and of the derived set of a set) in the struggle to determine the ultimate foundations on which general topology was built, during the first half of the 20th century.
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Zusammenfassung

Die Topologie hat ihre Wurzeln in reeller und komplexer Analysis, Bereiche, in denen wichtiger Gebrauch der miteinander verbundenen Begriffe "offene Menge", "abgeschlossene Menge", "Häufungspunkt einer Menge" gemacht wurde. Dieser Artikel untersucht, wie diese drei Begriffe im auslaufenden 19. und im beginnenden 20. Jahrhundert entstanden und sich entwickelten, insbesondere dank der Arbeiten von Weierstrass, Cantor und Lebesgue. Spezielle Aufmerksamkeit wird dabei den verschiedenen Fassungen des Satzes von Bolzano–Weierstrass gewidmet, die sich in den unveröffentlichten Vorlesungen von Weierstrass finden. Eine frühe, nicht weiter verfolgte und unveröffentlichte Einführung offener Mengen durch Dedekind wird untersucht und auch die Versuche von Peano und Jordan. Gleichzeitig untersuchen wir das Zusammenspiel der drei Begriffe (und auch die des Abschlusses einer Menge und der Ableitung einer Menge) in dem Bemühen während der ersten Hälfte des 20. Jahrhunderts, die bestmöglichen Fundamente zu finden, auf denen die allgemeine Topologie aufbaut.
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E-mail address: ghmoore@mcmaster.ca.

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1. Introduction

During the 20th century, real and complex analysis relied heavily on the concepts of open set, closed set, and limit point of a set. This article investigates the origins and development of these interrelated ideas as they led to a conception of topology based on them.[1] The earliest idea was that of the limit point of a set, due to Weierstrass but disseminated by Cantor, while that of a closed set (due to Cantor) arose somewhat later. The idea of an open set (with the exception of Dedekind's brief, unpublished work about it) came latest of all. The very slow diffusion of the concept of open set is surprising in view of its importance now.

After discussing how these ideas developed in analysis (where they were only seen as tools of analysis and not part of a separate subject of topology), we consider the evolution of the concept of topological space. Such spaces are now universally based on the concept of an open set. There was a period of evolution during which it was not clear what concept should be taken as primitive and what axioms should be assumed for such a space. This evolution of the notion of an abstract topological space began in 1904 with the introduction, by the French analyst Maurice Fréchet [1904], of his concept of an L-space, which took the limit of an infinite sequence as the primitive idea. Fréchet used L-spaces as an abstract framework for generalizing Weierstrass's theorem that a continuous real function on a closed interval attains its least upper bound on that interval.

Hausdorff's neighborhood spaces, which he introduced in [1914], did not immediately succeed in displacing their competitors: Fréchet's L-spaces, Fréchet's metric spaces, and Frigyes Riesz's spaces based on accumulation points. Indeed, Hausdorff made substantial contributions to metric spaces. Eventually there was adopted a concept of topological space that was more general than Hausdorff's and was based on the concept of open set. For a few decades there was a competition among concepts even more general than the one which eventually became dominant and which is still dominant today under the name of "topological space." In analysis, the concept of a metric space was central for several decades, particularly during the 1920s as Banach spaces (complete normed vector spaces, where the norm provides the metric), until displaced later by the more general concept of topological vector space; see Moore [1995, 275–285].

2. Weierstrass, limit points, and the Bolzano–Weierstrass Theorem

Although the concept of the limit point of a set was named and first published by Cantor [1872, 98], it had been invented by Weierstrass. It occurs repeatedly, as part of his statement of the Bolzano–Weierstrass Theorem, in his unpublished lectures delivered over two decades. It should be emphasized that Weierstrass did not see this theorem as a part of "topology" or "*analysis situs*" or any other part of geometry, but rather as a theorem in classical analysis. This theorem asserts that any infinite, bounded set in n-dimensional Euclidean space has a limit point. Despite the traditional name of this theorem, it is not found anywhere in the works of Bolzano, since the idea of the limit point of a set does not occur there. What is found in Bolzano [1817] is something close to the method which Weierstrass uses to prove the theorem, i.e., repeated interval subdivision; for a detailed discussion, see Moore [2000, 171–179].

The Bolzano–Weierstrass Theorem took a wide variety of forms in Weierstrass's work over two decades and illustrates how he used limit points. No historian has pointed out the differences between these forms or has analyzed these differences. We do so here.

The earliest known version of the theorem occurs in an 1865 lecture course entitled "Principles of the Theory of Analytic Functions" ("Prinzipien der Theorie der analytischen Functionen"), which survives in unpublished notes taken by Moritz Pasch and kept in his *Nachlass* at Giessen.[2] There he called this theorem a "lemma" ["Hülfssatz"]. It was expressed in two dimensions, and was intensional in form rather than extensional, referring to properties rather than to sets. Intriguingly, it used the concept of a neighborhood, which later would also be important in topology:

[1] The public use of the mathematical term "Topologie" began with Johann Listing's "Vorstudien zur Topologie" [1847]. Listing, who had been a student of Gauss, had already used the term privately in a letter to a friend in 1836; see Pont [1974, 41–42]. According to Pont, the term "Topologie" was rarely used before 1920, since during that earlier period the term "analysis situs" was employed instead. For a general history of topology, see Aull and Lowen [1997–2001], Epple et al. [2002], and James [1999]. I thank a referee for the reference to Epple et al. [2002].

[2] The list of Weierstrass's courses in his *Werke* gives this course a slightly different name: "Theorie der analytischen Functionen."

> If, in a bounded part of the plane, there are infinitely many points with a given property, then there is at least one point (inside that part or on its boundary) such that in every neighborhood of this point there are infinitely many points having the given property. [1865, 17]

Weierstrass's next extant version was in terms of a function as well as a property. In the summer semester of 1868 he gave a course entitled "Einführung in die Theorie der analytischen Functionen" ("Introduction to the Theory of Analytic Functions"), which is preserved as the lecture notes of Wilhelm Killing: "If a function has a definite property infinitely often within a finite domain, then there is a point such that in any neighborhood of this point there are infinitely many points with the property" [1868/1986, 77].

A third version of the theorem was expressed even more strongly in terms of functions. This occurred in the summer semester of 1874, when Weierstrass gave the course "Einleitung in die Theorie der analytischen Functionen" whose notes were taken by Georg Hettner:

> In the domain of a real magnitude x let another magnitude x' be defined but in such a way that it can assume infinitely many values which all lie between two definite limits. Then it can be shown that in the domain of x there is at least one place [Stelle] such that in any neighborhood, however small, of it there are infinitely many values of x'. [1874, 305]

Then Weierstrass stated and proved this form of the theorem for an n-dimensional real or complex space, apparently the first time that he had done so [1874, 313–320]. However, the proof of his original 1865 version for the plane could be carried over directly to n dimensions.

A referee of the present article inquired what Weierstrass used his Bolzano–Weierstrass Theorem for. A straightforward example of such a use, found in Weierstrass's 1876 article "On the Theory of Single-Valued Analytic Functions," has to do with essential singularities of complex functions. There he wrote:

> If for any single-valued function there exist in the interior of a bounded region [Bereich] infinitely many inessential singular points, then there is in the interior or on the boundary of the region at least one point p which is distinguished by the fact that in every one of its neighborhoods there is a singular point other than p, and hence p is necessarily an *essential* singular point of the function. [1876, 80]

In his 1878 lectures Weierstrass gave the theorem in a form expressed more in terms of sets than of functions: "In any discrete domain of a manifold, which contains infinitely many places, there is at least one place which is distinguished by the fact that in any neighborhood of it, however small, there occur infinitely many places of the domain" [1878/1988, 86]. This time the surviving notes were taken by Adolf Hurwitz.

Finally, in his lecture of 9 June 1886 there occurred the last time (of which we have a record) that Weierstrass stated the Bolzano–Weierstrass Theorem. He did so in a way that recalls his statement in terms of functions in 1874 (see above):

> If x is an unbounded variable magnitude, which—as it is said—forms a simple manifold and is represented geometrically by a straight line, and if in it another variable magnitude x' is defined in such a way that the number of defined places is infinite, then there is in the domain of x, for which x' is defined, at least one place in the neighborhood of which infinitely many defined places occur. Such a place can either belong to the defined places, or not. It the latter case it is called a "limit place" ["Grenzstelle"]. [1886/1988, 60]

Weierstrass never used Cantor's term "Grenzpunkt," but late in his career he introduced instead the term "Grenzstelle" (found in this quotation), which meant something different. For a Weierstrassian "Grenzstelle" of a set cannot belong to the set, while a Cantorian "Grenzpunkt" of a set may or may not belong to the set. It is intriguing that Weierstrass never gave a name to the concept which he had been using for twenty years in the Bolzano–Weierstrass Theorem. This concept was precisely what Cantor called the limit point ("Grenzpunkt") of a set.

