Available online at www.sciencedirect.com
SCIENCE DIRECT®

Historia Mathematica 33 (2006) 113–131

HISTORIA MATHEMATICA

www.elsevier.com/locate/yhmat

Beyond Cartesian limits: Leibniz's passage from algebraic to "transcendental" mathematics

Eberhard Knobloch *

Institut für Philosophie, Wissenschaftstheorie, Wissenschafts- und Technikgeschichte, Ernst-Reuter-Platz 7, 10587 Berlin, Germany

Available online 9 April 2004

Abstract

This article deals with Leibniz's reception of Descartes' "geometry." Leibnizian mathematics was based on five fundamental notions: calculus, characteristic, art of invention, method, and freedom. On the basis of methodological considerations Leibniz criticized Descartes' restriction of geometry to objects that could be given in terms of algebraic (i.e., finite) equations: "Descartes's mind was the limit of science." The failure of algebra to solve equations of higher degree led Leibniz to develop linear algebra, and the failure of algebra to deal with transcendental problems led him to conceive of a science of the infinite. Hence Leibniz reconstructed the mathematical corpus, created new (transcendental) notions, and redefined known notions (equality, exactness, construction), thus establishing "a veritable complement of algebra for the transcendentals": infinite equations, i.e., infinite series, became inestimable tools of mathematical research.

Zusammenfassung

Der Aufsatz behandelt Leibniz' Aufnahme von Descartes' „Geometrie". Die Leibnizsche Mathematik war auf fünf grundlegenden Begriffen aufgebaut: Kalkül, Charakteristik, Erfindungskunst, Methode, Freiheit. Leibniz' methodologische Betrachtungen zogen seine Kritik der cartesischen algebraischen Methoden nach sich, die das Gebiet der Geometrie definierten: „Descartes' Geist war die Grenze der Wissenschaft". Die Unvollkommenheit der Algebra (Lösung algebraischer Gleichungen höheren Grades) ließ Leibniz lineare Algebra entwickeln und eine Wissenschaft des Unendlichen entwerfen. Leibniz baute also das Gebäude der Mathematik neu auf, schuf neue Begriffe (transzendent) und definierte bekannte Begriffe neu (Gleichheit, Genauigkeit, Konstruktion). Auf diese Weise begründete er eine „wahre Ergänzung der Algebra für transzendente Größen": unendliche Gleichungen, das heißt unendliche Reihen, wurden unschätzbare Hilfsmittel der mathematischen Forschung.

* Fax: +49 30 3142 5962.
E-mail address: eberhard.knobloch@tu-berlin.de.

0315-0860/$ – see front matter
doi:10.1016/j.hm.2004.02.001

MSC: 01A45; 01A50; 15-30; 51-03

Keywords: Descartes; Leibniz; Linear algebra; Combinatorics; Geometry; Construction; Infinite series

---

Introduction: sources

Since 1976, seven volumes have appeared, comprising about three thousand printed pages of Leibnizian mathematical studies. They partly, mainly, or exclusively deal with algebra or related topics:

[Leibniz, 1976a]: combinatorics (340 pp.),
[Leibniz, 1976b]: arithmetic, algebra (200 pp.),
[Knobloch, 1980]: determinant theory (330 pp.),
[Leibniz, 1990]: geometry, number theory, algebra (954 pp.),
[Leibniz, 1993]: infinite series (160 pp.),
[Leibniz, 1996]: algebra (870 pp.),
[Leibniz, 2003]: infinite series and sequences (880 pp.).

These writings support by documentary evidence Leibniz's overwhelming interest in algebra and its relation to geometry.

The present article is based on these new, available sources.[1] It tries to show in what ways Leibniz's contributions to linear algebra and his understanding of Descartes' *Geometry* were motivated by imperfections in algebra and were profoundly influenced by Leibniz's new conception of mathematics.

1. From the theory of equations to linear algebra

In 1924, the French poet Paul Valéry praised the algebraic use of unknowns in the following way:

> Quelle idée plus digne de l'homme que d'avoir nommé ce qu'il ne sait point ? Je pus engager ce que j'ignore dans les constructions de mon esprit, et faire d'une chose inconnue une pièce de la machine de ma pensée. [Valéry, 1999]

Yet, in Leibniz's eyes algebra suffered from two imperfections:

(1) The algorithmic solution of the general algebraic equation of nth degree was still unavailable;
(2) Algebraic equations did not suffice to comprehend transcendental problems in geometry.

Like all of his contemporaries, Leibniz was convinced of the solvability of the first problem. His own attempts in this direction resulted in the emergence of determinant theory. His studies of transcendental

---

[1] The authors of the monograph *The Beginning and Evolution of Algebra* [Bashmakova and Smirnova, 2000] did not take notice of these volumes. Their only reference to Leibniz concerns his well-known letter to l'Hospital dating from 1693 and published in 1850 [Leibniz, 1850, pp. 236–241], in which Leibniz derived the solution of a system of three linear equations. In this respect, the monograph represents the state of affairs of 1850.

problems, such as the quadrature of conic sections, resulted in the invention of his differential and integral calculus.

In order to solve the quintic equation Leibniz generalized Cardano's approach by using a substitution of the form

$$x = a_1 + a_2 + a_3 + a_4.$$

His calculations resulted in essential results in the theory of symmetric functions and additive number theory [Knobloch, 1973] and led Leibniz to believe that the problem could be reduced to the solution of systems of linear equations. In May 1678 he wrote:

> I shall demonstrate that the labor of calculating is not difficult, because what is important is the fact that the quantities looked for are not multiplied by each other or by themselves. Hence every calculation arising from the equations used for elimination (destructitiae) is done exclusively by addition and subtraction of arbitrary quantities having only signs and known (sufficiently simple) numerical coefficients. This is neither laborious, nor difficult, nor prolix.
>
> Indeed, these tiny equations used for elimination cannot lead to confusion because they do not ascend either to rectangles or to powers, even if their number is large. For example, in the case of the [equation of the] fifth degree there are at most 284 equations serving for elimination or—if one uses an abbreviation—about 160 such equations. Yet, they are written down without any calculation and afterward the calculation derived from them is carried out by addition and subtraction alone.
>
> Hence to carry out the elimination calculation is no more difficult than to diligently write down 160 small lines, that is, the values of the unknowns. The value of every unknown can be written down at once without any calculation by the estimation of a glance.
>
> The writing down of the equations used for elimination does not require a great deal of attention. If somebody should follow the method prescribed by me, a mere description would do. In order to write down the values deduced from the equations no calculation is necessary. What is needed is attention in substituting. For that reason an attentive and industrious person who is not distracted could carry out the whole calculation for an equation of the fifth degree, within, I think, the space of one day, provided that everything has been rightly prepared. [Leibniz, 1976a, p. 113]

This long citation should make it clear that in Leibniz's opinion, progress in algebra depended on combinatorics [Knobloch, 1974], that algebra was subordinated to combinatorics. As he remarked in his decisive treatise on systems of linear equations dating from January 1684 and published in 1972:

> In this attempt I solved the problem, whereas earlier I always got stuck at some point. What is done here is an eminent example of the combinatorial art. [Knobloch, 1972, p. 167][2]

---

[2] Leibniz obtained essential results in three areas of determinant theory: inhomogeneous systems of linear equations, resultants of two polynomials, and elimination of a common variable from algebraic equations. These results are described in Knobloch [2001].

