HISTORIA MATHEMATICA 5 (1978), 43-75

ARCHIMEDES AND THE SPIRALS: THE HEURISTIC BACKGROUND

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SUMMARIES

In his work, The Method, *Archimedes displays the heuristic technique by which he discovered many of his geometric theorems, but he offers there no examples of results from* Spiral Lines. *The present study argues that a number of theorems on spirals in Pappus'* Collectio *are based on early Archimedean treatments. It thus emerges that Archimedes' discoveries on the areas bound by spirals and on the properties of the tangents drawn to the spirals were based on ingenious constructions involving solid figures and curves. A comparison of Pappus' treatments with the Archimedean proofs reveals how a formal stricture against the use of solids in problems relating exclusively to plane figures induced radical modifications in the character of the early treatments.*

Dans son livre, La méthode, *Archimède dévoile par quelles méthodes heuristiques il a découvert plusieurs de ses théorèmes de géométrie, mais aucun de ses exemples illustrent des résultats de son* Lignes spirales. *Dans la présente étude, nous soutenons que nombre de théorèmes sur les spirales dans le* Collectio *de Pappus reposent sur les premiers essais d'Archimède. Il s'ensuit que les découvertes d'Archimède sur les surfaces délimitées par une spirale et sur les propriétés des tangentes à une spirale sont basées sur des constructions ingénieuses impliquant des figures solides et des courbes. Une comparaison de l'approche de Pappus et des preuves d'Archimède révèle comment la restriction complète de l'usage des solides dans les problèmes reliés exclusivement aux figures planes induisit des modifications radicales du caractère des premiers essais.*

0315-0860/78/0051-0043$02.00/0

Pappus begins his presentation of a set of theorems on the Archimedean spiral with the following remark:

> *The geometer Conon of Samos put forward the theorem on the spiral described in the plane, while Archimedes proved it by means of a remarkable procedure.*
> [*Collectio* IV, 21]

The theorem in question states that the area enclosed by a full turn of the spiral is one-third that of the circle generated simultaneously. The proof Pappus provides [IV, 22] is indeed remarkable, transforming the determination of the area of the spiral into that of the volume of a cone. It would thus appear that this theorem originated in some form with Archimedes' respected friend Conon, but received its proof by Archimedes, in the manner to be presented by Pappus.

However, historians have not adopted this interpretation. T. L. Heath summarizes the accepted version thus:

> *Conon ...* [is] *cited as the propounder of a theorem about the spiral in a plane which Archimedes proved: this would, however, seem to be a mistake, as Archimedes says at the beginning of his treatise* [*On Spiral Lines*] *that he sent certain theorems without proofs, to Conon, who would certainly have proved them had he lived.* [1921, II, 359]

P. Ver Eecke extends this judgment: Pappus, so far removed from the time of Archimedes, knew of many of the Archimedean works probably only through excerpts and summaries. From Pappus' allegedly incorrect attribution of the invention of the spiral to Conon, Ver Eecke infers that Pappus could consult only defective versions of the *Spiral Lines*, in particular, lacking the prefatory letter. This also explains why Pappus' demonstration of the theorem on the area of the spiral [IV, 22] is "plus abrégée et un peu différente" from that of Archimedes [*SL*, prop. 24], even though Pappus' borrowing from Archimedes "sa méthode féconde d'exhaustion" is evident. In sum, Ver Eecke views Pappus' treatment of the spirals as little more than a small commentary on the Archimedean work, containing a few alternative proofs to some of its theorems [1933, I, xxviii-ix].

By contrast, F. Hultsch, editor of the Greek text of the *Collectio* (1875-78), is less harsh in his judgment of Pappus. While admitting that Pappus' proof proceeds differently, he perceives that Pappus' whole manner of argument stems from the "genius and authority" of Archimedes [1875, I, 237n, 239n].

Through an examination and comparison of the treatments of the spiral by Pappus and Archimedes we shall seek to make the following points: (1) The adverse comments by Heath and Ver Eecke are largely unwarranted, at best founded on assumptions which, if possible, are yet of little use for a historical analysis.

(2) Accepting that Pappus' treatment was based on an early Archimedean work on spirals, we can infer a possible heuristic background for the theorems on areas bound by spirals. (3) The theorem on the spherical spiral given by Pappus appears likewise to be of Archimedean origin. Remarks by Pappus and others on the properties of the conical and the cylindrical spirals will enable us to infer how Archimedes discovered the manner of drawing tangents to the plane spiral. (4) Certain features of the formal proofs in *Spiral Lines* will suggest Archimedes' reasons for instituting such a major revision of the earlier treatment of the spirals. These reasons include the availability of more advanced mathematical techniques and his own sensitivity to a formal stricture against the use of solids in the investigation of purely planar figures.

We thus hope to retrieve from Pappus an understanding of the development of Archimedes' study of spirals, from its heuristic stages to its complete formal presentation. Such a perspective is otherwise obscured by the formality of Archimedes' *Spiral Lines* and is not to be obtained from other works in the extant Archimedean corpus.

1. THE BASIS OF PAPPUS' TREATMENT OF THE SPIRALS

The interpretation of Pappus' treatment of the spirals proposed by Heath and Ver Eecke consists of three points: (a) that Pappus' statement about Conon is inconsistent with what Archimedes says in the preface to *Spiral Lines*; (b) that Pappus was working with defective source materials; and (c) that Pappus' treatment was intended to be a small commentary on Archimedes' formal treatise. We shall indicate the inaccuracy of each of these points.

First, there is nothing in Archimedes' preface incompatible with the attribution to Conon of the invention and prior study of the spiral. To see this, let us review what Archimedes says there. He first recalls to his correspondent Dositheus a list of theorems without proofs sent to Conon much earlier. Archimedes explains his delay in forwarding the proofs by his hope thereby to encourage other geometers to explore the areas represented by those theorems. Unfortunately, Conon had died before he had time to investigate the theorems; otherwise he would have worked out proofs and made many new discoveries besides, for his mathematical expertise was considerable. After deploring that no one in the "many years" since Conon's death had contributed anything to the study of these areas, Archimedes proceeds to name the theorems at issue. They include nine of the ten problems examined in a work he had already sent to Dositheus (and now extant as the second book *On the Sphere and Cylinder*); four theorems on conoids whose proofs Archimedes promises to send later (these are contained in the extant *Conoids and Spheroids*); and four theorems on the spiral, whose proofs are given in the communication at hand (namely, the extant treatise *On Spiral Lines*). Now, there is no suggestion in this that Conon had initiated the study of any of these theorems, nor is there any special association of Conon with the one theorem stated on the area of the spiral. It would thus be purely arbitrary to suppose, as Heath and Ver Eecke do, that Pappus or some earlier commentator had drawn such mistaken inferences from the preface. On the other hand, if Pappus knew from an independent source that Conon had introduced the study of the spiral, this does not contradict what Archimedes says in *Spiral Lines*.

Second, the gratuitous assumption that Pappus' sources are defective ought to astound us. In Pappus' *Collectio* we have received a wealth of mathematical materials, excerpts from works, many of which would otherwise have been lost entirely, and from authors, many of whom would otherwise be unknown to us. In the frequent instances where we can check him, Pappus possesses sources as good as the best extant and follows them closely. For instance, in Book VII so much detail is presented on the works in the *τόπος ἀναλυόμενος* (the so-called *Treasury of Analysis*) that remarkable restorations of several of them have been possible [1]. Pappus' proof of the manner of cutting the surface of a sphere into segments in a given ratio [V, 42] follows *verbatim* the Archimedean proof in *Sphere and Cylinder* [II, 3]. A lengthy development on isoperimetric plane figures, leading to the conclusion that the circle is the greatest of those plane figures having a given perimeter [V, 1-10] agrees virtually word for word with part of Theon of Alexandria's later commentary on Ptolemy [355-374]; as Theon cites one Zenodorus as his source, Pappus must also have been following Zenodorus. Pappus' silence is thus no indication that he is presenting original material. Pappus even transfers *verbatim* portions of one book into another, such as his own solution of the problem of the duplication of the cube [cf. III, 10 and VIII, 12]. When Eutocius later reproduces Pappus' solution in his commentary on Archimedes, he cites the "Mechanical Introduction," that is, Book VIII [III, 70-74]. This is remarkable in that it is Book III which contains Pappus' major exposition of the methods of duplicating the cube. Now, Eutocius provides many methods not given by Pappus; on Hero's method, given by both, their discussions are sufficiently different to indicate independent sources [2]. Thus, in the instance of Nicomedes' method, where Pappus and Eutocius agree in close detail, we may infer that they were following a common source, rather than assume that Eutocius was quoting from Pappus. The degree of agreement here indicates how closely Pappus follows his source. Many other such indications could be cited. We should thus feel ourselves moved to assume the integrity of Pappus' sources and his close adherence to them, and, conversely, to view as significant any marked discrepancies between the text in Pappus and that in the quoted original [3].

Third, Ver Eecke's assessment of Pappus' theorems on the spiral as a small commentary on Archimedes' treatise is simply inaccurate. As we shall discuss below, Pappus presents theorems not to be found in Archimedes' *Spiral Lines*, yet omits mention of all but four of the 28 theorems which *are* in it. Of these four, only one is important for itself, the theorem on the area of the spiral--but it is treated in a manner entirely different from the Archimedean. Significantly, Pappus passes over with a single word, "obvious," the entire convergence argument in this theorem. By contrast, Archimedes takes pains to effect this by a careful "exhaustion" treatment [4]. Thus, Pappus is hardly acting as a mere commentator on Archimedes. Moreover, given the excellence of the Archimedean treatise and, by comparison, the inelegance of the proofs provided by Pappus, one should wonder that Pappus--or, for that matter, any post-Archimedean author--sought to invent a treatment of the spiral along these lines. That is, it appears more likely that Pappus was following a work antedating the extant work *On Spiral Lines*.