3. Cantor, limit points, and derived sets

In 1872 Cantor gave a name ("Grenzpunkt") to the concept of limit point of a set. He did so in analysis, while extending his theorem on the uniqueness of representation of a real function by its trigonometric series from the case where the function was continuous to the case where it had certain sparse discontinuities.[3] His definition of limit point of a set P on a straight line was based on a definition of "neighborhood," which in turn was based on that of the "interior" of an interval:

> By a "limit point of a point-set P" I understand a point situated on the straight line in such a way that every neighborhood of the point contains *infinitely* many points of P. It may happen that the limit point also belongs to P. By the "neighborhood of a point" is meant any interval which contains the point *in its interior*. From this it is easy to prove that a point-set consisting of infinitely many points must have at least *one* limit point. [1872, 98]

This passage, which begins with the definition of a limit point, ends with the first published statement of the Bolzano–Weierstrass Theorem. Intriguingly, Cantor does not give any reference to where this theorem comes from. Nor does he state the theorem for a bounded set, although this is necessary to preserve the truth of the theorem.[4] Probably both of these facts are due to an oversight. Certainly in [1882, 149] he credited Weierstrass with having first "expressed, proved, and applied [this theorem] in the most inclusive way in the theory of functions."

In his 1872 article on trigonometric series, Cantor immediately used his new term "limit point" to define the "derived set" of a point-set P, i.e., the set of all limit points of P. He then iterated the operation of derived set with $P^{(n)}$ standing for the nth derived set of P. Like Weierstrass, Cantor did not see the Bolzano–Weierstrass Theorem as a part of "topology" or "*analysis situs*" or any other part of geometry, but rather as a theorem in classical analysis; the same was true for how Cantor regarded the concepts of limit point and derived set. It was extremely rare for Cantor to refer to topology or *analysis situs* at all.

Soon the concept of limit point had spread to Italy in the book of Ulisse Dini [1878] on the foundations of real analysis. There Dini stated the Bolzano–Weierstrass Theorem in a form close to that of Cantor [1872], but more precise: An infinite set G of points lying in an interval (a, b) has a limit point, which may or may not belong to G [Dini, 1878, 16]. A second source of the Bolzano–Weierstrass Theorem for Italian mathematicians was Pincherle's version of the lecture notes of Weierstrass's 1878 lectures which he attended [Pincherle, 1880, 237]. Here the theorem was stated and proved both for the real line and for n-dimensional space.

In 1880 Cantor iterated the operation of derived set into the transfinite by letting $P^{(\infty)}$ be the intersection of all $P^{(n)}$ for finite n. The set $P^{(\infty+1)}$ was defined to be the derived set of $P^{(\infty)}$ and this enabled him to define $P^{(\alpha)}$, where α was any polynomial in ∞ or in its exponential. In an unpublished letter of 21 June 1882 the Swedish analyst Gösta Mittag-Leffler asked Cantor if he could prove that a nowhere dense set P of real numbers had an empty derived set $P^{(\alpha)}$ for some such α. This was important to Mittag-Leffler because, as he wrote, "I am now actually able to represent analytically those single-valued [complex] functions whose singularities constitute sets of this kind," i.e., such that $P^{(\alpha)}$ was empty for some such α.[5] If Cantor had answered yes, then Mittag-Leffler would have been able to represent analytically any complex function having a nowhere dense set of singularities.

Unfortunately for Mittag-Leffler, Cantor answered no in his letter of 25 June, giving as example what became known as the "Cantor set," i.e., the set of all real numbers between 0 and 1 inclusive that can be represented in ternary notation without 1 occurring in any decimal place. This set was nowhere dense but was equal to its own derived set. In his [1883b] article, which Cantor dated "October 1882," he introduced the term "perfect set" for a point-set that was equal to its derived set. In that article he gave the first version of the Cantor–Bendixson Theorem: $P^{(1)}$ is the union of a perfect set and a countable set [1883b, 193]. He then defined a "continuum" as a perfect, connected set [1883b, 194]. (This essentially topological definition would be modified around 1930 to a compact, connected set; it would have a very beautiful and deep theory.)

In 1884 Cantor first defined the notion of closed set (as a set that contains all its limit points). He showed that any closed set P is the derived set of some set Q and also that the derived set of $A \cup B$ is the union of the derived set of A and the derived set of B [1884, 226].[6] This second property, but not the first, would later carry over to general topological spaces.

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[3] How Cantor used limit points and originated his concept of derived set to extend his uniqueness theorem for the representation of a real function by a trigonometric series has been studied in detail by various authors. See, e.g., Cavaillès [1962, 78–83], Dauben [1970; 1979, 41–46].

[4] Zermelo, in editing Cantor's collected works, inserted the word "beschränkte," or "bounded," about the set P.

[5] This letter, like all of those between Cantor and Mittag-Leffler, is now kept at the Institut Mittag-Leffler in Stockholm.

[6] Hawkins [1979, 72] has emphasized how Cantor's closed sets were important in Borel's theory of measure.

A good example of where the concept of open set would have been useful to Cantor is in what topologists now call the Countable Chain Condition: A topological space satisfies this condition if any collection of disjoint open subsets of the space is countable. In effect Cantor showed the earliest version of this for the real line, and expressed it as the following lemma: On an infinite straight line, if there is a collection of infinitely many intervals which overlap only at their endpoints, then this collection is countable [1883a, 161]. The previous year he had expressed, in a somewhat vague way, an analogous theorem for n-dimensional space: "In an n-dimensional continuous space A, let (a) consist of an infinite number of separated subdomains which overlap only on their boundaries; then the set (a) of such subdomains is always countable" [1882, 153]. Later this would be expressed by saying that in n-dimensional space the number of disjoint open sets is countable.

4. An abortive approach to open sets

Cantor never used the general idea of a open set, even on a straight line. Instead, he referred only to a point "interior to" an interval [1872, 98] or, a few years later, to "interior points" of a continuous point-set [1879, 135]. Nevertheless, Cantor's 1879 definition of interior point was close to that put forward by Giuseppe Peano in his book *Geometric Applications of the Infinitesimal Calculus* [1887].[7] Peano considered a point-set A (in a space of one, two, or three dimensions) and defined a point p to be *interior* to A if there is a positive number r such that all those points whose distance from p is less than r belong to A. In his next two definitions, Peano went beyond what Cantor had done and stated that a point p was said to be *exterior* to A if p is interior to the complement of A. Finally, p was said to be a *boundary point* of A if p was neither interior nor exterior to A. Peano realized that if A contains some but not all the points of space, then A necessarily has a boundary point, which may or may not belong to A [1887, 152–160].

Peano's ideas could easily have led to the concept of open set at the time, but in fact did not. He defined the boundary of a set as the collection of all its boundary points. Then, had he wished to do so, he could have defined a set to be open if it was identical to the set of all its interior points. But this did not occur to Peano [1887] any more than it did to Camille Jordan [1892], as we shall see in the next section.

It is ironic that ideas quite similar to Peano's and Jordan's had been developed many years earlier by Dedekind in an unpublished manuscript, which was first published in Dedekind's collected works in 1931, thanks to Emmy Noether. This brief manuscript, entitled "General Theorems about Spaces," began with the definition of what he calls a "Körper" (a concept different from what he called a Körper, or algebraic number field, in his algebraic research): "A system [i.e., set] of points p, p' ... forms a Körper if for each of its points p there is a length d such that all points whose distance from p is less than d belong to P. The points p, p' ... [are said to] lie within P" [Dedekind, 1931, 353]. Here Dedekind's Körper was precisely an open set in Euclidean space, presumably n-dimensional space. What did he do with his open set or Körper? He used it to define what it means for a point to lie outside a Körper P. From these two definitions, he then defined a "boundary point" ("Grenzpunkt") of P as a point that is neither in nor outside P; the "boundary" ("Begrenzung") of P was then defined to be the set of all boundary points of P. His final result was that the boundary of a Körper cannot be a Körper [1931, 354].

With that, Dedekind's brief manuscript ended. According to Noether, Dedekind's letter to Cantor of 19 January 1879 refers to this manuscript as one that Dedekind had written quite a few years earlier, at a time when he intended to publish Dirichlet's lectures on potential theory and to give a rigorous treatment of Dirichlet's Principle. Dedekind continued: "I had a few such definitions which seemed to me to give a good foundation. But I later put the whole thing aside, and at that time was only able to give an incomplete treatment since I was completely absorbed with reworking Dirichlet's [lectures on] number theory."[8] It is not surprising that Dedekind had no intention of publishing his manuscript since it made only the barest beginning. Yet it is intriguing to see that he had defined the concept of open set at a date when no one else even thought about doing so.

Both Dugac [1976, 108] and Ferreirós [1999, 139] view Dedekind as working in the context of metric spaces—a claim that is quite mistaken. No one had formulated the concept of metric space before Fréchet did so in [1906]. What is true is something quite different, namely that Dedekind's definitions and proofs, being expressed in terms of distance, can be carried over to an arbitrary metric space.

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[7] Hawkins [1979, 87] has remarked how at this time Peano was familiar with Cantor's concept of closed set.

[8] See Dedekind [1931, 355]. Dugac [1976, 107] dates Dedekind's manuscript as written about 1871, whereas Ferreirós [1999, 138] dates it as composed between 1863 and 1869.

5. The French reception of limit points

The concept of a limit point of a set entered France by two avenues. The first of these was opened by Henri Poincaré, who in his 1883 article on Kleinian groups mentioned and used several of Cantor's concepts, such as the derived set of P [1883, 78]. But Poincaré did so while leaving all of Cantor's terminology in German and not explaining the meaning. Naturally, no other French mathematician absorbed Cantor's ideas in this way.

