Historia Mathematica 10 (1983) 274-285

GREGORY'S CONVERGING DOUBLE SEQUENCE

A New Look at the Controversy between Huygens and Gregory over the "Analytical" Quadrature of the Circle*

BY CHRISTOPH J. SCRIBA
INSTITUT FÜR GESCHICHTE DER NATURWISSENSCHAFTEN,
MATHEMATIK UND TECHNIK,
UNIVERSITÄT HAMBURG,
D 2000 HAMBURG 13, FEDERAL REPUBLIC OF GERMANY

SUMMARIES

As is well known, upon publication of his Vera circuli et hyperbolae quadratura *(Padua 1667), James Gregory became involved in a bitter controversy with Christiaan Huygens over the truth of one of his major propositions. It stated that the area of a sector of a central conic cannot be expressed "analytically" in terms of the areas of an inscribed triangle and a circumscribed quadrilateral. Huygens objected to Gregory's method of proof, and expressed doubts as to its validity. As Gregory's iterative limiting process, employing an infinite double sequence, uses a combination of geometric and harmonic means, one may apply to it methods developed by the young Gauss for dealing with a similar process based on the combination of arithmetic and geometric means. This yields both the Leibnizian series for π/4 and the product found by Viète for 2/π, and thus serves to illuminate the structure of Gregory's procedure and the nature of Huygens' criticism.*

Wie bekannt, wurde James Gregory nach der Veröffentlichung der "Vera circuli et hyperbolae quadratura" (Padua 1667) in eine heftige Auseinandersetzung mit Christiaan Huygens über die Richtigkeit eines seiner zentralen Sätze verwickelt. Es ging um die Behauptung, der Sektor eines Mittelpunktkegelschnitts könne nicht "analytisch" durch die Flächen des einbeschriebenen Dreiecks und des umbeschriebenen Vierecks ausgedrückt werden. Huygens brachte Einwände gegen Gregorys

*Herrn Prof. Dr. Kurt Vogel zum 95. Geburtstag gewidmet. Revised version of lectures given at the Institute for the History and Philosophy of Science and Technology of the University of Toronto on May 15, 1978, and at the Symposium on the Life and Work of Christiaan Huygens, Amsterdam, August 22-25, 1979.

0315-0860/83 $3.00

Methode vor und zweifelte die Behauptung an. Da Gregorys Iterationsprozeß eine konvergente, durch Kombination eines geometrischen und eines harmonischen Mittels gebildete Doppelfolge benutzt, lassen sich die vom jungen Gauss für die Behandlung des arithmetisch-geometrischen Mittels entwickelten Methoden übertragen. Sie liefern sowohl die Leibniz-Reihe für π/4 wie das Viète-Produkt für 2/π und verdeutlichen so die Struktur von Gregorys Verfahren und die Natur der Huygensschen Kritik.

Cela est bien connu, suite à la publication de son Vera circuli et hyperbola *(Padoue, 1667), James Gregory se trouva engagé dans une vive controverse avec Christiaan Huygens à propos de la justesse de l'une de ses principales propositions. Celle-ci affirmait que l'aire d'un secteur d'une conique à centre ne peut s'exprimer "analytiquement" en terme de l'aire d'un triangle inscrit et d'un quadrilatère circonscrit. Huygens mettait en doute aussi bien la méthode de la preuve que la justesse du résultat. Puisque la procédé d'itération à la limite de Gregory, employant une suite double infinie, utilise une combinaison de moyennes géométriques et harmoniques, on peut lui appliquer des méthodes développées par le jeune Gauss destinées à des processus similaires basés sur une combinaison de moyennes arithmétiques et géométriques. On génère ainsi la série de Leibniz pour π/4 et le produit de Viète pour 2/π, et de la sorte l'on éclaire la structure de la méthode de Gregory et la nature de la critique de Huygens.*

1.

