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Two Leibnizian Manuscripts of 1690 Concerning Differential Equations

EMILY GROSHOLZ

Department of Philosophy, The Pennsylvania State University, University Park, Pennsylvania 16802

Leibniz was very interested in developing techniques for the solution of differential equations. In 1690 he elaborated two manuscripts in which he employed the technique of separating variables. Thus he had to evaluate the logarithm of negative numbers. The present article consists mainly of a critical edition, English translation, and a commentary on these two interesting manuscripts. © 1987 Academic Press, Inc.

Leibniz war sehr an der Entwicklung von Techniken zur Lösung von Differentialgleichungen interessiert. 1690 verfaßte er zwei Studien, wo er die Methode der Variablentrennung verwandte. Dies führte ihn auf den Logarithmus von negativen Zahlen. Der vorliegende Aufsatz besteht hauptsächlich aus einer kritischen Edition und einer englischen Übersetzung dieser zwei interessanten Handschriften, denen ein Kommentar beigegeben ist. © 1987 Academic Press, Inc.

Leibniz était très intéressé à développer des techniques pour la solution des équations différentielles. En 1690 il élabora deux études où il employa la technique de la séparation des variables. De cette manière il devait évaluer le logarithme des nombres négatifs. L'article présent consiste principalement en une édition critique, une traduction anglaise et un commentaire de ces deux manuscrits intéressants. © 1987 Academic Press, Inc.

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INTRODUCTION

Leibniz plays a central role in the early development of the theory of differential equations. At the end of his first publication on the calculus, "Nova methodus pro maximis et minimis . . ." [Leibniz 1684], he mentions a problem which Debeaune had posed to Descartes in 1638: "What curve has the property, that its ordinate y bears the same relation to its subtangent t as the difference of its abscissa x and ordinate y, to a given magnitude a?" This kind of problem, concerning the determination of a curve from a given property of its tangents, provided Leibniz with a good occasion to exhibit the power and simplicity of his new methods.

Descartes' solution to Debeaune's problem uses a fussy proof to achieve a pointwise construction of the curve using approximative methods [Scriba 1960–

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1962, 411–413]. Leibniz' solution, mentioned in a letter to Oldenburg for Newton dated August 27, 1676 [Leibniz 1676b], and worked out in leaf 3 [Leibniz 1676a], is much more straightforward. Essentially, he sees that the curve is the solution to the differential equation $dy/dx = (x - y)/a$ and recognizes it as a logarithmic curve. (For details of the proof, see [Hofmann 1972, 13–14, 15–18].)

For Leibniz, the differential and integral calculus is a method for studying curves, which embody relations between variable geometric quantities (abscissa, ordinate, radius, subtangent, tangent, normal, area between curve and the x-axis, and so forth), conceived as infinite sequences of terms induced by an infinite-sided polygon which approximates the curve; and differentials (differences between successive terms of those sequences) and sums (summations of successive terms) formed by the operators d and $\int$ [Bos 1974, 4–35]. The equations which express these relations are differential equations.

Leibniz was therefore centrally interested in developing techniques for the solution of differential equations. In the years following the publication of the two expositions of his new method [Leibniz 1684, 1686], one of Leibniz' central mathematical concerns was to develop such techniques. The present manuscripts, "Methodus pro differentialibus, ponendo $z = dy/dx$ et quaerendo dz, September 10, 1690" and "Methodus tangentium inversa per substitutiones (moderatas) assumendo $z = dy/dx$, September 11, 1690," are good examples of the investigations he undertook in 1690 upon his return from Italy. In these texts, he employs the technique of separating variables in ordinary differential equations; and he employs a technique for rewriting the form of homogeneous differential equations so that the resulting equation is then separable. At the end of the first manuscript, he gives a general method for treating such equations. Leibniz communicated some of these ideas to Huygens in the early 1690s, and Johann Bernoulli published an exposition of them in the *Acta Eruditorum* [Bernoulli 1694]. (See also [Kline 1972, 471–476].) Related problems continue to occur in Leibniz' correspondence with the Bernoullis, and in the *Acta Eruditorum*. For example, he publishes a solution to the catenary problem, finding the curve described by a flexible cord hanging freely from two points, in the *Acta* [Leibniz 1691], as did Huygens and Jakob and Johann Bernoulli; Bernoulli articulated the problem by means of the differential equation $dy = adx/\sqrt{(x^2 - a^2)}$. And a solution to the brachistochrone problem, finding the curve from point A to point B along which a body starting from rest under the influence of gravity, without friction or air resistance, will move most quickly, appears in the *Acta* [Leibniz 1697]. Johann Bernoulli, l'Hôpital, and Newton also offered solutions to this problem. Leibniz sees that the relevant curve is a cycloid [Bos 1980, 79–84]. In the late 1690s, he worked with the Bernoullis on a problem important for optics, that of orthogonal trajectories, finding a family of curves that cut a given family of curves orthogonally, for which he conceived the general problem and method [Kline 1972, 474–475].

On the second page of "Methodus pro differentialibus . . . ," Leibniz investigates the differential equation $y/x = -dy/dx$, which he recognizes as the defining condition of a family of hyperbolae. He forms the differential of the equation, eliminates terms involving y, separates the variables, and integrates term by term.

Since $\int dz/z = \log z$, this procedure leads him to write $\log y = \log x + \log z + \log(-1)$, and he must then evaluate the logarithm of a negative number.

In an interesting article, "The Controversy between Leibniz and Bernoulli on the Nature of the Logarithms of Negative Numbers," Peggy Marchi describes the debate which arose between Leibniz and Johann Bernoulli over the nature and evaluation of the logarithms of negative numbers in the early 1700s [Marchi 1974]. She states that this problem arose around 1702, when Bernoulli discovered that

$$\frac{adz}{b^2 + z^2} = \frac{1}{2}\frac{adz}{b^2 + ibz} + \frac{1}{2}\frac{adz}{b^2 + ibz}.$$

The present manuscript reveals that Leibniz had considered the problem at a much earlier date.

During the course of the Leibniz–Bernoulli debate, Leibniz objects to Bernoulli's claim that $\log x = \log(-x)$ and that $d \log x = -dx/-x$, i.e., that the curve of $\log x$ is symmetrical about the y-axis, on the grounds that this produces the result that $\log i = \log(-1) = 0$. This result is counterintuitive, since in general $\log x^2$ should be equal to $2 \log x$. In 1690, however, Leibniz had hypothesized that $\log(-1) = 0$ (though in a context where imaginary numbers are not explicitly treated).

Leibniz counters Bernoulli's proposal with the claim that the logarithms of negative numbers must be imaginary [Leibniz 1702]. We may imagine that Leibniz rejects Bernoulli's proposal as a position which Leibniz himself had considered and found to be a blind alley. Euler later shows that logarithms of negative numbers are imaginary, and that an infinite plurality of such logarithms corresponds to each number [Euler 1980, 15–18]. The problem of the logarithms of negative numbers is a good example of what Philip Kitcher [1983, 202–203] has called "language-induced question generation," where questions about members of a kind (in this case, numbers) arise in analogy with traditional questions about more familiar members of the kind.

A few comments on Leibniz' notation and on the textual apparatus may be useful to the reader. Leibniz uses a colon (:) to indicate division; thus $z = dy{:}dx$ means $z = dy/dx$. He uses a raised horizontal line, $\overline{\qquad}$ to indicate that the expression under the line should be bracketed; so $a\overline{dz{:}dx - xz}$ means $a(dz/dx - xz)$ and $d\overline{x}y$ means $(dx)y$. Occasionally he uses a tilde (˜) to indicate bracketing; so $dz\tilde{z}$, means $dz(z)$. Also, he sometimes uses a comma (,) to indicate that the preceding expression should be bracketed; so $dx + bdy,{:}dz$ means $(dx + bdy)/dz$. Often he encircles terms (sometimes indexing the circles by one, two, or more short strokes) as a bookkeeping device for keeping terms straight in complicated computations.

The passages inserted under a half-line are marginalia, and so in a sense should be considered part of the text. The passages inserted under a full line are those which Leibniz has deleted. The textual variants implied by these cancelings are indicated by numbers, letters, and iterated letters. Each phase of his thought is thereby reconstructed, with each phase replacing the preceding one and going beyond it: for example, *(1)*, *(2)*; *(3)*(*a*), *(3)*(*b*); *(3)*(*b*)(*aa*), *(3)*(*b*)(*bb*); and so forth. The symbol ⟨———⟩ indicates portions of the text which have become illegible.

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REFERENCES

Bernoulli, J. 1694. Modus generalis construendi omnes aequationes differentiales primi gradus. *Acta Eruditorum,* November, 435–437. Reprinted in *Opera omnia,* G. Cramer, Ed., Vol. 1, 1742, pp. 123–125. Lausanne/Geneva: Bousquet.

Bos, H. J. M. 1974. Differentials, higher-order differentials, and the derivative in the Leibnizian calculus. *Archive for History of Exact Sciences* **14,** 406–419.

——— 1980. Newton, Leibniz and the Leibnizian tradition. In *From the calculus to set theory: An introductory history,* I. Grattan-Guinness, Ed., pp. 49–93. London: Duckworth.

Euler, L. 1980. *Opera omnia,* Vol. IV, A5, A. P. Juškevič & R. Taton, Eds. Basel: Birkhäuser.

Hofmann, J. E. 1972. Über Auftauschen und Behandlung von Differentialgleichungen im 17. Jahrhundert. *Humanismus und Technik* **15,** 1–40.

Kitcher, P. 1983. *The nature of mathematical knowledge.* New York: Oxford Univ. Press.

Kline, M. 1972. *Mathematical thought from ancient to modern times.* New York: Oxford Univ. Press.

Leibniz, G. W. 1676a. Methodus tangentium inversa, July 1676. In [Leibniz 1899, 201–203].

——— 1676b. Letter to Oldenburg for Newton, August 27. In [Leibniz 1976, 558–586].

——— 1684. Nova methodus pro maximis et minimis . . . . *Acta Eruditorum,* December, 467–473. Reprinted in [Leibniz 1849–1855 5, 220–226].

——— 1686. De geometria recondita . . . . *Acta Eruditorum,* June, 292–300. Reprinted in [Leibniz 1849–1855 5, 226–233].

——— 1691. De linea, in quam flexile se pondere proprio curvat, ... *Acta Eruditorum,* June, 277–281. Reprinted in [Leibniz 1849–1855 5, 243–247].

——— 1697. Communicatio . . . . *Acta Eruditorum,* May, 201–206. Reprinted in [Leibniz 1849–1855 5, 331–336].

——— 1702. Letter to Johann Bernoulli, June 24. In [Leibniz 1849–1855 3, 702–705].

——— 1849–1855. *Mathematische Schriften,* C. I. Gerhardt, Ed. 7 vols. Halle: Asher. Reprinted, Hildesheim: Olms, 1962.

——— 1899. *Der Briefwechsel mit Mathematikern,* C. I. Gerhardt, Ed. Berlin: Mayer & Müller. Reprinted, Hildesheim: Olms, 1966.

——— 1976. *Sämtliche Schriften und Briefe: Mathematischer, naturwissenschaftlicher und technischer Briefwechsel,* Akademie der Wissenschaften der DDR, Ed., Ser. III, Vol. 1. Berlin: Akademie-Verlag.

Marchi, P. 1974. The controversy between Leibniz and Bernoulli on the nature of the logarithms of negative numbers. In: *Akten des II. Internationalen Leibniz-Kongresses* (Hannover, July 17–22, 1972), Bd. II. Wissenschaftstheorie und Wissenschaftsgeschichte, Wiesbaden, 1974, pp. 67–75.

Scriba, C. J. 1960–1962. Zur Lösung des 2. Debeauneschen Problems durch Descartes. *Archive for History of Exact Sciences* **1,** 406–419.

METHODUS PRO DIFFERENTIALIBUS, PONENDO $z = dy{:}dx$ ET QUAERENDO dz

September 10, 1690

Textual tradition: Leibniz concept: LH XXXV 13,1. Leaf 302. 1 sheet 2°. 2 pages

10 Septemb. 1690

Methodus pro differentialibus, ponendo $z = dy{:}dx$ et quaerendo dz.

Sit $zx \overset{(1)}{=} y$ et $zdx + xdz \overset{(2)}{=} dy$ et $z \overset{(3)}{=} y{:}x$ fit $dz \overset{(4)}{=} dy{:}x - ydx{:}xx$ tollamus y ex aeq. 4 per 1. et ex eadem per 2. seu per $x \overset{(5)}{=} \overline{-zdx + dy},{:}dz$, et fiet ex aeq. 4 $dz \overset{(6)}{=} dydz{:}\overline{-zdx + dy} - zdxdz{:}\overline{-zdx + dy}$ fiet $-zdxdz + dydz \overset{(7)}{=} dydz - zdxdz$ quae est aequatio identica.

Sit $z \overset{(1)}{=} xy + a$ et $dz \overset{(2)}{=} xdy + ydx$ et $z{:}y \overset{(3)}{=} x + a{:}y$ et $dz{:}y - d\bar{y}z{:}yy \overset{(4)}{=} dx - ady{:}yy$ ponamus $z \overset{(5)}{=} dy{:}dx$. Ex aeq. 2. et 5. fiet $dz{:}dx \overset{(6)}{=} xz + y$ per aeq. 1, 4, 6 tentemus tollere x et y per aeq. 6 est $y \overset{(7)}{=} dz{:}dx - xz$ hic valor substituatur in 1. fit $z \overset{(8)}{=} \overline{xdz{:}dx - x^2z}$. Idem valor y ex aeq. 7 substituatur in 4 fit $d\bar{z}\,\overline{dz{:}dx} - xz - d\bar{y}z \overset{(9)}{=} dx\,\overline{\overline{dz{:}dx}^2 - 2\,\overline{d\bar{z}{:}dx}xz + x^2z^2} - ady$ per aeqq. 8 et 9 tollatur x. Ex 8 est $xx - \overline{zd\bar{z}{:}dx}\,x + \tfrac{1}{4}\overline{zzd\bar{z}^2{:}d\bar{x}^2} \overset{(10)}{=} \tfrac{1}{4}zzd\bar{z}^2{:}d\bar{x}^2 - 1$ seu $2xd\bar{x} \overset{(11)}{=} \sqrt{zzd\bar{z}^2 - d\bar{x}^2} + zdz$ et ex aeq. 9 explicata x, fit

$$dz\cdot\overline{dz{:}dx} - d\bar{z}z\sqrt{zzd\bar{z}^2 - d\bar{x}^2}{:}2dx + zd\bar{z}^2{:}2dx$$

$$- dyz \overset{(12)}{=} dx\,\boxed{2}\,\overline{\overline{dz{:}dx - z{:}2dx\sqrt{zzd\bar{z}^2 - d\bar{x}^2} + zdz}}.$$

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1 Haec bona quatenus sed alibi melius posita vid. schedam in 8° 11 Septemb. 1690.