A second and much broader avenue was opened by Jules Tannery, who favorably reviewed the French translations of Cantor's articles [1884]. In his 1886 textbook on real functions Tannery, after discussing the limit of a sequence, introduced the notion of a limit point of a set. At the same time Tannery stated the Bolzano–Weierstrass Theorem, which he correctly credited to Weierstrass alone [Tannery, 1886, 42]. Tannery [1887] also reviewed Peano's book *Geometric Applications of the Infinitesimal Calculus* (see Section 4 above), where the concepts of interior point, exterior point, and boundary point of a set were defined. Probably Jordan learned of these concepts from Tannery's review.

In an 1892 article on definite integrals, Cantor's ideas were used by Jordan in a new and fruitful way. Working in n-dimensional space, Jordan introduced a metric ("écart") that was different from the usual distance function given by the Pythagorean theorem. Among topologists now, Jordan's metric is usually known as the taxicab metric. (By 1892, the only other use of a metric besides that given by the Pythagorean Theorem was Cayley's logarithmic metric for non-Euclidean geometry.) Jordan's metric assigns to two points $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ the distance

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|.$$

He then gave a definition of limit point that was different from Cantor's but that, two decades later, would become the standard definition: A point p is said to be a *limit point* of a set E if for any $\varepsilon > 0$ there is some point q of E other than p such that the distance between p and q is less than ε. Jordan followed Cantor in defining the derived set of E to be the set of all limit points of E. A set E would then be closed, in Cantor's terminology, if the derived set of E is a subset of E. Somewhat perversely, Jordan did not designate such a set as closed but as "perfect" (thus using a term of Cantor's with an incompatible meaning). Then Jordan defined the "interior points" of E to be those points in E that do not belong to the derived set of the complement of E [1892, 72].

With these definitions in hand Jordan would have been able, in a natural way, to define a set E to be "open" if E consisted of all and only the interior points of E. But he did not do so. Apparently he felt no need for such a concept. He was much more interested in defining the boundary points of a set E (those points interior neither to E nor to its complement) and showing that the set of boundary points of E is always nonempty and closed.[9]

Jordan included these ideas from his 1892 article the following year in the revised version of his *Cours d'analyse* [1893]. There was one substantial change, however. In the book he modified his definition of the limit point of a set E. Now p was said to be a limit point of E if p was the limit of a sequence of points belonging to E [1893, 19]. To distinguish this definition from his earlier one, we will say that his new definition is that of a "sequential limit point."[10] To prove that his new definition of sequential limit point is equivalent to his old definition of limit point requires the Axiom of Choice, which, however, had not yet been explicitly formulated; see Moore [1982, 18].

The treatment of point-sets in Tannery's and Jordan's textbooks had a substantial influence on the next generation of French mathematicians: Emile Borel, René Baire, and Henri Lebesgue. Borel's textbook, *Lessons on the Theory of [Complex] Functions*, was dedicated to Tannery and cited Jordan's book approvingly [Borel, 1898, 1]. All of these mathematicians saw the topological parts of their work as being components of analysis rather than as contributions to a separate subject of topology.

[9] See Jordan [1892, 73]. He did not notice that his argument for the existence of some boundary point of E required that both E and its complement be nonempty, something that Peano earlier had clearly understood.

[10] With Jordan's new definition, every point of E is a sequential limit point of E. However, he stated that a set E may contain points that are not sequential limit points of E, and so we alter his definition to require that all the terms of a sequence converging to a sequential limit point be distinct.

6. Limit points and connectedness

The idea of connectedness of a point-set was involved with that of a continuum in the mid-19th century, at a time before any notion of closed or open set had been proposed. As Wilder remarked [1978, 721], Bolzano's posthumous *Paradoxes of the Infinite* (1851) asserted that "a continuum is present when, and only when, we have an aggregate of simple entities (instants or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbor" [Bolzano, 1851/1950, 129]. Cantor strenuously objected to Bolzano's definition of a continuum since, under this definition, a set consisting of several separated continua would be a continuum. It may be, however, that Bolzano's definition of continuum led Cantor to formulate what he called "connectedness," i.e., a set M is connected if for every positive ε and every a and b in M there is some finite n and some points $p_1, p_2, \ldots, p_n$ such that the distances $ap_1, p_1p_2, \ldots, p_nb$ are all less than ε [Cantor, 1883b, 194].

Cantor's definition of connectedness shared some disadvantages with Bolzano's definition of continuum, although apparently no one pointed them out at the time. For instance, Cantor's definition made the set of all rational numbers connected, and likewise the set of all irrational numbers; in Bolzano's terms, these two sets would be continua. (Later, topologists would regard both of these sets as examples of "totally disconnected" sets.)

A quite different definition of connectedness was proposed by Jordan in his 1892 article on definite integrals. There, restricting himself to closed and bounded sets, he defined a set E in $\mathbb{R}^n$ to be connected ("d'un seul tenant") if and only if E cannot be partitioned into two closed and "separated" sets.[11] Jordan immediately went on to prove that a closed and bounded set is connected in his sense if and only if it is connected in Cantor's sense [1892, 75]. He repeated his discussion of such concepts (limit point, separated sets, closed set, connected set) in his *Cours d'analyse* [1893, 25–26].[12] It is clear that what we now view as topological concepts were seen by Jordan as parts of analysis and as tools to be used in analysis, rather than as a separate and distinct field of mathematics.

Both Cantor's and Jordan's definitions of connectedness were based on the notion of distance. In 1904, Arthur Schoenflies argued that it was desirable to have a "purely set-theoretic definition" of connectedness (although he did not make clear in what sense a definition involving distance was not set-theoretic) and gave what he regarded as such a set-theoretic definition for perfect sets in n-dimensional Euclidean space: a perfect set is connected if it cannot be partitioned into two perfect sets [1904, 209]. Schoenflies communicated his definition to Eduard Study, who informed him that Jordan had already formulated this definition.

The first definition of connectedness that was applicable to arbitrary sets in Euclidean n-space (and, as later recognized, in an arbitrary topological space as well) was presented by N.J. Lennes to the American Mathematical Society in December 1905. It was published in an abstract soon afterward [1906] but not published in an article until five years later: A point-set is connected if, when it is partitioned into nonempty sets B and C, at least one contains a limit point of the other [1911, 303]. Lennes was also working within analysis, giving in his 1911 article an application of "analysis situs" to the calculus of variations, but he cited Schoenflies' 1904 article about the desirability of a purely set-theoretic definition of connectedness. That article appears to have motivated Lennes to propose his own definition of connectedness [1911, 287].

7. Serious errors about closed sets

A number of serious errors about closed sets were made around the turn of the century, errors that were involved with the later concept of compactness. The first of these errors was due to Hurwitz. In a paper read to the first International Congress of Mathematicians, he claimed that if f is a one–one continuous function from a closed set P to a closed set Q, then the inverse of f is also continuous, giving that f is a homeomorphism [1898, 102]. Jordan had

[11] In his terminology, two sets A and B are "separated" if the greatest lower bound of the distance ab, where the point a belongs to A and b to B, is not zero.

[12] Wilder [1978, 722] argued that Cantor's definition of connectedness was unknown to Jordan, since he did not refer to Cantor explicitly in his book [1893]. But Wilder did not know that Jordan did refer to Cantor explicitly in his [1892] article discussing such concepts as limit points and introducing his concept of connected set ("d'un seul tenant"). Regarding Jordan's article, Hawkins wrote [1979, 93]; "It is difficult to document Jordan's familiarity with the work of his predecessors because he was not in the practice of acknowledging his indebtedness to them, a habit which put him on bad terms with Hermite.... It is clear, however, that Jordan was familiar with Cantor's ideas."

proved that this claim is true if P is both closed and bounded [1893, 53]. But Hurwitz's claim is false in general if P is closed but unbounded.[13]

Schoenflies [1900, 118] accepted Hurwitz's claim as true. The irony is that Schoenflies even cited the appropriate page of Jordan [1893], but did not notice that by leaving out the word "unbounded," he had made the claim false. This mistake was connected with Schoenflies' overemphasis on the importance of closed sets: "The most important point-sets from the theoretical viewpoint are the *closed* and the *perfect* sets. These are the ones most frequently encountered in analysis and geometry" [1900, 74].

A series of related errors followed. Schoenflies argued that the one–one continuous image of a perfect set P is perfect [1900, 117]. (But this can fail if P is unbounded.) His purported proof relied on the claim that an infinite set of points must have a limit point. (This is an erroneous form of the Bolzano–Weierstrass Theorem and fails, for example, if P is the set of natural numbers.) He made a similar but stronger claim about closed sets, but this is just as false as the claim about perfect sets. Then he argued that if a function is continuous at every point of a perfect set P, then the function is uniformly continuous on P [1900, 119]. But this fails if P is taken to be all nonnegative points on the real line and the function is $f(x) = x^2$. Last, he asserted that if P_n is closed and P_{n+1} is a nonempty subset of P_n for all positive integers n, then the intersection of all the P_n is nonempty [1900, 58]. If he had required that P_1 be bounded, then his assertion would have been true, but otherwise it is easily refuted. (Let P_n be the set of all real numbers not less than n.)