One might suppose, accepting this inference, that Pappus' text derived from a pre-Archimedean author, for instance, Conon, and that Pappus' allusion to Archimedes' "remarkable proof" referred to the extant *Spiral Lines*. But two things argue against this. First, Pappus *does* restrict Conon's contribution to the proposing of the theorem, reserving its proof to Archimedes. Indeed, the treatment presented by Pappus, for all its formal limitations, *is* accepted by him as a "proof" [IV, 22: δείκνυται]. Moreover, by treating this theorem at length in a formal work (*Spiral Lines*), Archimedes indicates the result as being an original discovery. For his works are not compilations of the discoveries of others; they are research communications, as he makes abundantly clear in the prefaces. His handling of works known from previous study, whether by himself or by others, is perfunctory: such theorems are grouped as preliminary lemmas, their proofs often omitted as "obvious," or as "already proved" in other works. Thus, to be consistent, we must take Archimedes to be the source of Pappus' proof of the area-theorem on the spiral. It is entirely in the character of Archimedes' correspondence with the Alexandrian mathematicians, that the *problem* of what the area enclosed by the spiral was should have been proposed by one of them, for instance Conon, to be solved by the others. This was, as we have seen, the purpose behind Archimedes' own list of theorems addressed to Conon. Moreover, the preface to an Archimedean work on the spiral sent to Conon would certainly have contained information of this sort, so to serve as the basis of Pappus' opening remark on the spiral.

Second, Pappus does not appear to have in hand the work we know as *Spiral Lines*. The striking differences between his treatment of the area of the spiral and that in *SL*, 21-28 have already been taken to indicate this. Further, the manner of drawing the tangents to the spiral, by which the spiral may be used to solve the problem of squaring the circle, is of central interest in *Spiral Lines*, 12-20. Yet in his copious remarks on the use of the *quadratrix* of Nicomedes in this connection, Pappus does not mention the property of the tangents of the spiral, not even when he compares the spiral and the quadratrix in the manner of their construction [IV, 33-34] or when he applies both curves for the multisection of angles [IV, 45-46]. To be sure, Pappus does twice mention the theorem on the tangent to the spiral [IV, 36, 54]. But in both instances the issue is Archimedes' inappropriate use of a "solid *neusis*," that is, of a construction involving the sections of solids, in the solution of a plane problem [5]. Yet Pappus' own resolution of the difficulty [IV, 54] is by his own classification a "solid" method, as it makes use of conic sections. Thus, historians have been puzzled as to what Pappus' objection precisely was [6]. The root of this puzzle is revealed when Pappus says that *others* have criticized Archimedes for an improper use of a solid construction and that *they* show how to find a straight line equal to a circle by means of the spiral [IV, 54]. That is, Pappus is dependent on sources commenting on and criticizing Archimedes' study of the tangents to the spiral; he does not have the Archimedean work itself. Indeed, given the confusion in his criticism, it is to be suspected that even his sources were commenting on a treatment of the tangent-theorem different from *SL*, 18.

These considerations lead us to infer that Pappus drew his discussion of the spiral from an Archimedean work addressed to Conon, and thus, as we have seen, antedating by "many years" the extant *Spiral Lines*. This readily accounts for the comparative crudeness of the treatment in Pappus. But more interestingly, this assumption enables us to view Pappus' treatment as representing an early stage of Archimedes' study of the spirals. Thus, we may hope to get closer to the heuristic notions on which Archimedes' extant formal treatment was based. With this objective in mind, we shall now survey the theorems presented by Pappus.

2. PAPPUS' AND ARCHIMEDES' THEOREMS ON THE PLANE SPIRAL

Subtle differences between the treatments by Pappus and by Archimedes are in evidence right from the start--in the very definitions introduced for the spiral. In the preface to *Spiral Lines* and again in the definition preceding prop. 12, the spiral is conceived as the path of a point moved "with equal speed itself to itself" (ἰσοταχέως αὐτὸ ἑαυτῷ) along a line, as the line simultaneously rotates "with equal speed" about a fixed end-point; the fixed point is termed the "origin of the spiral," the initial position of the line the "origin of the revolution."

As these motions may be continued indefinitely, we obtain a "first revolution," a "second," a "third," and so on, as the rotating line passes through its initial position. In Pappus, the point moves "uniformly" (ὁμαλῶς) from one end-point to the other of a finite line-segment *AB* "in the same time" as the segment makes one complete rotation "uniformly" about its fixed end-point *B*. Thus, although the curve traced out according to the two definitions is the same, there are variations in terminology. Moreover, Pappus' definition does not allow for continuation of the spiral beyond the first turn, whereas Archimedes devotes seven theorems to the higher turns in *Spiral Lines*. (Proclus recognizes the definition in the restricted form preserved by Pappus; he terms it "monostrophic," that is, "single-turned," and likens it to the circle and the cissoid because of its finite character [*In Euclidem*, 180, 187]. It is odd that for Pappus, unlike Proclus, the "monostrophic" spiral is a single turn of the cylindrical spiral [VIII, 28, p. 1110].) The definition provided by Pappus has another limitation: to construct the spiral in this manner, whether mechanically or conceptually, requires that two motions, one linear, the other circular, be completed in the same time. This in effect assumes solution of the quadrature of the circle; and in the case of another curve, the "quadratrix," which also assumes this synchrony, Sporus made this a ground for attacking Nicomedes' studies [Pappus IV, 31]. As we shall discuss below, both the spiral and the quadratrix were used to square the circle, and the construction of either can be reduced to that of the other [Pappus IV, 34]. Thus, we may infer a factor encouraging Archimedes to provide in *Spiral Lines* an altered definition which avoids the objectionable assumption of synchrony.

Pappus' term "uniformly" (ὁμαλῶς) is the same as that used by Autolycus (c. 300 B.C.) in his tract *On the Moving Sphere*. In fact, this is but one of a variety of indicators which link Pappus' treatment stylistically to Autolycus. In using ὁμαλῶς instead of the Archimedean ἰσοταχῶς, Pappus appears to prefer conformity with an earlier mathematical tradition. The issue is more complicated than this, however, since both terms are found in pre-Euclidean writing. For instance, in general discussions of the concepts of motion in the *Physics* Aristotle speaks of "uniform" motion; e.g., motion in a spiral is not "uniform" because the path of motion is not a "uniform" magnitude (in contrast with lines and circles which *are* "uniform" magnitudes) [*Physics*, 228b20]. In *de Caelo* [II, 4, 6] he terms "uniform" the motions of the heavenly bodies. But motion is termed ἰσοταχῆς whenever a comparison between motions is understood (cf. *Physics*, 216a20: the heavy and the light would fall "with equal speed" in a void space); when Aristotle defines constant motion as moving equal length in equal time, he terms it ἰσοταχῆς (249a13); and he employs this term throughout his detailed proofs of the impossibility of the existence of indivisible parts of magnitude or motion [*Physics*, VI, 1]. Such usage appears even earlier with Archytas (fr. 1): that when bodies move, "but not with equal speed," noise is produced.

While a full analysis of the usage of these terms and their relation to Archimedes cannot be given here, I believe the following view emerges. The term ἰσοταχής appears to be characteristic of Eudoxus' mathematical treatment of motion, the background to his system of planetary motion based on homocentric, uniformly rotating spheres, in the tract (now lost) *On Speeds* (περὶ ταχῶν). Eudoxus thus conforms with usage in prior studies of motion, as by Archytas, while Aristotle follows Eudoxus in the more mathematical arguments of the *Physics*. When Eudoxus' successors, e.g., Callippus, revised and improved his astronomical system, they also introduced changes of terminology, such as ὁμαλός for ἰσοταχής. It is the revised tradition which Aristotle follows in his discussions of the uniform circular motions (*de Caelo*), and the new terminology became fixed in the subsequent literature of spherics and astronomy (as by Autolycus and Theodosius). Pappus' treatment of the spiral, which we take to represent Archimedes' early studies, conforms with usage in the tradition of spherics. As far as Archimedes is concerned, this is not at all surprising. His father Phidias was an astronomer [cf. *Sand-Reckoner*, I, 9], as was his colleague Conon, and a firm grounding in spherics is implicit in most of the Archimedean works, notably in the books *On the Sphere and Cylinder*. Now, one should note that when Eudoxus' astronomical system was reworked, there was no need to change the mathematical foundation of motion which Eudoxus had established. Autolycus, for instance, assumes without proof several basic theorems on uniform motion, such as the fact that the distances traversed have the same ratio as the times. Indeed, Pappus does the same in his treatment of the spirals, as we shall see below. Thus, when Archimedes wished to formalize his treatment of the spirals in *Spiral Lines*, he could not use the works on spherics as a model, but had to return to the Eudoxean theory of motion. I thus view Archimedes' term ἰσοταχέως in *Spiral Lines* as a revival of Eudoxean usage, as Archimedes drew from and tightened the earlier treatments of proportion and motion in the course of formalizing his own studies in geometry.

Following the definition of the spiral, Pappus presents two lemmas. The first shows that if the spiral is cut by the line BZ, which continued meets at Γ the circle generated by the motion of A (see Fig. 1), then BZ is to AB as the circular arc $A\Delta\Gamma$ is to the whole circumference of the circle. He says this is "easy to see," since each line and its associated arc will be traversed in the same time. He then adds: "the motions are equally swift they to themselves (αὗται ἑαυταῖς ἰσοταχεῖς) so that they are also in proportion." This last remark thus introduces the term

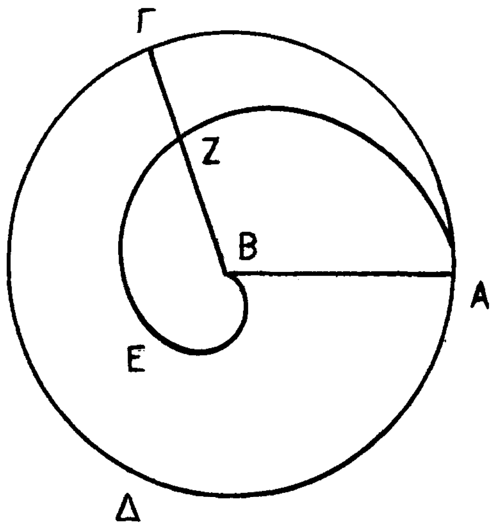

FIGURE 1

used in *Spiral Lines* and, in effect, completes the argument by a loosely worded appeal to *SL*, 2. It is tempting, in the light of our earlier comments, to detect here a remnant of Eudoxean terminology, introduced by Archimedes into his early study of the spiral. But for several reasons we do better to view the remark as a post-Archimedean interpolation. First, the phrase "motions equally swift they to themselves" in Pappus is an imprecise echo of the Archimedean phrase "the point moving with equal speed itself to itself." In *Spiral Lines*, the term "with equal speed" (ἰσοταχέως) is used adverbially, never as an adjective. Pappus' phrase "they to themselves" is supposed to mean "each to itself;" unfortunately, it might easily be understood as "each to the other," which would here be a plain error. Second, Pappus' inference that "they [sc. the motions] are also in proportion" is simply incorrect; it is not the motions which are in proportion, but the distances and the times. Third, a perfectly clear expression can be provided in accordance with the terminology of spherics: "since the two motions are uniform (ὁμαλαί), the times and distances traversed according to each are in proportion; and as the corresponding times are the same in both motions, the corresponding arcs and lines traversed are in proportion." It thus seems that the original of the version followed by Pappus left this argument incomplete and a later interpolator, possessed of an imperfect understanding of Archimedes' terminology, sought to remedy the gap by an allusion to *Spiral Lines*.