2–3 quaerendo dz. (*1*) $zz \overset{(1)}{=} xx + yy$ Ergo differentialiter $zdz \overset{(2)}{=} xdx + ydy$. rursus (*a*) $z \overset{(3)}{=} \overline{xx + yy}$. Ergo $dz = xdx + ydy$, (*b*) $z \overset{(3)}{=} xx + yy,{:}z$. Ergo $dz \overset{(4)}{=} 2xzdx + 2yzdy - \overline{xx + yy}\,dz,{:}zz$ (*aa*) Seu $z^2dz \overset{(5)}{=} 2xzdx + 2yzdy$ | $-zzdz$ *deleted* |. Ergo $z = 2xdx + 2ydx$. Ergo (*bb*) Iam ex aeq. 2 erat $z \overset{(5)}{=} xdx{:}dz + ydy{:}dz$ seu $zz \overset{(6)}{=} x^2d\bar{x}^2 + 2xydxdy + yyd\bar{y}^2,{:}d\bar{z}^2$ unde in aeq. 4 substituendo | valores ex *inserted* | aeq. 5 et 6 | ductis ex aeq. 2 *inserted* | fit $dz \overset{(7)}{=} \overline{2xdx + 2ydy}\cdot\overline{xdx + ydy}{:}dz - dz\overline{xx + yy},{:}\overline{x^2d\bar{x}^2 + 2xydxdy + y^2d\bar{y}^2}{:}d\bar{z}^2$. Iam ex 6 et 1 fit $xx + yy \overset{(8)}{=} xxd\bar{x}^2 + 2xydxdy + yyd\bar{y}^2{:}d\bar{z}^2$. (*aaa*) Sit $xy \overset{(9)}{=} v$ et $x + y \overset{(10)}{=} \omega$ (*bbb*) Et ex. aeq. (*aaaa*) 10 et 7 fiet zz (*bbbb*) 8 (*aaaaa*) fiet $dz = 2\omega\omega$ (*bbbbb*) fiet (*ccccc*) | ex *editor deletes* | 7 (*aaaaaa*) $dz =$ (*bbbbbb*) $\overline{2xdx + 2ydy}\cdot\overline{xdx + ydy}\,dz{:}\overline{x^2d\bar{x}^2 + 2xydxdy + y^2d\bar{y}^2} \overset{(9)}{=} 2dz$. Videndum an per aeqq. 8 et 9 tolli possit aliqua adhuc litera ut x. (2) Sit $zx \overset{(1)}{=} y$ *L*

16 $-dyz$ (*1*) $= dx\,\boxed{2}\,\overline{dz{:}dx} - \overline{z{:}2dx}\sqrt{zzd\bar{z}^2 - d\bar{x}^2 + zdz}$

(*2*) $= dx\,\boxed{2}\,dz{:}dx - \overline{\overline{z{:}2dx}\sqrt{zzd\bar{z}^2 - d\bar{x}^2 + zdz}}$ (*3*) $\overset{(12)}{=} dx$ *L*

Ita habetur aequatio in qua solum extant dx, dy, z et dz, seu in effectu praeter literam z extabunt $dy{:}dx$ et $d\bar{z}{:}d\bar{x}$, tollatur $dy{:}dx$ quia $= z$. Ergo restabit solum $d\bar{x}$ cuius valor habetur per dz et z itaque hoc posito res est reducta ad quadraturas. Nisi scilicet quod unum vereor explicando $dy{:}dx$ per z. etiam dz evanescat. Sed huic malo fortasse mederi licebit, non penitus tollendo dy sed in partibus ubi impedit summabilitatem, ut si esset $ad\bar{y} + bd\bar{x} = dz{\cdot}\bar{z}$ ubi licuisset facere $az + b = dz\bar{z}{:}dx$ seu $dx = dz\bar{z}{:}\overline{az + b}$. Imo video si semel licet tollere x et y remanente dz, ut credo quia aeq. 5. moderate usi sumus, utique postea non potest tolli dz sublata dy, quia alioqui dx restaret sola, adeoque evanesceret, et haberetur z definite quod est absurdum.

Resumamus exemplum superius: $zx \overset{(1)}{=} y$, $z \overset{(2)}{=} d\bar{y}{:}d\bar{x}$ differentialis ipsius 1 est $zdx + xdz \overset{(3)}{=} dy$ et rursus ex 1 est $x \overset{(4)}{=} y{:}z$ cuius differentialis est $dx \overset{(5)}{=} d\bar{y}z - ydz{:}zz$. et ex 5 et 2 fit $1 \overset{(6)}{=} \left(\frac{dy}{dx} =\right) z - y\ dz{:}dx$. Ex aeq. 3 habemus x sine y, seu, $x \overset{(7)}{=} dy{:}dz - zdx{:}dz$ et ex 6 est $y \overset{(8)}{=} zdx{:}dz - dz{:}dx$ quos valores 7 et 8 substituendo in aeq. 1 fit: $zd\bar{y}{:}dz - zzdx{:}dz \overset{(9)}{=} zdx{:}dz - dz{:}dx$ seu $zdydx - zzdx^2 \overset{(9)}{=} zd\bar{x}^2 - d\bar{z}^2$ et divisis omnibus per $d\bar{x}^2$ et pro $dy{:}dx$ ponendo z fiet $(zz - zz) \overset{(10)}{=} z - d\bar{z}^2{:}d\bar{x}^2$ seu $d\bar{x} = dz{:}\sqrt{z}$. Verendum ne subsit error in calculo.

Resumamus: $zx \overset{(1)}{=} y$ $z \overset{(2)}{=} dy{:}dx$ differentialis ipsius aeq. 1 est $zdx + xdz \overset{(3)}{=} dy$. Ex aeq. 1 fiat $x \overset{(4)}{=} y{:}z$ et huius aeq. 4 differentialis erit $dx \overset{(5)}{=} dy{:}z - dzy{:}zz$ itaque supra in aeq. 5 male calculavi pro $dy{:}z$ ponendo dy ex 5 et 2 fit $1 \overset{(6)}{=} 1 - dzy{:}d\bar{x}zz$ quod videtur esse absurdum fit enim $dzy{:}d\bar{x}zz \overset{(7)}{=} 0$. quod significat locum esse ad rectam ubi z necessario est constans, et ideo $dz \overset{(8)}{=} 0$ qui successus egregius.

Sit $z \overset{(1)}{=} yx$. sit $y \overset{(2)}{=} z{:}x$ fit $dy \overset{(3)}{=} dz{:}x - d\bar{x}\ z{:}xx$ seu $xxdy \overset{(4)}{=} d\bar{z}x - d\bar{z}z$ et $dz \overset{(5)}{=} ydx + xdy$ ubi fiet $dz{:}dx \overset{(6)}{=} y + xz$ posito $z \overset{(7)}{=} dy{:}dx$. Ex aeq. 4 est $x^2 - \overline{dz{:}dy}\ x + \tfrac{1}{4}d\bar{z}^2{:}d\bar{y}^2 \overset{(8)}{=} \tfrac{1}{4}d\bar{z}^2{:}d\bar{y}^2 - d\bar{z}z{:}dy$ seu $x \overset{(7)}{=} \sqrt{\tfrac{1}{4}d\bar{z}^2{:}d\bar{y}^2 - d\bar{z}z{:}dy}$

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28 $y{:}x{::}dy{:}dx$. locus est ad rectam. Si foret $y{:}x = -dy{:}dx$ foret ad Hyperbolam.

36–37 $x \overset{(3)}{=} dy - zdx,{:}dz$

37–38 $y \overset{(5)}{=} -\overline{zzdx + dyz}{:}dz$ fit ex $1{\cdot}3{\cdot}5 \cdot \overline{dy - z^2dx}{:}dz = -zzdx + dyz$ unde nil novi sed nec debuit imo hinc sublata dy fit z constans.

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43 $y + xz$ (*1*) posito $z \overset{(7)}{=} dy{:}dx$ et ob $x =$ (*2*) posito z *L*

$+ \frac{1}{2}dz{:}dy =$. Iam $y \overset{(8)}{=} dz{:}dx - xz \overset{(9)}{=} z{:}x$. Ergo fit $\overline{dz{:}dx}\, x - x^2z \overset{(10)}{=} z$ seu $xx - \overline{dz{:}\overline{dxz}}\, x + \frac{1}{4}dz^2{:}\overline{dx^2\, zz} \overset{(11)}{=} 1 + \frac{1}{4}dz^2{:}dx^2zz$ seu $x \overset{(12)}{=} \sqrt{1 + \frac{1}{4}dz^2{:}\overline{dx^2z^2}} + \frac{1}{2}dz{:}\overline{dxz}$ quos duos valores 7 et 12 aequando, priore prius multiplicato per zz seu $dy^2{:}dx^2$ fit $\sqrt{\frac{1}{4}dz^2{:}dx^2 - dzz^2{:}dx}$ $+ \frac{1}{2}zdz{:}dx$ $\overset{(13)}{=} z^2\sqrt{1 + \frac{1}{4}dz^2{:}\overline{dx^2z^2}}$ $+ \frac{1}{2}zdz{:}dx$ seu $\frac{1}{4}dz^2{:}dx^2 - dzz^2{:}dx = z^4 + \frac{1}{4}dz^2z^2{:}dx^2$ quae est aequatio quaesita. Atque ita tandem videor desideratum artificium obtinuisse.

$yx = dy{:}dx$. $\frac{1}{2}xx = \int \overline{dy{:}y}$. Generaliter sit aeq. (1) inter z. x. y. posita $z \overset{(2)}{=} dy{:}dx$. quaeratur valor ipsius x ex aeq. 1 dabit aeq. (3) habebitur eius differentialis (4) in qua aeq. pro $dy{:}dx$ saltem alicubi substituatur z fit aeq. (5) in qua (ut et in 4) datur y sine x similiter quaeratur valor y fit aeq. (6) cuius differentialis (7) in qua x sine y, valores y et x ex aeqq. 5 et 7 substituantur in aeq. 1 habetur aequatio (8) inter z. dz. dx. dy. tollatur dy quia $\overset{(2)}{=} zdx$ et habetur aeq. (9) reducta ad quadraturas.

Si sit $y{:}x = dy{:}dx$ aequatio est ad Rectam, sed si fiat: $y{:}x = -dy{:}dx$ aequatio est ad Hyperbolam nam fit $xdy + ydx = 0$ adeoque $xy = aa$. Videamus ergo an Methodo nostra praesente huc veniri possit. Sit $dy{:}dx \overset{(1)}{=} z$ et sit $y{:}x \overset{(2)}{=} -dy{:}dx$ aequatio ad curvam quaesitam, et ex 1 et 2, fiet $y \overset{(3)}{=} -xz$. Ergo eius differentialis $dy \overset{(4)}{=} -xdz - zdx$ seu $x \overset{(5)}{=} -dx{:}dz - zdx{:}dz$. Ita habetur valor ipsius x sine y. Rursus ex 3 fit $x \overset{(6)}{=} -y{:}z$ cuius differentialis fit $dx \overset{(7)}{=} -d\overline{yz} + d\overline{zy},{:}zz$. Seu $y \overset{(8)}{=} zzdx{:}dz + dyz{:}dz$ qui est valor ipsius y sine x. Iam hos valores literarum x et y in aequationibus 5 et 6 inventos, substituendo in aeq. 3 fit $zzdx{:}dz + d\overline{yz}{:}dz = zd\overline{y}{:}dz + zzdx{:}dz$ quae est aequatio identica unde discimus nihil.

Itaque rem resumamus, et prius moderata substitutione ipsius z in locum sui valoris utamur $dy{:}dx \overset{(1)}{=} z$ $y{:}x \overset{(2)}{=} -dy{:}dx$. Ergo per 1 et 2 fit $y \overset{(3)}{=} -xz$ et $dy \overset{(4)}{=} -xdz - zdx$ quam aequationem dividamus per dx, et in valorem ipsius $dy{:}dx$ substituamus z per aeq. 1. fiet $dy{:}dx \overset{(5)}{=} -xdz{:}dx - z$ seu $z \overset{(6)}{=} -xdz{:}dx - z$ seu $2zdx \overset{(7)}{=} -xdz$ seu $a - \int \overline{dx{:}x} \overset{(8)}{=} 2\int \overline{dz{:}z}$. Ergo datur relatio inter x et z per quadraturas adeoque et relatio inter x et $-y{:}x$

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45 Ergo fit (*1*) $xx - \overline{dz{:}\overline{dxz}}x = $ (*2*) $\overline{dz{:}dxx}$ *L*

47 7 et 12 (*1*) necessario ascendemus (*2*) aequando *L*

60 Methodo (*1*) ista ad hanc (*2*) nostra *L*

63 fit (*1*) $z = -y{:}x$ (*2*) $x \overset{(6)}{=} -y{:}z$ *L*

63–64 cuius differentialis *L inserts*

71 aeq. 1. (*1*) fiet $2z \overset{(5)}{=} -xdz{:}dx$. Atque ita iam tum solutio habetur etiamsi non log (*2*) fiet *L* per aeq. 3. hoc est relatio inter x et y. Iam per 3 est $\log y \overset{(9)}{=} \log x + \log z + \log \overline{-1}$. Iam posito $\log 1 \overset{(10)}{=} 0$ fit $\log -1 \overset{(11)}{=} 0$. habemus ergo $\log z \overset{(12)}{=} \log y - \log x$. Iam ex aeq. 8 est $a - \log x \overset{(13)}{=} 2 \log z$. ergo ex 12 et 13 fit a $\textcircled{-\log x} \overset{(14)}{=} 2 \log y - \textcircled{2} \log x$. Seu $b^a = y^2{:}x$. Quod falsum itaque alicubi error in calculo.

Resumamus $dy{:}dx \overset{(1)}{=} z$ $y{:}x \overset{(2)}{=} -dy{:}dx$ $z \overset{(3)}{=} dy{:}dx$ Ergo per 1 et 2 fit $y \overset{(4)}{=} -xz$. Cuius differentialis erit $dy \overset{(5)}{=} -xdz - zdx$ quam dividendo per dx fit $dy{:}dz \overset{(6)}{=} -xdz{:}dx - z$ seu per 3 fit $z \overset{(7)}{=} -xdz{:}dx - z$ seu fit $2zdx \overset{(8)}{=} -xdz$. Seu $zdx + xdz + zdx \overset{(9)}{=} 0$. Iam $zdx = dy$ per 3, unde ex aeq. 9 fit $zdx + xdz + dy = 0$ seu $xz = -y$ ut ante. Probus igitur est calculus usque ad aeq. 8. Ergo ex aeq. 8 fit $2 \int \overline{dx{:}x} \overset{(10)}{=} a - \int \overline{dz{:}z}$ in eo ergo erratum est in prioris calculi aeq. 8 quod ibi numerus 2 fuit praefixus ipsi $\int \overline{dz{:}z}$. Ex 10 fit $2 \log x \overset{(11)}{=} a - \log z$. Iam $\log z \overset{(12)}{=} \log y - \log x + \log \overline{-1}$. Sed $\log \overline{-1} \overset{(13)}{=} 0$ posito $\log 1 \overset{(14)}{=} 0$. Ergo ex 12 fit $\log z \overset{(15)}{=} \log y - \log x$ quo valore substituto in aeq. 11 fit $2 \log x \overset{(16)}{=} a - \log y + \log x$. Ergo $\log x \overset{(17)}{=} a - \log y$ seu $\log x$ et $\log y \overset{(18)}{=} a$. Ergo $xy \overset{(19)}{=} b^a$. posito ipsius b logarithmum esse unitatem. Et ita deprehensum est Hyperbolam posito satisfacere aequationi propositae 2. quod est verissimum. Itaque hac Methodo discimus aliquid. Et hactenus una tantum differentiali usi sumus redeundo ergo ad aeq. 5. caeteris quae postea scripta sunt quasi non scriptis. Iam quaeramus et modum inveniendi valorem ipsius y sine x. Nempe $x \overset{(20)}{=} -y{:}z$ per 4. ergo $dx \overset{(21)}{=} -d\bar{y}z + d\bar{z}y,{:}zz$ seu $zzdx \overset{(22)}{=} -d\bar{y}z + d\bar{z}y$ quam dividendo per $d\bar{x}$ fit $zz \overset{(23)}{=} \overline{-dy{:}dx}\, z + d\bar{z}y{:}dx$ seu per 3. $zz \overset{(24)}{=} -zz + d\bar{z}y{:}dx$. Seu $2zzdx \overset{(25)}{=} ydz$ seu $y \overset{(26)}{=} 2zz\, d\bar{x}{:}dz$ qui est valor inventus per

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75 $\log -1 = e$. Ergo—[*text stops*]

79 *Deleted* $fx = yy$ $\quad fdx = 2ydy \quad dy{:}dx = f{:}2y{::}-ydx \quad$ fit $fx = -2yy$ male

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74 x et y. (*1*) Iam $\log z = \log y + \log x$ (*2*) $\log x = \log$ (*3*) Iam per *L*

77 $b^a =$ (*1*) b^{2y-x} (*2*) $y - x$ (*3*) yx (*4*) $y^2{:}x$ *L*

80 erit (*1*) $d\bar{y} \overset{(5)}{=} -xdz + zdy$, quam dividendo per dx fit (*2*) dy *L*

85 ibi (*1*) litera 2 (*2*) fuit praefixa (*3*) numerus *L*

88–89 $a - \log y$ (*1*) quod significat (*2*) seu *L*

89 $xy \overset{(19)}{=}$ (*1*) numero cuius (*2*) b^a *L*

94 modum (*1*) tollendi y (*2*) inveniendi *L*

<u>substitutionem</u> <u>moderatam</u>. Sed ex aeq. 5 in qua nulla substitutio facta est habemus $x \overset{(27)}{=} -dy{:}dz - zd\bar{x}{:}dz$ quos duos valores ex 26. 27 substituendo in aeq. 4 evanescit dz, et fit $\textcircled{2}zzdx \overset{(28)}{=} d\bar{y}z$ $\textcircled{+ zzdx}$ Unde prodit $z = dy{:}dx$ ut ante. Itaque substitutio quam credebamus moderatam non fuit. At supra fuit, sufficit ergo uno modo obtineri aliquid per substitutionem moderatam.