Four years later, Schoenflies was aware that if a perfect connected set M in Euclidean space is mapped one–one and continuously, then the image of M is also connected. But he believed that if the mapping was continuous but not one–one, then the image of M could fail to be connected [1904, 209]. In this Schoenflies was mistaken. A decade earlier Jordan had shown that the continuous image of a closed, bounded, connected point-set is connected [1892, 79–80].

8. Where are the open sets?

If there is one theorem in analysis that is now inseparably linked to open sets, it is the result that in English-speaking countries and in Germany is called the Heine–Borel Theorem but that in France is called the Borel–Lebesgue Theorem. The latter name is more appropriate, but the history of this theorem is quite complicated and so we defer it to another occasion. The theorem uses a concept originally called "bicompactness" when first formulated during the 1920s,[14] but now universally known as "compactness": A set E is said to be *compact* if, given any family S of open sets that covers E, some finite subset of S covers E. The theorem then states that every closed and bounded set of real numbers is compact. The concept of an open set is essential to stating compactness, and hence is essential to the theorem in question in full generality.

Nevertheless, when Emile Borel stated the first version of this theorem in his doctoral dissertation, he did not mention open sets. Rather, this first version was the following, seen as a part of his development of measure theory and Borel-measurable sets:

> If on a line [segment] there are infinitely many partial intervals such that each point of the line [segment] is interior to at least one interval, then one can determine effectively a *finite number* of intervals, chosen from the given intervals and having the same property (i.e., every point of the line [segment] is interior to some one of them). [1895, 51]

For convenience, let us call this original form of the result "the Borel Theorem," and let us call "the Heine–Borel Theorem" the modern form that states that every closed and bounded set of real numbers is compact. Almost 30 years passed between Borel's first statement of his theorem and the modern statement.

Nevertheless, the concept of an open set was first stated in print some four years after Borel proved his theorem. This concept was first published by René Baire in his doctoral dissertation while discussing semicontinuous real functions. After defining a "closed sphere" S and the "open sphere" S' with the same center and radius in n-dimensional Euclidean space, he continued:

[13] On the history of Hurwitz's error, see Moore [2007, 338–340].

[14] It was called bicompactness by Aleksandrov and Urysohn to distinguish it from Fréchet's concept of compactness.

> Given *any* point of S', there is a sphere of positive radius having this point as center, all of whose points belong to S'. More generally, I call any set of points possessing this property an *open domain of n dimensions*. [1899, 6–7]

He then proceeded to use a sequence of open domains to prove a theorem about the conditions under which, in a given open domain, all the upper semicontinuous functions have the same least upper bound.

And, strange to say, that was the only thing for which Baire used open sets in his dissertation. Cantor's perfect sets were much more central to that work than were open sets. Indeed, the heart of Baire's dissertation was the theorem that a discontinuous function f can be represented by a infinite series of continuous functions if and only if f is pointwise discontinuous relative to every perfect set [1899, 62]. Like Lebesgue, Baire used topological ideas as tools to get results in analysis, and had no particular interest in those ideas as part of a separate discipline called topology.

The name "open" set was originated by Lebesgue in his doctoral dissertation of 1902, whose essential aim was to introduce Lebesgue measure (as an extension of the Borel-measurable sets) and the Lebesgue integral. Explicitly influenced by Jordan's *Cours d'analyse* of 1893, Lebesgue adopted from that book the definitions of an interior point of a set and of the boundary of a set [1902, 231]. He then defined a set on a straight line to be "open" ("ouvert") if it did not contain any point of its boundary. It followed, he added, that every point of an open set E is an interior point of E and that the complement of an open set is closed. This enabled him to show that an open set is Borel-measurable, i.e., a Borel set. (This was necessary because Borel himself had defined the Borel-measurable sets as the class of sets obtained by starting with closed intervals and then closing under the operations of complement and of countable union, and had shown in particular that the closed sets are Borel-measurable [1898, 49].) Then Lebesgue extended the above considerations, including open sets, to n-dimensional Euclidean space [1902, 232–234]. In what followed in the plane he used more often than his open sets what he called a "domain" (in French, "domaine"), i.e., the interior of a simple closed curve [1902, 235]. We will have occasion in Section 15 below to return to this term "domain" and the corresponding German word "Gebiet."

It is ironic that Lebesgue, having formulated the general concept of an open set in n-dimensional Euclidean space, generalized the Borel Theorem (as defined above) to uncountable covers without at the same time generalizing it so that those covers could consist of arbitrary open sets rather than just intervals. But that is what occurred in his 1904 book on the Lebesgue integral. Even in the second edition of that book in 1928, Lebesgue still stated his form of the Borel Theorem in the following way, without using open sets:

> If there is a family Δ of intervals such that every point of an interval (a, b), containing both a and b, is interior to at least one of the Δ, then there exists a family formed of a *finite* number of the intervals Δ and enjoying the same property (every point of (a, b) is interior to one of them). [1928, 112]

Nevertheless, the Borel Theorem was generalized in 1905 to sets that are both open and connected by the Hungarian mathematician Frederic Riesz. In an article published in the *Comptes rendus* of the Paris Academy of Sciences, Riesz wrote: "We will call a *domain* any connected set, no member of which is a limit point of its complement" [1905, 226]. He then expressed this general form of the Borel Theorem for n-dimensional Euclidean space as follows: If each point of a closed set is interior to at least one point in a set S of domains, then there is a finite subset of S having the same property.[15] Only later did it become clear that connectedness had nothing in particular to do with the question and that the theorem remained true even when connectedness was dropped as a requirement.

In 1905 Lebesgue, in the course of publishing an extensive treatment of analytically representable functions, gave a much more detailed treatment of open sets (as part of the first detailed treatment of the hierarchy of Borel sets) than anyone had done previously. His article relied heavily on Baire's classification of real functions. For each countable ordinal α, a real function f was said to be of *Baire class* α if f was the pointwise limit of an infinite sequence of functions belonging to Baire classes lower than α but did not belong to a Baire class lower than α. Corresponding to these Baire classes of real functions were levels of Borel sets. In this regard Lebesgue introduced two hierarchies of Borel sets. A subset E of the real numbers was said to be of class F_α if E was equal to the set S of all real x such that $a \leqslant f(x) \leqslant b$ for some real a and b, and some function f belonging to Baire class α but E was not equal to any such S if f belonged to a Baire class lower than α. Likewise, a subset E of the real numbers was said to be of class O_α if E was equal to the set S of all real x such that $a < f(x) < b$ for some real a and b and some function f belonging to

[15] Unfortunately, his theorem is false as stated. For the theorem to be true, the closed set must be assumed to be bounded.

Baire class α, but E was not equal to any such S if f belonged to a Baire class lower than α. Lebesgue used F_α, where F stood for "fermé" (that is, closed), because F_0 was the class of closed sets. Analogously, he used O_α, where O stood for "ouvert" (that is, open), because O_0 was the class of open sets [1905, 157]. He established that the intersection of a countable family of sets at most of class F_α is at most of class F_α. Likewise, the union of a countable family of sets at most of class O_α is at most of class O_α. In particular, the union of a countable family of open sets is open. (In fact, this remains true even if the family is uncountable, but he did not state this or appear to be aware of it.)

The relationship between the two hierarchies F_α and O_α was the following: A set of class F_α is of class $O_{\alpha+1}$ at most (e.g., that F_0 is of class O_1 means that any closed set is the intersection of a countable family of open sets). Likewise, a set of class O_α is of class $F_{\alpha+1}$ at most (in particular, any open set is the union of a countable family of closed sets). Consequently, the union of a countable family of sets at most of class F_α is at most of class $F_{\alpha+2}$, and the intersection of a countable family of sets at most of class O_α is at most of class $O_{\alpha+2}$. Thus Lebesgue had worked out all the basic structure of the classes of Borel sets, later to prove so useful in the more general framework of metric spaces [1905, 159–164].

Despite Lebesgue's ground-breaking work about open sets, they diffused rather slowly through the mathematical community. In England, W.H. and G.C. Young defined a point-set to be "open" if it was not closed—a definition that is incompatible with Lebesgue's, in which a point-set is open only if its complement is closed. Likewise, they called an interval "open" if it did not include at least one of its endpoints [1906, 19, 15]. E.W. Hobson, in his *Theory of Functions of a Real Variable*, adopted the Young's terminology for open set (i.e., non-closed) but used the more modern terminology for "open interval" (i.e., not containing its endpoints) [1907, 84, 119]. Two decades later, Hobson accepted the modern usage of open set [1927, 78], but took it from de la Vallée Poussin [1916, 10] rather than from Baire or Lebesgue.

9. Fréchet and general topology

The generalization of Euclidean spaces to more abstract spaces began with Maurice Fréchet's introduction of L-spaces in [1904]. The motivation behind these abstract spaces was to extend Weierstrass's theorem that a continuous function (of one or several real variables) on a closed and bounded interval attains its least upper bound; the question of Dirichlet's Principle for minima was also on his mind. Fréchet's abstract spaces, called L-spaces, were based on the primitive idea of the limit of an infinite sequence, along the pattern laid out by Jordan [1893] for sequential limit points. An L-space was a set X together with a function $F : S \to X$, where S was some set of infinite sequences of members of X. If s was a member of S, then $F(s)$ was said to be the "limit" of the sequence s. There were only two axioms:

(1) If s is a constant sequence whose value is a, then the limit of s is a.
(2) If the limit of a sequence s is b, then the limit of any subsequence of s is also b.

