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NOTE

GÖDEL'S CONTRIBUTION TO THE JUSTIFICATION OF LEIBNIZ' NOTION OF THE INFINITESIMALS

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It is said that Kurt Gödel reflected very much on the ideas of Leibniz; moreover, in Gödel's opinion, the future of mathematics would rest heavily on the development and application of infinitesimals. However, it is well known that the foundations of the calculus based on the notion of infinitesimals was not accepted for a long time. The rejection of such "numbers" with absolute value smaller than every positive real number led to the so-called ε-δ analysis. But today, infinitesimals have been justified by A. Robinson's Nonstandard Analysis. Insofar as one of Gödel's results can be shown to be essentially incorporated in the foundations of Nonstandard Analysis, one can ask whether and to what extent Gödel has contributed to the justification of infinitesimals. Indeed, in the present formulations of Nonstandard Analysis, one cannot dispense with the axiom of choice. In view of Gödel's relative consistency result, this is no disadvantage (as it would be without this eminent theorem). After sketching the construction principles of Nonstandard Analysis, this note goes on to discuss in this context the role of Gödel's result.

Nonstandard Analysis can be presented as an axiomatic system, and this axiomatic system has well-defined models. To indicate a fairly simple model: the starting point is the field of the reals $\mathbb{R}$ together with a free ultrafilter $\mathfrak{F}$ in the power set of the natural numbers $\mathbb{N}$ as index set, and there is an $\mathfrak{F}$-relative cartesian structure $\mathbb{R}^{\mathbb{N}}(\mathfrak{F})$ -- with the underlying set $\mathbb{R}^{\mathbb{N}}$ -- which is a commutative ring with unit element and zero divisors. One can define an equivalence relation $\overset{F}{\sim}$ on $\mathbb{R}^{\mathbb{N}}$ which is a congruence relation with respect to the operations and relations in $\mathbb{R}^{\mathbb{N}}(\mathfrak{F})$; the hyperreal numbers $\mathbb{R}^*$ are then obtained as the underlying set of the quotient structure $\mathcal{R}^*$ $(\mathbb{R}^* := \mathbb{R}^{\mathbb{N}}/_{\underset{\sim}{F}})$. This model of Leibniz' infinitesimals is an ordered field and has the following properties: In $\mathbb{R}^*$ there are copies of the reals $\mathbb{R}_{\mathcal{R}^*}$ and copies of the natural numbers $\mathbb{N}_{\mathcal{R}^*}$ isomorphic to the original structures, so they may be identified with them. $\mathbb{N}_{\mathcal{R}^*}$ is a proper subset of the hypernatural numbers $\mathbb{N}^*$. There is a hypernatural number Ω greater than every number of $\mathbb{N}_{\mathcal{R}^*}$ and $\mathbb{R}_{\mathcal{R}^*}$; hence $\mathcal{R}^*$ is a non-Archimedean field. But to every hyperreal number $r^* \in \mathbb{R}^*$ there is a hypernatural number $n^* \in \mathbb{N}^*$ greater than r^*, so $\mathcal{R}^*$ is hyper-archimedean. There is no hypernatural number $n^* \in \mathbb{N}^* \setminus \mathbb{N}_{\mathcal{R}^*}$ between the natural numbers. $\mathbb{N}^* \setminus \mathbb{N}_{\mathcal{R}^*}$ has no greatest lower bound in $\mathbb{R}^*$. One can formulate principles of hypermathematical and hypercomplete induction

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and a principle of smallest hypernatural number, but restricted to subsets of $\mathbb{N}^*$ definable by a first-order formula. ($\mathbb{N}, \mathbb{N}^*, \mathbb{F}$ -- the finite numbers in $\mathbb{R}^*$ -- are not definable in that way.) Further important subclasses are the infinitesimals $\mathbb{J}$ and $\mathbb{J}_0$. $\mathbb{J}_0$ is an ideal in the commutative ring $\mathbb{F}$ with unit element. The hyperreals $\mathbb{R}^*\backslash\mathbb{F}$ are precisely the inverses of the infinitesimals. $\mathbb{R}^*\backslash\mathbb{F}$ has no greatest lower bound in $\mathbb{R}^*$. The so-called transfer principle generates a quite close connection between the completely ordered field $\mathcal{R}$ of the reals and the field $\mathcal{R}^*$ of the hyperreals. All first-order sentences are valid in $\mathcal{R}$ if and only if they are valid in $\mathcal{R}^*$. The transfer principle is a corollary of the important ultraproduct theorem of Łoś, which is proved by induction on the complexity of the formulas using the axiom of choice. Related to the transfer principle is the solution principle: it states that if each real solution of the finite set of equations and/or inequalities Γ_1 is also a solution of the finite set of equations and/or inequalities Γ_2, then each hyperreal solution of Γ_1 is also a solution of Γ_2. The solution principle is not equivalent to the full transfer principle, but to the transfer principle restricted to quantifier-free formulas. In $\mathcal{R}^*$ an equivalence relation $\overset{\mathbb{J}_0}{\sim}$ is defined, called "infinitesimally close." Every finite number, especially every element of $\mathbb{F}\backslash\mathbb{R}_{\mathcal{R}^*}$, is infinitesimally close to a real number. The equivalence classes of $\overset{\mathbb{J}_0}{\sim}$ are called monads. The monad of 0^* consists of 0^* and the infinitesimals. In the monad of a finite number r^* there is exactly one real number, called the standard of r^*. The notion of "standard" corresponds to the notion of limit in ordinary calculus, and one can prove analogous theorems.