Sit $z \overset{(1)}{=} dy{:}dx$ et $dy{:}dx \overset{(2)}{=} yx$. $z \overset{(3)}{=} yx$ fiet $dz \overset{(4)}{=} ydx + xdy$ seu $dz{:}dx \overset{(5)}{=} y + xz$. rursus $y \overset{(6)}{=} z{:}x$ ex 3, fiet $x^2dy \overset{(7)}{=} d\bar{z}x - d\bar{x}z$. seu $xx - \overline{d\bar{z}{:}d\bar{y}}x + \overline{\tfrac14 d\bar{z}^2{:}d\bar{y}^2} \overset{(8)}{=} \sqrt{\tfrac14 d\bar{z}^2{:}d\bar{y}^2 - zdx{:}dy}$ seu $x \overset{(9)}{=} \sqrt{\tfrac14 d\bar{z}^2{:}d\bar{y}^2 - zdx{:}dy} + \tfrac12 d\bar{z}{:}d\bar{y}$. ex aeq. 5 erat $y \overset{(10)}{=} dz{:}dx - xz$. ubi substituendo valorem ipsius x ex 9 fit $y \overset{(11)}{=} d\bar{z}{:}dx - z\sqrt{\ldots} - \tfrac12 zd\bar{z}{:}dy$. et hos valores aeq. 9 et 11 substituendo in aeq. 3 fit $z \overset{(12)}{=} \overline{dz{:}dx}\,\sqrt{\ldots} + \tfrac12 d\bar{z}^2{:}d\bar{y}d\bar{x} - \tfrac14 d\bar{z}^2{:}d\bar{y}^2 + zdx{:}dy - \tfrac14 d\bar{z}^2{:}d\bar{y}^2 - \overline{dz{:}d\bar{y}}\,\sqrt{\ldots}$ tollendo $d\bar{y}$ ope z fiet $\sqrt{\ldots} \overset{(13)}{=} \sqrt{\tfrac14 d\bar{z}^2{:}zz - d\bar{x}^2}{:}dx$ et fit $z \overset{(14)}{=} \overline{dz{:}d\bar{x}^2}\,\sqrt{\tfrac14 d\bar{z}^2{:}zz - d\bar{x}^2} + \tfrac12 d\bar{z}^2{:}zd\bar{x}^2 - \tfrac12 d\bar{z}^2{:}zzd\bar{x}^2 + 1 - dz\sqrt{\tfrac14 d\bar{z}^2{:}zz - d\bar{x}^2}{:}zd\bar{x}^2$. seu $zzzd\bar{x}^2 \overset{(15)}{=} z^2dz\sqrt{\ldots} + \tfrac12 d\bar{z}^2z - \tfrac12 d\bar{z}^2 + z^2d\bar{x}^2 - d\bar{z}z\sqrt{\ldots}$. Iam compendii causa sit $z^3 - zz \overset{(16)}{=} mzz$ et $zz - z \overset{(17)}{=} mz$ et $z - 1 \overset{(18)}{=} m$ et ex 15 fiet $\cancel{m}zzd\bar{x}^2 \overset{(19)}{=} \cancel{m}zd\bar{z}\sqrt{\ldots} + \tfrac12 \cancel{m}\, d\bar{z}^2$ vel $m = 0$ seu $z = 1$. Sed hoc misso pro $\sqrt{\ldots}$ seu pro $\sqrt{\tfrac14 d\bar{z}^2{:}zz - d\bar{x}^2}$ scribendo $\frac{1}{2z}\sqrt{d\bar{z}^2 - zzd\bar{x}^2}$. ex aeq. 19 fiet $2zzd\bar{x}^2 \overset{(20)}{=} d\bar{z}\sqrt{d\bar{z}^2 - zzd\bar{x}^2} + d\bar{z}^2$ vel $2zzd\bar{x}^2 - d\bar{z}^2 \overset{(21)}{=} d\bar{z}\sqrt{\ldots}$ unde quadrando fit $4z^4d\bar{x}^4 - 4zzd\bar{x}^2d\bar{z}^2 \textcircled{+d\bar{z}^4} \overset{(22)}{=} \textcircled{d\bar{z}^4} - zzd\bar{x}^2d\bar{z}^2$, et divisis omnibus per $zzd\bar{x}^2$, fit $4z^2d\bar{x}^2 \overset{(23)}{=} 3d\bar{z}^2$. seu $dx \overset{(24)}{=} \overline{d\bar{z}{:}z}\sqrt{3{:}2}$. Sed quia vereor ne subsit error in calculo sequenti scheda sequentis diei 11. septembr. 1690. repetemus.

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98 Substitutiones moderatas deprehendi hic non prodesse quia postremo plane tollenda *dy*.

104 $\tfrac12 xx = \int dy{:}y$.

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100 dz, (*1*) et fit $-dy - zdx \overset{(28)}{=} -zdy - zzdx$ (*2*) et fit *L*

107 $-xz$. (*1*) Et hos valores (*a*) substituendo fit (*b*) ex 9 et 10 substituendo in 3, fit $z =$ (*2*) ubi *L*

108 $y \overset{(11)}{=}$ (*1*) $z\sqrt{\tfrac14 \ldots} + \tfrac12 dz{:}dy$ (*2*) $d\bar{z}{:}dx$ *L*

110 tollendo $d\bar{y}$ (*1*) fiet $\sqrt{\ldots} =$ (*a*) $\tfrac14$ (*b*) $\sqrt{\tfrac14 dz}$ (*2*) ope z *L*

120 scheda (*1*) eiusdem diei. repetemus. 10 (*2*) sequentis *L*

METHODUS TANGENTIUM INVERSA PER SUBSTITUTIONES (MODERATAS) ASSUMENDO $z = dy{:}dx$

September 11, 1690

Textual tradition: Leibniz concept: LH XXXV 13,1. Leaves 300–301. 1 sheet 2°. 3 pages

11 Sept. 1690

Methodus tangentium inversa per substitutiones (moderatas,) assumendo $z = dy{:}dx$. Initia inventa in scheda praecedenti in fol. (est demiplagula) 10 Septemb. 1690.

Resumamus exemplum praecedentis schedae quia forte error in calculo, et majoris securitatis causa adhibeamus numeros: $z \overset{(1)}{=} dy{:}dx$ et $dy{:}dx \overset{(2)}{=} yx$ fit $z \overset{(3)}{=} yx$ et huius differentialis $d\bar{z} \overset{(4)}{=} xd\bar{y} + yd\bar{x}$. Rursus $y \overset{(5)}{=} z{:}x$, cuius differentialis $d\bar{y}xx \overset{(6)}{=} xdz - zdx$. Tollamus $d\bar{y}$ ex aeq. 6. dividendo eam per xdz, fiet $d\bar{y}xx{:}xd\bar{x} \overset{(7)}{=} xd\bar{z}{:}xdx - zdx{:}xdx$ seu $zx \overset{(8)}{=} dz{:}dx - z{:}x$. Seu fiet $zxxd\bar{x} + zdx \overset{(9)}{=} xdz$. seu $\int \overline{d\bar{z}{:}z} + a \overset{(10)}{=} \frac{1}{2}xx + \int \overline{dx{:}x}$ seu $\log z - \log x \overset{(11)}{=} \frac{1}{2}xx$. Iam $\log z - \log x \overset{(12)}{=} \log y$ per 5. Ergo denique fit $\log y \overset{(13)}{=} \frac{1}{2}xx$ quod est verum, nam ob aeq. 2. fit $\int \overline{d\bar{y}{:}y} + b \overset{(14)}{=} \frac{1}{2}xx$. hoc est $\log y \overset{(13)}{=} \frac{1}{2}xx$ ut ante. Et ita usi sumus una solum differentiali 6, videamus an liceat uti et altera 4, tollendo in ea dy fiet $d\bar{z}{:}dx \overset{(15)}{=} zx + y$ seu ex 5 $dz{:}dx \overset{(16)}{=} zx + z{:}x$ et prodit idem. Quid si tollere velimus x et dx, relicta y et dy. Nempe in aeq. 4 dividamus per dy fiet $d\bar{z}{:}dy \overset{(17)}{=} x + ydx{:}dy$ seu ex 1. tollendo $dx{:}dy$, et ex 5 tollendo x ex aeq. 17. fiet $d\bar{z}{:}dy \overset{(18)}{=} z{:}y + y{:}z$ seu $yzdz \overset{(19)}{=} zzdy + yydy$. Quod quidem verum est sed non nisi aptum ad solutionem. Iam similiter quaeramus valorem ipsius y ope aequationis novae ita ut prodeat sine x, sumendo ex aeq. 5, $x \overset{(20)}{=} z{:}y$ fiet: $yyd\bar{x} \overset{(21)}{=} ydz - zdy$ et tollendo dx per aeq. 1. fit $yydy \overset{(22)}{=} zydz - zzdy$. Ergo ex aeq. 19 et 22 aequando duos valores ipsius $yydy$ fit $zydz - zzdy \overset{(23)}{=} yzdz - zzdy$. Quae est aequatio

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125 Quae hic bona ut et in scheda 10 Septemb. Haec sunt in scheda 11 Septemb. in 4.° melius posita, et quicquid hic bonum in pauca contractum. NB. puto nihil referre substitutio sit moderata an immoderata. Ita est, nihil refert.

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132 $xdz - zdx$. *(1)* Hic iam nullis opus est differentialibus novis, et proinde sequentia licet praestare per Numeros. Tollendo z in aeq. 6. fit $dyxx = xdz - dy$. in 4 et 5 tollamus dy et ex 4 fiet: $d\bar{z}{:}dx \overset{((1))}{=} xz + y$ et ex 6 | dividendo per xdx *inserted* | fit $dz{:}dx \overset{((2))}{=} d\bar{y}xx{:}d\bar{x}x + z{:}x$ seu $d\bar{z}{:}dx = zx + z{:}x$ seu dz *(2)* Tollamus *L* identica unde discimus nihil. Quae res non parum turbat, et dubitare facit, an methodus haec nostra semper procedat.

Sit $z \overset{(1)}{=} dy{:}dx \overset{(2)}{=} ax + by \overset{(3)}{=} z$. Ipsius aequationis 3 differentialis est $adx + bdy \overset{(4)}{=} dz$, et tollendo $d\bar{x}$, fiet $ad\bar{y}{:}z + bdy \overset{(5)}{=} dz$. Seu $y \overset{(6)}{=} \int \overline{d\bar{z}{:}\overline{a{:}z + b}} + c$. Itaque soluta est aequatio in qua $axdx + bydx \overset{(7)}{=} dy$. Nam fit $y \overset{(8)}{=} \int \overline{zdz{:}\overline{a+bz}} + c$ quae pendet ex quadratura Hyperbolae.

Sit $z \overset{(1)}{=} dy{:}dx$ et $ax + by \overset{(2)}{=} czx + ezy$ fiet $y \overset{(3)}{=} \overline{ax - czx},{:}\overline{ez - b}$. Huius aeq. 3 differentialis erit $d\bar{y}$, $\boxed{2}\ \overline{ez - b} \overset{(4)}{=} \overline{ez - b}\ \overline{adx - czdx - cxdz} - \overline{ax - czx}\ ed\bar{z}$ et tollendo dy per 1 faciendoque compendii causa $-ez + b = n$ et $-cz + a = m$ fiet $zn{:}dx \overset{(5)}{=} mndx - ncxdz - emxdz$ et $me + nc$ sit $\overset{(6)}{=} f$ fiet $x \overset{(7)}{=} mndx - znn{:}dx,{:}fd\bar{z}$ et eodem modo $y \overset{(8)}{=} mndy - zmm{:}dy,{:}fdz$, et tollendo dy, per zdx, fiet $y \overset{(9)}{=} mnzdx - mm{:}dx,{:}fdz$. Quos valores x et y ex 7 et 9 substituendo in aeq. 2. evanescit $d\bar{z}$ nec quicquam lucramur. Iam talis aequatio resolvi potest qualis est 2. quia ibi x et y per se solae servant legem homogeneorum. Scribamus ergo $z \overset{(1)}{=} d\bar{y}{:}dx$ et $h + ax + by \overset{(2)}{=} czx + ezy$. fiat $y \overset{(3)}{=} h + ax - czx,{:}\overline{ez - b}$ compendii causa $cz - a \overset{(4)}{=} m$ et $ez - b \overset{(5)}{=} n$ fit $y \overset{(6)}{=} h - mx,{:}n$ ergo fit $dm \overset{(7)}{=} cdz$ et $dn \overset{(8)}{=} edz$ ergo ex 6, 7, 8 fiet $d\bar{y}nn \overset{(9)}{=} -mdx - cxdz - hedz + emxdz$ ubi rursus $em - c \overset{(10)}{=} p$ et $cn - e \overset{(11)}{=} q$ fiet $dynn + mdx + hedz,{:}p \overset{(12)}{=} x$. Et pro dy ponendo zdx, fiet $x \overset{(13)}{=} \overline{znn + m}\,dx + hed\bar{z},{:}pdz$. Iam ad imitationem aequationis 12 statim scribere possumus $y \overset{(14)}{=} d\bar{x}mm + n\,\overset{zdx}{(dy)} + hcdz,{:}qdz$. Quos duos valores literarum x et y substituendo in aeq. 2 vel eius loco in aeq. 15, quae 4 et 5 est $h \overset{(15)}{=} mx + ny$, tunc fiet $hpqdz \overset{(16)}{=} qm\,\overline{\overline{znn + m}dx + hedz} + pn\,\overline{\overline{zn + mm}\,dx + hcdz}$, ubi sufficit videri an maneat dz quod fiet modo non sit $pq \overset{((17))}{=} emq + cnp$.

154–155 Notandum artificium ut x et y tractentur eodem modo, ita contrahitur calculus.

171 ((. . .)) sic noto aequationes quae non omnino, sed tentamenti causa assumuntur.