2. Descartes vs Leibniz: the realm of geometry

What can be called the realm of geometry? Leibniz developed a conception which differed decisively from that of Descartes. He did this in the course of a silent, yet enormously creative conversation with the French mathematician.

Descartes had explained:

> Aptius quidquam afferre nescio, quam ut dicam, quod puncta omnia illarum, quae *Geometricae* appellari possunt, hoc est, quae sub *mensuram* aliquam *certam* et *exactam* cadunt, necessario ad puncta omnia lineae rectae, *certam* quandam *relationem* habeant, quae per *aequationem* aliquam, omnia puncta respicientem, *exprimi* possit.
>
> [The most appropriate thing I can say is that all points of those curved lines which are usually called *geometric*, that is those which are subject to any *determined* and *exact measure*, necessarily bear a certain *determined* relation to all points of a straight line which can be *expressed* by some *equation* regarding all points.] [Descartes, 1659–1661, vol. I, p. 21, emphasis added]

Leibniz thoroughly discussed this Cartesian characterization of geometrical lines in October 1674 [Leibniz, 2003, No. 38_{12}] in his hitherto unpublished extraordinarily interesting and long inquiry *De serierum summis et de quadraturis plagulae quindecim* (15 sheets on sums of series and quadratures) [Leibniz, 2003, No. 38].

First of all, Leibniz accepted Descartes' axiom:

exactness implies geometricity.

In his historical study of geometrical exactness, Bos [2001, p. 405] observes that the dissatisfaction with the Cartesian interpretation of geometrical exactness was caused by its exclusion of nonalgebraic curves. In his 1674 inquiry Leibniz still excluded helical curves (space curves on a cylinder), on the grounds that spirals cannot be exactly described because they depend on two motions being independent from one another. Human beings are not able to give their determined proportion (certa proportio). Such spirals can only be described by a divine art, by means of an intelligence whose distinct thoughts are realized in time intervals which are smaller than any arbitrarily given time. This, he thought, did not apply even to angels.

Yet a family of curves, the trochoids, are described by a continuous motion and thus are exactly described. Hence Descartes was obligated to explain in what respect their description is not exact. While he was right in blaming the ancients for having confused the conchoids and cissoids with helical curves and spirals respectively, he must in turn be blamed for having confused trochoids and evolutes with spirals and helical curves respectively. Leibniz disputed Descartes' notion of exactness [Bos, 2001, p. 336].

Leibniz considered a parabola AMG′ unrolling on the tangent AB. A point F of the axis of the parabola is describing the parabolic trochoid FG′ [Leibniz, 2003, No. 38_{12}] (see Fig. 1). In [Leibniz, 2003, No. 38_{13}] he studied the "parabolic trochoid here invented for the first time" (trochois parabolica hic primum inventa). It is described by the focus B of the generating parabola (parabola genitrix) unrolling on the tangent FTM (see Fig. 2). He was still unaware that the described line is the catenary.

Leibniz carried on a dialogue with a fictitious interlocutor:

> But you will say, then we have realized the quadrature of the circle. I deny that we have a quadrature as desired. You say: 'By means of the circular trochoid, a moving curve, we have exactly a straight line which is equal to the circumference of the circle.' 'Yes,' Leibniz answered, 'if exactly, then certainly also geometrically. But not only a geometrical but also an analytical quadrature of the circle is required.'

This is the crucial point: Descartes restricted geometrical lines to analytical lines, thus depriving science of a necessary aid (scientiam auxilio necessario privat). Leibniz believed that Descartes did this for a secret reason: he wanted to be able to boast of having given a method for obtaining all geometrical curves and their tangents.

Leibniz disagreed emphatically. Such nonanalytical curves are no less geometrical just because they are exact. Also, they have excellent uses. Examples of useful curves are Huygens's cycloid and his own parabolic trochoid. He explicitly said [Leibniz, 2003, No. 38_{13}]: "We must admit that nonanalytical lines are necessary in geometry. For there are problems that can be solved, that is, by calculation without the

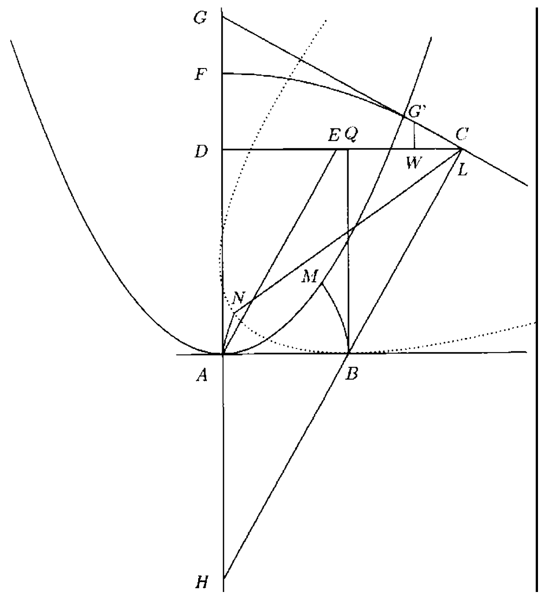


Fig. 1. Parabolic trochoid 1 [Leibniz, 2003, 489].