With the notion of "standard" such fundamental concepts as differentiability, differential quotient, continuity, Riemannian integral, etc., are definable. The well-known theorems arising from these notions are provable in Nonstandard Analysis. The construction of a nonstandard field $\mathcal{R}^*$ out of the field of the real numbers $\mathcal{R}$, as sketched above, is suitable for generalization: proceeding from a structure S, a sequence of structures $(S_n)_{n\in\mathbb{N}}$ can be constructed the union of which is called the standard universe $S_\omega (S_\omega = \bigcup_{n\in\mathbb{N}} S_n)$. In a quite similar way one can define recursively a nonstandard universe $(S_\omega)^*$ which corresponds to S_ω like $\mathcal{R}^*$ to $\mathcal{R}$ [Davis 1977]. This enlargement from "standard" to "nonstandard" -- "real" to "hyperreal," "standard" to "nonstandard" universe -- is reflected in the extension of set theory by adding new language particles (called ε^*, "standard," "internal") and new axioms to ZF[AC] involving these new language particles [e.g., Nelson 1977], but it does not fit in with the intentions of this note to go into further detail.

Having sketched how Leibniz' idea of infinitesimals may be justified within the context of Nonstandard Analysis by means of model theory as developed by Robinson [1974], let us discuss the question of Gödel's -- at least indirect -- contribution to this subject. If one examines the formal proofs for the theorems indicated above which form the basis of Nonstandard Analysis, one sees that the axiom of choice is used in at least two places. For example, using Zorn's lemma which, as is well known, is equivalent to the axiom of choice, the ultrafilter theorem (i.e., the theorem on the extensibility of proper filters to ultrafilters in every Boolean power set algebra) is proved; BPUFE is equivalent to BUFE (i.e., the theorem on the extensibility of proper filters to ultrafilters in every Boolean algebra). (For details see Jech [1973, 15-17].) BUFE is equivalent with the Boolean Prime Ideal Theorem BPI (the theorem that every Boolean algebra has a prime ideal); BPI is equivalent with RPIE (the theorem on the extensibility of proper ideals to prime ideals in a commutative ring with unit element)[see Banaschewski 1983]. RPIE is equivalent with RPI (the theorem that every commutative ring with unit element has a prime ideal; likewise RMIE (the theorem on the extensibility of proper ideals to maximal ideals in a commutative ring with unit element) is equivalent with RMI (the theorem that every commutative ring with unit element has a maximal ideal): if R is a commutative ring with unit element and $\mathcal{J}$ is a proper ideal on R, then the quotient structure $R/\mathcal{J}$ is likewise a commutative ring with unit element, for the zero element in $R/\mathcal{J}$ holds $[0] = \mathcal{J}$; presupposing RPI, $R/\mathcal{J}$ has a prime ideal K. With $\mathcal{J}^*: = \{x \varepsilon R \mid [x] \varepsilon K\}$, it holds $\mathcal{J}/\mathcal{J} = \{[0]\} \subseteq K = \mathcal{J}^*/\mathcal{J}$ and with $x \varepsilon \mathcal{J}$, $[x] \varepsilon \mathcal{J}/\mathcal{J} \subseteq K$, $x \varepsilon \mathcal{J}^*$. Thus $\mathcal{J} \subseteq \mathcal{J}^*$ and $\mathcal{J}^*$ is a proper prime ideal; therefore RPI implies RPIE; the reverse implications hold trivially. The same is true when "maximal" is substituted for "prime": RMI is equivalent to RMIE. RMIE is equivalent with the axiom of choice AC [see Hodges 1979, 285]. According to the fact that in every commutative ring with unit element every maximal ideal is a prime ideal, RPIE is implied by RMIE, but RPIE does not imply RMIE: if RPIE -- being equivalent with BPI -- would imply RMIE -- being equivalent with AC --, then BPI would imply AC; but BPI does not imply the axiom of choice AC [Halpern and Levy 1971]; therefore RPIE does not imply RMIE; so the problem raised by Dana Scott [1954, 390] whether RPIE is equivalent with MPIE must be answered negatively. However, the other problem posed by Dana Scott, whether or not RMIE is equivalent with AC, has been answered positively [Hodges 1979].