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156 $-emxdz$ (*1*) seu $x = \overline{mndx - zmm{:}dx{:}dz\overline{mc + en}}$ (*2*) et $me + nc$ *L*

160 lucramur. (*1*) Itaque aliud (*2*) Sed aliunde (*3*) Itaque sit (*4*) Iam talis *L*

162 $czx + ezy$. (*1*) fiet $adx + bdy = czdx + e$ (*2*) fiet $adx + bdy = czdx + z$ (*3*) fiat $y \overset{(3)}{=}$ *L*

164–165 $+emxdz$ (*1*) ubi rursus $em + c,{:}nn =$ (*2*) ubi rursus $em + c \overset{(10)}{=} p$ et similiter (*3*) ubi rursus (*a*) $em + c,{:}nn \overset{(10)}{=} p$ (*b*) $em - c \overset{(10)}{=} p$ *L*

Quod experiamur, fingendo sit $a = 1$ et $b = 2$ et $c = 3$ et $e = 4$, et $z = 5$, ergo per aeq. 4 fiet $m = 14$ et per aeq. 5 fit $n = 18$ et per 10 fit $p = 53$ et per 11 fit $q = 50$ ergo $p \cdot q = 53 \cdot 50$ et $emq = 4 \cdot 14 \cdot 50$ et $cnp = 3 \cdot 18 \cdot 53$ ergo deberet esse in numeris veris $53 \cdot 50 = 4 \cdot 14 \cdot 50 + 3 \cdot 18 \cdot 53$, seu deberet esse $53 \cdot 25 = 2 \cdot 14 \cdot 50 + 3 \cdot 9 \cdot 53$ quod fieri non potest quia $2 \cdot 14 \cdot 50$ non potest dividi per 53. Similiter, si $d\bar{z}$ destrueretur seu si foret $pq = emq + cnp$, deberet etiam fieri ob reliqua $qmz\overline{nn + m} \overset{((18))}{=} pnz\overline{n + mm}$. Iam $znn + m = 5 \cdot 18^2 + 14 = 1634$ et $zn + mm = 90 + 14^2 = 286$ fit $14 \cdot 50 \cdot 1634 = 18 \cdot 53 \cdot 286$. Quod etiam fieri non potest una enim pars dividitur per 7. altera non item. Sed terminos actu ipso explicemus.

$$pq = cemn + ce - eem - ccn$$
$$53 \cdot 50 = 34 \cdot 14 \cdot 18 + 3 \cdot 4 - 4 \cdot 4 \cdot 14 - 3 \cdot 3 \cdot 18$$

$$mn = cezz + ab - aez - bcz$$
$$14 \cdot 18 = 3 \cdot 4 \cdot 5 \cdot 5 + 1 \cdot 2 - 1 \cdot 4 \cdot 5 - 2 \cdot 3 \cdot 5$$

Ergo fit $pq =$ $cceezz + abce - aceez - bccez$
$+ ce - ceez + aee - ccez + bcc =$
$emq + cnp =$ $cceezz - aceez + eea - bccez + abce - eecz$
$+ cceezz - bccez + ccb - aceez + abce - ccez.$
⟨———⟩ dz in $-zccee\overline{zz + cceez + ee}, = qmz\,\overline{nn + m} + pnz\,\overline{n + mm}, dx.$
Quae posterior aequationis pars adhuc ⟨———⟩ foret explicanda, ut fieri facile potest, sed non ⟨———⟩ quia x et y per se tunc servant leges homogenorum. sufficit rem esse in potestate modo ⟨———⟩ non succedit altera nostra methodus.

$$mmn = ccez^3 - 2acez^2 + a^2ez - bccz^2 + 2abcz - aab.$$
$14^2 18$ 0 3 2 0 6 7

$$mnn = ceez^3 - 2bcez^2 + b^2cz - aeezz + 2abez - abb.$$
$14 \cdot 18 \cdot 18$ $3 \cdot 16 \cdot 125 - 2 \cdot 2 \cdot 3 \cdot 4 \cdot 25 + 4 \cdot 3 \cdot 5 - 1 \cdot 16 \cdot 25 + 2 \cdot 2 \cdot 4 \cdot 5$ 4

$$mm = cczz - 2acz + aa. \qquad nn = eezz - 2bez + bb.$$
7 0 6 1 $18 \cdot 18$ $16 \cdot 25 - 2 \cdot 2 \cdot 4 \cdot 5 + 4$
~~0~~ ~~4~~ ~~1~~ ~~4~~

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172 Debebam ponere $e = 6$, foret $n = 2m$.

196 $4 = 0 + 3 + 1 \qquad 0 = 3 + 2 + 7 + 6$

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172 experiamur, (*1*) ex $pq \overset{(18)}{=} ecmn - eecn - eccn$ (*2*) $ecmn - ccm - ccn + cc$ (*3*) fingendo sit *L*

$$\begin{array}{ccccccccc} mq & = & ccnz & + & ae & - & acn & - & cez \\ 14\cdot 50 & & 9\cdot 18\cdot 5 & + & 1\cdot 4 & - & 1\cdot 3\cdot 18 & - & 3\cdot 4\cdot 5 \\ 7 & & 0 & & +4 & & -0 & & 3 \end{array} \qquad pn = eemz + bc - bem - cez$$

Ut calculum absolvamus, invendiendus valor quantitatis p, $mmn + nn$ et q, $mm + mnn$. invenimus autem esse

$$mmn = ccez^3 - 2acez^2 + a^2ez - bccz^2 + 2abcz - a^2b$$
$$mnn = ceez^3 - 2bcez^2 + b^2cz - aeez^2 + 2abez - ab^2$$
$$mm = cczz - 2acz + aa$$
$$nn = eezz - 2bez + bb$$
$$p = cez - ae - c \qquad q = cez - bc - e$$

per compendium faciamus

$$\begin{array}{ll} r = ee - bcc - 2ace & t = a^2e + 2abc - 2be \\ \overset{\frown}{-26} \quad +16 - 18 - 24 & \overset{\frown}{0} \quad 4 + 12 - 16 \end{array}$$

$$\begin{array}{ll} s = cc - aee - 2bce & v = b^2c + 2abe - 2ac \\ \overset{\frown}{-55} \quad +9 - 16 - 48 & \overset{\frown}{22} \quad 12 + 16 - 6 \end{array}$$

$$\begin{array}{ll} w = bb - a^2b & \psi = ae + c \\ \overset{\frown}{2} \quad 4 - 2 & \overset{\frown}{7} \end{array}$$

$$\begin{array}{ll} \mu = aa - ab^2 & \omega = bc + e \\ \overset{\frown}{-3} \quad 1 \quad -4 & \overset{\frown}{10} \end{array}$$

fiet $\underset{14^2\cdot 18}{mmn} + \underset{18^2}{nn} = ccez^3 + \underset{0}{rz^2} + \underset{7}{tz} + \underset{+2}{w}$, in $\underbrace{cez - \psi}_{53}$

dat $$\begin{array}{ccccccc} c^3e^2z^4 & + & cerz^3 & + & cetz^2 & + & cewz \\ & - & cce\psi z^3 & - & r\psi z^2 & - & t\psi z \quad - \quad w\psi \\ 0 & & \not 6 & & \not 5 & & \not 3 \quad\quad \not 4 \end{array}$$

202–204 *Deleted:* $mmn + mm = ccez^3 - 2aecz^2 + a^2ez - bc^2z^2 + 2abcz - a^2b + cczz - 2acz + aa$ mult. per $p = em - c = cez - ae - c$ seu $mmn + mm = ccez^3 + \overline{cc - bc^2 - 2acez^2 + a^2e + 2abc} - 2acz + aa - a^2b$. Sit $cc - bcc - 2ace = \underset{(-42)}{r}$ et $a^2e + 2abc - 2ac = \underset{(10)}{t}$ et $ee - aee - 2bce = \underset{(48)}{s}$ et $b^2c + 2abe - 2be = \underset{(12)}{y}$ et $a^2 - a^2b = \underset{(-1)}{w}$ $ae + c = \underset{(7)}{\mu}$ et $b^2 - abb = \underset{(0)}{\psi c}$ + $e = \underset{(10)}{\omega}$. fiet $\underset{53\cdot 14^2\cdot 18}{pmmn} + \underset{53\cdot 14^2}{pmm} = c^3e^2z^4 + cerz^3 + cetz^2 + cewz - ce\mu z^3 - v\mu z^2 - t\mu z - w\mu$.

$$mm + mnn = ceez^3 + sz^2 + vz + \mu, \text{ in } \underbrace{cez - \omega}_{50} \text{ dat}$$

$$\begin{array}{llllll} 14^2 & 0 & 6 \;+\; 2 \;+\; 2 & 6 & 50 \\ 7 & & & & \end{array}$$

$$\begin{array}{lllll} cce^3z^4 + & cesz^3 & + cevz^2 & + ce\mu z & \\ & - ceewz^3 & - s\omega z^2 & - v\omega z & - \mu\omega \\ 0 & 6 & 1 & 7 & 3 \end{array}$$

Iam in z^4 nihil destrui potest, addamus in unum coefficientes z^3, fiet

$$z^3 \text{ in } \begin{cases} \overbrace{\textcircled{ce^3} - bc^3e - {}^3\textcircled{2}accee}^{+\cancel{cer}} \quad \overbrace{-\textcircled{accee} - \textcircled{c^3e}}^{-\cancel{cee\phi}} \\ \overbrace{\textcircled{c^3e} - ace^3 - {}^3\textcircled{2}bccee}^{+\cancel{ces}} \quad \overbrace{\textcircled{-bccee} - \textcircled{ce^3}}^{-\cancel{cee\omega}} \end{cases}$$

seu fit z^3 in $-ce\overline{bc^2 + ae^2 + 3ace + 3bce}$.

$$zz \text{ in } \begin{cases} \overbrace{\textcircled{+a^2ce^2} + {}^3\textcircled{2}abc^2e - \textcircled{2bcee}}^{+\cancel{eer}} \\ \\ \overbrace{\textcircled{-ae^3} + \textcircled{abc^2e} + {}^3\textcircled{2}aacee - cee + \textcircled{bc^3} + \textcircled{2acce}}^{-\cancel{r\phi}} \\ \overbrace{\textcircled{+b^2c^2e} + {}^3\textcircled{2}abce^2 - \textcircled{2ac^2e}}^{+\cancel{cev}} \\ \overbrace{\textcircled{-bc^3} + \textcircled{abce^2} + {}^3\textcircled{2}bbcce - cce\textcircled{+\ ae^3}\ \textcircled{+\ 2bcee}}^{-\cancel{s\omega}} \end{cases}$$

seu fit zz in $ce \begin{cases} 3abc + 3aae - c \\ 3abe + 3bbc - e \end{cases}$

$$z \text{ in } \begin{cases} \overbrace{b^2ce \; - a^2bce}^{+\,\not{cew}} \\[6pt] \overbrace{-a^3e^2 - {}^3\textcircled{2}\,a^2bce + \textcircled{2}\,aabee - a^2ce - 2abcc + 2bce}^{-\,\not{t\psi}} \\[6pt] \overbrace{aace \; - \; ab^2ce}^{+\,\not{cep}} \\[6pt] \overbrace{-b^3c^2 - {}^3\textcircled{2}\,ab^2ce + \textcircled{2}\,abbce - b^2ce - 2abee + 2ace}^{-\,\not{t\psi}} \end{cases}$$

$$\text{seu } z \text{ in } \begin{cases} -a^3e^2 + a^2bce - 2abcc + 2bce \\ -b^3c^2 + ab^2ce - 2abee + 2ace \end{cases}$$

$$z^0 \text{ in } \begin{cases} \overbrace{-abbe + a^3be - bbc + a^2bc}^{-\,\not{w\psi}} \quad \overbrace{-aabc + ab^3c - aae + ab^2e}^{-\,\not{\mu\varpi}} \end{cases}$$

$$\text{seu } z^0 \text{ in } \begin{cases} +a^3be - bbc \\ +ab^3e - aae \end{cases}$$

Est error in calculo. Haec omnia non procedunt nec licet simul tollere generaliter x et y. Ergo tandem fit, $d\bar{x} =$

$$hd\bar{z} \text{ in } \frac{\begin{matrix} -2cceezz + ccee & & +ce \\ bcce & z & -abce \\ acee & & \end{matrix}}{\begin{matrix} +c^3ee \\ +cce^3 \end{matrix} z^4 \; -ce \begin{Bmatrix} +bc^2 \\ +ae^2 \\ +3ace \\ +3bce \end{Bmatrix} z^3 \; +ce \begin{Bmatrix} +3abc \\ +3abe \\ +3a^2e \\ 3b^2c \\ -c \\ -e \end{Bmatrix} z^2 \begin{matrix} -a^3e^2 \\ -b^3c^2 \\ +a^2bce \\ +ab^2ce \\ -2abcc \\ -2abee \\ +2ace \\ +2bce \end{matrix}\Bigg\} z \; \begin{matrix} +a^3be \\ +ab^3e \\ -aae \\ -bbc \end{matrix}}$$

249–251 $\textcircled{-a^2bce}\,(1) - a^2bce - 2ab^2c^2 + 2b^2ce - a^2ec - 2abce + 2bce\,(2) - a^3e^2$ L

Cuius ope solvetur aequatio $h + ax + by = czx + ezy$ posito $z = dy{:}dx$ sed si absit h vel c, vel e non procedit. Modo calculus rectus est, cui non fido, nisi in numeris peregerim.

Interim sumamus exemplum $h \overset{(2)}{=} czx + ezy$ $z \overset{(1)}{=} dy{:}dx$ $y \overset{(3)}{=} h{:}z - cx$. Ergo $dy \overset{(4)}{=} -hdz{:}zz - cdx$ et pro dy ponendo zdx fit $zdx \overset{(5)}{=} -hdz{:}zz - c\overline{dx}$ seu fit $d\bar{x} \overset{(6)}{=} -hd\bar{z}{:}\overline{zz\,\overline{z + c}}$ atque ita habetur solutio ex quadratura Hyperbolae.

Rursus sit $z \overset{(1)}{=} dy{:}dx$ et $h + ax \overset{(2)}{=} ezy$. Ergo fit eius differentialis $adx \overset{(3)}{=} ezdy + eydz$ seu $ad\bar{x} \overset{(4)}{=} ezzdx + eydz$ ex 1 et 3 et fit $y \overset{(5)}{=} adx{:}dz - ezzdx{:}dz$. Iam supra ex aeq. 2 fit $y \overset{(6)}{=} h + ax,{:}ez$ quos valores aequando fit $dx{:}\overline{h + ax} \overset{(7)}{=} d\bar{z}{:}\overline{ez\,\overline{a - ezz}}$.

Sit $zdx \overset{(1)}{=} dy$ et $h + by \overset{(2)}{=} ezx$ ergo fit $bdy \overset{(3)}{=} ezdx + exdz$. Ergo $bzdx \overset{(4)}{=} ezdx + exdz$ et fit $dx\overline{b - e}{:}x \overset{(5)}{=} dz{:}z$ et habetur solutio.

Sit $ax + by \overset{(1)}{=} czx + ezy$ et $zdx \overset{(2)}{=} dy$. Sit $cz - a \overset{(3)}{=} m$ et $ez - b \overset{(4)}{=} n$ fiet $dm \overset{(5)}{=} dz \overset{(6)}{=} dn$ et ex aeq. 1 fit $mx + ny \overset{(7)}{=} 0$ ergo $x \overset{(8)}{=} -ny{:}m$ et $mmdx \overset{(9)}{=} -ndy - ydz + nydz$ seu $y \overset{(10)}{=} \overline{mm + nz}\,dx,{:}\overline{-dz + ndz} \overset{(11)}{=} -mx{:}n$. Ergo fit $dx{:}mx \overset{(12)}{=} dz\overline{n - 1}{:}n\overline{mm + nz}$ et pari iure $dy{:}nx \overset{(13)}{=} dz\overline{m - 1}{:}m\overline{nn + mz}$. Sin fuisset $mx + ny \overset{(14)}{=} h$ seu $x \overset{(15)}{=} h - ny{:}m$ fiet $m^2dx = \overline{-ndy - ydz}\,m + \overline{h - ny}\,dz$, et pro dy ponendo zdx fit $m^2dx + mnzdx + hdz,{:}\overline{1 - m}\,dz$ ($=y$) $\overset{(16)}{=} h - mx,{:}n$ unde colligo si semel detur $dx{:}dz$ per x et z non posse dari adhuc semel generaliter alioqui ⟨— non esset —⟩ Itaque Methodus ista non procedit ⟨———⟩ tollat y, dx et x.

Possumus etiam assumere, ut z non sit $= dy{:}dx$ sed aliquid praeterea ut sit $h + ax + by \overset{(1)}{=} cxdy{:}dx + eydy{:}dx$. Sit $z \overset{(2)}{=} \overline{cx + ey}\,dy{:}dx$ seu $dy \overset{(3)}{=} zdx{:}\overline{cx + ey}$ et fiet $h + ax + by \overset{(4)}{=} z$ adeoque fiet $adx + bdy \overset{(5)}{=} dz$ et per 3 fit $a\,\overline{cx + ey}\,dx + bzdx \overset{(6)}{=} \overline{cx + ey}\,dz$. Iam ex aeq. 4 est $y = z{:}b - h{:}b - ax{:}b$. Ergo fit $acxdz + \overline{aez{:}b}\,dx - \overline{aeh{:}b}\,dx + \overline{a^2ex{:}b}\,dx + bzdx = cxdz + \overline{aez{:}b}\,dz - \overline{aeh{:}b}\,dz + \overline{aaex{:}b}\,dz$ sed nil hinc lucrum.