Thus, in Pappus this lemma is termed "easy to see" and his brief discussion contains at best the germ of a satisfactory proof. By contrast, Archimedes demonstrates an analogous theorem [*SL*, 14]: that for two arbitrary lines drawn to the spiral the lines and arcs marked off are in proportion. As in Pappus, Archimedes identifies the arcs and lines traversed by two points moving, the one along a circle, the other along a line, "each with equal speed itself to itself;" that the claimed proportion among lines and arcs holds is thus "clear." He adds, "for this has been proved outside in the preliminaries." In fact, the required step is proven in *SL*, 2 via a complete argument based on the Euclidean definition of proportion [*Elements*, V, Def. 5]. A comparison of these treatments by Pappus and Archimedes thus weighs against Ver Eecke's view that the one is a "commentary" on the other, but rather encourages us to view Pappus' version as an informal argument, subsequently amplified in *Spiral Lines*.

After this lemma, Pappus states without proof a second "obvious" property of the spiral: that any lines drawn from the origin to the spiral as to contain equal angles exceed each other by an equal amount. Archimedes' statement in *SL*, 12 is similar, but he provides a full proof. He makes clear that a sequence of rays is intended, drawn at equal angles and thus increasing in length arithmetically. The treatment is careful, referring back to *SL*, 1, which itself applies Archimedes' axiom on the continuity of magnitude, stated explicitly in the preface. From the formal point of view, this is hardly to be dismissed as "obvious," and it would be extraordinary for a commentator with Archimedes' proof before him so to dismiss it. On the other hand, in the initial stages of thinking through the properties of the spiral and their proof, this one might well be taken as evident. This thus conforms to our general view: that Pappus' version is an early draft by Archimedes of some of the results now extant in *Spiral Lines*, and that the preliminary theorems in the extant treatise were added by him in the process of formalizing the proofs.

Pappus is now ready to give the main theorem: that the area bounded by the spiral and the original line AB is one-third that of the circle which contains the spiral. We may outline his proof as follows: let the spiral be drawn as in Fig. 2 and let a rectangle $KN\Lambda\Pi$ be drawn with diagonal $K\Lambda$ as in Fig. 3. From the circle containing the spiral an arbitrary integral fractional part $A\Gamma$ of the circumference is marked off and the same part of the side of the rectangle $K\Pi$ is marked off as KP. The ray $B\Gamma$ meets the spiral at Z and the circular arc ZH is drawn; thus the sector BZH is inscribed in the portion of the area BZA bounded by the spiral. In turn, the rectangle $KPTN$ is completed and the line $M\Omega$ drawn to the diagonal. From the first lemma it follows that the radii BZ, $B\Gamma$ have the same ratio as the lines $T\Omega$, TP. Now, the sectors BZH, $B\Gamma A$ will have the ratio of the squares of these radii, as they are similar plane figures. Moreover, if the rectangle $K\Pi\Lambda N$ is conceived to revolve around its side $N\Lambda$ as axis, so generating a cylinder, the rectangles $KPTN$, $M\Omega TN$ will likewise generate cylinders, and the volumes of these latter will be in the ratio of the squares of the lines TP, $T\Omega$ (being, respectively, the radii of the circular bases of cylinders having equal height). Thus, the ratio of the areas of the sectors BZH, $B\Gamma A$ is the same as the ratio of the volumes of the cylinders generated by $M\Omega TN$, $KPTN$. In the same manner, an arc $\Gamma\Delta$ equal to $A\Gamma$ is marked off, and a line PX equal to KP; then, the ratio of the sectors $BE\Theta$, $B\Delta\Gamma$ will be the same as the ratio of the cylinders generated by $\Xi O\Phi T$, $PX\Phi T$. This procedure is to be continued (although Pappus complicates the diagram no further), so that the ensemble of sectors (literally: "all the sectors") inscribed within the spiral has to the circle the same ratio which the ensemble of cylinders inscribed within the cone (generated by the revolution of the triangle $K\Lambda N$) has to the entire cylinder generated by $K\Pi\Lambda N$. The same holds for the ensemble of sectors circumscribed about the spiral and the ensemble of cylinders circumscribed about the cone (Pappus does not provide details here). From this it is "obvious" that the area bounded by the spiral has to the area of the circle the same ratio which the cone has to the cylinder, namely, one-third.

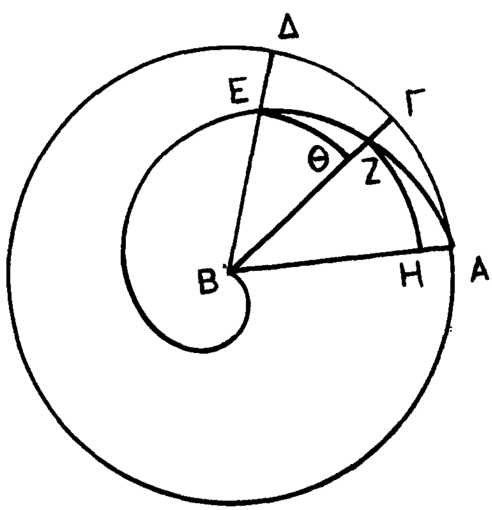

FIGURE 2

Pappus' earlier assessment of this procedure as "remarkable" appears entirely justified. By means of familiar elementary theorems on similar figures and the notion of solids generated by revolution, the area of the spiral is determined.

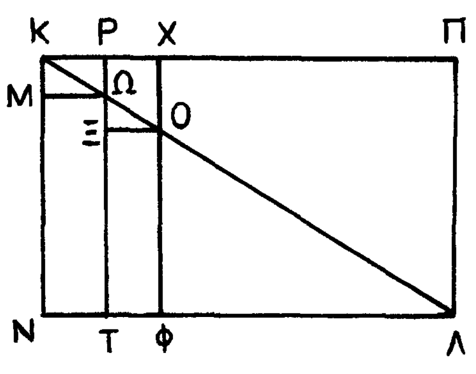

FIGURE 3

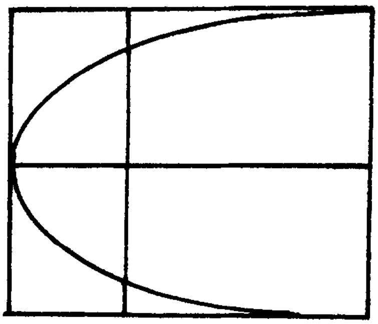

FIGURE 4

As we shall see, the procedure employed is surprisingly unlike that given in *Spiral Lines* [7]. In fact, the closest analogues are to be found in *The Method*, where Archimedes presents the heuristic arguments behind his theorems on the volumes and centers of gravity of solids of revolution and related figures, but none on the spiral. In the *Method* the evaluation of a volume as the sum of constituent parallel elements is commonplace; but unlike the theorem in Pappus, the elements of volume are indivisibles. In prop. 4, for instance, Archimedes compares the paraboloid segment and the cylinder which contains it by establishing a proportion involving the associated circular sections of each (Fig. 4). Archimedes speaks here of the sections "filling up" the volumes. When he formalizes this theorem in *Conoids and Spheroids*, prop. 21, the circular sections are replaced by narrow cylinders whose aggregates bound the segment above and below (Fig. 5). The expression for the aggregate, "all the cylinders," recalls the similar phrase which appears in Pappus, and in fact is usual in Archimedean theorems on area and volume. As another instance, in *Method*, prop. 12-15, Archimedes determines, first heuristically, then formally, the volume of a cylinder cut obliquely by a plane through a diameter of its base. In prop. 14 this is done via indivisibles and a reduction of the volume-problem to the area of a parabolic segment: a proportion is established linking "all the triangles" which comprise a solid prism and all the surfaces which comprise the solid to be measured to "all the lines" which comprise a rectangle and all the lines which comprise the parabolic area. In prop. 15 the formally correct proof is given, in which the solids are bounded by aggregates of prisms and the plane figures by aggregates of rectangles. In relation to the spiral-theorem in Pappus, these are interesting not only for their similar subdivision of volumes, but also for their transformation of the given problem to one of a different dimension.

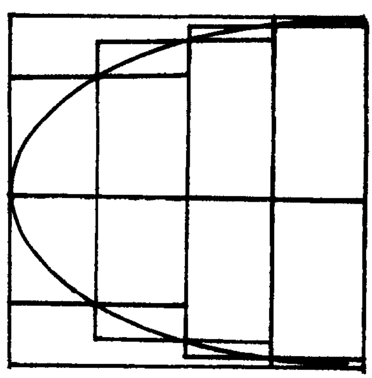

FIGURE 5

Comparison with the theorems in *The Method* reveals how far the treatment in Pappus still is from confronting the demands of a fully formal argument. Asserting as "obvious" the convergence of the bounding aggregates to the cone might seem to rely on the Euclidean treatment of the cone [*Elements* XII, 10], but this is not possible in the present case, since Euclid conceives the cone as the limit of inscribed rectilinear pyramids, not aggregates of cylinders. Thus, an independent convergence argument must be given. On the model of the convergence theorem for conoids [*CS*, 19], it would take this form: the circumscribed and inscribed aggregates are considered together (Fig. 6); their difference is seen to equal one of the sections of the large enclosing cylinder. If, then, the axis of the cylinder is successively bisected, the cylindrical section will eventually become less than any preassigned magnitude. Thus, the circumscribed and inscribed aggregates may be constructed as to differ by less than a preassigned amount. It follows that the aggregates converge to the cone. To complete the convergence argument for the spiral, we employ the fact that each aggregate of cylinders has to the whole cylinder the same ratio that a corresponding aggregate of sectors has to the whole circle. By an indirect argument, the assumption that the ratios of volumes and the ratios of areas tend to different limits leads to contradiction. Thus, the aggregates of sectors must converge to the spiral and have the same ratio to the whole circle as the cone to the cylinder, namely, one-third.