The ultrafilter theorem BPUFE, as we have already stated, is used to develop the foundations of Nonstandard Analysis, and is not provable in either the NBG or ZF system without the axiom of choice. If the axiom of choice were used only at this point in establishing Nonstandard Analysis, then the latter could be established in NBG[BPI] or ZF[BPI]. But it can be shown that the axiom of choice is also used in proving the ultraproduct theorem of Łoś. (For certain generalizations, the axiom of choice is also required.) In fact, as is well known, the theorem of Łoś together with BPI is equivalent to the axiom of choice [Howard 1975, 426-428]. But neither BPI -- as stated already -- [Halpern and Levy, 1971], nor the theorem of Łoś, by themselves, imply the axiom of choice [see

Blass, 1977, 329-331]. Therefore, BPI does not imply Łoś' theorem, nor does Łoś' theorem imply BPI. Indeed, in the present foundations of Nonstandard Analysis, the axiom of choice cannot be dispensed with.

But on the basis of Gödel's result that the consistency of Neumann-Bernays-Gödel set theory (NBG) is equivalent to the consistency of NBG plus the axiom of choice -- which, incidentally, implies at once that NBG is consistent if and only if NBG plus the prime ideal axiom BPI is consistent -- the use of the axiom of choice is no disadvantage (in contrast to the pre-Gödel era!), and alternative theories which may be developed without that axiom have no claim to preference by this fact alone.

Thus Gödel's result represents an important contribution to Nonstandard Analysis. Moreover, because the ultrafilter theorem is equivalent to Gödel's theorem on the completeness of first-order logic [see Henkin 1954], his result, seen from the viewpoint of Nonstandard Analysis, appears in a new light. Gödel has, therefore, made an important contribution toward the justification of Leibniz' idea of infinitesimals.

REFERENCES

Banaschewski, B. 1983. *Journal of the London Mathematical Society* 27, 193-202.

Blass, A. 1977. A model without ultrafilters. *Bulletin de la Académie Polonaise de Science, Série Mathématique*, 25, 329-331.

Christian, C. C. 1983. Der Beitrag Gödels für die Rechtfertigung der Leibniz'schen Idee von den Infinitesimalien. *Sitzungsberichte der Österreichischen Akademie der Wissenschaften, mathem.-naturw. Klasse* 191.

Davis, M. 1977. *Applied Nonstandard Analysis*. New York: Wiley.

Gödel, K. 1966. *The consistency of the continuum hypothesis*. Annals of Mathematics Studies, Princeton, N.J.: Princeton Univ. Press.

Halpern, J. D., and Levy, A. 1971. *Proceedings of symposia in pure mathematics*, Vol. XIII, Part 1, Providence, R.I.: Amer. Math. Soc.

Henkin, L. 1954. Metamathematical theorems equivalent to the prime ideal theorems for Boolean algebras: Preliminary report. *Bulletin of the American Mathematical Society* 60, 388.

Hlawka, E. 1968. Leibniz als Mathematiker. *Philosophia Naturalis* 10, 146-168.

Hodges, W. 1979. Krull implies Zorn. *Journal of the London Mathematical Society* (2) 19, 285-287.

Jech, Th. 1973. *The axiom of choice*, pp. 15-17. Amsterdam: North-Holland.

Laugwitz, D. 1978. *Infinitesimalkalkül*. Mannheim/Wien/Zürich: Bibliographisches Institut.

Luxemburg, W. A. J., and Stroyan, K. 1976. *Introduction to the theory of infinitesimals*. New York: Academic Press.
Meisters, G. H., and Monk, J. D. 1973. Construction of the reals via ultrapowers. *Rocky Mountain Journal of Mathematics* 3, 141-158.
Nelson, E. 1977. Internal set theory: a new approach to nonstandard analysis. *Bulletin of the American Mathematical Society* 83, 1165-1198.
Richter, M. M. 1972. Nonstandard Analysis and general compactification. *Habilitationsschrift*, Tübingen 1972.
Robinson, A. 1974. *Nonstandard Analysis*, 2nd ed. New York: American Elsevier.
Schmieden, C. and Laugwitz, D. 1958. Eine Erweiterung der Infinitesimalrechnung. *Mathematische Zeitschrift* 69, 1-39.
Scott, D. 1954. Prime ideal theorems for rings, lattices, and Boolean algebras: Preliminary report. *Bulletin of the American Mathematical Society* 60, 390.