Then Fréchet defined p to a the "limit point" of a subset A of X if p is the limit of some sequence of distinct elements of A. And A was defined to be "closed" if it contained all its limit points, i.e., precisely as Cantor had defined it for point-sets. Fréchet said that a function f on X was "continuous" provided that it was sequentially continuous in Jordan's sense; i.e., if the limit of the sequence $a_1, a_2, \ldots$ was p, then the limit of the sequence $f(a_1), f(a_2), \ldots$ was $f(p)$. He took one of Schoenflies' (false) theorems for point-sets (i.e., if P_n is closed and P_{n+1} is a nonempty subset of P_n for all positive integers n, then the intersection of all the P_n is nonempty) and let this be the definition for a subset of an L-space to be "compact" (but which we will call "S-compact," for Schoenflies, in order to avoid confusion with the modern notion of compactness) [1904, 849]. The culmination of his article was the following generalization of Weierstrass's theorem:

(3) A continuous real-valued function on a closed and S-compact set (in an L-space) is bounded and attains its least upper bound.

He concluded by observing that, in an L-space, a set E is S-compact if and only if any infinite subset of E has a limit point in the space; i.e., E satisfies the Bolzano–Weierstrass Theorem. (We have here a process that occurred repeatedly: what is a theorem in the Euclidean context was turned into the definition of a property for a more general kind of space.) He pointed out that, for finite-dimensional spaces, S-compactness implied boundedness. But he remarked that with an appropriate definition of limit, the theorem could be extended to a space with denumerably many dimensions [1904, 850].

Fréchet's next article underlined the motivation behind his introduction of L-spaces:

> Just as, in mechanics, a preparatory theory of vectors allows us to avoid repeating the same proofs for forces, for moments, and for velocities, etc.—in just this way it would be advantageous to separate out from the theories of sets of points, of functions, of curves, etc., and the real-valued functions defined on them, those properties which are shared by these theories. [1905, 27]

His L-spaces were an attempt to isolate those properties, an attempt that grew from analysis and was intended to develop a tool for analysis. But the question then arose as to whether, in his L-spaces, every derived set is closed. It would turn out a year later that, for certain function spaces, the answer was no.

In his doctoral dissertation of 1906, Fréchet explored in depth both his L-spaces and introduced his new concept of metric space [1906, 30]. Every metric space is an L-space but not conversely. In a metric space, every derived set is closed, but not in every L-space. Although he defined the Cantorian concepts of limit point, closed set, and perfect set for his L-spaces, as well as the idea that a point is interior to a set, he did not introduce open sets for any of his spaces.

A key concept of Fréchet's dissertation was what he called a compact set. He defined a set A to be "compact" if any infinite subset of A has a limit point, which may or may not belong to A [1906, 6]. We will refer to his concept as "Fréchet-compactness" in order to avoid confusion with the modern concept of compactness. This allowed him to generalize the Heine–Borel Theorem to metric spaces.

10. Hausdorff and general topology

The idea of an open set in an abstract space (as opposed to n-dimensional Euclidean space, where the idea was due to Baire and Lebesgue) was originated by Felix Hausdorff in the context of his topological spaces. However, what Hausdorff called a topological space is a more specialized idea than what is now universally called a topological space. What he used as a primitive idea is "neighborhood of a point." To avoid ambiguity, we will call his spaces "neighborhood spaces." Hausdorff defined a neighborhood space to be a set E, whose members were called "points," together with a collection of subsets of E. These subsets were called neighborhoods and were subject to four axioms:

(A) Every point x belongs to at least one neighborhood of x, and every neighborhood of x contains x.
(B) If U and V are neighborhoods of x, then there is some neighborhood W of x such that $W \subseteq U \cap V$.
(C) If a point y belongs to a neighborhood U of x, then there is some neighborhood V of y such that V is a subset of U.
(D) If x and y are distinct points, then there is a neighborhood U of x and a neighborhood V of y such that U and V are disjoint. [1914, 213]

Immediately after giving his axioms for a topological space, Hausdorff defined what he meant by an "interior point" of a subset A of a topological space. Namely, x is an interior point of A if some neighborhood of x is a subset of A. And x was said to be a boundary point of A if x belongs to A but is not an interior point of A. Then a set A was defined to be an open set ("Gebiet") if all of its points are interior points [1914, 214–215]. Finally, he showed that the union of any family (countable or uncountable) of open sets is open and that the intersection of finitely many open sets is open. (By showing that the union of any uncountable family of open sets is open, he went beyond what Lebesgue had done in 1905 with open sets.)

It should be noticed that Hausdorff's neighborhoods do not necessarily correspond to neighborhoods when, as is the case today, the concept of open set is taken as primitive for topological spaces. In particular, for a neighborhood space E containing at least two points, the whole space E need not be a neighborhood of any point, for if the only neighborhood of any point x is just $\{x\}$, then all Hausdorff's axioms are satisfied, although the whole space X is not an open set. However, if the modern concept of topological space with "open set" is taken as the primitive idea, then a neighborhood of x is defined to be any set V such that x belongs to some open subset of V. Hence the whole space is then open and is a neighborhood of each of its points, in contradiction to our example of a Hausdorff neighborhood space. (What for Hausdorff was the set of neighborhoods of a space is today called a "neighborhood base" for the space.)

After defining open sets in terms of neighborhoods, Hausdorff turned to accumulation points and closed sets. He defined p to be an accumulation point ("Häufungspunkt") of a set B if every neighborhood of p contained infinitely many points of B. (This was in complete agreement with Cantor's concept of limit point.) A set was defined to be closed if it contained all its accumulation points. Then Hausdorff showed that the intersection of any family of closed sets was closed and that the union of any finite number of closed sets was closed [1914, 219–225].

Hausdorff adopted from Fréchet the name "compact" (which, above, we called "Fréchet-compact") for any set each of whose infinite subsets has an accumulation point. What Hausdorff called the "Borel Theorem" was the proposition that if a closed and Fréchet-compact set M is a subset of the union of an infinite sequence S of open sets, then M is already a subset of the union of some finite subset of S [1914, 231].

Hausdorff's second axiom of countability (i.e., the set of all neighborhoods is countable) had various consequences for the open and closed sets. One of these consequences was what is now known as the countable chain condition: any set of disjoint open sets is countable. Another was that the set of all open sets has the same cardinality as the set of all closed sets, namely that of the set of all real numbers. Furthermore, this axiom implied a sharper form of the Borel Theorem, with S having any infinite cardinality rather than necessarily being countable [1914, 270–272].

In a metric space, as Hausdorff showed, the structure of Borel sets followed the basic lines of Lebesgue's 1905 article. In particular, in a metric space (but not in an arbitrary topological space), every closed set was the intersection of some countable family of open sets, and every open set was the union of some countable family of closed sets [1914, 306].

11. Survival of the fittest

Despite the appearance of Hausdorff's definition of a "topological" space in terms of axioms on the neighborhoods, there was no consensus over the next decade that topology should deal with his neighborhood spaces rather than with some other concept of abstract space. Instead there was a rivalry among various concepts of abstract space.

In 1922 the young Polish topologist Kasimierz Kuratowski published an article "On the Operation $\overline{A}$ in Analysis Situs," based on his doctoral dissertation of two years earlier. In this article he used this operation—the closure of a set—as the sole primitive idea. Interestingly, Kuratowski referred to Fréchet's L-spaces but not to Hausdorff's neighborhood spaces [Kuratowski, 1922, 183]. (In later work, Kuratowski would refer to both [1933, ix].) For convenience, we will refer to a space of the sort introduced by Kuratowski as a "closure space" (he gave it no name). A closure space is a set X together with a function (closure) from subsets of X to subsets of X, satisfying four axioms: for any subsets A and B of X,

(1) the closure of $A \cup B$ is the union of the closure of A and the closure of B;
(2) A is a subset of the closure of A;
(3) the closure of the empty set is the empty set;
(4) the closure of the closure of A is the closure of A [1922, 182].

A set was defined to be closed if it was identical to its closure and to be open if it was identical to the complement of the closure of its complement [1922, 188]. Although Kuratowski did not mention the fact, his closure spaces were identical to Hausdorff's topological spaces if the last of the latter's axioms was dropped (i.e., the axiom stating that any two distinct points are contained in disjoint neighborhoods). Kuratowski did point out that his closure spaces were more general than Fréchet's L-spaces (which were based on axioms for the limit of a sequence). In fact, these closure spaces were the first form of what are now universally called "topological spaces."

At the end of his article Kuratowski gave an alternative axiomatization, based on a different and more Cantorian primitive idea, that of the "derived set" A' of a set A. For convenience, we refer to a space of this sort as a "derived space." Such a space was a set X that was required to satisfy four axioms: If A and B are any subsets of X, then

(1) $(A \cup B)' = A' \cup B'$;
(2) $X' = X$;
(3) $0' = 0$;
(4) A'' is a subset of A'.

As he remarked, any derived space is a closure space if the closure of A is taken to be $A \cup A'$ [1922, 198]. He seemed to believe that any closure space is a derived space if A' is taken to be the set of all those points p in X belonging to the closure of the set $A - \{p\}$. However, this is easily seen to be false if X contains exactly one point; for then X can be a closure space but cannot be a derived space.