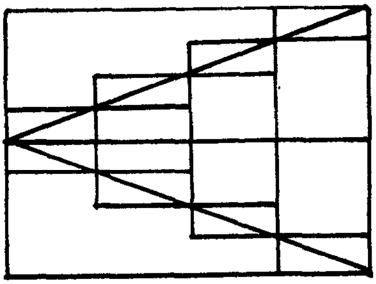

FIGURE 6

From the formal viewpoint, therefore, the conclusion asserted by Pappus is far from "obvious." He has not introduced the notions required for the indirect proof of convergence: preassigned differences of area and volume, subdivision of the circle or the cylinder into sufficiently many parts so that the difference between constructed figures circumscribing and inscribed in the spiral and the cone will be less than those preassigned, and so on. In fact, the consideration of the inscribed cone in separation from the circumscribed suggests that convergence is being viewed in the Euclidean manner of "approximation," rather than in the Archimedean manner of "compression" [8]. If so, the convergence argument would turn out to be even more complicated than sketched above. This lends further support to our view that the treatment in Pappus derives from an early stage of Archimedes' mathematical studies, when he would be more apt to apply the methods of Euclidean geometry, rather than the methods of his mature works. At any rate, the present theorem shows no concern over formal matters--a fact which would be surprising if it had been composed after Archimedes had worked out the formally correct proof in *Spiral Lines*.

Archimedes' presentation of the theorem in *SL*, 24 differs strikingly from that in Pappus [IV, 22], both in conception and in attention to formal detail. As in Pappus, the spiral is bounded above and below by aggregates of sectors. But in a prior theorem [*SL*, 21] Archimedes has shown how to construct the sectors so that the difference of the circumscribed and inscribed aggregates shall be less than a preassigned magnitude; since that difference equals the area of one of the sectors of the circle, the construction follows by successive bisection of one quadrant. Now, the radii of the sectors increase in arithmetic progression (from *SL*, 12). Thus, the sectors increase as the sequence of square integers. Via an arithmetic lemma [*SL*, 10] to the effect that the sum of the squares 1, 4, ..., n^2 is greater than $(1/3)n^3$ but less than $(1/3)(n+1)^3$, it follows that the inscribed aggregate is less than one-third the whole circle, while the circumscribed aggregate is greater. If, then, the area enclosed by the spiral is assumed not to equal one-third the circle, the bounding aggregates may be constructed to differ by less than their difference. In conjunction with the result on the sums of the squares, the contradiction follows, first that the third of the circle both exceeds and is exceeded by the circumscribed aggregate, next that it both exceeds and is exceeded by the inscribed aggregate. As this is impossible, the spiral must equal one-third the circle.

Among the many changes between the version in Pappus and this one, those occasioned by formal necessity are easily understood. But why Archimedes should have introduced the argument based on arithmetic manipulations--more complicated than the above indicates--in preference to the ingenious solid construction is far from obvious. It will emerge from what follows later that a formal convention is the likely explanation: namely, that in a plane investigation, such as the determination of an area, the use of solids is viewed as inappropriate. We have noted already that Archimedes' construction of the tangent to the spiral had been criticized for unnecessarily employing solid methods. The same objections would certainly have been raised against their use in the theorems on area.

Having established the area bound by the whole spiral, Pappus asserts the analogous theorem for sectors of it. "We will prove," he says, that if any line *BZ* is drawn in the spiral, the area between it and the spiral is one-third the circular sector of radius *BZ* and contained between *BZ* and the initial line of the spiral [IV, 23]. The proof is omitted, however, presumably for its close resemblance to that already given. In *SL*, 26, Archimedes provides the complete formal proof of a related theorem on the sectors of the spiral, Pappus' theorem being a special case of it.

The remaining two theorems in Pappus are actually only corollaries to the area-theorems and do not appear in *Spiral Lines*. The first is that the ratio of a sector of the spiral (bounded, as before, by the line *BZ*) to the whole spiral (terminated by the line *AB*) is the same as that of the cube on *BZ* to the cube on *AB* [IV, 24]. The proof is effected via the manipulation of compound ratios. From the two prior theorems each area is the third part of a sector or of the whole circle, respectively; hence, their ratio equals that of the sector to the whole circle. Now, sectors are in the ratio compounded of the squares of their radii and the lengths of the bounding arcs. In the present case, we compound the ratio of the square on *BZ* to the square on *AB* with the ratio of *BZ* to *AB* (since, by Pappus' first lemma, the arcs are proportional to the terminating lines) and obtain the ratio of the cube on *BZ* to the cube on *AB*.

A noteworthy feature of this proof is its phrasing in terms of the ratio between cubes. We should have expected instead the expression "triplicate ratio of *BZ* to *AB*," that is, $(BZ:AB)^3$ rather than $BZ^3:AB^3$, in accordance with virtually unanimous usage by Euclid (as in *Elements* XII) and by Archimedes. But the phrasing in terms of cubes does occur in Archimedes, in one theorem of *Plane Equilibria* [II, 10] and in the alternative proof to a theorem from *Sphere and Cylinder* [II, 8]. In the former, two segments of the same parabola, cut off by parallel chords, are shown to be in the ratio of the cubes of their ordinates (i.e., the respective half-chords) by an argument similar to that in Pappus. In the latter, proving an inequality on the segments of the surface of a sphere, there is a manipulation of compound ratio also in the syle of the Pappus theorem [9]. As observed by Heiberg [*Archimedes* I, 217n] this alternative proof is neither clearer nor briefer than the principal proof. Why, then, should it have been included by Archimedes? Significantly, its use of cubes and other solid terms is not adopted in the main proof. In line with our remarks above, the alternative proof can be understood as an earlier version of the proof which was subsequently replaced owing to the stricture against solid methods in plane or surface problems. In addition, as Archimedes' studies of the sphere and of the centers of gravity of plane figures appear to have been done considerably before the composition of such mature treatises as *Spiral Lines*, this unusual phrasing of proportions serves to support our view that Pappus' theorems on the spiral are drawn from an early Archimedean work [10].

From the theorem on cubes, Pappus draws the "obvious" corollary that the areas bound by successive quadrants of the spiral are in the ratio 1 : 7 : 19 : 37 (that is, areas *BΛE*,

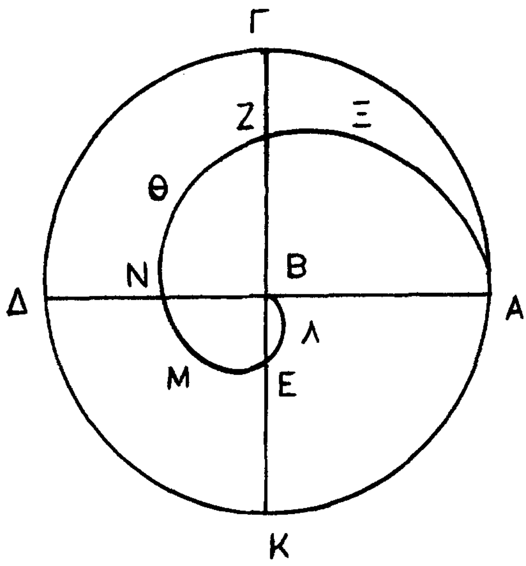

FIGURE 7

BEMN, *BNΘZ*, *BZΞA* in Fig. 7). Here, Pappus' second lemma is applied: since the terminating lines are at equal angles, their lengths are in arithmetic progression, namely, as 1 : 2 : 3 : 4. No further detail is provided; but from the previous theorem, it is clear that the areas bound by the spiral between each line and the initial position will be as 1 : 8 : 27 : 64. Taking the consecutive differences produces the areas of the quadrants, as claimed.

Although Archimedes does not include this property in *Spiral Lines*, he does prove a similar theorem for the areas bound between successive higher turns of the spiral [*SL*, 27]. If the area contained under the first turn is 1, then that between it and the second turn will be 6, that between the second and third 12, between the third and fourth 18, and so on in arithmetic progression. The proof is effected by the manipulation of plane expressions only, based on an extension of the principal area-theorem to the higher turns of the spiral [*SL*, 25]. Nevertheless, it is readily seen how the result in Pappus leads to Archimedes' theorem. The lines terminating the successive turns of the spiral are in the ratio 1 : 2 : 3 : 4 : etc. Forming the cubes and taking consecutive differences yields the ratio of the areas of the spiral between each line and the initial position, namely, 1 : 7 : 19 : 37 : etc. Hence, the areas bound between consecutive turns of the spiral will be as the differences of these terms, 1 : 6 : 12 : 18 : etc. The proof for the general turn requires but a single straightforward identity of differences among cubes [11]. Such observations could well have marked the heuristic stage of Arcimedes' study of the higher turns of the spiral [12]. But, as before, the use of solid terms in a plane problem and the generation of overlapping areas through rotation would present difficulties for a formal treatment along these lines. Thus, it is not surprising to find that the proofs in *Spiral Lines* became more complicated, through the effort to avoid these formal objections.

It is interesting that only in this theorem on the areas of the quadrants of the spiral does Pappus use his lemma on the arithmetic increase of the radii drawn at equal angles. Whereas this same lemma (in the form of *SL*, 12) was the foundation of Archimedes' proof of the principal area-theorem [*SL*, 24], as we have seen, Pappus' version of the proof, making use of the cone, did not require it. Thus, the theorem on the quadrants [IV, 25] may well have performed the additional service of suggesting the way to revise the proof of the area-theorem. For it makes explicit the need for a comparison of the radii drawn at equal angles, and this, in conjunction with the proportions of square terms employed throughout Pappus' version of the principal area-theorem, points to the relevance of the sum of consecutive square numbers for the preparation of an alternative proof.