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290 $d\overline{-n{:}m} = -d\overline{n{:}m}$. $d\overline{n{:}m} = d\bar{n}m - d\bar{m}n,{:}mn$. ergo $d\overline{-n{:}m}$ $d\bar{n}m + d\bar{m}n{:}mn$. $d\overline{-n{:}m}$

284–285 $-ezzdx{:}dz$. (*1*) quem valorem substituendo in aeq. 3 (*2*) Iam *L*

289 $zdx \overset{(2)}{=} dy$. (*1*) fit $adx + bdy \overset{(3)}{=} czdx + cxdz + ezdy + eydz$. seu $adx + bzdx \overset{(4)}{=} czdx + cxdz + ez^2dx + eyd\bar{z}$. $x \overset{(5)}{=} \overline{ez - b}y{:}-\overline{cz - a}$ seu $x \overset{(6)}{=} ny{:}-m$ fit (2) Sit $cz - a$ *L*

296 generaliter *L inserts*

Tale quid in mentem venit sit aequatio, verbi gratia $h + ax + by \overset{(1)}{=} cxdy{:}dx + ey\,dy{:}dx$ seu $ax + by \overset{(2)}{=} cxdy{:}dx + ey\,dy{:}dx - h$ fiat $z \overset{(3)}{=} cxdy{:}dx + ey\,dy{:}dx - h,{:}x.$ fiet $ax + by \overset{(4)}{=} xz$, adeoque $ax + by \overset{(4)}{=} xz$ seu $adx + bdy = zdx + xdz.$

$ax + by \overset{(1)}{=} xy + z$ fit $adx + bdy \overset{(2)}{=} xdy + ydx + dz$. ex aeq. 1 est $y \overset{(3)}{=} ax - z,{:}\overline{x - b}$ ergo fit $\overline{adx + bdy - dz}\;\overline{x - b} - dy\,x.\;\overline{x - b} = \overline{ax - z}dz.$ $\overline{cz - a}\,x + \overline{ez - b}\,y \overset{(1)}{=} 0 \overset{(2)}{=} mx + ny$ fit $x = 0 - ny{:}m$ et fiet $-mndy - mydz + nydz, = mmdx$ et fit $mmdx + mndy,{:}\overline{ndz - mdz} = y$ et $nndy + mndx,{:}mdz - ndz = x$, et ambos valores substituendo in aeq. 2 fiet $mnndy + mmndx - mmndx - mnndy = 0$ et ita evanescit et dz.

Sit $mx + ny \overset{(1)}{=} h$ fit $x \overset{(2)}{=} -ny{:}m + h{:}m$ et $mmdx \overset{(3)}{=} -mndy - mydz + nydz - hdz$. Ergo $y \overset{(4)}{=} mmdx + mndy + hdz,{:}\overline{n - m}\,dz$. Et similiter $x \overset{(5)}{=} nndy + mndx + hdz,{:}\overline{m - n}\,dz$. Ergo hos valores substituendo in aeq. 1 fiet $mnndy + mmndx + mhdz + mmndx - mnndy - nhdz =$ $hmdz - hndz.$

Quae rursus est identica. Itaque nihil sic lucramur nec possumus tollere simul x et y. Itaque aliud in mentem venit, ubi praesens artificium ponendi $z = dy{:}dx$ et quaerendi $d\bar{z}$ combino cum alio artificio seu observatione, qua deprehendi semper posse aequationem differentialem resolvi, quando x et y per se solae servent homogeneitatem. Quod si ergo adsit aliqua constans vel plures, primum semper plures constantes reducemus ad unam. Sint enim a, b, c, et cetera. pro b ponere possum βa, et pro c ponere possumus κa. ita ut β et κ sint numeri, sola vero a sit linea. Sit ergo $dy \overset{(1)}{=} zdx$. Et sit aeq. (2) proposita, inter $x.\ y.\ z.\ a.$ Huius quaerantur

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304 verbi gratia (*1*) $h + ax = cxdy{:}dx + eydy{:}dx$ (*2*) $h + ax + by$ *L*

306 xz, (*1*) seu $a + by{:}x = z$ (*2*) seu $a + bv = z$ posito $v = y{:}x$ (*3*) adeoque *L*

309 ergo fit (*1*) $\overline{adx + bdy}, \overline{x - b} - xdy -$ (*2*) $\overline{adx + bdy - dz}$ *L*

309–310 $\overline{ax - z}dz.$ (*1*) $ax \overset{(1)}{=} xy + z$ fit $adx \overset{(2)}{=} zdy + ydz + dz$ (*2*) $ax + by \overset{(1)}{=} czx + ezy$ fit $adx + zxy$ (*3*) $ax + by = zxy$ (*4*) $\overline{cz - a}x$ *L*

309 $mx + ny.$ (*1*) $z = dy{:}dx$ (*2*) fit x *L*

311 $mmdx$ (*1*) ubi pro y substituendo (*2*) et fit $y = mmdx - mndy$ (*3*) et fit *L*

313 et ita (*1*) tolluntur ambae (*2*) evanescit *L*

320 Itaque aliud (*1*) artificium (*2*) in mentem *L*

326 linea. (*1*) Sit (*2*) Ergo aeq. (*3*) Sit ergo $z = xdx$. Et sit (*4*) Sit *L* differentiales duae, (3) et (4), quas combinando cum ipsa 2, habebimus duas aequationes (5) et (6) in quibus aberit a, iam ope aequationum 5 et 6 tollatur y fit aeq. (7). ex qua ope aeq. 1 tollatur dy. habebitur aeq. (8) in qua erunt solum z, x, dz, dx. servantibus legem homogeneorum ipsis z et x adeoque solubilis erit per quadraturas. Sed quia verendae destructiones res reipsa tentanda. Sit $zdx \overset{(1)}{=} dy$ et $h \overset{(2)}{=} cxz + eyz$ ubi h est constans quae sola turbat homogeneitatem nam, z est ratio, et c atque e, sunt ut numeri. fit $cxdz + czdx + eydz + ezdy \overset{(3)}{=} 0$ quae est sine h. rursus $h{:}z \overset{(4)}{=} cx + ey$. fit $-hdz{:}zz \overset{(5)}{=} cdx + edy \overset{(6)}{=} -cx - ey, d\bar{z}{:}z$. Habemus ergo duas aequationes in quibus abest a, in quibus tollendo dy per aeq. 1. fit ex aeq. 3 $cxdz + czdx + eydz + ezzdx \overset{(7)}{=} 0$ et ex aeq. 6 fit $zcdx + ezzdx + cxdz + eydz \overset{(8)}{=} 0$ quae duae aequationes 7 et 8 coincidunt inter se, itaque nihil hac ratione lucramur.

333 res reipsa tentanda. (*1*) Sit $h = zy +$ (*2*) $h \overset{(1)}{=} cxz + eyz$ fit $h \overset{(2)}{=} exz$ (*3*) Sit $zdx \overset{(1)}{=} dy$ *L*

ENGLISH TRANSLATION OF "METHODUS PRO DIFFERENTIALIBUS, PONENDO $z = dy{:}dx$ ET QUAERENDO dz"

September 10, 1690

Textual tradition: Leibniz concept: LH XXXV 13, 1. Leaf 302.
1 sheet 2°. 2 pages

September 10, 1690.[1] A method for differentials, positing $z = dy/dx$ and seeking dz.

Let $zx \overset{(1)}{=} y$ and $zdx + xdz \overset{(2)}{=} dy$ and $z \overset{(3)}{=} y/x$, which yields $dz \overset{(4)}{=} dy/x - ydx/x^2$. Let us eliminate y from equation 4 by means of equation 1, and from this, by means of equation 2 or through $x \overset{(5)}{=} (-zdx + dy)/dz$, there will result from equation 4

$$dz \overset{(6)}{=} \frac{dydz}{-zdx + dy} - \frac{zdxdz}{-zdx + dy},$$

which yields $-zdxdz + dydz \overset{(7)}{=} dydz - zdxdz$, which is an identical equation.

Let $z \overset{(1)}{=} xy + a$ and $dz \overset{(2)}{=} xdy + ydx$ and $z/y \overset{(3)}{=} x + a/y$ and $dz/y - dyz/y^2 \overset{(4)}{=} dx - ady/y^2$. Let us posit that $z \overset{(5)}{=} dy/dx$. From equations 2 and 5 there will result $dz/dx \overset{(6)}{=} xz + y$. By means of equations 1, 4, and 6, let us try to eliminate x and y. Through equation 6, there is $y \overset{(7)}{=} dz/dx - xz$. Let this value be substituted in equation 1; this yields $z \overset{(8)}{=} xdz/dx - x^2z$. Let the same value for y from equation 7 be substituted in equation 4; this yields

$$dz\left(\frac{dz}{dx} - xz\right) - (dy)z \overset{(9)}{=} dx\left(\frac{dz^2}{dx^2} - 2xz\frac{dz}{dx} + x^2z^2\right) - ady.$$

Through equations 8 and 9, let x be eliminated. From equation 8, there is

$$x^2 - \frac{zdz}{dx}x + \frac{1}{4}\frac{z^2dz^2}{dx^2} \overset{(10)}{=} \frac{1}{4}\frac{z^2dz^2}{dx^2} - 1$$

or $2xdx \overset{(11)}{=} \sqrt{z^2dz^2 - dx^2} + zdz$, and from equation 9, with x thus expanded, there results

$$dz\frac{dz}{dx} - zdz\frac{\sqrt{z^2dz^2 - dx^2}}{2dx} + \frac{zdz^2}{2dx} - (dy)z$$
$$\overset{(12)}{=} dx\left(\frac{dz}{dx} - \frac{z}{2dx}(\sqrt{z^2dz^2 - dx^2} + zdz)\right)^2.$$

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[1] To an extent this is good, but it has been set forth better elsewhere; see the page in 8° for September 11, 1690.

Thus an equation is obtained in which only dx, dy, z, and dz occur, or in effect, besides the letter z there will occur dy/dx and dz/dx. Let dy/dx be eliminated, since it is equal to z. Thus only dx will remain, whose value is obtained through dz and z. And so, with this established, the matter is reduced to quadratures. Unless, of course, by expressing dy/dx through z, even dz will vanish, which is the only thing I fear, but perhaps for this problem it will be possible to give the remedy of not entirely eliminating dy, except in certain places where it impedes summability, just as if it were the case that $ady + bdx = dz(\bar{z})$, whence it had been possible to make $az + b = dz(\bar{z})/dx$ or $dx = dz(\bar{z})/(az + b)$. On the contrary, I observe that if it is permissible to eliminate x and y at once, with dz remaining, as I believe, because we have used equation 5 moderately, then thereafter dz cannot be eliminated, when dy is taken away, because otherwise dx alone would remain and what is more, it would vanish, and z would be precisely determined, which is absurd.

Let us resume the example given above: $zx \overset{(1)}{=} y$, $z \overset{(2)}{=} dy/dx$.[2] The differential of equation 1 is $zdx + xdz \overset{(3)}{=} dy$, and again, from 1, there is $x \overset{(4)}{=} y/z$, whose differential is $dx \overset{(5)}{=} dy(z) - ydz/z^2$. And from equations 5 and 2 there results $1 \overset{(6)}{=} z - ydz/dx$. From equation 3 we get an expression for x without y, or $x \overset{(7)}{=} dy/dz - zdx/dz$, and from equation 6 there is

$$y \overset{(8)}{=} z\frac{dx}{dz} - \frac{dz}{dx}.$$

Substituting these values from equations 7 and 8 in equation 1 yields

$$\frac{zdy}{dz} - \frac{z^2dx}{dz} \overset{(9)}{=} z\frac{dx}{dz} - \frac{dz}{dx}$$

or $zdydx - z^2dx^2 \overset{(9)}{=} zdx^2 - dz^2$; and with all this divided through by dx^2, and taking z for dy/dx, there results

$$\boxed{z^2 - z^2} \overset{(10)}{=} z - \frac{dz^2}{dx^2} \quad \text{or} \quad dx = \frac{dz}{\sqrt{z}}.$$

It is feared that there is an error in the calculation.

[2] $y/x = dy/dx$. The locus is of the straight line. If it were $y/x = -dy/dx$ it would be of the hyperbola.

Let us resume: $zx \overset{(1)}{=} y$, $z \overset{(2)}{=} dy/dx$. The differential of equation 1 is $zdx + xdz \overset{(3)}{=} dy$.[3] From this equation, let $x \overset{(4)}{=} y/z$. The differential of this equation 4 will be $dx \overset{(5)}{=} dy/z - dz(y)/z^2$, and thus above, in equation 5, I have calculated badly, taking dy for dy/z.[4] From equations 5 and 2 there results $1 \overset{(6)}{=} 1 - dz(y)/dx(z^2)$, which appears to be absurd for it yields $dz(y)/dx(z^2) = 0$, which indicates that the locus is that of a straight line; whence z is necessarily constant, and thus $dz \overset{(8)}{=} 0$, which is an extraordinary outcome.

Let $z \overset{(1)}{=} yx$, let $y \overset{(2)}{=} z/x$, which yields $dy \overset{(3)}{=} dz/x - (dx)z/x^2$ or $x^2dy \overset{(4)}{=} (dz)x - (dz)z$ and $dz \overset{(5)}{=} ydx + xdy$, whence there will result $dz/dx \overset{(6)}{=} y + xz$, it having been posited that $z \overset{(7)}{=} dy/dx$. From equation 4 there is

$$x^2 - \frac{dz}{dy}x + \frac{1}{4}\frac{dz^2}{dy^2} \overset{(8)}{=} \frac{1}{4}\frac{dz^2}{dy^2} - \frac{(dz)z}{dy}$$

or

$$x \overset{(7)}{=} \sqrt{\frac{1}{4}\frac{dz^2}{dy^2} - \frac{(dz)z}{dy}} + \frac{1}{2}\frac{dz}{dy}.$$

Now $y \overset{(8)}{=} dz/dx - xz$ and $y \overset{(9)}{=} z/x$. Thus there results

$$\frac{dz}{dx}x - x^2z \overset{(10)}{=} z$$

or

$$x^2 - \frac{dz}{zdx}x + \frac{1}{4}\frac{dz^2}{dx^2z^2} \overset{(11)}{=} 1 + \frac{1}{4}\frac{dz^2}{dx^2z^2}$$

or

$$x \overset{(12)}{=} \sqrt{1 + \frac{1}{4}\frac{dz^2}{dx^2z^2}} + \frac{1}{2}\frac{dz}{dxz}.$$

By setting these two values for x equal, from equations 7 and 12, the foregoing having been multiplied first by z^2 or dy^2/dx^2, there results

$$\sqrt{\frac{1}{4}\frac{dz^2}{dx^2} - \frac{z^2dz}{dx}} + \frac{1}{2}z\frac{dz}{dx} \overset{(13)}{=} z^2\sqrt{1 + \frac{1}{4}\frac{dz^2}{dx^2z^2}} + \frac{1}{2}z\frac{dz}{dx}$$

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[3] $x \overset{(3)}{=} (dy - zdx)/dz$.

[4] $y \overset{(5)}{=} (-z^2dx + dyz)/dz$ yields, from 1, 3, and 5, $(dy - z^2dx)/dz = -z^2dx + dyz$, whence nothing new but it was not needed; on the contrary here, with dy removed, it makes z constant.

or

$$\frac{1}{4}\frac{dz^2}{dx^2} - \frac{dzz^2}{dx} = z^4 + \frac{1}{4}\frac{dz^2z^2}{dx^2},$$

which is the equation sought. And thus in the end I seem to have obtained the desired theoretic result.

$yx = dy/dx$. $\frac{1}{2}x^2 = \int dy/y$. In general let equation 1 be given in terms of z, x, and y, positing that $z \overset{(2)}{=} dy/dx$. Let the value of x be sought from equation 1; this will give equation 3, and 4 will be obtained from its differential. In equation 4 let z be substituted for dy/dx, at least in some places; this yields equation 5, in which, as in 4, y is given without x. Likewise, let the value of y be sought; this yields equation 6, in whose differential, equation 7, x is present without y. Let the values of y and x from equations 5 and 7 be substituted in equation 1. Equation 8 is obtained in terms of z, dz, dx, and dy; let dy be removed because it is equal to zdx (equation 2), and equation 9, reduced to quadratures, is obtained.