A year later in the eminent journal *Mathematische Annalen*, the Austrian topologist Heinrich Tietze published an article entitled "Contributions to General Topology. I. Axioms for Various Forms of the Neighborhood Concept" [1923]. Hausdorff had not used the adjective "open" ("offen") to describe a set but only the noun "Gebiet." Tietze adopted the adjective "offen" from the book *Vorlesungen über reelle Funktionen* by Constantin Carathéodory, an analyst who worked in Germany [1918, 40] and who referred to an "offen Menge" or "open set" rather than to a "Gebiet" [Tietze, 1923, 292].

Carathéodory, after proving that in n-dimensional Euclidean space a set is closed if and only if its complement contains only interior points, had added,

> This duality between closed point-sets and those which consist purely of interior points is, as we will see, a very deep one. We will best express it if we characterize sets which possess only interior points with a name that is explicitly related to the word "closed." We wish to name as "open" those point-sets which consist entirely of interior points. We can do so all the more unobjectionably since we will later prove that a point-set cannot at the same time be both open and closed.... [1918, 40]

When he did so later (§213), he correctly excluded the cases where the point-set was either empty or the whole space, since in those cases a point-set is indeed both open and closed. As it happened, there are topological spaces (and even subspaces of the real line) with subsets that are both open and closed—precisely those subspaces that are not connected.

Very probably, there is an additional reason that Carathéodory invented a new term for open set rather than using Hausdorff's term "Gebiet." And this is that in his book Carathéodory reserved the term "Gebiet" for a set that was both open and connected. (Was Carathéodory aware of Lebesgue's use of the term "open set" more than a decade earlier? This is uncertain.)

Tietze wished to find a set of axioms for a neighborhood space that were expressed with the concept of open set, rather than neighborhood, as the primitive idea. He did so by staying as close to Hausdorff's axioms (A)–(D) as possible. Thus Tietze defined what we will call an O-space as a set E whose members are said to be points, subject to the four conditions

(A^o) Every point x belongs to at least one open set.
(B^o) If U and V are open sets with at least one point in common, then $U \cap V$ is an open set.
(C^o) If every point x of A is a member of some open subset of A, then A is open.
(D^o) If x and y are distinct points then there is an open set U containing x and an open set V containing y such that U and V are disjoint. [1923, 294]

Tietze observed that in a neighborhood space the set of all open sets (i.e., those subsets of the space containing only interior points) is an O-space. Conversely, if in an O-space all the open sets containing a point x are regarded as neighborhoods of x, then the family of all sets that are neighborhoods of some point of the space is a neighborhood space.

The primary aim of Tietze's article was to formulate and investigate separation axioms that were stronger than Hausdorff's axiom (D). Since the book of Aleksandrov and Hopf [1935], Hausdorff's axiom (D) has been known as T_2, while Tietze's three stronger separation axioms are now generally known as T_3, T_4, and T_5. These four separation axioms state the existence of two disjoint open sets containing, respectively, (T_2) two distinct points, (T_3) a point and a nonempty closed set not containing it, (T_4) two disjoint nonempty closed sets, and (T_5) two disjoint nonempty sets neither of which contains a limit point of the other [1923, 301].

Two years later, the Russian topologist Pavel Aleksandrov, published in the same journal an article on topology in Euclidean spaces. There he gave a definition of Hausdorff's neighborhood spaces in terms of open set (for which Aleksandrov used the term "Gebiet"), a definition that was much simpler than Tietze's:

(1) the intersection of two open sets is open, and the union of any set of open sets is open;
(2) any two distinct points are contained in disjoint open sets [1925, 298].

(If Aleksandrov's second axiom was dropped, his first axiom would not yield precisely the topological spaces, since then there might be no open sets, or the only open set might be the empty set and not the whole space.)

12. Topological spaces: Sierpiński and Kuratowski

A year after Aleksandrov's article, and also in *Mathematische Annalen*, the Polish topologist Waclaw Sierpiński published a detailed analysis of how the idea of "derived set" A' of A might be used as a basis for topology, placing his work in the context of all that done in the previous twenty years:

> Since Fréchet's thesis we have seen various attempts to base topology (analysis situs) on one or another primitive concept, e.g. on that of limit (Fréchet), accumulation point (F. Riesz), distance or neighborhood (Hausdorff, Fréchet), and closure (Kuratowski). The purpose of this article is to show how topology could be developed by taking as primitive the concept of derived set. [1926, 321]

For a given set E, this operation of "derived set" was assumed to be a function from the set of all subsets of E to the set of all subsets of E. Sierpiński's first goal was to show that quite a bit of topology could be obtain merely by letting this operation be an arbitrary such function. By using such an arbitrary operation, he could define for any subset of E what it meant to be a closed set, a set dense-in-itself, a perfect set, a connected set, as well as what it meant for a function $f: E \to E$ to be continuous at a point or to be a homeomorphism. He was able to prove that the continuous image of a connected set is connected and that being dense-in-itself is a topological invariant. Then he considered what additional consequences would follow if this arbitrary operation of "derived set" were assumed to be "monotonic"; i.e., if $A \subseteq B \subseteq E$, then $A' \subseteq B' \subseteq E$. With this assumption it followed that the intersection of any family of closed sets is closed, that the union of any nonempty family of sets, each dense-in-itself, is dense-in-itself, that the closure of a connected set is connected, and finally that the union of a family of connected sets, each pair of which have a element in common, is connected [1926, 321–334].

In the final sections of his article Sierpiński took a different approach and let the notion of "closed set" be the sole primitive idea. He then considered an arbitrary nonempty set E and an arbitrary set F of subsets of E, subject only to the following two axioms:

(1) The set E is closed.
(2) The intersection of any set of closed sets is closed.

Thus members of F were to be designated as "closed." He called the set E together with such a set F of subsets an F-space. Then he proved that a set E is an F-space if and only if E has a monotonic operation of derived set (where the derived set of A is taken to be the set of all elements p of E such that every closed set including $A - \{p\}$ contains p). Likewise he gave a set of axioms for the closure of a set such that E satisfied these axioms if and only if E was an F-space (where a set was taken to be closed if it was equal to its closure).

Sierpiński's F-spaces (and their equivalents in terms of derived set or of closure) were a broader class than Hausdorff's neighborhood spaces and than Kuratowski's closure spaces. That is, every Hausdorff neighborhood space was a Kuratowski closure space (i.e., a topological space in today's parlance), and every Kuratowski closure space was a Sierpiński F-space; but there were F-spaces which were not closure spaces, and there were closure spaces which were not neighborhood spaces. Sierpiński remarked that if one added to the axiom of monotonicity for derived sets the assumptions that

(1) $(A \cup B)' \subseteq A' \cup B'$ and
(2) $\{p\}'$ is empty for every point p,

then it followed that the union of two closed sets is closed, that Cantor's Theorem is true (i.e., if there is an infinite descending sequence of nonempty closed Fréchet-compact sets, then the intersection of this sequence is nonempty), and that the Borel Theorem is true (i.e., if a closed and Fréchet-compact set A is covered by a countable set S of open sets, then A is covered by a finite subset of S) [1926, 336]. If one added to these three axioms the assumption that every derived set is closed, then one obtained what Fréchet called H-spaces.

Soon afterward, Sierpiński published in Polish a volume entitled *Topologia ogólna* (*General Topology*) [1928], which was translated into English in Canada as his [1934]. In this book he considered himself to be following the lead of Fréchet (rather than of Hausdorff) and let the primitive concept be "open set." In his 1928 preface he wrote: "The axiomatic development based on the concept of an open set (as a basic concept) seemed to us simpler and more intuitive than other axiomatic treatments which will be mentioned" [1934, iii]. Sierpiński introduced a small number of axioms for open sets in his first chapter and then added one or more axioms in each of the six following chapters. The axioms with which he began, for an arbitrary set K, were these:

(1) The empty set is an open set.
(2) K is an open set.
(3) The union of any family of open subsets of K is open. [1934,1]

With these axioms he deduced the theorems that he had proved for F-spaces in [1926].

In Chapter 2 Sierpiński added two more axioms:

(4) If p and q are distinct elements of K, then there is an open set containing p but not containing q.
(5) The intersection of two open sets is an open set. [1934, 28]

His axioms (1)–(3) and (5) were those that eventually became the standard definition for a topological space. His axiom (4) was a separation axiom (now called T_1) that had been formulated by Fréchet [1921, 366] in terms of neighborhoods and was weaker than Hausdorff's separation axiom (D).

Sierpiński's next three axioms (in Chapters 3, 4, and 5, respectively) were a version of Hausdorff's second axiom of countability

(6) There is a countable set S of open sets $W_1, W_2, \ldots$ such that every open set is the union of some subset of S [1934, 40],

and then two separation axioms (T_2 and T_3, respectively), which enabled him to prove Urysohn's famous lemma. This lemma gave conditions under which, for any two disjoint closed sets A and B in a topological space, there is a continuous function taking the value 0 on A and the value 1 on B [Sierpiński, 1934, 53, 65, 70].

In his last two chapters Sierpiński took a different approach. Now he began from the concept of metric space, proving Urysohn's metrization theorem (which gave conditions under which a topological space is also a metric space) and discussing Baire's zero-dimensional space. Sierpiński's last axiom, added to those for metric spaces, was the assumption that every bounded set is Fréchet-compact. He showed that this assumption was equivalent to requiring the Bolzano–Weierstrass Theorem to hold: every infinite bounded set has at least one limit point [1934, 118]. This assumption enabled him to deduce deep results on the hierarchy of Borel sets in a metric space.