This comparison of the treatments by Pappus and by Archimedes has revealed numerous discrepancies which are not well understood under Ver Eecke's assumption that Pappus' treatment is merely a commentary on Archimedes. Pappus appears to introduce needless variations of order and terminology, he includes results not to be found in the Archimedean work, and he captures none of the formal sophistication of the Archimedean proofs. On the other hand, these difficulties disappear when we view Pappus' version as based on an earlier stage of Archimedes' study of the spiral. We can thereby begin to perceive how Archimedes discovered the theorems on the areas bound by spirals and what considerations affected his devising the formal proofs in *Spiral Lines*. We shall now turn to discussions by Pappus on other forms of the spiral and show how these may indicate the heuristic thought behind Archimedes' study of the tangents to the spiral.

3. SPHERICAL, CONICAL AND CYLINDRICAL SPIRALS

We have surveyed Pappus' discussion of the "spiral in the plane." Later in the *Collectio* he takes up the construction of a related curve, the "spiral on the sphere," [IV, 35] and, by a proof which bears striking resemblance to his treatment of the plane spiral, he determines the area bound by the spherical form [13]. Given a sphere with a great circle $K\Lambda M$ described about the polar point Θ, the quadrant of a great circle ΘNK is taken as reference line. If now the quadrant of the circle $\Theta O\Lambda$ is conceived to rotate about Θ from the initial position ΘNK, the spiral on the sphere will be the path of a point moving from Θ toward Λ in such a manner that at each position O, the arc $K\Lambda$ has to the circumference of the great circle the same ratio that the arc ΘO has to the arc $\Theta\Lambda$ (Fig. 8). Pappus claims that if a quadrant of a circle $\Delta A\Gamma$ is drawn with radius $A\Delta$, equal to that of the great circle of the sphere, and if its chord $A\Gamma$ is drawn, then the area between the spiral and the terminal position of the rotating generator has to the area of the hemisphere the same ratio that the area of the segment has to the area of the sector (Fig. 9). The proof is remarkably like that of the area

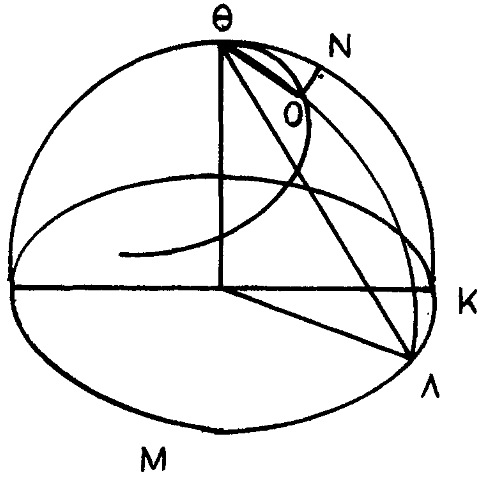

FIGURE 8

of the plane spiral. An arbitrary part $K\Lambda$ of the circumference of the great circle is taken, the point O determined, and a spherical sector ΘNO marked off. At the same time, the sector $A\Gamma Z$ is drawn (being equal in area to sector $\Delta A\Gamma$) and an arc ZE marked off as the same part of ZA that $K\Lambda$ is of the great circle. The sector $Z\Gamma E$ is drawn, its radius $E\Gamma$ meeting the original circle at B. Finally, the small sector ΓBH is completed. Now, arc $\Theta\Lambda$ = arc $A\Gamma$ and arc ΘO = arc $B\Gamma$. It follows that the corresponding chords are also equal. From Archimedes' theorem on the surface of the segment of the sphere [*SC* I, 42], it is known that the spherical sector ΘNO has to the spherical sector $\Theta K\Lambda$ the ratio of the square on chord ΘO to the square on chord $\Theta\Lambda$, which is the same as the ratio of the plane sector $H\Gamma B$ to the plane sector $Z\Gamma E$. From here, Pappus' remarks become very condensed. The same proportion holds for each subsequent part of the great circle. "We will show," he says, that "all the sectors in the hemisphere equal to $\Theta K\Lambda$, which are the whole surface of the hemisphere" have to the aggregate of circumscribed sectors the same ratio that "all the sectors in $AZ\Gamma$ equal to $EZ\Gamma$, that is, the whole sector $AZ\Gamma$," have to the aggregate of sectors which circumscribe the segment $AB\Gamma$. "In the same manner it will be shown" that the analogous proportion obtains for the inscribed figures; "so that also" the surface of the hemisphere has to the area bound by the spiral the same ratio which the sector $A\Delta\Gamma$ has to the segment $AB\Gamma$. Since the hemispherical surface is eight times the sector [cf. *SC* I, 33], it follows that the area bound by the spiral and the terminal quadrant of the great circle is eight times the segment $AB\Gamma$.

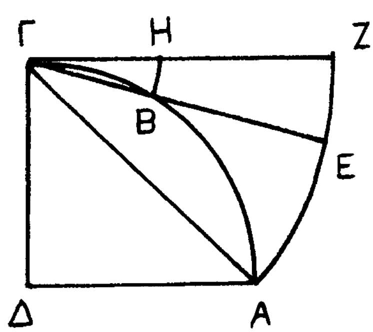

FIGURE 9

This proof, like that of the area-theorem for the plane spiral, is really but a sketch. On numerous points the two treatments are comparable: for instance, marking off an arbitrary portion of the entire circumference; establishing proportions between the figures in two separate diagrams; constructing the circumscribing ensembles apart from the inscribed ensembles. In neither case is the passage to the limiting spiral explained; nor is it precisely clear how this was to be effected--whether by a Euclidean "approximation" argument (as the separation of the circumscribing and inscribed figures suggests) or by an Archimedean "compression" procedure. Moreover, they agree on points of terminology, in particular, speaking of "all the figures" to designate the bounding aggregates--this being, as we have said, commonplace in Archimedean treatments. They are even linked by the common use of future tense ("we will show," or "it will be shown") to indicate proofs which *can* be given, but which in fact are omitted. This usage is common in Autolycus, for instance, and is found often in Archimedes' *Spiral Lines* [props. 1, 14, 15, 16, 17, 19, 20, 25, 27], *Sphere and Cylinder* [I, props. 6, 44] and elsewhere. (When in the *Method*, prop. 1, he closes the heuristic argument by saying, "we will place in order the geometric proof which we have discovered and published earlier," [ed. Heiberg, II, 438] Archimedes may thus be indicating that the informal proof *can* or *must* be followed by a formal version, without actually intending to do so here, as Heiberg assumes.)

A major difference between the theorem on the plane spiral and that on the spherical spiral is the technical level assumed. Whereas the former requires only such theorems on areas and volumes which may be found in Euclid's Books VI and XII, the spherical curve requires Archimedes' theorems on the surface of the segments of the sphere [*SC* I, 33, 42]. Now, this might be supposed to compel us to date the study of the spherical spiral after the composition of *Sphere and Cylinder*. But, in fact, it does not. As indicated earlier [3], both Hero and Pappus had access to a book called by the same title, but differing from the one we know. Pappus presents many theorems from it in Book V of the *Collectio*, among them versions of the two theorems required for the spiral theorem [V, 30-31]. An indication that the spiral theorem in fact refers to this version, rather than the extant Archimedean treatise we know, is that it speaks of the "pole" [πόλος] of the segment, as do Pappus and Hero, rather than the "vertex" [κορυφή] which is usual in the Archimedean treatises. Moreover, on the grounds of the methods employed in the theorems presented by Pappus, it is apparent that his version of *Sphere and Cylinder* represents an earlier stage of Archimedes' treatment of these materials. It would thus appear that Pappus' theorem on the spherical spiral, arguably an Archimedean work from its close resemblance to the theorem on the plane spiral, derives from an early period of Archimedes' researches, but after his first presentation of his results on the sphere and cylinder.

Beyond the spherical spiral, the ancients also recognized the conical spiral [Proclus, 111]. This may be conceived as the path of a point which, as it generates the spiral in the plane is simultaneously moving perpendicularly to the plane with a uniform motion. It is thus the vertical projection of the plane spiral onto an isosceles cone, or the intersection of that cone with the vertical cylindrical surface generated by the spiral. It is this last conception which appears in Pappus [IV, 34]. Among its properties, the area bound by the spiral and the projection of the initial line may readily be found. (This is not cited by Pappus.) Archimedes determined the surface of the cone [*SC* I, 14] and it is cited in this form by Pappus [V, 23]. But Hero [*Metrica*, I, 37] has preserved a different conception of the area of the cone which makes it evident: if the cone is cut along a radius and unfolded, it becomes a plane circular sector. From a theorem attributed by Hero to Archimedes and related to the first proposition of *Dimension of the Circle*, it is known that the area of the sector is one-half the radius times the bounding circular arc; thus, the area of the cone is half the slant-height times the circumference of the base. Now, if the spiral is described on the cone, one may show easily that when it is unfolded, as above, the conical spiral will become the familiar plane spiral. Thus, the area bound by it and by the conical spiral is the same, namely, one-third the surface of the cone.

The ancient studies of the conical spiral must surely have included theorems like the one above, although none of this type has been preserved in Pappus. But one property of the conical spiral does appear in Pappus, and it has interesting implications for the problem, not of areas, but of tangents to the spiral. In IV, 33-34 Pappus shows how the spiral can be related to another plane curve, the "quadratrix," by means of projections involving the conical spiral and the cylindrical spiral. Now, the principal property of interest of the quadratrix is its use to construct a rectilinear area equal to a given circle [IV, 30-32]. This curve is defined as the intersection of a line $B\Gamma$ (in Fig. 10) moving vertically from B to A with a uniform motion and a radius AB, equal to $B\Gamma$ and rotating uniformly through a quadrant in the same time. Thus, for any point Z on the curve, $Z\Theta:AB$ = arc $E\Delta$:arc $B\Delta$. As Pappus proves, the terminal position H of the curve BZH so generated is such that the lines AB, AH have the same ratio as the arc $B\Delta$ and the line AB. From this, the circle-quadrature is evident. Now, Pappus relates two objections by Sporus of Nicaea. First, that the construction

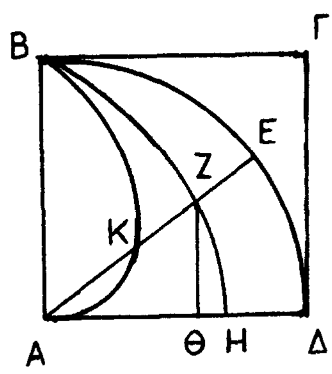

FIGURE 10

requires the synchrony of two motions, one circular and one linear, and thus appears to beg the principle; second, the terminal point *H* of the curve is not determinate, since here the two moving lines coincide. In presenting the solid construction by which the quadratrix and the spiral are related, Pappus seeks effectively to circumvent the former criticism. But the latter objection still holds: the terminal position of the quadratrix can be specified only as the limiting point of curves determined by intersecting surfaces.