In 1668/1669 Christiaan Huygens was involved in a mathematical dispute with the Scottish mathematician James Gregory. Gregory, born in 1638, had gone to Italy in 1664, where at Padua he studied mathematics especially. Here the geometrical school of Cavalieri was still flourishing. Before returning to Britain, Gregory had published two books, in 1667 and 1668. We are interested here in the first book, whose title is *Vera circuli et hyperbolae quadratura*, which purported to give the true quadrature of the circle and the hyperbola [Gregory 1667]. It was published at Padua in a limited edition, and was reprinted at Venice in the following year as an appendix to Gregory's second book, the *Geometriae pars universalis*.

Gregory sent a complimentary copy of the first edition to Huygens on September 28, (October 8, N.S.), 1667. In an accompanying letter [1] he claimed that his method of dealing with the problem of squaring the circle was better than that of Gregorius à Sancto Vincentio in his voluminous work [Gregorius à St. Vincentio 1647], and he asked Huygens for his comments. He was particularly interested in these since Huygens himself had written a book on the same problem, entitled *De circuli magnitudine inventa* [Huygens 1654].

Although Huygens never replied directly to Gregory, he published a review of Gregory's book in the following summer, in July of 1668, in the French *Journal des Sçavans* [2]. This review recognized the importance of the work, but it contained a number of criticisms and also pointed to the fact that some of Gregory's propositions had already appeared in Huygens' work.

When Gregory saw this review, he immediately wrote a letter in reply; this was published in the same month in the *Philosophical Transactions* [3]. He corrected the mistake which Huygens had discovered, but at the same time refused to accept the critique concerning his central proposition. This proposition was a very audacious one for a mathematical work of the second half of the 17th century, for Gregory claimed that the circle cannot be squared "analytically," as he called it. By this he meant that π cannot be obtained by a finite sequence of the five basic operations--addition, subtraction, multiplication, division, and root extraction. But while Huygens was not convinced that the proof was correct (and perhaps he was not even convinced that the claim of the proposition in question was true), Gregory in his reply insisted upon the validity of his demonstration.

After having seen Gregory's reply in the *Philosophical Transactions*, Huygens published a second article on November 12, 1668, in the *Journal des Sçavans* [4]. Apart from some detailed points of criticism, which do not concern us here, and the repetition of his suggestion that some of Gregory's propositions were identical with his own theorems, Huygens insisted again on the main point at issue: whether the area of a circle can be derived "analytically" from the radius. In his opinion, this was still an open question.

Gregory, upon seeing this second article of Huygens about his book, was infuriated and in return published a strong attack on Huygens--this time in his new book *Exercitationes geometricae* [Gregory 1668]. Obviously the charge of plagiarism had hurt him deeply and caused him to make rather offensive remarks about the Dutch mathematician.

Since both adversaries belonged to the Royal Society, several of its members were drawn into this conflict. John Wallis in particular drew up a very detailed report about Gregory's "Vera quadratura" for the Society's president, Lord Brouncker [5]. Further details of these discussions are not given here; however, it must be said that at the time the controversy ended without a definite result: Gregory did not give up his claim to have demonstrated the impossibility of an "analytical" quadrature of the circle [6], while Huygens and those members of the Royal Society who had sufficient mathematical capability remained doubtful. The whole polemic has been briefly but competently summarized by E. J. Dijksterhuis [7] in [Turnbull 1939]. The *Vera quadratura* in general is reviewed by M. Dehn and E. Hellinger [8] much more profoundly than was done by G. Heinrich at the beginning of the century [Heinrich 1901]. Thirty years ago J. E. Hofmann studied Gregory's approximations to the sector of a central conic [Hofmann 1950], but he did not discuss the convergent double sequence and Gregory's principal objective: the impossibility of an "analytical" quadrature.

2.

In this article ideas developed by Gauss for dealing with the sequence based on the arithmetic-geometric mean [Gauss 1917, 1927] are applied to Gregory's double sequence. It is shown that, although Huygens was right in rejecting Gregory's arguments as a mathematical proof, at the core of Gregory's mathematical procedure was a gold mine for obtaining deeper results--results which ultimately would open a way for just that proof which Gregory envisaged with deep insight into the structure of his process, although, with the methods of his time, a true demonstration could not yet be given.