Shortly before this translation of Sierpiński's book was published, Kuratowski also authored a book on topology, written in French [1933]. In it he used as a basis his axioms from his 1922 article, with one addition and one deletion. The additional axiom stated that, for any point p in the space, $\{p\}$ is identical to its closure [1933, 15]. The deletion was his axiom (2) stating that, given any subset A of the space, A is a subset of the closure of A. For this axiom could now be proven from his other axioms together with his additional axiom. Later in the book he adopted a fourth axiom (Tietze's separation axiom T_4) and then a fifth (Hausdorff's axiom of second countability) [1933, 95, 100]. Thanks to Urysohn's metrization theorem, the spaces satisfying all five axioms were homeomorphic to separable metric spaces. The book ended with theorems on projective sets in complete, separable metric spaces.

Thus both Sierpiński's and Kuratowski's books ended with investigations of complete, separable metric spaces. The difference between the two books was reflected in Sierpiński's last axiom, which asserted that every infinite bounded set has at least one limit point. While this axiom was satisfied by Euclidean spaces if they were finite-dimensional, it ruled out certain infinite-dimensional metric spaces such as Hilbert space, even though they were complete and separable.

13. Topological spaces in real analysis and combinatorial topology

Intriguingly, topological spaces were soon used as a framework for real analysis by a mathematician whose contact with the Polish topologists was minimal. This was Hans Hahn of the University of Vienna, who authored a book [1932] on real functions and devoted its long second chapter to topological and metric spaces. In order to define a topological space, he took as his primitive concept "open set," and cited Tietze [1923]. But Hahn's axioms for open sets were different from Tietze's and included the three axioms of Aleksandrov [1925], discussed above. (Hahn's other two axioms were immediate consequences of Aleksandrov's, but he made no mention of Aleksandrov's article.) It is likely that Hahn had rediscovered Aleksandrov's axioms on his own.

It was less surprising that topological spaces would find uses in combinatorial topology. But this occurred rather slowly at first. Oswald Veblen's book in English about analysis situs [1922], like Solomon Lefschetz's book [1924] in French on the same subject, was concerned with n-cells in n-dimensional Euclidean space, and made no mention of topological spaces. Lefschetz took the continuity of a function to be understood. By contrast, Veblen defined a transformation f of X (a cell and its boundary) into Y (a cell and its boundary) to be continuous if, for any subset A of X and every limit point p of A, the point $f(p)$ is a limit point of the image of A [1922, 2]. (Veblen's definition has the odd consequence, which he apparently did not notice, that a constant function is not continuous.)

In a few years the situation had shifted in combinatorial topology. Lefschetz, in his English book *Topology* [1930], which superseded his earlier French book, began by considering Hausdorff's "neighborhood spaces" [1930, 4]:

> *Topology* or *Analysis Situs* is usually defined as the study of properties of spaces or their configurations invariant under continuous transformations. But what are *spaces* and their *continuous* transformations?
>
> Whatever a space may be it is difficult to reconcile it with anything conforming with our "spatial" intuition unless it is endowed with the following property ...: With each point there goes a portion of the space in which it is imbedded. [1930, 1]

These portions of a space R he called "neighborhoods," and at first he required of them only Hausdorff's first axiom: Every point p of R belongs to at least one neighborhood of p, and every neighborhood of p contains p. He observed that "it is surprising how far it is possible to go even on such a slender foundation as Axiom I" [1930, 2]. In particular, he used this axiom to define the concepts of interior point of a set, boundary point of a set, closed set, open set, closure of a set, limit point of a set, and dense subset. What was odd, however, is that he defined a transformation of such a space R into a space S to be continuous if and only the image of a set open in R was open in S [1930, 3]. This disagreed with Hausdorff's definition of continuous function, and once again had the consequence that a constant function is not continuous. But, like Veblen's odd definition above, Lefschetz's resulted in a definition of homeomorphism that is equivalent to the now standard definition.

Lefschetz mentioned Hausdorff's three other axioms only in passing, and concluded his discussion of neighborhood spaces [1930, 6] by Urysohn's metrization theorem that a second-countable Hausdorff space is metrizable if and only if it is satisfies Tietze's axiom T_3.

In 1942 Lefschetz returned to the subject with an enlarged textbook, *Algebraic Topology*, which began with a much more thorough treatment of topological spaces, based on the now standard axioms for open sets. Therefore the book took as its basis a more general kind of space than Hausdorff's neighborhood spaces [Lefschetz, 1942, 5–6]. What he had in 1930 called a "continuous transformation" he now, correctly, called an "open transformation" [1942, 7]. He next defined a transformation to be "continuous" if its inverse is an open transformation—in accord with Hausdorff's and the modern definition. Connectedness, compactness, homotopy, and deformations were all treated in the general context of an abstract topological space.

In Germany, what we now call topological spaces first made their appearance in a textbook of combinatorial topology with the *Lehrbuch der Topologie* by Herbert Seifert and William Threlfall [1934], who were acquainted with the books of Lefschetz [1930] and Kuratowski [1933]. Like Lefschetz, they began with a "neighborhood space" more general than Hausdorff's and satisfying only his first axiom.

But they immediately added a second axiom that was not true in general of Hausdorff's concept of neighborhood. Their second axiom states that, for any subset V of the space M, if U is a neighborhood of p and if U is a subset of V, then V is a neighborhood of p [1934, 21]. (In modern terminology, what Hausdorff called the set of neighborhoods of a space is now called a "neighborhood base," and the neighborhoods satisfy the second axiom of Seifert and

Threlfall.) In an endnote, they observed that this very general concept of neighborhood space is adequate for defining the fundamental topological concepts of open and closed sets, of continuous function and homeomorphism. But, they emphasized, in their book the concept of neighborhood space was merely a station on the way to the more important combinatorial concept of a "complex" [1934, 316]. Thus, in regard to topological spaces, the difference between Seifert and Threlfall's *Lehrbuch* and the earlier book of Lefschetz [1930] was minimal.

By contrast, the book *Topologie* by Aleksandrov and Hopf [1935] made topological spaces central to a vision of topology that included both abstract general topology and the more special combinatorial topology. Like Lefschetz [1930] and Seifert and Threlfall [1934], Aleksandrov and Hopf [1935] began with a concept of topological space that was more general than Hausdorff's neighborhood spaces and also more general than the topological spaces used today. Aleksandrov and Hopf were influenced by Kuratowski's [1922] axioms for the closure operation $\overline{A}$. But unlike Kuratowski, they required only that this operation assign to any subset A of the space some subset $\overline{A}$ of the space; it was not required that A have any particular relation to $\overline{A}$. They called such a space a "general-topological space," which they distinguished from a "topological space." In such a general-topological space they defined what it meant for a subset to be closed or open [1935, 26].

After discussing metric spaces and giving examples, they defined what they called a "topological space" as a general-topological space that also satisfied the axioms for closure that Kuratowski had introduced in his [1922] article. They then showed that any such topological space satisfied all of the axioms of Hausdorff's neighborhood spaces except possibly for the last separation axiom (which stated that any two distinct points are contained in disjoint neighborhoods) [1935, 37–43]. Before turning to the algebraic topology of complexes, they discussed in great detail all of the separation axioms T_2, T_3, T_4, and T_5, as well as the weaker separation axioms T_1 (due to Fréchet) and T_0 (still later, and due to Kolmogorov) [1935, 51–77]. This discussion culminated in Urysohn's metrization theorem (stated above). Finally, they recognized Fréchet-compact topological spaces as particularly important and devoted a chapter to them [1935, 83–122].

In England, the first book to discuss topology at length was *Elements of the Topology of Plane Sets of Points* [1939] by M.H.A. Newman. Essentially the entire book was formulated in terms of metric spaces. He briefly discussed what he called "abstract topological spaces," in which what he called the five fundamental topological concepts (derived set, closure of a set, closed set, open set, and convergent sequence) have no restrictions placed on them by axioms. By contrast, what he called "topological spaces" took one of the five concepts as basic, assumed some axioms about it, and then gave definitions of the other four concepts in terms of the basic one. This distinction between "abstract topological spaces" and "topological spaces" corresponds to what Aleksandrov and Hopf [1935] called "general-topological spaces" and "topological spaces." And Newman does cite their book. Newman discussed only one class of spaces more general than the metric spaces. These were what he called "convergence spaces," based on convergent sequences and Fréchet's L-spaces [1939, 48–49]. But in his endnotes he defined "topological spaces" using Kuratowski's axioms for closure [1939, 207].

In his book's second edition [1951], Newman continued to work in metric spaces, but defined his "topological spaces" in terms of the four now-standard axioms for open sets [1951, 204]. This changes appears to be due to the influence of Bourbaki's approach to topology, which Newman cites (see below).

By contrast, the book *Analytic Topology* by the American topologist Gordon Whyburn used neighborhoods to define a topological space. He employed Hausdorff's original axioms, except for replacing Hausdorff's T_2 axiom by the weaker T_1 axiom [Whyburn, 1942, 1–2]. But he then quickly added the assumptions that the space is T_3 and second-countable, which had the effect of only considering separable metric spaces.