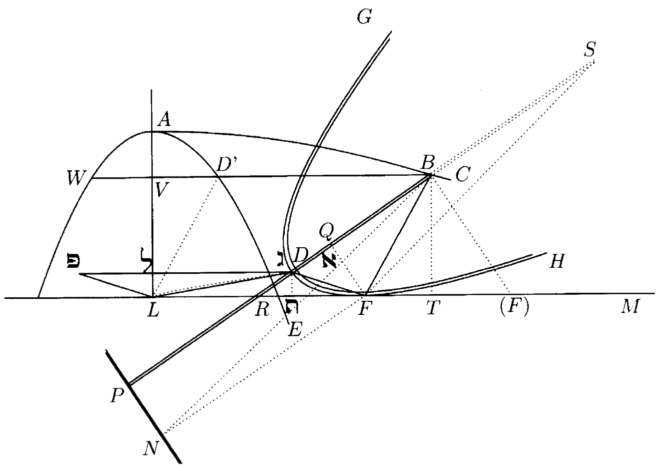
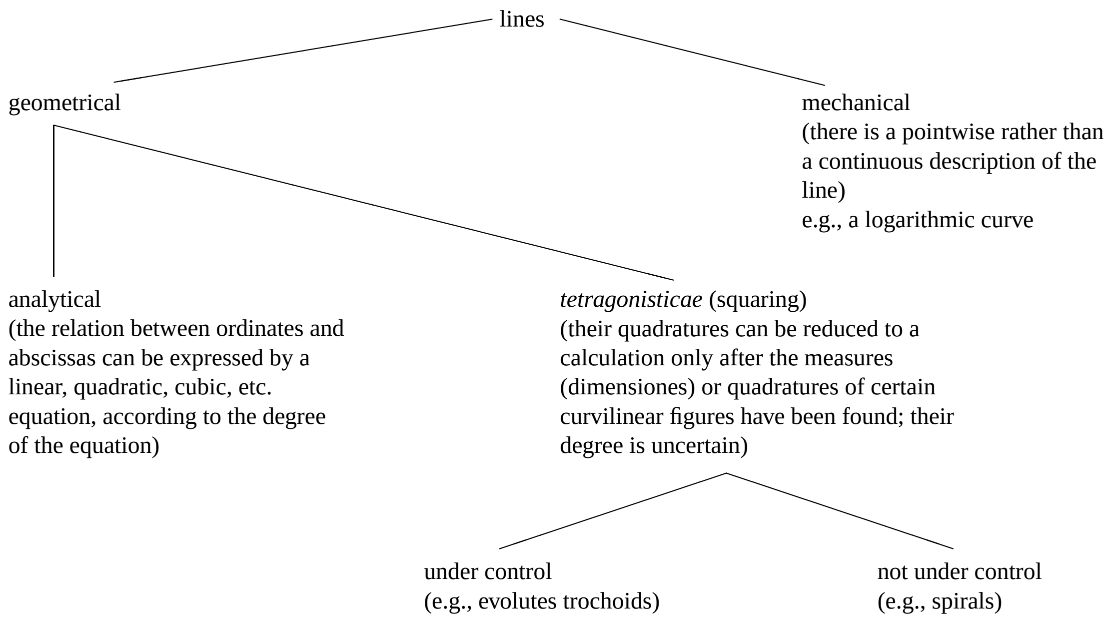


Fig. 2. Parabolic trochoid 2 [Leibniz, 2003, 497].

existence of a geometrical construction by analytical lines; for example, consider the problem of dividing a ratio into a ratio of two irrational numbers. Such problems can be solved only by means of nonanalytical lines.” By “dividing a ratio into a ratio of two irrational numbers” Leibniz and the mathematicians of his time meant the insertion of an irrational mean proportional between two given numbers a, b. The logarithmic curve provided the general solution of this problem. In other words, according to Leibniz, exactness is indispensable, but analyticity is not. What counts is the solvability of problems. For Leibniz, solvability was a central notion of mathematics.[3]

For Leibniz the boundary between geometric and nongeometric lines is not fixed once and for all. It can happen that a nongeometrical line becomes geometrical when a way of describing it is found (for example, the logarithmic curve) and that a nonanalytical line becomes analytical (for example, the trochoid of the quadratocubic parabola $y^2 = x^3$). Only after Hendrik van Heuraet had found its measure (dimensio) did it become analytical. (Leibniz was referring to Heuraet’s *Letter on the transformation of curved lines into right lines* [Heuraet, 1659].)

In other words, Descartes adhered to a mathematically fixed, closed, static realm of geometry, while Leibniz adhered to a mathematically changing, open, dynamical realm of geometry in which the classification of lines depends on our current knowledge. The following diagram describes Leibniz’s classification of lines as of October 1674:



It is worth mentioning that spirals are, so to speak, candidates for the realm of geometrical lines.

Just one year later, Leibniz seemed to have changed this classification regarding the logarithmic curve. In April/May 1675 he wrote his *Geometrical contemplation on the diminution of motion which represents*

[3] Such an approach reminds one of Hilbert’s conviction of the solvability of every mathematical problem, of his saying that in mathematics there is no *ignorabimus* [Hilbert, 1900, p. 1102].

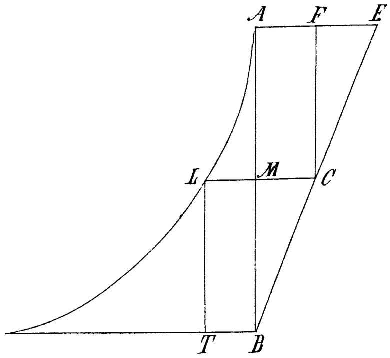


Fig. 3. “Physical” construction of logarithms [Leibniz, 1689, Fig. 15].

the logarithms by a wonderful artifice of nature (De detrimento motus contemplatio geometrica quae mirabili naturae ingenio repraesentat logarithmos) [Rivaud, 1914–1924, No. 946]. He published it only in 1689 [Leibniz, 1689]. His method consists in “a physical way of constructing logarithms which cannot be exactly constructed by the common geometry” (see Fig. 3).

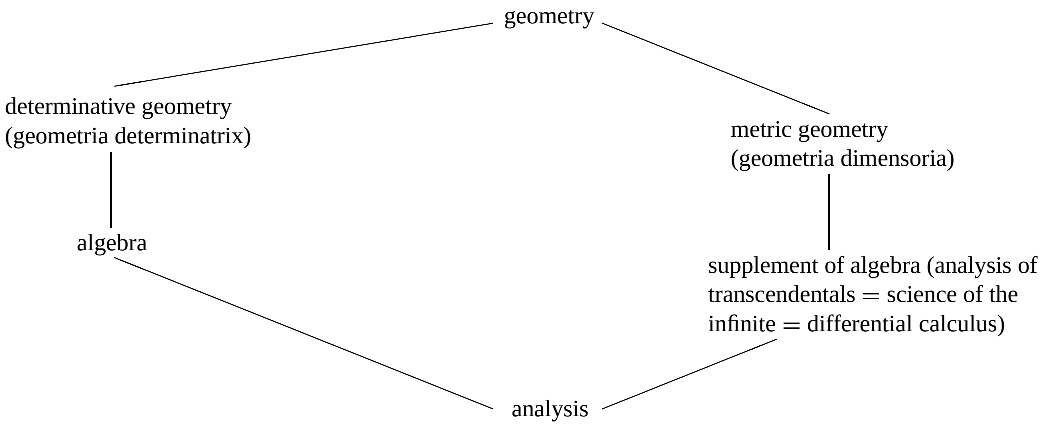
A body undergoes a uniform motion, which is retarded in proportion to the distance traversed. Let AE be the initial velocity at the beginning, AB the distance to be traversed. Let AM be the distance traversed, MB the distance still to be traversed, MC the remaining velocity, and FE the velocity which has been lost. If the remaining distances MB = LT are to each other like numbers, then the spent times ML = TB are to each other like their logarithms. For if the elements of the distance belong to a geometrical sequence, then the remaining distances belong to the same geometrical sequence, and consequently also the remaining velocities. Hence the increments of time are equal, the times belong to a corresponding arithmetical sequence, that is, they form the logarithms.