But more interesting than these remarks by Pappus are some he omits. If, in Fig. 10, accompanying the quadratrix, the plane spiral is traced from *B* to *A* (where the genesis is the reverse of the usual, so that *A* is actually the "origin"), the tangents to both curves at *B* will be one and the same line. Moreover, the intercept of this mutual tangent with the terminal line *AΔ* extended equals the length of the quadrant *BEΔ* (a known property of the tangent to the spiral), so that we obtain an alternative means of solving by the quadratrix the very problem for which it was named. In view of Pappus' interest in these curves, it is surprising that such properties should be omitted. Did the ancients remain unaware of these facts? Perhaps. But if we take the far more likely view that they did perceive them, we can discover the heuristic insight behind Archimedes' theorems on the tangents to the plane spiral.

From Pappus' discussion of the conical spiral [IV, 34] the following relations are known: if the spiral *BHA* in the plane is projected vertically onto a right isosceles cone, the curve described is a conical spiral. In Fig. 11, the latter is the path of the moving point *K* where *BH* = *KH* [14]. If, in its turn, the conical spiral is projected horizontally onto the vertical cylindrical sheet whose base lies along the quadrant *AΔΓ*, the cylindrical spiral *Γθ* results [15]. Now, the cylindrical spiral is also formed as the distortion of the diagonal of a rectangle into a space-curve when the rectangle is folded around to form a cylinder. With reference to Fig. 12, it follows that the tangent *ED* drawn to the cylindrical spiral at *E* will intercept the plane at its base to produce a line segment *BD* equal to the corresponding arc *AB* of the plane projection of the spiral. Moreover, when one of these curves is projected onto another, the tangent to the first will be projected onto the tangent to the second (since the generating motions in the plane of projection are the same). Thus, in Fig. 11, the tangent to the conical spiral at *K* will project onto the tangent to the cylindrical spiral at *θ*. Where the two curves meet at *M*, their tangents have intercepts in the plane of the plane spiral each equal to the arc *AΔΓ* of the circle. Similarly, the tangent to the plane spiral at *H* projects onto the tangent of the conical spiral at *K*. The latter lies in a plane which intersects the base plane in the line *BN*, perpendicular to the radius *BH* of the

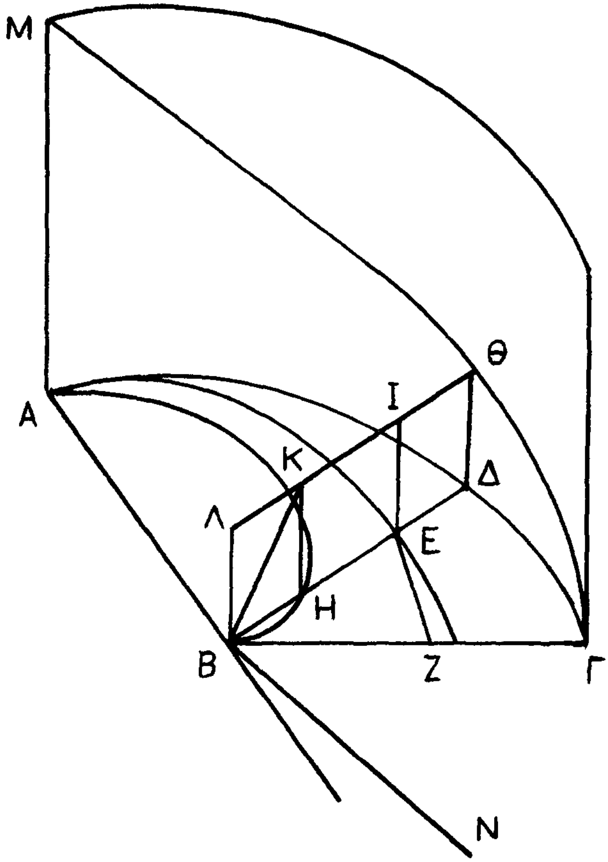

FIGURE 11

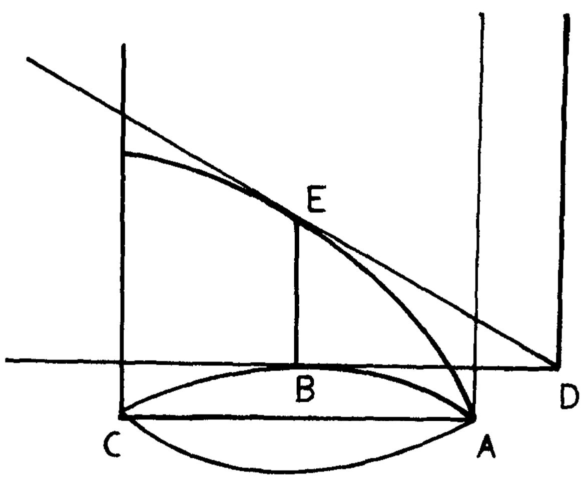

FIGURE 12

plane spiral [16]. The tangents at *A* and *M* will thus meet on the line $B\Gamma$. From before, the intercept along $B\Gamma$ equals the arc $A\Delta\Gamma$. Thus, we obtain the property of the tangent drawn to the plane spiral at *A*: its intercept with $B\Gamma$ equals the arc $A\Delta\Gamma$ [17].

This construction may seem complicated. But it entails nothing beyond the capabilities of the Greek geometers and, in fact, accords well with the methods favored in many of their theorems. As instances of the Greeks' phenomenal intuition, one can recall Archytas' use of intersecting solids to duplicate the cube, Eudoxus' generation of the hippopede to simulate planetary motion by compounding spherical revolutions, Apollonius' extensive inquiry into the sections of cones, and numerous other feats of solid geometry. Indeed, Pappus' discussion of the conical spiral reveals an ease working with the intersections of solids that makes the tangent-construction seem relatively straightforward. To a modern eye, trained in the algebraic tradition of analysis, it might seem remarkable that this property of the tangent might be discovered through such a spatial image, rather than one, say, drawing on the composition of the two motions by which the spiral is generated. This latter type of argument has been proposed by Heath, for instance, and also serves well to make the tangent-property clear [18]. But the Greek mathematicians seem to have had difficulty with the notion of instantaneous velocity, on which it depends, and may thus have missed seeing it.

Accepting a solid construction of the type just given as the basis of Archimedes' discovery of the property of the tangent to the spiral, the objections to his procedure raised by Pappus now become clearer. As in the examination of the area-theorem, it is the actual introduction of solids which is inappropriate, since these problems are plane. Upon reconsideration, Archimedes could devise the *neusis*-constructions given in *Spiral Lines*, so to eliminate the solids. Apparently, Pappus, in replacing the *neuses* by the intersection of two conic sections, a parabola and a hyperbola [IV, 54], did not perceive that he was reintroducing a solid method into the problem. But he derived his classification of geometric problems--into plane, solid and linear--from Apollonius [cf. *Collectio* III, 7; IV, 36; VII, 662]. Indeed, Apollonius' extensive work, the two books on *Neuses*, arranged systematically all those instances which could be effected by plane methods, that is, constructions involving only circles and lines [Heath 1921, II, 189-192]. The confusion in Pappus is thus accountable under the assumption that the source from which he drew his objections to Archimedes' *neuses* and his own alternative construction via conic sections was composed soon after Archimedes' early studies of the spiral, but before Apollonius' studies. In the extant *Spiral Lines* Archimedes introduces the constructions by *neuses* without concern for their alternative execution, whether by plane or by solid means [19].

It would thus appear that in his own treatment he was mindful of the stricture against the use of actual solids in problems of this type, yet unaware of Apollonius' classification of *neuses*.

As presented above, the construction of the tangents to the spiral via the conical and cylindrical spirals has not yet referred to the quadratrix. Since at the initial point the motions which begin its generation are the same as those which generate the spiral in the plane, it follows that a single line is tangent to both curves at that point [20]. In this connection, it is interesting to examine a remark by Simplicius (drawn from Iamblichus) on the use of curved lines to solve the problem of squaring the circle:

> *Archimedes constructed* [a solution] *to the problem by means of the helicoid* (?) *curve, and Nicomedes by means of the curve called specifically the quadratrix, and Apollonius by means of a certain curve which he himself names sister of the cochlioid, but this is the same as that of Nicomedes, and Carpus by means of a certain curve which he calls simply of double motion, and many others in a variety of ways.* [21]

Now, Pappus' construction [IV, 33] produces the quadratrix via projection from the cylindrical spiral. As Apollonius called this spiral the "cochlias," it would thus appear that this relation prompted Apollonius to refer to Nicomedes' quadratrix as the "sister of the cochlioid" [sc. the cylindrical spiral]. The curve would thus have acquired its name "quadratrix" only later, with Nicomedes. In fact, this curve has another basic property: its use to trisect any rectilinear angle. Its invention appears to date back to Eudoxus' colleague Dinostratus [22], and Pappus describes how to use this curve, as well as the plane spiral, not only to trisect a given angle, but to solve the general problem of angle-division [IV, 45-46]. In view of this, it seems plausible that Apollonius was the source of Pappus' solid construction, designed to relate the tangent properties of the plane spiral and the "sister of the cochlioid," the latter then being known only as Dinostratus' angle-trisector. Under this view, the discovery which led to renaming this curve the "quadratrix"--namely that the limiting position (point H in Fig. 10) enabled an alternative solution to the quadrature problem--was made somewhat later by Nicomedes [23].

4. ARCHIMEDES' FORMAL TREATMENT OF THE SPIRALS

We have argued that Pappus' discussions of spirals can be taken as a guide to understanding the development of Archimedes' study from its heuristic stage to its formal stage. The book *Spiral Lines* itself contains internal indications of stages in its composition.