14. Topological spaces based on open sets

As the group of French mathematicians collectively known as Nicolas Bourbaki was deciding how to treat general topology in the years 1935–1938, they began with a mix of concepts taken from Fréchet, Riesz, Weyl, Hausdorff, and Aleksandrov. To some extent, the Bourbakists hoped that the book of Aleksandrov and Hopf [1935] would settle the organizational question for them. The 1936 draft submitted to Bourbaki by André Weil began with metric spaces and insisted on the need to go on to more general spaces, formulated in terms of Kuratowski's closure operator, as Aleksandrov and Hopf had done [Beaulieu, 1989, 348]. By contrast, Claude Chevalley insisted on the need "to put general topological spaces before metric spaces, since the notion of a metric appears more and more ridiculous to me" [Chevalley in Beaulieu, 1989, 351]. Weil's draft presupposed that the topological space satisfied T_0, but devoted a lot of space to the equivalence of various sets of axioms in terms of open sets, closed sets, closure, and neighborhoods. Again, Chevalley expressed disagreement. He insisted on starting firmly with the axioms for open sets, and consigning all the others to an appendix. The group decided to opt for axioms in terms of open sets, which were utilized in defining continuity, rather than the closure axioms [Beaulieu, 1989, 353]. (This is mildly ironic, since continuity could also be formulated in terms of closure rather than open sets.) Weil's next draft later in 1936 used the axioms for open sets and did not assume any separation axiom—neither T_0 nor a stronger one.

In the first published edition [1940] of his chapter "Structures topologiques" Bourbaki used the concept of open set as the sole primitive idea, and as sole axioms a slight variant on the first of those that Aleksandrov had used in 1925: the intersection of any finite set of open sets is open, and the union of any set of open sets is open [Bourbaki, 1940, 1]. He then noted that from these two axioms it follows that the empty set is open (as the union of the empty family of sets) and the whole space is open (as the intersection of the empty family of sets; this requires a particular convention as to how the intersection of the empty family of sets is to be understood) [1940, 1–2]. Weyl had already proposed this convention in his first 1936 draft. In the second edition of this chapter [Bourbaki, 1951] and its later publication in a complete book, these axioms for open sets were unchanged.

However, after introducing filters as a form of convergence more general than sequences, Bourbaki assumed Hausdorff's axiom T_2. This was because the axiom T_2 is equivalent, in a topological space, to the proposition that a filter cannot have more than one limit point—a proposition on which Bourbaki insisted [1940, 32; 1951, 47].

In the United States the most influential topology textbook for several decades (beginning in 1955) was undoubtedly John L. Kelley's *General Topology*, which, he wrote, "I have, with difficulty, been prevented by my friends from labeling it: What Every Young Analyst Should Know" [1955, v]. In particular, Kelley based his treatment on the modern concept of a topological space (with no separation axiom presupposed), which takes the concept of open set as the sole primitive idea.

Kelley was familiar with Bourbaki's work and adopted precisely Bourbaki's two axioms for open sets [1955, 37]. Later treatments of topology would usually spell out explicitly the two further axioms that the empty set and the whole space are open. These four axioms for a topological space, expressed using open sets alone, then became standard. Many textbooks on general topology appeared in the later decades of the twentieth century, and they all used these same four axioms. As far as general topology was concerned, the competition for which concept was most fundamental ("survival of the fittest") had ended with the modern definition of a topological space based on open sets. The still more general spaces considered by Sierpiński, Aleksandrov, and Hopf fell by the wayside as historical curiosities.

15. Coda: Weierstrass, analytic functions, and open sets

When in 1914 Hausdorff defined the concept of "Gebiet," or open set, in a neighborhood space, he added an intriguing footnote:

> This does not coincide with the usual terminological usage, introduced by Weierstrass, according to which a Gebiet is also required to be connected (§7). For us, a *Weierstrassian* Gebiet is a connected Gebiet, and according to the usual terminology our Gebiet would in general be a set of Gebiets or a union of Gebiets. However, the concept of a set without boundary points is so fundamental that it definitely deserves its own noun. [1914, 215]

It is unclear why Hausdorff associated Weierstrass's use of the word "Gebiet" with an open, connected set, since Hausdorff does not cite any article by Weierstrass with this usage.

There is, however, ample evidence that Weierstrass used a different word to describe what is now called an open, connected set in n-dimensional Euclidean space. For in his article "Zur Functionenlehre" Weierstrass used the word "Continuum" with this meaning [1880, 721].[16] Gösta Mittag-Leffler [1884, 2] explicitly adopted this usage of "continuum" from him. Moreover, in Weierstrass's 1886 lecture course on analytic functions he used the word "Continuum" in this sense [1886/1988, 65]. R. Siegmund-Schultze, who annotated the text of that course, wrote the following about

[16] There was a brief period, evident in his 1878 lectures, in which he used "Continuum" in the sense of an open set which need not be connected [1878/1988, 83]. But even there he was primarily interested in the connected components of this open set. By 1880, his usage of "Continuum" for connected, open set is fixed and continues for the rest of his career.

Weierstrass's usage: "By a 'continuum' Weierstrass understands in the following an open and connected point-set in $\Re^n$, or what corresponds to a 'Gebiet' in today's usage" [Weierstrass, 1886/1988, 240]. Finally, in the 1901 article on analytic functions of a complex variable in the German *Encyklopädie der mathematischen Wissenschaften*, W.F. Osgood defined a "Kontinuum" as an open and connected set in the complex plane, remarking that this was in accord with Weierstrass's usage [1901, 9]. As late as the 1928 edition of his textbook on complex analysis, Osgood called a set in the plane a "two-dimensional continuum" if it was path-connected and all of its points were interior points [1928, 162].

Weierstrass's usage of "Continuum" makes sense when we understand how he defined a complex analytic function. He considered an "Element" of such a function to be a power series in $z - a$, where this power series converged in the interior of some circle with center a. Then he considered a circle overlapping with this one but having a different center b such that a power series in $z - b$ converged in this circle and agreed in value with the power series in $z - a$ wherever the interiors of the two circles overlapped [1868/1986, 72].

Weierstrass also used the term "Gebiet" with the meaning of the domain of a function, this time for an unbounded real variable. All the values of the function that could be reached in the interior of such a sequence of overlapping intervals he called a "Continuum," and proceeded analogously in n-dimensional space using overlapping box neighborhoods instead of circles or intervals. He noted that it is easy to transfer this process to a complex variable [1878/1988, 83].

Nevertheless, it is clear that over time the usage changed so that the German word for an open and connected set (in complex analysis) became "Gebiet." Our conjecture is that Cantor's usage of "Continuum" as a connected, perfect set eventually displaced Weierstrass's usage of the same word within complex analysis, and the term "Gebiet" in German (and "region" in English) was substituted for "Continuum" with Weierstrass's meaning of connected, open set. This tension between Weierstrass's meaning of continuum (open and connected set) and Cantor's meaning of continuum (perfect and connected set) is apparent in the letters that Cantor and Mittag-Leffler exchanged in 1883, where Mittag-Leffler inclined toward Weierstrass's usage. But in his letter of 11 March 1883 to Cantor, Mittag-Leffler explicitly adopted Cantor's definition of continuum.

A half-century later, this new usage was well established among German-speaking analysts. Their new usage probably originated with Carathéodory in his influential book *Vorlesungen über reelle Funktionen*, where he defined an open connected point-set as a "Gebiet" and a closed connected point-set containing more than one point as a "Kontinuum" [1918, 208, 222]. Interestingly, in the second edition of his book on set theory Hausdorff had changed his own terminology to agree (almost) with Carathéodory's, whose book he cited. For Hausdorff, an open connected point-set was now a "Gebiet" and a closed, connected point-set was a "Kontinuum" [1927, 150–151]. And his book *Reelle Funktionen* the Austrian analyst Hahn defined an open, connected set as a "Gebiet," while he defined a compact, connected set as a "Kontinuum" [1932, 102]. (Hahn's change in the definition of continuum, from being closed to being compact, would later become standard in topology.)

We can get some sense of how these matters were viewed in France by an article which Arnaud Denjoy published in 1910, in which he quoted two words in German: "Let us call a *continuum* (Gebiet) a set which only contains interior points and which is connected, and let us call a *domain* (Bereich) the sum of a continuum and its boundary" [1910, 138]. Denjoy's use of continuum agrees with that of Weierstrass and the early Mittag-Leffler. In this same period Borel defined a "domaine de Weierstass" as an open, connected set in the complex plane [1917, 10]. Somewhat earlier, Jordan used the term "domaine" for a closed set having a nonempty interior [1893, 22].

In the United States the term "region" is now in common use in complex analysis for an open, connected set. It is used this way, for example, by the Harvard analyst Lars Ahlfors [1979, 57] and the Berkeley analyst J.E. Marsden [Marsden and Hoffman, 1987, 70]. However, the usage of "region" in English to mean an open, connected set is by no means universal. Thus Einar Hille of Yale University, in his book *Analytic Function Theory*, defined a "region" as an open set together with a subset (possibly empty) of its boundary, and defined a "domain" as a path-connected open set: "The reader is warned that many authors use the term 'region' for what we call a domain; others make no distinction between the two terms" [1959, 29].

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