In 1693 Leibniz alluded to this “construction which has admixed something physical” in his *Supplement to the measuring geometry or the most general realization of all quadratures by means of a motion*; *and in a similar way a multiple construction of a line according to a given tangent condition* [Leibniz, 1693]. In this paper he described the gradually growing realm of geometry:

> In order to construct transcendental quantities an application or adaptation of curves to straight lines has hitherto been used, as occurs in the description of the cycloid or in the case of the unwinding of a thread, or leaf, tied up with a line or surface. ... Should someone wish to describe geometrically the spiral of Archimedes or the quadratrix of the ancients, that is, by means of an exact, continuous motion, he will easily do that by a certain adaptation of a straight line to a curve, such that the rectilinear motion is adapted to the circular. [Leibniz, 1693, 295]

We see that while Descartes excluded these curves from geometry Leibniz insisted on including them. His reasons are that lines described in such a way are exact, have very useful properties, and are applicable to transcendental quantities. Exactness of the method of construction is the decisive criterion for inclusion within geometry. If a construction is easy and useful, then it becomes part of practice.

In 1693, at the latest, Leibniz had in mind a bipartite geometry corresponding to a bipartite analysis:



3. Descartes and Leibniz: differences and similarities

While Descartes' and Leibniz's mathematical programs differed fundamentally, their style of mathematical writing reveals some striking similarities. Descartes had written that

> cum ratio, quae inter rectas et curvas existit, non cognita sit, nec etiam ab hominibus (ut arbitror) cognosci queat ; nihilque inde, quod exactum atque certum est, concludere possimus. [Descartes, 1659–1661, vol. I, p. 39]
>
> since the ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, no conclusion based upon such ratios can be accepted as rigorous and exact. [Descartes, 1954, p. 91]

The contributions of mathematicians like Christopher Wren, William Neile, Pierre de Fermat, and James Gregory to rectifications of curves are well known [Kline, 1972, p. 354f.]. They obtained their results before Leibniz began his mathematical studies in Paris. To some extent, their work rendered Descartes' invalid. Nevertheless, the Cartesian demand for exactness remained a challenge. Leibniz took this demand seriously when he wrote about his arithmetical quadrature of the circle by means of an infinite series:

> Hactenus appropinquationes tantum proditae sunt verus autem valor nemini quod sciam visus nec a quoquam aequatione exacta comprehensus est, quam hoc loco damus, licet infinitam, satis tamen cognitam, quoniam simplicissima progressione constantem uno velut ictu mens pervadit. [Leibniz, 1993, p. 8]
>
> Up to now, only approximations have been produced. But to my knowledge the true value has not been envisaged by anybody, and has not been expressed by anybody by means of an exact equation such as we are giving here. This is, admittedly, an infinite equation, but one that is readily perceived in view of its easy law of formation; the mind rushes through it, so to speak by a single stroke.

Leibniz consciously took up the Cartesian expressions *exacta* and *cognita*. We will return to this issue in Section 6. What is important here is the fact that Leibniz denied Descartes' assertion, while acknowledging Descartes' own criteria and using his own terminology. Later, he repeatedly criticized Descartes for having made his own mind the limit of geometry and of science [Knobloch, 1999, p. 221].

Yet both mathematicians believed that they had initiated a broad field of research without, however, having explicitly set down everything that there was to do. Descartes finished his Geometry by saying:

> ... it is only necessary to follow the same general method to construct all problems, more and more complex, ad infinitum; for in the case of a mathematical progression, whenever the first two or three terms are given, it is easy to find the rest.
>
> I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery. [Descartes, 1659–1661, vol. I, p. 106; Descartes, 1954, p. 290]

When Leibniz illustrated earlier the use of logarithms in the above cited treatise, he interrupted his explanations by saying:

> Their use could also be shown in the solution of equations and in many other questions. But whoever will have understood that, will easily notice what large field of invention lies open. And I prefer to leave it to others in what they might successfully exercise themselves than to try in vain by an obscure diligence that I seem to have said everything. [Leibniz, 1993, p. 125]

4. Leibniz's mathematics: aims, notions, terminology, objects, operations, problems

We present some general conclusions regarding Leibniz's mathematics, in order to better understand his mathematical approach and his criticism of Descartes. Leibniz's main aims were fruitfulness, certainty, conclusiveness, and universality. He explicitly adhered to them in his treatise on the arithmetical quadrature of conic sections [Leibniz, 1993]. For that reason, he always set the highest value on five key notions: calculus, characteristic, art of invention, method, and freedom. (I have shown elsewhere how these notions guided his mathematical thinking [Knobloch, 1999].)

A calculus implies universality and fruitfulness. Something should always be calculated in such a way that uniformity and justice (iustitia) or symmetry are preserved. A calculus might result in a powerful algorithm, as is the case for the differential calculus. The "characteristic art" (ars characteristica) is the art of creating and applying suitable signs. A characteristic makes it possible to examine calculations without additional computing, reveals laws of formations, and makes possible new insights. Leibniz's determinant theory and his differential calculus are excellent examples in order to illustrate these implications of calculus and characteristic. The art of invention expresses the dynamical principle of increasing human knowledge, a principle to which Leibniz subordinated all his mathematical efforts. It is evident that he adhered to an open, dynamical conception of geometry. Methods should comply with all four conditions, that is to say, fruitfulness, certainty, conclusiveness, and universality. Hence he always esteemed methods more than special solutions of particular problems. His famous first publication of his differential calculus was entitled "New Method" (Nova methodus). Yet, in spite of his preference for universal methods he knew very well that special procedures permitted abbreviations and simplifications. Finally, freedom of thought was a necessary presupposition for the art of invention and the introduction of new methods. Leibniz valued intellectual liberty (as Copernicus did before him or Georg Cantor after) and invoked this liberty when he treated curves as lines in the form of infinitangular polygons.