If we let $y/x = dy/dx$, this is the equation for the straight line, but if we let $y/x = -dy/dx$, this is the equation for the hyperbola, for it yields $xdy + ydx = 0$ and moreover $xy = a^2$. Let us see therefore if it is possible for our present method to be brought to bear on this. Let $dy/dx \overset{(1)}{=} z$ and let $y/x \overset{(2)}{=} -dy/dx$, the equation of the curve sought; from equations 1 and 2 there results $y \overset{(3)}{=} -xz$. Thus the differential of 3 is $dy \overset{(4)}{=} -xdz - zdx$ or $x \overset{(5)}{=} -dx/dz - zdx/dz$. The value of x without y is thus obtained in this way. Again from equation 3 there results $x \overset{(6)}{=} -y/z$, whose differential yields $dx \overset{(7)}{=} (-dy(z) + dz(y))/z^2$, or $y = z^2dx/dz + dy(z)/dz$, which is the value of y without x. Now these values of the terms x and y discovered in equations 5 and 6 yield, when substituted into equation 3,

$$z^2\frac{dx}{dz} + \frac{(dy)z}{dz} = \frac{zdy}{dz} + z^2\frac{dx}{dz},$$

which is an identical equation, from which we learn nothing.

And thus let us take up the matter again, and make use of the previous moderate substitution of z in the place where it occurs. Let $dy/dx \overset{(1)}{=} z$, $y/x \overset{(2)}{=} -dy/dx$. Thus through equations 1 and 2 there results $y \overset{(3)}{=} -xz$ and $dy \overset{(4)}{=} -xdz - zdx$. Let us divide this equation through by dx, and substitute z for the occurrence of dy/dx, by means of equation 1. This yields $dy/dx \overset{(5)}{=} -xdz/dx - z$

or $z \overset{(6)}{=} -xdz/dx - z$ or $2zdx \overset{(7)}{=} -xdz$ or $a - \int dx/x \overset{(8)}{=} 2\int dz/z$. Thus the relation between x and z is given by means of quadratures, and moreover, through equation 3, the relation between x and $-y/x$. This is the relation between x and y. Now through equation 3 there is $\log y \overset{(9)}{=} \log x + \log z + \log(-1)$. Now, having posited that $\log 1 \overset{(10)}{=} 0$, this yields $\log(-1) \overset{(11)}{=} 0$.[5] Therefore, we have $\log z \overset{(12)}{=} \log y - \log x$. Now from equation 8 there is $a - \log x \overset{(13)}{=} 2 \log z$. Thus from equations 12 and 13 there results $a - \log x \overset{(14)}{=} 2 \log y - 2 \log x$. Or $b^a = y^2/x$, which is false, and there is thus an error somewhere in the calculation.

Let us resume: $dy/dx \overset{(1)}{=} z$, $y/x \overset{(2)}{=} -dy/dx$, $z \overset{(3)}{=} dy/dx$. Thus through equations 1 and 2 there results $y \overset{(4)}{=} -xz$, whose differential will be $dy \overset{(5)}{=} -xdz - zdx$, which, divided through by dx, yields $dy/dz \overset{(6)}{=} -xdz/dx - z$ or, by means of equation 3, yields $z \overset{(7)}{=} -xdz/dx - z$ or $2zdx \overset{(8)}{=} -xdz$, or $zdx + xdz + zdx \overset{(9)}{=} 0$. Now $zdx = dy$ by equation 3, whence from equation 9 there results $zdx + xdz + dy = 0$ or $xz = -y$ as before. The calculation is therefore sound up to this point, equation 8. Thus from equation 8 there results $2\int dx/x \overset{(10)}{=} a - \int dz/z$, in which therefore the error occurs in the foregoing calculation, because there the number 2 was put in front of $\int dz/z$. From equation 10 there results $2 \log x \overset{(11)}{=} a - \log z$. Now $\log z \overset{(12)}{=} \log y - \log x + \log(-1)$. But $\log(-1) \overset{(13)}{=} 0$, given that $\log 1 \overset{(14)}{=} 0$. Thus from equation 12 there results $\log z \overset{(15)}{=} \log y - \log x$. With this value substituted into equation 11, we obtain $2 \log x \overset{(16)}{=} a - \log y + \log x$. Thus $\log x \overset{(17)}{=} a - \log y$ or $\log x + \log y \overset{(18)}{=} a$. Thus $xy \overset{(19)}{=} b^a$, positing that the logarithm of b is unity. And thus it is grasped that the hyperbola satisfies the proposed equation 2, which is most true. And thus by this method we learn something. And thus far we have used only one differential equation, returning thus to equation 5. The equations written after equation 5 are as if not written. Now let us seek as well a means of discovering a value for y without x. In fact, $x \overset{(20)}{=} -y/z$ through equation 4. Thus $dx \overset{(21)}{=} (-(dy)z + (dz)y)/z^2$ or $z^2dx \overset{(22)}{=} -dyz + dzy$; dividing this by dx yields

$$z^2 \overset{(23)}{=} \frac{-dy}{dx} z + \frac{(dz)y}{dx}$$

or through equation 3,

$$z^2 \overset{(24)}{=} -z^2 + \frac{(dz)y}{dx}.$$

[5] log −1 = e. Thus −

Or

$$2z^2dx \overset{(25)}{=} ydz \qquad \text{or} \qquad y \overset{(26)}{=} 2z^2\frac{dx}{dz},$$

which is the value discovered through moderate substitution.[6] But from equation 5, in which no substitution was made, we have $x = -dy/dz - zdx/dz$. By substituting these two values from equations 26 and 27 in equation 4, dz vanishes and we get $2z^2dx \overset{(28)}{=} dyz + z^2dx$, whence there results $z = dy/dx$ as before, and thus the substitution which we believed to be moderate was not. With regard to the above, it is therefore sufficient for something to be obtained in one way by moderate substitution.

Let $z \overset{(1)}{=} dy/dx$ and $dy/dx \overset{(2)}{=} yx$, $z \overset{(3)}{=} yx$.[7] This will yield $dz \overset{(4)}{=} ydx + xdy$ or $dz/dx \overset{(5)}{=} y + xz$. Again $y \overset{(6)}{=} z/x$ from equation 3, which will yield $x^2dy \overset{(7)}{=} (dz)x - (dx)z$ or

$$x^2 - \frac{dz}{dy}x + \frac{1}{4}\frac{dz^2}{dy^2} \overset{(8)}{=} \sqrt{\frac{1}{4}\frac{dz^2}{dy^2} - \frac{zdx}{dy}}$$

or

$$x \overset{(9)}{=} \sqrt{\frac{1}{4}\frac{dz^2}{dy^2} - \frac{zdx}{dy}} + \frac{1}{2}\frac{dz}{dy}.$$

From equation 5 there was $y \overset{(10)}{=} dz/dx - xz$, whence substituting the value of x from equation 9 yields

$$y \overset{(11)}{=} \frac{dz}{dx} - z\sqrt{\ldots} - \frac{1}{2}\frac{zdz}{dy},$$

and by substituting these values from equations 9 and 11 in equation 3, there results

$$z \overset{(12)}{=} \frac{dz}{dx}\sqrt{\ldots} + \frac{1}{2}\frac{dz^2}{dydx} - \frac{1}{4}\frac{dz^2}{dy^2} + \frac{zdx}{dy} - \frac{1}{4}\frac{dz^2}{dy^2} - \frac{dz}{dy}\sqrt{\ldots}.$$

By eliminating dy with the help of z, there will result $\sqrt{\ldots} \overset{(13)}{=} \sqrt{dz^2/4z^2 - dx^2/dx}$ and this makes

[6] Moderate substitutions are here discovered not to be useful, because in the end dy was completely eliminated.

[7] $\frac{1}{2}x^2 = \int dy/y$.

$$z \overset{(14)}{=} \frac{dz}{dx^2}\sqrt{\frac{1}{4}\frac{dz^2}{z^2} - dz^2} + \frac{1}{2}\frac{dz^2}{zdx^2} - \frac{1}{2}\frac{dz^2}{z^2dx^2} + 1 - dx\left(\sqrt{\frac{1}{4}\frac{dz^2}{z^2} - dx^2/zdx^2}\right).$$

Or $z^3dx^2 \overset{(15)}{=} z^2dz^2\sqrt{\ldots} + \frac{1}{2}dz^2(z) - \frac{1}{2}dz^2 + z^2dx^2 - dz^2\sqrt{\ldots}$. Now for the sake of conciseness, let $z^3 - z^2 \overset{(16)}{=} mz^2$ and $z^2 - z \overset{(17)}{=} mz$ and $z - 1 \overset{(18)}{=} m$, and from equation 15 there will result $mz^2dx^2 \overset{(19)}{=} mzdz\sqrt{\ldots} + \frac{1}{2}mdz^2$ or $m = 0$ or $z = 1$. But with the latter having been eliminated by means of writing $\frac{1}{2}z\sqrt{dz^2 - z^2dx^2}$ for $\sqrt{\ldots}$ or for $\sqrt{\frac{1}{4}(dz^2/z^2) - dx^2}$, from equation 19 there will result $2z^2dx^2 \overset{(20)}{=} dz\sqrt{dz^2 - z^2dx^2} + dz^2$. Or $2z^2dx^2 - dz^2 \overset{(21)}{=} dz\sqrt{\ldots}$, whence by squaring we obtain $4z^4dx^4 - 4z^2dx^2dz^2 + dz^4 \overset{(22)}{=} dz^4 - z^2dx^2dz^2$, and with all this having been divided through by z^2dx^2, there results $4z^2dx^2 \overset{(23)}{=} 3dz^2$, or $dx = (dz/z)(\sqrt{3}/2)$. But because I fear an error may remain in the calculation, we will take the matter up again, in the following pages of the following day, September 11, 1690.

ENGLISH TRANSLATION OF "METHODUS TANGENTIUM INVERSA PER SUBSTITUTIONES (MODERATAS) ASSUMENDO $z = dy{:}dx$"

September 11, 1690

Textual tradition: Leibniz concept: LH XXXV 13,1. Leaves 300–301. 1 sheet 2°. 3 pages

September 11, 1690.[8] The inverse method of tangents by (moderate) substitutions, assuming $z = dy/dx$. The beginnings are worked out in the preceding page, in the folio (it is a half-sheet) dated September 10, 1690.[9]

Again let us take up the example from the preceding page, because there is perhaps an error in the calculation, and for the sake of greater confidence, let us apply numbers: $z \overset{(1)}{=} dy/dx$ and $dy/dx \overset{(1)}{=} yx$ yields $z \overset{(2)}{=} yx$; the differential of this is $dz \overset{(4)}{=} xdy + ydx$. Again, $y \overset{(5)}{=} z/x$, whose differential is $dyx^2 \overset{(6)}{=} xdz - zdx$. Let us remove dy from equation 6 by dividing it through by xdz; this will yield

$$\frac{dy(x^2)}{xdx} \overset{(7)}{=} \frac{xdz}{xdx} - \frac{zdx}{xdx} \quad \text{or} \quad zx \overset{(8)}{=} \frac{dz}{dx} - \frac{z}{x}.$$

[8] The things which here are sound, as also in the page from September 10, are here in the page from September 11 better set forth; and whatever here is sound is expressed economically.

[9] I think that it does not make any difference whether the substitution is moderate or not. Thus it is, it does not matter.

Or it will yield $zx^2dx + zdx \overset{(9)}{=} xdz$, or $\int dz/z + a \overset{(10)}{=} \frac{1}{2}x^2 + \int dx/x$ or $\log z - \log x \overset{(11)}{=} \frac{1}{2}x^2$. Now $\log z - \log x \overset{(12)}{=} \log y$, by equation 5. Thus there finally results $\log y \overset{(13)}{=} \frac{1}{2}x^2$, which is true, for equation 2 yields $\int dy/y + b \overset{(14)}{=} \frac{1}{2}x^2$; this is $\log y \overset{(13)}{=} \frac{1}{2}x^2$ as before. And thus we have used only one differential equation, 6. Let us see if it is also permissible to use the other equation, 4; removing dy from this equation will yield $dz/dx \overset{(15)}{=} zx + y$ or, from equation 5, $dz/dx \overset{(16)}{=} zx + z/x$, and it comes out the same. What if we wish to remove x and dx, leaving y and dy? Indeed, in equation 4 let us divide through by dy, which will yield $dz/dy \overset{(17)}{=} x + ydx/dy$ or, eliminating dx/dy (by means of equation 1) and x (by means of equation 5) from equation 17, will yield $dz/dy \overset{(18)}{=} z/y + y/z$ or $yzdz \overset{(19)}{=} z^2dy + y^2dy$. This indeed is true, but not particularly suited to a solution. Now in like fashion let us seek a value for y by means of a new equation; so that it produces a result without x, taking over from equation 5, let $x \overset{(20)}{=} z/y$, which will yield $y^2dx \overset{(21)}{=} ydz - zdy$, and, with dx removed by means of equation 1, yields $y^2dy \overset{(22)}{=} zydz - z^2dy$. Thus, equating the two values for y^2dy from equations 19 and 22 yields $zydz - z^2dy \overset{(23)}{=} zydz - z^2dy$, which is an identical equation from which we learn nothing. This outcome is not a little unsettling, and makes us wonder if our method always yields results.

Let $z \overset{(1)}{=} dy/dx \overset{(2)}{=} ax + by \overset{(3)}{=} z$. The differential of equation 3 is $adx + bdy \overset{(4)}{=} dz$, and with dx eliminated, will yield $ady/z + bdy \overset{(5)}{=} dz$. Or $y \overset{(6)}{=} \int dz/(a/z + b) + c$. And thus the equation is solved in which $axdx + bydx \overset{(7)}{=} dy$, for it yields $y \overset{(8)}{=} \int zdz/(a + bz) + c$, which depends on the quadrature of the hyperbola.

Let $z \overset{(1)}{=} dy/dx$ and $ax + by \overset{(2)}{=} czx + ezy$, which will yield $y \overset{(3)}{=} (ax - czx)/(ez - b)$. The differential of equation 3 will be $dy(ez - b)^2 \overset{(4)}{=} (ez - b)(adx - czdx - cxdz) - (ax - czx)(edz)$, and eliminating dy through equation 1, and setting $-ez + b = n$ and $-cz + a = m$ for the sake of abbreviation, will yield $zn/dx \overset{(5)}{=} mndx - ncxdz - emxdz$; and let $me + nc \overset{(6)}{=} f$, which will yield $x \overset{(7)}{=} (mndx - zn^2/dx)/fdz$ and by the same token $y \overset{(8)}{=} (mndy - zm^2/dy)/fdz$,[10] and eliminating dy through zdx, will yield $y \overset{(9)}{=} (mnzdx - m^2/dx)/fdz$. When these two values for x and y, from equations 7 and 9, are substituted in equation 2, dz drops out and we do not gain anything. Now, an equation such as (2) can be

[10] Note the technical trick, that x and y are derived in the same way; thus the calculation is shortened.

solved because there x and y, by themselves, obey the law of homogeneity. Let us therefore write $z \overset{(1)}{=} dy/dx$ and $h + ax + by \overset{(2)}{=} czx + ezy$. Let that yield $y \overset{(3)}{=} (h + ax - czx)/(ez - b)$; for the sake of abbreviation, let $cz - a \overset{(4)}{=} m$ and $ez - b \overset{(5)}{=} n$, which yields $y \overset{(6)}{=} (h - mx)/n$; which therefore yields $dm \overset{(7)}{=} cdz$ and $dn \overset{(8)}{=} edz$. Therefore, from equations 6, 7, and 8 there will result $dy(n^2) \overset{(9)}{=} -mdx - cxdz - hedz + emxdz$, whence again $em - c \overset{(10)}{=} p$ and $cn - e \overset{(11)}{=} q$ will yield

$$\frac{dy(n^2) + mdx + hedz}{p} \overset{(12)}{=} x.$$

And with zdx taken for dy, this will yield

$$x \overset{(13)}{=} \frac{(zn^2 + m)dx + hedz}{pdz}.$$

Now, in imitation of equation 12, we are at once able to write

$$y \overset{(14)}{=} \frac{xm^2 + ndy + hcdz}{qdz}.$$

Substituting these two values for x and y into equation 2, or rather, in place of it, equation 15, which is (by equations 4 and 5) $h \overset{(15)}{=} mx + ny$, will yield $hpqdz \overset{(16)}{=} qm((zn^2 + m)dx + hedz) + pn((zn + m^2)dx + hedz)$, where it suffices to determine whether dz remains, which will happen provided that it is not the case that $pq \overset{((17))}{=} emq + cnp$.[11]

Let us explore this condition, supposing that $a = 1$, $b = 2$, $c = 3$, $e = 4$, and $z = 5$;[12] therefore through equation 4 it will yield $m = 14$, and through equation 5, $n = 18$, and through equation 10, $p = 53$, and through equation 11, $q = 50$. Therefore $p \cdot q = 53 \cdot 50$ and $emq = 4 \cdot 14 \cdot 50$ and $cnp = 3 \cdot 18 \cdot 52$; therefore it would be necessary that, in actual numbers, $53 \cdot 50 = 4 \cdot 14 \cdot 50 + 3 \cdot 18 \cdot 53$ or $53 \cdot 25 = 2 \cdot 14 \cdot 50 + 3 \cdot 9 \cdot 53$, which cannot happen, because $2 \cdot 14 \cdot 50$ cannot be divided by 53. Similarly, if dz were to be eliminated, or if $pq = emq + cnp$, it would also be necessary that, because of the remaining terms, $qm(zn^2 + m) \overset{((18))}{=} pn(zn + m^2)$. Now $zn^2 + m = 5 \cdot 18^2 + 14 = 1634$ and $zn + m^2 = 90 + 14^2 =$

[11] ((. . .)) Thus I note equations which are not assumed to hold in general, but only for the sake of a thought experiment.