One such indicator is Archimedes' use of a special "lemma" in his proofs of convergence. In *SL*, 21-23 (as in *Conoids and Spheroids*, 19-20) he shows how the difference between circumscribed and inscribed aggregates can be made smaller than a preassigned magnitude via the continual bisection of a given magnitude until it becomes less than that preassigned. Archimedes thus relies upon the principle proved in *Elements* X, 1, that continual bisection of a given finite magnitude will eventually produce a remainder less than any preassigned magnitude. But in other works, *Quadrature of the Parabola* (prop. 16), *Sphere and Cylinder* I (props. 2, 6, 33-34, 42-43) and in the opening sections of *Spiral Lines* (props. 1, 4), this lemma is understood in a different form. In the prefaces to these works, Archimedes states it as an explicit assumption: that, given two unequal magnitudes, their difference may by finite multiplication be made to exceed any preassigned magnitude of the same type. In view of his care in making this assumption explicit, and in view of his uneasiness in requiring such an assumption (*QP*, preface), it is odd that he discontinues its use, reverting to the Euclidean lemma in the area-theorems of *Spiral Lines* and in the later book on *Conoids*. This is especially the case, since more efficient proofs of convergence in *SL*, 21-23 and *CS*, 19-20 are obtainable by means of Archimedes' version of the lemma. But our survey of Archimedes' early studies of the spiral has suggested that the theorems on area were begun considerably before the writing of *Spiral Lines*, at a time when the elementary methods were a stronger influence on his style and before he had discovered the need for such formal improvements as a new axiom of convergence. Of course, the Euclidean form is valid. Such early proofs as *SL*, 21 and *CS*, 19 might retain their original form when they were incorporated into the later formal treatises. But once his own more exact and efficient form of the axiom of convergence had been formulated, Archimedes would surely frame the proofs of new theorems around it [24]. That this happened with the tangent-theorems in *Spiral Lines* also agrees with our view that the *neusis*-constructions in the formal treatment were a later modification following criticisms of the initial studies which employed solids.

We might find additional confirmation of the claim that the tangent-theorems were worked out after the area-theorems (in the form preserved by Pappus) from the fact that when Archimedes lists the principal results proved in *Spiral Lines* in the preface to that work, he gives first place to *SL*, 24--the area-theorem central to Pappus' treatment [IV, 22]. Archimedes next cites the major tangent-theorem [*SL*, 18], the theorem on the areas bound by consecutive turns of the spiral [*SL*, 27], and a theorem on the portions of area divided by the spiral in a sector [*SL*, 28]. Now, these theorems in the preface formed a section of a list of theorems sent much earlier to Conon. Consequently, it appears from our discussion of Pappus' theorems that this list announced not new discoveries, but new proofs--at least as far as *SL*, 24 was concerned, and perhaps also *SL*, 18. But the formal proofs now extant were considerably later than the early studies and profited from Archimedes' notable advances in technique, the nature of which we have seen through our comparison of the treatments by Pappus and by Archimedes.

Second, we recall that Pappus' treatment of the area-theorem opened with two lemmas: (1) on the proportionality between the angles and the radii of the spiral and (2) on the arithmetic increase of radii drawn at equal angles. Pappus offers no proof of (2), treating it as "obvious;" and , in fact, it is an immediate corollary to (1). It is thus surprising that Archimedes gives complete and independent proofs of both, in reverse order, lemma (2) as *SL*, 12, lemma (1) as *SL*, 14. The explanation, I believe, is to be found in a change in the theory of proportions he was using. In the proof of *SL*, 14 appeal is made to *SL*, 2 in which the Euclidean definition of proportion [*Elements* V, Def. 5] is understood. But there are indications that in earlier studies he employed a different theory, more closely related to the anthyphairetic approach developed in the mid-fourth century [25]. In this theory there is characteristically a division into commensurable and incommensurable cases. Under it, Archimedes' theorem on the proportionality between angles and radii in the spiral would first require a proof for commensurable angles, equivalent to *SL*, 12. Then the incommensurable case would follow, in which the assumption of non-proportionality could be reduced to the commensurable case to obtain a contradiction. In the later revision, Archimedes could draw from the Euclidean theory to produce a general theorem on motions: that the distances covered in equal times according to two uniform motions have a constant ratio [*SL*, 2]. The theorem on the lines in the spiral [*SL*, 14] is merely a special case of this, so that it can be accepted as "clear," with a passing reference to "the preliminaries outside" for justification. The commensurable case of the earlier version could be retained virtually unaltered as a separate theorem [*SL*, 12], being in this form useful for the proof of the area-theorem [*SL*, 24]. Thus, the ordering of these theorems in *Spiral Lines* appears to betray a stage when the proofs were effected under the older proportion theory, subsequently revised under the Euclidean. The theorems in Pappus then represent an even earlier pre-formal stage, when the proportionality in lemma (1) could be passed over altogether as "easy to see."

5. SUMMARY AND CONCLUSION

Our argument concerning Archimedes' early study of the spiral has been founded on the assumption, confirmed in one instance after another, that Pappus possessed sources we no longer have and quoted from them *verbatim*, sometimes at length, even when not saying explicitly that he was doing this. But in the case of the theorems on the area of the spiral he does give the impression of following a source: he assigns to Conon a role in their formulation and pronounces on the "remarkable procedure" employed in Archimedes' proof. This source was not the extant *Spiral Lines*--and, in fact, Pappus seems not to have had access to that work at all. Comparison of the excerpts with related sections of the extant work recommends viewing it as an early Archimedean effort, more heuristic than formal, yet ingenious in its conceptions, particularly in the use of the cone and the cylinder to determine the ratio of the areas of the spiral and its enclosing circle.

Although Pappus attributes the initiation of the study of the spiral to Conon, it is not likely that Conon had worked out the theorem on the area of the spiral, as that is the project of the Archimedean proofs. Rather, one may infer that Conon had introduced the spiral for another purpose: to effect the division of any angle in a given ratio, that is, the general form of the problem of the trisection of the angle. This is a use which Pappus makes of the spiral [IV, 46] and which is easily deduced from an Archimedean theorem [*SL*, 14] included among the results leading up to Pappus' version of the area-theorem [IV, 21]. Archimedes' silence on this basic property of the spiral in *Spiral Lines* would be virtually unaccountable, save under the view that he could assume this to be a familiar result due to another mathematician. Conon's definition of the spiral, perhaps accompanied by a statement of the problem of determining what the area bound by the spiral was, might thus have served to stimulate Archimedes' researches.

In addition to the area-theorem of the plane spiral, solutions of the area bound by the spherical and conical spirals were possible under the same methods, and Pappus has preserved a treatment of the former which leaves the impression of being another early Archimedean study [IV, 35]. Once the space curves have been introduced, a route is opened to the discovery of the properties of the tangents to the spiral. An intricate construction, linking the plane, conical and cylindrical spirals with the quadratrix is given by Pappus [IV, 33-34], and although we have argued this as derived from Apollonius, it may represent the manner of Archimedes' early investigation of the tangents. The extensive use of solids in the study of plane curves was deemed improper, however, and this formal stricture appears to have encouraged Archimedes to devise alternative *neusis*-constructions to prove the tangent-properties. In addition, it led Apollonius to classify the different types of *neuses* and to show which *neusis*-problems were solvable by plane methods.

With few exceptions, the Archimedean proofs are ruthlessly formal and precise. From the seventeenth century onward the complaint has frequently been made that Archimedes and other ancient geometers deliberately hid their heuristic methods. But after Heiberg retrieved and published *The Method* early in this century, this view has been seen as inaccurate. While great emphasis was indeed set on formal rigor, geometers like Archimedes made free use of heuristic devices, such as the introduction of mechanical notions and the manipulation of indivisibles, and were eager to communicate them. A review of Pappus' theorems on the spiral, accepted as an extract from early Archimedean studies, shows how a different set of heuristic devices--a modification of the technique of indivisibles and the introduction of auxiliary solids--formed the basis of Archimedes' discoveries on the spiral. But proofs along such lines could not satisfy the formal conditions recognized by the Alexandrian geometers of that time. Just as Apollonius allowed the circulation of a hurriedly written and uncorrected draft of the *Conics*, but subsequently deemed it wise to prepare a revised edition [*Conics*, I, preface], so it would appear that Archimedes' formal treatises, *Spiral Lines* and others, were reworkings of earlier versions known at Alexandria [26]. Through the materials preserved by Pappus we can better appreciate the manner and the motives under which the heuristic thought leading to discoveries about the spirals was transformed into formal demonstrations.

NOTES

The following conventions are followed in citing mathematical authors: (1) Pappus--by the book and chapter in the *Collectio*, ed. Hultsch; (2) Archimedes--by the proposition in the edition of Heiberg; *SL* = *Spiral Lines*, *SC* I = *Sphere and Cylinder*, Book I; *CS* = *Conoids and Spheroids*; (3) Euclid--by the book and proposition of the *Elements*, ed. Heiberg; (4) Apollonius, Theon, Proclus, Eutocius--by the page of the cited editions.

1. Ver Eecke cites the following works reconstructed on the basis of the account in Pappus' Book VII: of Apollonius, "Section of an Area" [1933, I, lix], "On Determinate Section" [*ibid.*, lxi], the two books on "Neuses" [lxv], "On Contacts" [lxix], and "Plane Loci" [lxxv]; of Euclid, the "Porisms" [lxxxiii-viii]. Cf. also the commentary by Heath [1921 I, 435-437; II, 179-192].
2. Hero gave his construction in at least two different places: the *Mechanica* and the *Belopoeica*; cf. I. Thomas [1939 I, 267n].
3. An instance of this will be important for our discussion later. In V, 30 Pappus provides an alternative proof of the theorem on the surface of a segment of the sphere, proved by Archimedes as *SC* I, 42. However, Pappus' statement of the theorem, as well as his proof, differs from Archimedes' statement; in particular, Pappus speaks of the "pole" of the

segment, rather than the "vertex." One might suppose that Pappus had produced the alternative proof himself, to complement the familiar Archimedean treatment; indeed, Pappus refers frequently to an Archimedean work he calls "The Book on the Sphere and Cylinder." However, when Hero cites this same theorem, from the book of the same title, he uses the phraseology of Pappus, not Archimedes [*Metrica* I, 39]. Since Hero lived over two centuries before Pappus, he could certainly not be quoting from Pappus. Moreover, Pappus could not derive his far more extensive materials from Hero. Rather, both must have been drawing from an earlier work, known under the name of Archimedes and the title "Sphere and Cylinder," but markedly different from the work we know by that title. Comparison of the extracts produced by Pappus in *Collectio*, Book V with the extant treatise reveals the source for the former to be an early Archimedean work, as I argue in detail in a study in progress. Thus, Pappus' theorems on the sphere and cylinder in Book V bear to the extant formal work by Archimedes a relation similar to that proposed in the present paper for Pappus' theorems on the spiral and Archimedes' *Spiral Lines*.