Leibniz's basic objection to the geometry of Grégoire de St. Vincent, Guldin, Cavalieri, and especially Descartes, was rooted in his conviction that their geometry was confined within certain limits (certi limites) [Leibniz, 1686, 232; Sasaki, 2003, 276]. In his eyes Descartes' method lacked fruitfulness, universality, and freedom of thought. For this reason he reinvented the mathematical corpus, creating a new terminology, new objects, and new operations, thus inevitably encountering new problems. His new terminology concerned new meanings of old notions or new categories created by him. The first category included

algebra, which to him was subordinated to the combinatorial art (Section 1),
construction, which became equivalent with "general quadrature" (Section 6),
equal, which became equivalent with "the difference is smaller than any given quantity" [Leibniz, 1993],
geometrical, which became equivalent with "exact" (Section 2).

The second category concerned the term *transcendental*. He used this expression from the autumn of 1673 [Leibniz, 2003, No. 23] and applied it to curves, figures, problems, and equations (Section 5). His new objects included

infinitely small or large quantities (smaller or larger than any given quantity) [Leibniz, 1993],
infinite equations, subject to all algebraic operations (Section 5),
determinants (Leibniz spoke about "aequationes resultantes" without using the term "determinant") (Section 1),
indeterminate equations, such as $x^x - x = 24$ [Leibniz, 1987, 844, 846, 889, 898, 903].

In April/May 1673 he spoke of "new algebraic arts" (novae artes algebraicae) [Leibniz, 2003, No. 17_1] emerging from the study of roots of binomials. Among the countless new problems envisaged by Leibniz, infinite equations played a fundamental role. The main question was how to compute the sums of infinitely many terms, a subject to which we turn in the next section.

5. The infinite and infinite series

From the very beginning of his mathematical studies in Paris Leibniz dealt with the infinite and tried to identify its peculiarities, nature, way of handling, and utility. His thinking was guided by two convictions:

(1) There is no exception to Euclid's common notion: the whole is greater than the part;
(2) The same rules hold in the domain of the infinite as in the domain of the finite (law of continuity).

5.1. Peculiarities

Very early in his investigations Leibniz noticed some peculiarities of the infinite. At the turn of the years 1672 to 1673 he wrote the remarkable study *De progressionibus et de arithmetica infinitorum* (On progressions and the arithmetic of the infinites) [Leibniz, 2003, No. 7]. In it he stated:

> Cognitio infiniti apex est humanae subtilitatis.
>
> The recognition of the infinite is the summit of human acuteness.

And somewhat later:

> Non est dubitandum, quin aliquae series constantes licet ex numeris rationalibus, aequentur numeris surdis, quod investigandum.
>
> There cannot be any doubt that some series are equal to irrational numbers though they consist of rational numbers. This must be investigated.

In 1674 his research led him to an arithmetical quadrature of the circle [Ferraro and Panza, 2003, 20],

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

a result that expresses an irrational number in terms of a series of rational numbers. It is apparent that such a result will not hold for finite series: it is a peculiarity of the infinite.

5.2. Nature

On sheet 10 (No. 38_{10}) of his inquiry into sums of series and quadratures Leibniz subtracted the area CFGBC from the area ACBEMA between the hyperbola GBE, the x-axis AC, the y-axis AF and the ordinate CB, shown in Fig. 4.

Because of the symmetry of the figure, CFGBC can be considered as part of the infinitely long area ACBEMA. Putting $\mathrm{DE} = 1/(1-y)$, $\mathrm{HL} = 1/(1+y)$, he obtained

$$\mathrm{DE} = 1 + y + y^2 + y^3 + \cdots,$$
$$\mathrm{HL} = 1 - y + y^2 - y^3 \pm \cdots$$

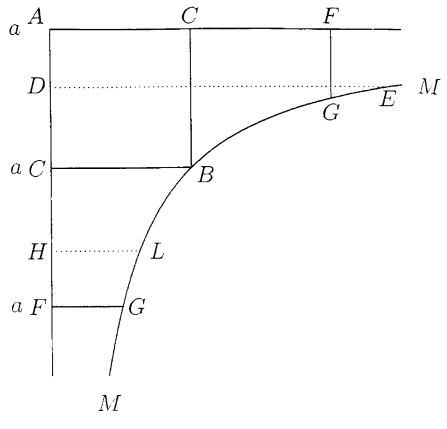


Fig. 4. Hyperbolic areas [Leibniz, 2003, 467].

or as the sums of all DE or all HL, respectively ("integrating" termwise):

$$\text{ACBEMA} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots,$$
$$\text{CFGBC} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \mp \cdots.$$

Subtracting the second series from the first he obtained

$$1 - 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \text{ACBEMA}.$$

Leibniz commented:

> Quod satis mirabile est, et ostendit, summam ipsius series $1 \cdot \frac{1}{2} \cdot \frac{1}{3}$ etc.
> esse infinitam ...
> Hoc argumento concluditur, infinitum non esse totum ; nec nisi fictionem, alioqui enim foret pars aequalis toti.
> This is quite wonderful and demonstrates that the sum of the series
> $1, \frac{1}{2}, \frac{1}{3}$ etc. is infinite ... This argument leads to the conclusion that the infinite is not whole and only a fiction. For otherwise a part would be equal to the whole.

In other words, the actual infinite cannot exist because its existence would lead to an exception to Euclid's axiom.

5.3. Way of handling

In April or May 1673 Leibniz wrote his three-part *De progressionibus intervallorum tangentium a vertice* (On progressions of the intervals of tangents from the vertex) [Leibniz, 2003, No. 17]. He explained why the whole of the alternating series

$$1 - 2 + 4 - 8 + 16, \text{etc.},$$

must be finite. He envisaged two possibilities. If one were allowed each time to subtract the preceding term from the following and to add these differences, the whole would become infinite:

$$1 + (4 - 2) + (16 - 8) + \cdots.$$

If one were allowed each time to subtract the following term from the preceding one and to add these differences, the whole would become less than nothing, less than the whole infinite:

$$1 - \Big[(2 - 4) + (8 - 16) + \cdots.$$

> Nunc fere cum neutrum liceat, aut potius cum non possit determinari utrum liceat, natura medium eligit, et totum aequatur finito.

> Now normally nature chooses the middle if neither of the two is permitted, or rather if it cannot be determined which of the two is permitted, and the whole is equal to a finite quantity.

Following this reasoning, ignorance of what is permitted is said to imply the choice of the middle, which in this case leads to a finite answer.[4] It is worth mentioning that Leibniz did not speak of a sum in this case. “Pretending that a series was equal to a finite quantity was not the same as asserting that it had a sum,” as Ferraro and Panza put it when dealing with 18th-century analysis [Ferraro and Panza, 2003, p. 21].