[12] I should have set $e = 6$, and let $n = 2m$.

286, which yields $14 \cdot 50 \cdot 1634 = 18 \cdot 53 \cdot 286$; which also is not possible as an outcome, for one side is divisible by 7, the other not. But let us explicate the terms in detail.

$$\begin{array}{ccccccccc} pq & & cemn & + & ce & - & e^2m & - & c^2n \\ 53 \cdot 50 & = & 34 \cdot 14 \cdot 18 & + & 3 \cdot 4 & - & 4 \cdot 4 \cdot 14 & - & 3 \cdot 3 \cdot 18 \end{array}$$

$$\begin{array}{ccccccccc} mn & = & cez^2 & + & ab & - & aez & - & bcz \\ 14 \cdot 18 & = & 3 \cdot 4 \cdot 5 \cdot 5 & + & 1 \cdot 2 & - & 1 \cdot 4 \cdot 5 & - & 2 \cdot 3 \cdot 5 \end{array}$$

Thus this yields $pq = c^2e^2z^2 + abce - ace^2z - bc^2ez + ce - ce^2z + ae^2 - c^2ez + bc^2 = emq + cnp = c^2e^2z^2 - ace^2z + e^2a - bc^2ez + abce - e^2cz + c^2e^2z^2 - bc^2ez + c^2b - ace^2z + abce - c^2ez$. ⟨———⟩ dz multiplied by $-zc^2e^2(z^2 + c^2e^2z + e^2) = (qmz(n^2 + m) + pnz(n + m^2))dx$. The foregoing part of the equation would need to be explained, as can easily be done, but not ⟨———⟩ because then x and y by themselves obey the laws of homogeneity; it suffices that the matter be in force, only ⟨———⟩ our alternate method does not succeed.

$$\begin{array}{ccccccc} m^2n & = ce^2z^3 - & 2acez^2 + & a^2ez - & bc^2z^2 + & 2abcz - & a^2b. \\ 14^2 18 & 0 & 3 & 2 & 0 & 6 & 7 \end{array}$$

$$\begin{array}{ccccccccccccc} mn^2 & = & ce^2z^3 & - & 2bcez^2 & + & b^2cz & - & ae^2z^2 & + & 2abez & - & ab^2. \\ 14 \cdot 18 \cdot 18 & & 3 \cdot 16 \cdot 125 & - & 2 \cdot 2 \cdot 3 \cdot 4 \cdot 25 & + & 4 \cdot 3 \cdot 5 & - & 1 \cdot 16 \cdot 25 & + & 2 \cdot 2 \cdot 4 \cdot 5 & & 4 \end{array}$$

$$\begin{array}{ccccc} m^2 = & c^2z^2 & - 2acz & + a^2. \\ 7 & 0 & 6 & 1 \end{array}$$

$$\begin{array}{ccccccc} n^2 & = & e^2z^2 & - & 2bez & + & b^2. \\ 18 \cdot 18 & & 16 \cdot 25 & - & 2 \cdot 2 \cdot 4 \cdot 5 & + & 4 \\ \not{0} & & \not{4} & & \not{1} & & \not{4} \end{array}$$

$$\begin{array}{ccccccccc} mq & = & c^2nz & + & ae & - & acn & - & cez \\ 14 \cdot 50 & & 9 \cdot 18 \cdot 5 & + & 1 \cdot 4 & - & 1 \cdot 3 \cdot 18 & - & 3 \cdot 4 \cdot 5 \\ 7 & & 0 & + & 4 & - & 0 & & 3 \end{array} \qquad pn = e^2mz + bc - bem - cez$$

So that we may finish the calculation, the value of the quantity $p(m^2n + n^2)$ and of $q(m^2 + mn^2)$ must be determined. But we have determined it to be

$$m^2n = c^2ez^3 - 2acez^2 + a^2ez - bc^2z^2 + 2abcz - a^2b$$
$$mn^2 = ce^2z - 2bcez^2 + b^2cz - ae^2z^2 + 2abez - ab^2$$
$$m^2 = c^2z^2 - 2acz + a^2$$

$$n^2 = e^2z^2 - 2bez + b^2$$
$$p = cez - ae - c$$
$$q = cez - bc - e$$

As abbreviations, let us write

$r = e^2 - bc^2 - 2ace$	$t = a^2e + 2abc - 2be$
$-26 \quad +16 - 18 - 24$	$0 \quad 4 + 12 - 16$
$s = c^2 - ae^2 - 2bce$	$v = b^2c + 2abe - 2ac$
$-55 \quad +9 - 16 - 48$	$22 \quad 12 + 16 - 6$
$w = b^2 - a^2b$	$\psi = ae + c$
$2 \quad 4 - 2$	7
$\mu = a^2 - ab^2$	$\omega = bc + e$
$-3 \quad 1 - 4$	10

This will yield

$$\begin{array}{l} mn^2 + n^2 = c^2ez^3 + rz^2 + tz + w\,(cez - \psi), \\ 14^2{\cdot}18 \quad 18^2 \quad 0 \qquad\qquad + 2 \quad 53 \end{array}$$

which gives

$$\begin{array}{ccccc} c^3e^2z^4 + & cerz^3 & + cetz^2 & + cewz & \\ & - c^2e\psi z^3 & - r\psi z^2 & - t\psi z & - w\psi \\ 0 & \not{6} & \not{5} & \not{3} & \not{4} \end{array}$$

$$\begin{array}{l} m^2 + mn^2 = ce^2z^3 + sz^2 + vz + \mu\,(cez - \omega), \\ 14^2 \quad 0 \qquad 6 \quad + 2 \quad + 2 \quad 6 \quad 50 \end{array}$$

which gives

$$\begin{array}{ccccc} c^2e^3z^4 + & cesz^3 & + cevz^2 & + ce\mu z & \\ & - ce^2\omega z^3 & - s\omega z^2 & - v\omega z & - \mu\omega. \\ 0 & 6 & 1 & 7 & 3 \end{array}$$

Now as far as the coefficients of z^4 are concerned, nothing can be reduced. Let us add together the coefficients for z^3; it will yield z^3 multiplied by $((ce^3 - bc^3e - 2ac^2e^2) + (-ac^2e^2 - c^3e) + (c^3e - ace^3 - 2bc^2e^2) + (-bc^2e^2 - ce^3))$ or $(cer - c^2e\psi + ces - ce^2\omega)$; or it yields z^3 multiplied by $-ce(bc^2 + ae^2 + 3ace + 3bce)$. And z^2 is multiplied by $((a^2ce^2 + 2abc^2e -$

$2bce^2) + (-ae^3 + abc^2e + 2a^2ce^2 - ce^2 + bc^3 + 2ac^2e) + (b^2c^2e + 2abce^2 - 2ac^2e) + (-bc^3 + abce^2 + 2b^2c^2e - c^2e + ae^3 + 2bce^2))$ or $(cet - r\psi + cev - s\omega)$; or it yields z^2 multiplied by $ce\ (3abc + 3a^2e - c + 3abe + 3b^2c - e)$. And z is multiplied by $((b^2ce - a^2bce) + (-a^3e^2 - 2a^2bce + 2a^2be^2 - a^2ce - 2abc^2 + 2bce) + (a^2ce - ab^2ce) + (-b^3c^2 - 2ab^2ce + 2ab^2ce - b^2ce - 2abe^2 + 2ace))$ or $(cew - t\psi + ce\mu - t\psi)$; or z multiplied by $(-a^3e^2 + a^2bce - 2abc^2 + 2bce - b^3c^2 + ab^2ce - 2be^2 + 2ace)$. And z^0 is multiplied by $((-ab^2e + a^3be - b^2c + a^2bc) + (-a^2bc + ab^3c - a^2e + ab^2e))$ or $(-w\psi - \mu\omega)$; or z^0 is multiplied by $(a^3be - b^2c + ab^3e - a^2e)$.

There is an error in the calculation. All these things do not turn out properly, nor is it justified in general to remove x and y at the same time. Thus in the end it yields $dx = hdz$ multiplied by $([(-2c^2e^2z^2 + c^2e^2) + bc^2e + ace^2]z + ce - abce)/((c^3e^2z^4 + c^2e^3)z^4 - [ce(bc^2 + ae^2 + 3ace + 3bce)]z^3 + [ce(3abc + 3abe + 3a^2e + 3b^2c - c - e)]z^2 + (-a^3e^2 - b^3c^2 + a^2bce + ab^2ce - 2abc^2 - 2abe^2 + 2ace + 2bce)z + (a^3be + ab^3e - a^2e - b^2c))$.

By means of this, the equation $h + ax + by = czx + ezy$ is solved, positing that $z = dy/dx$; but if h or v or e are removed, it does not work out. Provided that the calculation is correct, in which I would not trust, except that I had gone through it in numbers.

Anyhow, as an example let us take up $h \overset{(2)}{=} czx + ezy$, $z \overset{(1)}{=} dy/dx$, $y \overset{(3)}{=} h/z - cx$. Thus $dy \overset{(4)}{=} -hdz/z^2 - cdx$, and taking zdx for dy, it yields $zdx \overset{(5)}{=} -hdz/z^2 - cdx$ or it yields $dx = -hdz/z^2(z + c)$ and thus the solution is acquired, through the quadrature of the hyperbola.

Again, let $z \overset{(1)}{=} dy/dx$ and $h + ax \overset{(2)}{=} ezy$. Thus the differential of this yields $adx \overset{(3)}{=} ezdy + eydz$ or $adx \overset{(4)}{=} ez^2dx + eydz$ from equations 1 and 3 and yields $y \overset{(5)}{=} adx/dz - ez(2dx/dz)$. Now from equation 2 above, we get $y \overset{(6)}{=} (h + ax)/ez$; equating these two values for y yields $dx/(h + ax) \overset{(7)}{=} dz/ez(a - ez^2)$.

Let $zdx \overset{(1)}{=} dy$ and $h + by \overset{(2)}{=} ezx$; thus it yields $bdy \overset{(3)}{=} ezdx + exdz$. Thus $bzdx \overset{(4)}{=} ezdx + exdz$ and this yields $dx(b - e)/x = dz/z$ and the solution is obtained.

Let $ax + by \overset{(1)}{=} czx + ezy$ and $zdx \overset{(2)}{=} dy$. Let $cz - a \overset{(3)}{=} m$ and $ez - b \overset{(4)}{=} n$; this will yield $dm \overset{(5)}{=} dz = dn$ and, from equation 1, we get $mx + ny \overset{(7)}{=} 0$. Thus $x = -ny/m$ and $m^2dx \overset{(9)}{=} -ndy - ydz + nydz$ or

$$y \overset{(10)}{=} \frac{(m^2 + nz)dx}{-dz + ndz} \overset{(11)}{=} \frac{-mx}{n}.$$

Thus there results

$$\frac{dx}{mx} \overset{(12)}{=} \frac{dz(n-1)}{n(m^2+nz)}$$

and by a similar procedure,

$$\frac{dy}{nx} \overset{(13)}{=} \frac{dz(m-1)}{m(n^2+mz)}.$$

If it had been that $mx + ny \overset{(14)}{=} h$ or $x \overset{(15)}{=} h - ny/m$, it will make $m^2dx = (-ndy - ydz)m + (h - ny)dz$, and taking zdx for dy, yields

$$m^2dx + \frac{mnzdx + hdz}{(1-m)dz} = y \overset{(16)}{=} \frac{h - mx}{n},$$

whence I gather, if dx/dz is given at once through x and z, that it cannot still be given at once in the general case ⟨———⟩ And thus this method does not work out ⟨———⟩ let y, dx, and x remove . . .

We are also able to assume that z is not $= dy/dx$, but is something else. Hereafter, let

$$h + ax + by \overset{(1)}{=} cx\frac{dy}{dx} + ey\frac{dy}{dx}.$$

Let

$$z \overset{(2)}{=} (cx + ey)\frac{dy}{dx} \quad \text{or} \quad dy \overset{(3)}{=} z\frac{dx}{cx + ey}$$

and this will yield $h + ax + by \overset{(4)}{=} z$ and what is more it will make $adx + bdy \overset{(5)}{=} dz$ and from equation 3 there results $a(cx + ey)dx + bzdx \overset{(6)}{=} (cx + ey)dz$. Now from equation 4 there is

$$y = \frac{z}{b} - \frac{h}{b} - \frac{ax}{b}.$$

Thus there results

$$acxdz + \left(\frac{aez}{b}\right)dx - \left(\frac{aeh}{b}\right)dx + \left(\frac{a^2ex}{b}\right)dx + bzdx = cxdz + \left(\frac{aez}{b}\right)dz - \left(\frac{aeh}{b}\right)dz + \left(\frac{a^2ex}{b}\right)dz.$$

but there is nothing profitable from this.

A certain thing comes to mind; for the sake of abbreviation let the equation be

$$h + ax + by \overset{(1)}{=} cx \frac{dy}{dx} + ey \frac{dy}{dx},$$

or

$$ax + by \overset{(2)}{=} cx \frac{dy}{dx} + ey \frac{dy}{dx} - h;$$

let it yield

$$z \overset{(3)}{=} \frac{cx(dy/dx) + ey(dy/dx) - h}{x}.$$

This will yield $ax + by \overset{(4)}{=} xz$, and moreover $ax + by \overset{(4)}{=} zx$ or $adx + bdy = zdx + xdz$.

$ax + by \overset{(1)}{=} xy + z$ yields $adx + bdy \overset{(2)}{=} xdy + ydx + dz$. From equation 1 there is $y \overset{(3)}{=} (ax - z)/(x - b)$, and thus there results $(adx + bdy - dz)(x - b) - dyx(x - b) = (ax - z)dz$. $(cz - a)x + (ez - b)y \overset{(1)}{=} 0 \overset{(2)}{=} mx + ny$ yields $x = 0 - ny/m$ and this will yield $-mndy - mydz + nydz = m^2dx$, and this yields

$$m^2dx + \frac{mndy}{ndz - mdz} = y \qquad \text{and} \qquad n^2dy + \frac{mndx}{mdz - ndz} = x,$$

and substituting both these values in equation 2 will yield $mn^2dy + m^2ndx - m^2ndx - mn^2dy = 0$ and thus also dz vanishes.

Let $mx + ny \overset{(1)}{=} h$, which yields $x \overset{(2)}{=} -ny/m + h/m$ and $m^2dx \overset{(3)}{=} -mndy - mydz + nydz - hdz$. Thus

$$y \overset{(4)}{=} \frac{m^2dx + mndy + hdz}{(n - m)dz}.$$

And similarly,

$$x \overset{(5)}{=} \frac{n^2dy + mndx + hdz}{(m - n)dz}.$$

Thus, substituting these values in equation 1 will yield $mn^2dy + m^2ndx + mhdz + m^2ndx - mn^2dy - nhdz = hmdz - hndz$.