4. It is remarkable that Ver Eecke interprets the "procedure" (ἐπιβολή) which Pappus praises as used in Archimedes' proof [IV, 21; quoted above] as a reference to Archimedes' method of exhaustion [1933, I, xxviii; cf. 182n]. For the heart of that method is the indirect argument justifying the assertion of a property for a limiting case, and this is entirely lacking in Pappus' proof. Moreover, ἐπιβολή ought to refer to the overall procedure or general conception, literally the "plan of attack." Hultsch interprets the term as referring to the conception (*ratio*) of points moving uniformly [*Collectio*, 235]. But this is essential for the very definition of the curve and would hardly have been absent from Conon's treatment of it. Rather, as we shall show, the "procedure" or "conception" was the ingenious use of an equation between the ratios of areas and of volumes to effect the proof of the theorem on the area bound by the spiral [IV, 22].

5. A construction by *neusis*, or "inclination" (that is, by means of a marked ruler) involves placing a line segment of given length between two given curves so that the segment inclines toward a given point (i.e., the point lies on the extension of the line segment). Such constructions are an important device in Archimedes' proofs on the tangents to the spiral in *Spiral Lines*. On these constructions, see Heath [1897, ch. V] and Dijksterhuis [1957, 133-140].

6. For discussions of this question, see Heath [1897, ciii-v; 1921 II, 68, 556-561], Dijksterhuis [1957, 138], and Ver Eecke [1933 I, 209n].

7. It is interesting that A. Czwalina has proposed as the heuristic basis of Archimedes' theorem on the spiral an argument

precisely like that just given: namely, the reduction of the determination of the area of the spiral to that of the volume of the cone [1922, 61-71]. Czwalina calls the conception "algebraic," apparently because the link between the two problems lies in their common solution via the summation of second-powers. Unfortunately, the points he raises in support of his view are largely unpersuasive. Moreover, he is unaware of the strongest point in its favor: that Pappus presents this same proof in association with the name of Archimedes.

8. These are the terms for Archimedes' convergence-proofs used by Dijksterhuis [1957, 130-133]. In a study now in preparation, I am employing this distinction of "approximation" vs. "compression" as one criterion for separating earlier and later parts of the Archimedean corpus.

9. One may note further that "composite ratio" appears in the *Elements*, notably in VI, 23. The terminology for proportion used by Pappus [IV, 22, 24] and in the Archimedean theorems at issue here ("as *A* is to *B*, so *C* is to *D*") is standard in Euclid (146 occurrences in Book XII alone). It is often used by Archimedes (see Heiberg's index); but he tends to prefer an alternative expression: "*A* has the same ratio to *B* that *C* has to *D*."

10. As indicated in note 3, I will argue that a substantial work on the sphere and cylinder appeared as an earlier version of the formal work we know by that title. The placement of *Plane Equilibria*, II, has been debated: Heiberg favoring an early placement [Archimedes III, xc; cf. II, 53n]; Arendt arguing a much later date [1913, 296-301]. In the study in progress, referred to in note 8, I argue an early date for this work, more in line with Heiberg's view.

11. Namely, $[(n+1)^3 - n^3] - [n^3 - (n-1)^3] = 6n$. A proof of this identity can be effected easily by means of the geometric techniques of *Elements* II, as extended to the solid case.

12. Heath [1897, 180-182] attempts a different heuristic argument for the area of the higher turns of the spiral [*SL*, 27], based on a generalization of the principal area-theorem [*SL*, 24]. However, Archimedes nowhere states the theorem in the form employed by Heath (contrast *SL*, 26).

13. Besides the text in Hultsch's edition and the translation by Ver Eecke, see also the text with translation by I. Thomas [1941, II, 580-587] and Heath's discussion [1921, II, 382-385].

14. The diagram in Fig. 11 combines the two diagrams of Pappus: that in IV, 34 where *BHA* is the plane spiral and *K* the tracer of the conical spiral; and that in IV, 33 where θ traces the cylindrical spiral. In both diagrams, the point *E* traces the quadratrix and *I* a spiral whose plane projection is the quadratrix. It is noteworthy that the lettering in these two diagrams is consistent.

15. Apollonius devoted a work to the cylindrical spiral

("cochlias") as we learn from Pappus [VIII, 28] and Proclus [p. 105]; cf. Heath [1921 I, 231-232; II, 193] and Heiberg [Apollonius II, 117].

16. This follows since the tangent to the conical spiral at K lies in the plane which contains BK and is perpendicular to the plane of BHK; as the plane of the tangent intersects the plane of the spiral BHA in the line BN, the tangent must meet the plane of the spiral in the same line.

17. The property of the tangent drawn to any point of the plane spiral follows in the same fashion. Let there be drawn the cylindrical spiral which meets the conical spiral at K and has radius BH and its initial point on the line $B\Gamma$ in the plane of BHA. As seen above, the line from H to the point of interception of the tangent to the cylindrical spiral with the plane of BHA will equal the arc of the circle of radius BH and angle $HB\Gamma$. The tangent drawn to the conical spiral at K will cut off from BN a segment of the same length. Hence, the tangent to the plane spiral at H, being the plane projection of the tangent to the conical spiral at K, will intercept the same line BN, cutting off the same segment--equal to the arc of angle $HB\Gamma$ in the circle of radius BH. This is the relation stated in the Archimedean theorems *SL*, 18-20.

18. According to Heath [1921 II, 557-558], Archimedes "must have ... divined the result" in such a manner. To be sure, the Greeks knew of the mechanical principle of the parallelogram of motions (cf. the Aristotelian *Mechanica* 2, 848b10). But, apparently, instantaneous velocity was a notion which eluded them (cf. the arguments on the impossibility of motion in an instant in Aristotle's *Physics* VI, 1-2). The heuristic argument I have proposed, by contrast, avoids instantaneous velocity by referring the tangents of the plane and conical spirals to that of the cylindrical spiral, the latter being "obvious," at least in a heuristic context.

19. *Neuses* are employed in *SL*, 5-9, to be applied in *SL*, 18-20. See Heath [1897, chapter V] and Dijksterhuis [1957, 133-140].

20. In Fig. 10 the tangents drawn at B to the spiral BKA and the quadratrix BZH are one and the same line. Its intercept with the line $A\Delta$ extended equals the arc $BE\Delta$.

21. Simplicus, *In Aristotelis Physica* [ed. J. L. Heiberg I, 60]; text reproduced by I. Thomas [1939 I, 334]. (Translation is mine.) The "cochlioid" (or "cochlias") is the cylindrical spiral of Apollonius, while its "sister" is the quadratrix of Nicomedes. I take Carpus' curve to be the cylindrical spiral also. According to Heath [1921 I, 232], Tannery claimed, but without evidence, that it was the cycloid. My view draws support from a question raised by Proclus [*In Euclidem*, 105, 112]: is the cylindrical spiral a "simple" or a "mixed" curve? Apollonius had shown that this spiral, like the line and the

circle, is "homoeomeric;" that is, any part of it can be superimposed precisely over any other part. Some had therefore classed the spiral as a third "simple" curve. But Proclus (apparently here following Geminus) argues that the two classifications are different; since the spiral is generated through two dissimilar simple motions, one linear, the other circular, the curve must be called "mixed." I infer that Carpus, in light of Apollonius' study, called the cylindrical spiral the "simple curve of double motion." (This may require altering ἁπλῶς of the text to ἁπλῆν.) As Pappus draws from Carpus biographical information on Archimedes [VIII, 1026]--namely, that he wrote only one mechanical book, the *Sphaeropoeia* ("On Sphere-Making"), and as both Pappus and Proclus appear to report through the mediation of Geminus (first century B.C.), any dating of Carpus between the times of Apollonius and Geminus is consistent with my remarks.

22. Pappus names Dinostratus and Nicomedes as having employed the quadratrix [IV, 30]; in this same connection, Proclus names Nicomedes and a mathematician Hippias [pp. 272, 356]. The standard view that this is Hippias of Elis, the Sophist of Socrates' generation [cf. Heath 1921, I, 225-226] is incredible. It entirely overlooks the technical level of geometry ca. 400 B.C. and the significant demands required for the investigation of the quadratrix. Details cannot be provided here, but a far more probable view is to place the Hippias of the quadratrix close to the time of Nicomedes, that is, late in the third century B.C.

23. This passage is interpreted in an entirely different way by Heath [1921 I, 225, 231-232]. He wishes to take Apollonius' "sister of the cochlioid" itself to be the "cochlias" (cylindrical spiral); moreover, the curve of Nicomedes to which it is likened is not, he believes, the quadratrix, but rather the "conchoid," another curve studied by Nicomedes. This requires some questionable philological moves and is ultimately senseless, since, as Heath admits, the conchoid of Nicomedes was not used for squaring the circle and the "sisterhood" of the "conchoid" and the "cochlias" would thus rest on nothing more than the similarity of their names. Moreover, Heath does not relate Pappus' constructions of these curves [IV, 33-34] to this passage, even in his own discussion of Pappus [*op. cit.*, II, 380-382]. He thus missed the way to a far more straightforward interpretation of Simplicius' passage, as given here.

24. Archimedes' lemma and its relation to the Euclidean lemma are discussed in detail in the paper in progress [8].

25. On the anthyphairetic proportion theory and its connection with fourth-century studies of incommensurable magnitudes, see Knorr [1975, ch. VIII/II-III and Appendix B]. The technical features of the older theory can be retrieved from mathematical arguments in Archimedes and contemporary authors (paper in progress).

26. The mathematicians were not alone at Alexandria in engaging in heated debates on formal style. In a famous episode, the followers of the poet Callimachus ridiculed Apollonius of Rhodes for violating the canons of syle in his verse epic, the *Argonautica*. Apollonius eventually regained his good reputation after publishing a suitably revised second edition of the work [cf. *Oxford Classical Dictionary* 1970, "Apollonius of Rhodes"].

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