Approximately two years later, Leibniz proved the first convergence criterion for infinite series in the history of mathematics, namely for the alternating series. He did this in his treatise on the arithmetical quadrature of conic sections, published only in 1993 [Leibniz, 1993, Prop. 49]. In order to facilitate understanding, I replace his letters A, b, c, d, etc. by s and indexed letters a_k, $k = 1, 2, 3, \ldots$. Leibniz’s criterion reads as follows:

> Let $s = a_1 - a_2 + a_3 \pm \cdots$ be such that the terms a_i finally become smaller than any given quantity (Leibniz’s equivalent terminology would be: finally become infinitely small). Then the following inequalities expressed in words hold,
>
> $$\begin{aligned} &a_1 > s && \text{and} \quad a_1 - s < a_2 \\ &a_1 - a_2 < s && \text{and} \quad s - (a_1 - a_2) < a_3 \\ &a_1 - a_2 + a_3 > s && \text{and} \quad (a_1 - a_2 + a_3) - s < a_4 \\ &a_1 - a_2 + a_3 - a_4 < s && \text{and} \quad s - (a_1 - a_2 + a_3 - a_4) < a_5, \ \text{etc}, \end{aligned}$$
>
> or a partial sum s_n ending with an addition is larger than s, a partial sum ending with a subtraction is smaller than s. The error or the difference is smaller than the immediately following term.

If we use modern symbolism, the procedure can be described by $|s_n - A| < a_{n+1}$, whereby a_{n+1} becomes smaller than any given quantity. Or: the sequence of the partial sums s_n has the limit s. Leibniz’s proof runs as follows: he compares the partial series consisting of terms with even or odd indices, respectively,

$$a_{2k+1} < a_{2k} \quad \text{or} \quad \sum_{k=1}^{\infty} a_{2k} > \sum_{k=1}^{\infty} a_{2k+1}, \tag{1}$$

because the sequence of the a_k is monotonically decreasing. Now

$$a_1 - \sum_{k=1}^{\infty} a_{2k} + \sum_{k=1}^{\infty} a_{2k+1} = s.$$

---

[4] Leibniz is not quite consistent here, since the second series reduces to 1 − (minus infinity) = plus infinity, just as the first series did. He apparently intended to mean that the series could be rewritten in the form $(1 - 2) + (4 - 8) + (16 - 32) + \cdots$, which is equal to minus infinity. Hence the middle choice between the two possibilities of plus infinity and minus infinity is a finite quantity.

To obtain s, we have more to subtract than to add. Hence

$$s_1 = a_1 > s.$$

In the same way Leibniz demonstrated that

$$s_3 > s, \qquad s_5 > s, \qquad \text{etc.,}$$
$$a_{2k+1} > a_{2k+2} \quad \text{or} \quad \sum_{k=1}^{\infty} a_{2k+1} > \sum_{k=1}^{\infty} a_{2k+2}. \tag{2}$$

Now

$$a_1 - a_2 + \sum_{k=1}^{\infty} a_{2k+1} - \sum_{k=1}^{\infty} a_{2k+2} = s.$$

To obtain s we have more to add than to subtract. Hence

$$s_2 < s.$$

In the same way Leibniz demonstrated that

$$s_4 < s, \qquad s_6 < s.$$

Assertion:

$$a_1 - s < a_2. \tag{3}$$

According to (1), (2) $a_1 > s$

$$a_1 - a_2 < s$$

or

$$a_1 - s < a_1 - (a_1 - a_2) \quad \text{or} \quad a_1 - s < a_2.$$

In the same way Leibniz proved that

$$s - (a_1 - a_2) < a_3$$

because $a_1 - a_2 < s$ and $s < a_1 - a_2 + a_3$ or

$$s - (a_1 - a_2) < (a_1 - a_2 + a_3) - (a_1 - a_2)$$

or

$$s - (a_1 - a_2) < a_3, \quad \text{etc.}$$

What is important here is the fact that Leibniz *did not calculate with inequalities* (which we would do today) *but compared differences*. Therefore, he needed two columns, although he could have just changed the inequalities in the left column. In other words, in his mathematical thinking series are new mathematical objects, while inequalities are not. He showed that the difference between s and the partial sums s_n becomes smaller than any given quantity for a sufficiently high index n, in order to equate s with the sum of the series. That is, he implicitly used the modern convergence definition,

$$|s_n - s| < \varepsilon$$

for sufficiently large n and arbitrarily small $\varepsilon > 0$.

It is useful to continue the comparison between Leibniz's approach and the modern procedure. While Leibniz compared partial sums of terms with even or odd indices, respectively (see steps (1), (2)), a modern proof compares such partial sums with themselves:

(1) $s_{2n+2} > s_{2n}$, that is, s_{2n} is monotonically increasing.
(2) $s_{2n+3} < s_{2n+1}$, that is, s_{2n-1} is monotonically decreasing.
(3) $s_{2n+1} > s_{2n}$.
(4) Hence s_{2n} is a sequence that is bounded above and s_{2n+1} is a sequence that is bounded below.
(5) Therefore both sequences must have a limit point.
(6) This limit point is the same for both sequences because the difference a_n "converges" to zero.

Leibniz did not have at his disposal the underlying topology. Yet, steps (4) and (6) describe exactly the situation of James Gregory's "convergent" double series which Leibniz especially studied in June 1676 [Leibniz, 2003, No. 64; Gregory, 1668, definitions].

5.4. *Utility*

Leibniz again and again underlined the overwhelming importance of the infinite: it is the source of the transcendental quantities like logarithms [Leibniz, 1693, p. 294].[5] He called these quantities transcendental because they transcended every algebraic equation [Leibniz, 1686, pp. 228f.], adding: "Finally I found a veritable complement of algebra for the 'transcendentals,' that is, my calculus of indefinitely small quantities, which I also call differential or summing calculus" [Leibniz, 1686, pp. 232f.]. In other words, his new analysis corresponding to the geometry of transcendentals is the science of the infinite.

6. Quadrature or construction: more or less geometrical

When in 1668 Nicolaus Mercator calculated areas under the hyperbola by means of infinite series he still called his treatise *Logarithmotechnia: sive methodus construendi logarithmos nova, accurata, et facilis* (Logarithmotechnia: or new, accurate, and easy method of constructing logarithms) [Mercator,

[5] This remark reminds one of Anaximander's saying that the apeiron (the unrestricted) is the beginning and the element of the being of things [Diels/Kranz fragment 12 A 9, B1].