This again is an identical equation, and thus in this way we glean nothing, nor are we able to remove x and y at the same time. And thus something else comes to mind, wherein there is the earlier technical strategy, taking $z = dy/dx$, and seeking dz in combination with another strategy or observation, whereby it is always possible to grasp and resolve a differential equation when x and y by themselves obey [the law of] homogeneity. Because if therefore one or more constants are present, first we shall always reduce the many constants to one. For let there be a, b, c, et cetera. For b I am able to take βa and for c, κa; as thus β and κ are numbers, while a alone in truth is a line. Therefore let $dy \overset{(1)}{=} zdx$ and let equation 2 be the equation proposed in terms of x, y, z, and a. Let there be sought two differentials of the latter equations 3 and 4. Having combined these with equation 2, we will have two equations, 5 and 6, in which a will not be present. Now by means of equations 5 and 6, let y be removed, which yields equation 7, from which, by means of equation 1, let dy be removed. Equation 8 will result, in which there will figure only z, x, dz, and dx, with z and x obeying the law of homogeneity, and moreover it will be soluble through quadratures. But because there are the feared eliminations, let the thing itself be attempted. Let $zdx \overset{(1)}{=} dy$ and $h \overset{(2)}{=} cxz + eyz$, where h is constant, which alone disturbs the homogeneity for z is a ratio, and c and e are like numbers. This yields $cxdz + czdx + eydz + ezdy \overset{(3)}{=} 0$, which is an expression without h. Again, $h/z \overset{(4)}{=} cx + ey$, which yields

$$\frac{-hdz}{z^2} \overset{(5)}{=} cdx + edy \overset{(6)}{=} (-cx - ey)\frac{dz}{z}.$$

We have thus two equations in which a is not present, in which, removing dy by equation 1, there results from equation 3, $cxdz + czdx + eydz + ez^2dx \overset{(7)}{=} 0$, and, from equation 6, we get $zcdx + ez^2dx + cxdz + eydz \overset{(8)}{=} 0$. These two equations, 7 and 8, are just the same, and thus we learn nothing by this line of reasoning.

NOTES

7–16 Leibniz investigates, not very successfully, a family of curves defined by the condition $dy{:}dx = xy + a$.

10 Equation 8 should be $z = xdz/dx - x^2z + a$.

12–13 Equation 10 should be $x^2 - (dz/zdx)x + \frac{1}{4}dz^2/z^2dx^2 = \frac{1}{4}dz^2/z^2dx^2 - 1 + a$.

13 Equation 11 should be $2xdx = \sqrt{\frac{1}{4}(dz^2/z^2dx^2) - 1 + a}\;2dx + dz/z$.

15–16 $\boxed{2}\overline{\quad\quad}$ indicates that the expression under the line should be squared. Equation 12 should be

$$dz\left(\frac{dz}{dx}\right) - dz\cdot z\left(\sqrt{\frac{1}{4}\frac{dz^2}{z^2dx^2} - 1 + a} + \frac{dz}{2zdx}\right) - dyz = dx\left(\left(\frac{dz}{dx}\right)^2 - 2\frac{dz}{dx}z\left(\sqrt{\frac{1}{4}\frac{dz^2}{z^2dx^2} - 1 + a} + \frac{dz}{2zdx}\right) + z^2\left(\sqrt{\frac{1}{4}\frac{dz^2}{z^2dx^2} - 1 + a} + \frac{dz}{2zdx}\right)^2\right) - ady,$$

which is hardly illuminating.

28f. Leibniz investigates the curves defined by the condition $y/x = dy/dx$, which is, as he indicates in the margin, the family of straight lines.
30 Equation 5 should be $dx = (dyz - ydz)/z^2$.
30 Equation 6 should be $1 = 1 - ydz/z^2dx$. From this Leibniz might have concluded that $dz = 0$, but instead he just continues to recombine terms, which involves many divisions by dz. He circles dy/dx to indicate that it is immediately replaced by z.
31–34 Thus, Equation 8 would be $y = 0$, and Equation 9 would be $dy - dx = 0$, which is meaningless. By Equation 10, Leibniz realizes he has made a mistake.
36–41 Leibniz runs through the calculation again, discovering his mistake at Equation 5. He observes that one might infer from the new Equation 7 that $dz = 0$, and that this is appropriate, since z ought to be constant for straight lines.
Marginal note to lines 37–38: This equation should be $(zdy - z^2dx)/dz = (-z^2dx + dyz)/dz$, and this is just an identical equation.
42–50 Leibniz is investigating the family of curves defined by the condition $dy/dx = yx$. His computational errors lead him down a blind alley, so that he does not see that his result is only an identical equation.
42 Equation 4 should be $x^2dy = dzx - dxz$.
44 Equation 8 should be $x^2 - (dz/dy)x + \frac{1}{4}dz^2/dy^2 = \frac{1}{4}dz^2/dy^2 - dxz/dy$.
44–45 Equation 7 should be $x = \sqrt{\frac{1}{4}dz^2/dy^2 - dxz/dy} + \frac{1}{2}dz/dy$.
46 Equation 11 should be $x^2 - (dz/dxz)x + (\frac{1}{4}dz^2/dx^2)z^2 = -1 + \frac{1}{4}dz^2/dx^2z^2$.
46–50 Equation 12 should be $x = \sqrt{-1 + \frac{1}{4}dz^2/dx^2z^2} + \frac{1}{2}dz/dxz$. When the corrected versions of Equations 7 and 12 are equated, the result is an identical equation.
58–59 Leibniz defines the family of straight lines by the differential equation $y/x = dy/dx$, and the family of hyperbolae by the differential equation $y/x = -dy/dx$.
65–67 Leibniz' method yields only an identical equation.
68–74 Leibniz introduces the term "moderated substitution" in cases where he uses z for dy/dx.
72 Equation 8 should be $2\int dx/x = a - \int dz/z$.
74–78 Clearly Leibniz realizes that $\int dz/z$ is $\ln z = \ln(-y/x)$, for in what follows he explores the relationship, from Equation 3 below, $z = -y/x$, $\log z = \log(y) - \log(x) + \log(-1)$. This of course raises the problem of the logarithms of negative numbers. The erroneous Equation 11, $\log(-1) = 0$, along with Leibniz' mistake at Equation 8, leads to Equations 12–24, resulting in the equation $b^a = y^2/x$ (b is the logarithmic base), where Leibniz realizes that he has gone wrong.
81 Equation 6 should be $dy/dx = -xdz/dx - z$, but in Equation 7 he compensates for this error.
83–85 Leibniz catches the error he made in Equation 8.
85–92 He continues his computation, still assuming that $\log(-1) = 0$, and ends up with the correct conclusion $xy = b^a$, which is, he says, the equation for the hyperbola and "most true; and thus from this method we learn something."
93–101 Leibniz goes back over the same ground, finding expressions for x which do not involve y (Equation 27) and for y which do not involve x (Equation 26). This results only in another uninformative identical equation, Equation 28, given that $z = dy/dx$. Leibniz somehow blames this on the fact that his substitution was not moderated.
104f. Leibniz goes back to his consideration of the equation $dy/dx = yx$, perhaps because the expression he arrived at earlier was not especially informative. As in the earlier case, here computational errors lead him astray; so that he does not see that he has produced only another identical equation. The family of curves in question here ($y = ke^{x^2/2}$) is a fairly esoteric one, so it is not surprising that Leibniz had no intuitive grasp of what he was looking for, to guide him through the labyrinth of computation.
105–107 Equation 8 should be

$$x^2 - \frac{dz}{dy}x + \frac{1}{4}\frac{dz^2}{dy^2} = \frac{1}{4}\frac{dz^2}{dy^2} - \frac{dxz}{dy}.$$

However, he compensates for this mistake in Equation 9. Leibniz' habit of elegantly completing squares is nicely illustrated here.

109–110 Equation 12 should be

$$z = \frac{dz}{dx}\sqrt{\frac{1}{4}\frac{dz^2}{dy^2} - z\frac{dx}{dy}} + \frac{1}{2}\frac{dz^2}{dxdy} - z\left(\frac{1}{2}\frac{dz^2}{dy^2} - \frac{zdx}{dy} + \frac{dz}{dy}\sqrt{\frac{1}{4}\frac{dz^2}{dy^2} - z\frac{dx}{dy}}\right).$$

111–113 This mistake carries over to Equations 14 and 15. Equation 14 should be

$$z = \frac{dz}{dx^2}\sqrt{\frac{1}{4}\frac{dz^2}{z^2} - dx^2} + \frac{1}{2}\frac{dz^2}{zdx^2} - z\left(\frac{1}{2}\frac{dz^2}{z^2dx^2} - 1 + \frac{dz}{zdx^2}\sqrt{\frac{1}{4}\frac{dz^2}{z^2} - dx^2}\right).$$

This collapses to the identical equation $z = z$.

113–115 Unaware of his error, Leibniz believes he has something interesting in Equation 15, which he transforms and simplifies by a clever change of variable.

115–116 Leibniz claims that $\sqrt{\frac{1}{4}(dz^2/z^2) - dx^2}$ can be rewritten; the expression he gives, however, should be $(1/2z)\sqrt{dz^2 - 4z^2dx^2}$. This error carries through Equations 20–24.

132–133 Leibniz writes xdz, but he means xdx; he produces Equation 7 by dividing both sides of Equation 6 by xdx.

132–137 Starting with Equation 6 and rearranging and integrating terms (he seems to drop a constant of integration in Equation 11), Leibniz arrives at Equation 13, $\log y = \frac{1}{2}x^2$. Here he is only one step away, assuming the logarithmic base to be e, from the modern solution to his differential equation, $y = ke^{x^2/2}$. Of Equation 13 he says, "est verum"; yet he seems to be looking for something more, since he continues to play around with the equations, according to the method expounded at the end of 302r.

137–142 Starting with Equation 4, he uses the variable quantity z to eliminate x and dx, which results in Equation 19, which he finds unhelpful.

142–148 Similarly, he starts with Equation 5 and eliminates x and dx, which results only in an identical equation. This leads him to wonder about the general usefulness of his method.

149–152 Leibniz explores the differential equation $dy/dx = ax + by$ (Equation 2) using the variable quantity z and integrating. He claims that his result, Equation 8, depends on the quadrature of the hyperbola, but does not elaborate.

153f. Leibniz begins his exploration of the equation $ax + by = czx + ezy$ ($z = dy/dx$ as usual), which will extend through two pages, in extraordinary and inconclusive detail. He resorts to a kind of combinatoric procedure, and his calculations are often erroneous. All the same, what follows is interesting with reference to his methodology.

154 As before, $\boxed{2}\, ez - b$ means $(ez - b)^2$.

156–160 Equation 5 should be $zn^2dx = -mndx + ncxdz - emxdz$, and therefore Equation 6 should be not $f = me + nc$ but $f' = -me + nc$. I reconstruct Equation 7 accordingly as

$$x = \frac{mndx + zn^2dx}{f'dx},$$

and Equation 8 as

$$y = \frac{mndy + (m^2/z)dy}{(-f')dz}$$

and Equation 9 as

$$y = \frac{mnzdx + m^2dx}{(-f')dz}.$$

In any case, as Leibniz rightly observes, all the dz terms drop out when Equations 7 and 9 are plugged into Equation 2, and nothing comes of it.

161–167 Leibniz decides to revise the equation slightly; here Equation 2 is $h + ax + by = czx + ezy$. Again for the sake of abbreviation, he sets $m = cz - a$ and $n = ez - b$ in the course of finding an expression involving dy. If

$$y = \frac{h + ax - czx}{ez - b},$$

then

$$dy = \frac{a - cz}{ez - b}\, dx + (-cxdz)\left(\frac{1}{ez - b}\right) + \left(\frac{-edz}{(ez - b)^2}\right)(h + ax - czx).$$

Equation 9 should therefore be $dyn^2 = -mndx - cnxdz - hedz + emxdz$. By the same token, his abbreviations, $p = em - c$ and $q = cn - e$ in Equations 10 and 11 should be $p' = em - cn$ and $q' = cn - em$. My reconstruction of Equation 12 is thus

$$x = \frac{dyn^2 + mndx + hedz}{p'dz},$$

and of Equation 14,

$$y = \frac{dxm^2 + nmdy + hcdz}{q'dz}.$$

168–171 The important point is that $q' = -p'$, so that when you plug Equations 12 and 14 (reconstructed) into Equation 2, $h + ax + by = czx + ezy$, or, $h = mx + ny$, you get, instead of Leibniz' Equation 16, $p'dzh = m((zn^2 + mn)dx + hedz) - n((zmn + m^2)dx + hcdz)$. Since $p' = em - cn$, this collapses to $0 = 0$. Leibniz' miscalculation leads him instead into the ensuing combinatorial wild goose chase.

170f. Leibniz notes that the dz terms in his Equation 16 will drop out if $pq = emq + cnp$ (Equation 17); and in this case will leave Equation 18, which should be, however, $qm(zn^2 + m) = -pn(zn + m^2)$. He uses his combinatorial method to determine if the condition of Equation 17 holds, and concludes that it does not. In the remainder of the page, he explores (in the margins) Equation 18 by his combinatorial method, and concludes "non succedit altera nostra methodus." But he takes up the problem again on the next page.

205–275 Leibniz again takes up the material he was exploring at the end of the preceding page, in particular, Equation 16, $hpqdz = qm((zn^2 + m)dx + hedz) + pn((zn + m^2)dx + hcdz)$. Somehow in the process certain terms have dropped out, leaving only

$$dx = hdz\frac{pq}{q(mn^2 + m^2) + p(m^2n + n^2)}.$$

Unpacking the terms p, q, m, and n, assigning various abbreviations, and evaluating some of the terms (inconclusively) by his combinatorial methods, he arrives at a full expression of the latter equation at lines 263–275.

276–278 Leibniz seems to have some confidence in the foregoing calculation (although at line 262 he notes that there is an error), because he has checked it by numbers. We have seen, however, that it was a blind alley.

279–282 Leibniz takes up the equation $h = czx + ezy$, where $z = dy/dx$ as usual. From Equation 3, $h = zy + czx$; Leibniz does not seem to notice that this is incompatible with Equation 2. Once again Leibniz claims that the solution stems from the quadrature of the hyperbola; one could go on to integrate both sides of Equation 6 (x variables on the left-hand side, z variables on the right).

283–286 Leibniz does the same thing with the equation $h + ax = ezy$. Equation 5 should be $y = adx/edz - ez^2dx/edz$ and Equation 7 should be $dx/(h + ax) = edz/ez(a - ez^2)$.

287–288 Leibniz treats the equation $bdy = ezdx + exdz$ similarly.

289–297 Leibniz returns to his consideration of the equation $ax + by = czx + ezy$. Equation 9 should be $m^2dx = -nmdy - ymdz + nydz$. He equates the values for y in Equations 10 and 11 to get Equations 12 and 13. In Equation 14, he goes back to the equation $h + ax + by = czx + ezy$, but again his manipulations of it are inconclusive: "methodus ista non procedit."

298–303 Leibniz continues to consider the equation he was examining at the end of the preceding page, but now he assumes that $z = (cx + ey)(dy/dx)$, rather than simply dy/dx. The unnumbered equations at lines 302–303 should be $acxdx + ae(z/b)dx - ae(h/b)dx + dx + ae(x/b)dx + bzdx = cxdz + e(z/b) - e(h/b)dz - ae(x/b)dz$. The result is inconclusive.

304–307 In the same context, Leibniz sets

$$z = \left(cx \frac{dy}{dx} + ey \frac{dy}{dx} - h\right) x^{-1},$$

which at least simplifies the form of the differential equation.
308–309 Leibniz examines the equation $ax + by = xy + z$. The unnumbered equation at line 309 should be $(adx + bdy - dz)(x - b) - dy \cdot x \cdot (x - b) = (ax - z)dx$.
310–313 Leibniz returns to $ax + by = czx + ezy$, and his change of variable, $cz - a = m$, $ez - b = n$. Here he forms the differential equation of lines 290–291 correctly (still assuming $dn = dz = dm$); he uses it to find values for x and y, which, plugged into Equation 2, yield only an identical equation.
314–320 Leibniz tries the same approach with $h + ax + by = czx + ezy$, and once again gets only an identical equation. The equation at lines 317–318 should be $(mn^2dy + m^2ndx) + mhdx - m^2ndx - mn^2dy - nhdz = hmdz - hndz$. He notes that he cannot eliminate x and y at the same time.
320–332 Leibniz reviews his method of solving differential equations by separating variables and then integrating.
333f. Assuming once again that $z = dy/dx$, Leibniz considers the equation $h = cxz + eyz$, where h is constant, and forms its differential equation. But the two equations which he arrives at, 7 and 8, are just the same equation, and so again the result is inconclusive.