1668], not "method of calculating or determining." Leibniz cited it seven times in his work on the arithmetical quadrature of conic sections. In it he deduced his famous infinite converging alternating series for $\pi/4$ mentioned in Section 5 [Leibniz, 1993, Prop. 32]. The 51st and last theorem asserted that this arithmetical quadrature is the most geometrical quadrature, that is possible. In other words, a more geometrical quadrature would consist of a finite formula. There is not such a formula or—in Cartesian terms—algebraic equation. (In 1974, the historian Joseph Hofmann wrote that Leibniz believed it was impossible to represent π as the quotient of two finite numbers [Hofmann, 1974, p. 95]. This is saying much too little: Leibniz claimed to have demonstrated it.)

> First of all, the equation
>
> $$\text{area of circle : circumscribed square} = \text{arc of the quadrant : diameter} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \text{ etc. : } 1$$
>
> is an exact equation. Yet, it is readily perceived because the mind rushes through it so to speak by a single stroke (unus ictus) [Leibniz, 1993, p. 80] (cf. Section 3).

"Unus ictus" was a key notion for Leibniz. At the end of 1672, he had written about "one of the most difficult problems which can be devised" [Leibniz, 2003, No. 7]:

> Let there be given a monotonically decreasing series having a finite sum. To find another monotonically decreasing series (of finite or infinite sum) whose differences are the terms of the preceding series. This series can be used to find the sum of the given series because the beginning of the series so found will be the sum of the given series. Whoever will have solved this problem will have led geometry to an admirable perfection by, so to say, a single stroke (uno velut ictu).

According to Leibniz, the given result in the theory of series would lead by "a single stroke" to the perfection of geometry.

Second, in order to prove Theorem 51 Leibniz behaved like a partner in a critical dialog. Let a be the arc of the circle, t the corresponding length of the tangent, 1 the length of the radius. Let us assume, said Leibniz, that there is a more geometrical relation (relatio)—he again consciously used a Cartesian expression [Descartes, 1659–1661, vol. I, p. 21]—between the arc and the tangent, and that this relation is expressed by a finite formula instead of the equation given above. For example, it might be expressed by one of the following equations:

(1) $ct + ma = b$,

(2) $ct + dt^2 + eta + na^2 + ma = b$,

(3) $ft^3 + dt^2 + ct + eta + gt^2 + gt^2a + hta^2 + pa^3 + na^2 + ma = b$,

etc.

Assume without loss of generality that the third equation is the one which holds for our problem. Let us suppose that a given arc BO must to be divided into 11 equal parts. This can be done by means of the third equation because we are looking for the tangent of the arc which is the 11th part of the arc BO. Once the tangent has been found, the arc or angle will be divided into 11 parts.

Let $a/11$ be the eleventh part of the arc, ϑ the corresponding tangent. Equation (3) expresses the general relation between an arbitrary arc and its tangent. Hence it also expresses the relation between $a/11$ and ϑ. Leibniz replaces a by $a/11$, t by ϑ, and thus obtains

$$c\vartheta + d\vartheta^2 + f\vartheta^3 + \frac{h\vartheta a^2}{11^2} + \frac{e\vartheta a}{11} + \frac{pa^3}{11^3} + \frac{na^2}{11^2} + \frac{ma}{11} = b.$$

By means of this algebraic equation we will find the unknown ϑ. Only the third power of ϑ occurs, and so it is a matter of a cubic problem. Hence the division of an arbitrary angle into 11 parts will be a cubic problem. Instead of 11 we could take an arbitrary number, so that the division of an angle into an arbitrary number of parts will be only a cubic problem. This is absurd because, according to Viète's treatise on angular sections [Viète, 1615], we know that for the division of an angle into equal parts an equation of ever-higher degree is necessary. Hence the general angle-division problem is neither a cubic problem nor a problem of a determined finite degree. Whatever polynomial equation in t is taken, it does not suffice for the division of an angle into a prime number of equal parts which is larger than the exponent of the largest power of t.

Hence such an equation cannot express either a general relation between arc and tangent or between arc and sine. The same holds if we replace the arc by a sector or segment. Hence there is no comprehensive, analytical quadrature of the circle and its parts that is *more geometrical* than Leibniz's quadrature.

The same argumentation can be applied to the quadrature of the hyperbola. For if there existed a single equation of determined degree for the quadrature of the hyperbola or for the relation between a number and its logarithm, then we could find arbitrarily many mean proportionals by means of a single equation of determined degree.

Leibniz concluded by saying: "Hence there is no general quadrature or construction (quadratura generalis sive constructio), serving for an arbitrary given part of the hyperbola or circle and therefore also for an ellipse, which is more geometrical than our quadrature."

Leibniz's whole proof was elaborated along the lines of Descartes' terminology.

7. Epilogue

Leibniz's scientific credo is to be found on the 16th sheet of his inquiry into sums of series and quadratures [Leibniz, 2003, No. 38_{16}]:

> Malo enim bis idem agere, quam semel nihil.
>
> For I prefer to do the same twice instead of doing nothing once.

It is the bequest of a restless scholar.

Acknowledgment

The paper was presented at the 2003 Barcelona "History of Algebra" conference. I thank Abe Shenitzer for polishing my English.

References

Primary sources

Descartes, R., 1659–1661. Geometria anno 1637 gallice edita, postea autem una cum notis Florimondi de Beaune in latinam linguam versa et commentariis illustrata opera atque studio Francisci a Schooten. Nunc demum ab eodem diligenter recognita locupletioribus commentariis instructa, multisque egregiis accessionibus exornata. 2 vols. Elzevir, Amsterdam.

Descartes, R., 1954. The Geometry, translated from the French and Latin by David Eugene Smith and Marcia L. Latham. Dover, New York. [Republication of 1925 edition.]

Diels, H., Kranz, W., 1966. Die Fragmente der Vor sokratiker, griechisch und deutsch. 12. Unveränderte Auflage. vol. I. Weidmann, Dublin/Zurich.

Gregory, J., 1668. Vera circuli et hyperbolae quadratura. Heredes Pauli Frambotti, Padua.

Heuraet, H.v., 1659. Epistola de transmutatione curvarum linearum in rectas. In: [Descartes, 1659–1661, vol. I, pp. 517–520].

Knobloch, E., 1980. Der Beginn der Determinantentheorie, Leibnizens nachgelassene Studien zum Determinantenkalkül, Textband. Gerstenberg, Hildesheim.

Leibniz, G.W., 1686. De geometria recondita et analysi indivisibilium atque infinitorum. Acta Eruditorum (July), 292–300. Also in: G.W. Leibniz, 1858. Mathematische Schriften, vol. V, edited by C.I. Gerhardt, H.W. Schmidt Halle. Reprint: Olms, Hildesheim, 1962, pp. 226–233.

Leibniz, G.W., 1689. Schediasma de resistentia medii et motu projectorum gravium in medio resistante. Acta Eruditorum (January), 38–47. Also in: G.W. Leibniz, 1860. Mathematische Schriften, vol. VI, edited by C.I. Gerhardt, H.W. Schmidt. Halle. Reprint: Olms, Hildesheim, 1962, pp. 135–144.

Leibniz, G.W., 1693. Supplementum geometriae dimensoriae, seu generalissima omnium tetragonismorum effectio per motum: similiterque multiplex constructio lineae ex data tangentium conditione. Acta Eruditorum (September), 385–392. Also in: G.W. Leibniz, 1858. Mathematische Schriften, vol. V, edited by C.I. Gerhardt, H.W. Schmidt. Halle. Reprint: Olms, Hildesheim, 1962, pp. 294–301.

Leibniz, G.W., 1850. Mathematische Schriften, vol. II, edited by C.I. Gerhardt. Natt/Asher, London/Berlin.

Leibniz, G.W., 1976a. Die mathematischen Studien zur Kombinatorik, Textband, im Anschluß an den gleichnamigen Abhandlungsband zum ersten Mal nach den Originalhandschriften, edited by E. Knobloch. Steiner, Wiesbaden.

Leibniz, G.W., 1976b. Ein Dialog zur Einführung in die Arithmetik und Algebra, nach der Originalhandschrift, edited with an introduction and commentary by E. Knobloch. Frommann–Holzboog, Stuttgart, Bad Canstatt.

Leibniz, G.W., 1987. Sämtliche Schriften und Briefe, Reihe 3, Band 2, 1676–1679 Mathematischer, naturwissenschaftlicher und technischer Briefwechsel, edited by Heinz-Jürgen Hess. Akademie-Verlag, Berlin.

Leibniz, G.W., 1990. Sämtliche Schriften und Briefe, Reihe 7, Band 1, 1672–1676 Geometrie—Zahlentheorie—Algebra (1. Teil), edited by E. Knobloch, W.S. Contro. Akademie-Verlag, Berlin.

Leibniz, G.W., 1993. De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis, critical edition and commentary by Eberhard Knobloch. Abhandlungen der Akademie der Wissenschaften in Göttingen, Mathematisch-physikalische Klasse 3. Folge, Nr. 43. Vandenhoeck & Ruprecht, Göttingen.

Leibniz, G.W., 1996. Sämtliche Schriften und Briefe, Reihe 7, Band 2, 1672–1676 Algebra (2. Teil), edited by E. Knobloch, W.S. Contro, with the assistance of N. Gädeke. Akademie-Verlag, Berlin.

Leibniz, G.W., 2003. Sämtliche Schriften und Briefe, Reihe 7, Band 3, 1672–1676 Differenzen, Folgen, Reihen, edited by S. Probst, E. Knobloch, N. Gädeke. Akademie-Verlag, Berlin.

Mercator, N., 1668. Logarithmotechnia: sive methodus construendi logarithmos nova, accurata, et facilis antehac communicata, anno sc. 1667. Nonis Augusti: cui nunc accedit vera quadratura hyperbolae, et inventio summae logarithmorum. Mosis Pitt, London.

Viète, Fr., 1615. Ad angulares sectiones theoremata katholikotera demonstrata per Alexandrum Andersonum. Paris. Also in: Francisci Vietae opera mathematica, in unum volumen congesta, ac recognita, opera atque studio Francisci a Schooten. Bonaventura/Abraham Elzevir, Leiden. Reprint: Olms, Hildesheim, New York, 1970, pp. 286–304.

Secondary literature

Bashmakova, I.G., Smirnova, G.S., 2000. The Beginnings and Evolution of Algebra, translated from the Russian by A. Shenitzer with the editorial assistance of D.A. Cox. Mathematical Association of America, Washington, DC.

Bos, Henk J.M., 2001. Redefining Geometrical Exactness, Descartes' Transformation of the Early Modern Concept of Construction. Springer-Verlag, New York/Berlin/Heidelberg/London/Paris/Tokyo/Hong Kong/Barcelona/Budapest.

Ferraro, G., Panza, M., 2003. Developing into series and returning from series: a note on the foundations of eighteenth-century analysis. Historia Math. 30, 17–46.

Hilbert, D., 1900. Mathematische Probleme, Vortrag, gehalten auf dem internationalen Mathematiker-Kongress zu Paris. Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 253–297. Partial English translation in: Ewald, W. (Ed.), From Kant to Hilbert: A Source Book in the Foundations of Mathematics, vol. II. Clarendon, Oxford, 1996, pp. 1096–1105.

Hofmann, J.E., 1974. Leibniz in Paris 1672–1676, His Growth to Mathematical Maturity. Cambridge Univ. Press, Cambridge, UK.

Kline, M., 1972. Mathematical Thought from Ancient to Modern Times. Oxford Univ. Press, New York.

Knobloch, E., 1972. Die entscheidende Abhandlung von Leibniz zur Theorie linearer Gleichungssysteme. Studia Leibnitiana 4, 163–180.

Knobloch, E., 1973. Die mathematischen Studien von G.W. Leibniz zur Kombinatorik. Steiner, Wiesbaden.

Knobloch, E., 1974. Unbekannte Studien von Leibniz zur Eliminations- und Explikationstheorie. Arch. Hist. Exact Sci. 12, 142–173.

Knobloch, E., 1999. "Im freiesten Streifzug des Geistes" (liberrimo mentis discursu): Zu den Zielen und Methoden Leibnizscher Mathematik. In: Nowak, K., Poser, H. (Eds.), Wissenschaft und Weltgestaltung: Internationales Symposium zum 350. Geburtstag von Gottfried Wilhelm Leibniz vom 9. bis 11. April 1996 in Leipzig. Olms/Hildesheim, Zürich/New York, pp. 211–229.

Knobloch, E., 2001. Déterminants et élimination chez Leibniz. Rev. Histoire Sci. 54, 143–164.

Rivaud, A. (Ed.), 1914–1924. Catalogue critique des manuscrits de Leibniz, fascicule II (Mars 1672–Novembre 1676). Société Française d'Imprimerie et de Librairie, Poitiers. Reprint: Olms/Hildesheim/Zürich/New York, 1986.

Sasaki, Ch., 2003. Descartes' Mathematical Thought. Kluwer Academic, Dordrecht/Boston/London.

Valéry, P., 1999. Alphabet. Edition établie, présentée et annotée par Michel Jarrety. Librairie générale française, Paris.