Gregory's starting point, in line with the classical tradition, was the construction of a double sequence of inscribed and circumscribed polygons to a sector of a circle (Fig. 1). Let the radius be r, the sectorial angle (at the center A) 2α, the chord $\overline{BC} = 2c$. The tangent lines at the endpoints of the sectors B and C meet at D, and $\overline{AD}$ will intersect the circle at E and the chord at F. With $\overline{AF} = a$, $\overline{FD} = b$ we have $r^2 = a(a + b) = a^2 + c^2$. Then the initial inscribed polygon I_0 will be the triangle ABC, the

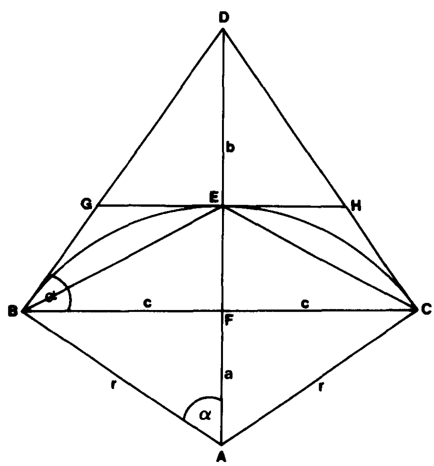

Figure 1

initial circumscribed polygon C_0 the quadrilateral $ABDC$, and we have

$$I_0 = ac, \qquad C_0 = (a + b)c. \tag{1}$$

To obtain the next pair of polygons I_1, C_1, Gregory adds the tangent line at E and the chords $\overline{BE}$, $\overline{EC}$, obtaining $I_1 = ABEC$ and $C_1 = ABGHC$. Since

$$\frac{a + b}{r} = \frac{r}{a} = \frac{a + b - r}{r - a} = \frac{\overline{DE}}{\overline{EF}} ,$$

$$\frac{C_0}{I_1} = \frac{I_1}{I_0} = \frac{C_0 - C_1}{C_1 - I_1}$$

or

$$I_1 = \sqrt{C_0 I_0} , \qquad C_1 = \frac{2C_0 I_1}{C_0 + I_1} . \tag{2}$$

This leads, in general, to

$$I_{n+1} = \sqrt{C_n I_n} , \qquad C_{n+1} = \frac{2C_n I_{n+1}}{C_n + I_{n+1}} . \tag{3}$$

Gregory showed that his double sequence is "convergent" (in fact, it is in this book that he coined the term "convergent"). From the geometrical construction it is clear that the limit or "terminatio," as he called it, will represent the area of the given sector.

Obviously, for the purpose of computing the actual area to a high degree of accuracy, the terms of the double sequence soon become too clumsy, of which Gregory was well aware. He therefore showed in his book how they can be approximated. Of much greater moment, however, were the theoretical conclusions which he based on the structure of the sequence, as given in (3).

One can determine the limit or "terminatio" T of such a sequence, Gregory argues correctly, if one can find a function M ("quantitas," he says) that is composed from the first pair of the sequence in the same way as from the second (or any following) pair:

$$M(I_0, C_0) = M(I_n, C_n) = M(T, T) = T . \tag{4}$$

As an easy example, in his reply to Huygens published in the *Philosophical Transactions* of July 1668 [3] Gregory mentioned the arithmetic-harmonic mean

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} , \qquad b_{n+1} = \frac{a_n + b_n}{2} ,$$

where

$$M(a_n, b_n) = \sqrt{a_n b_n} = M(a_0, b_0) = \sqrt{a_0 b_0} \ ,$$

so that

$$M(T, T) = \sqrt{T \cdot T} = T = \sqrt{a_0 b_0} \ ,$$

i.e., the termination (or limit) in this case is the geometric mean.

With respect to the sequence (3) in question, however, he claimed it to be impossible to give such a function that would be invariant with respect to the order n of the terms "analytically"--that is, using only the five basic operations +, −, ×, ÷, and $\sqrt[k]{\ }$. To make his argument plausible, for it was not a proof, as Huygens rightly objected, he gave the following parameters for the first two pairs:

$$\begin{array}{lll} \text{If} & I_0 = u^2(u + v), & C_0 = v^2(u + v), \\ \text{then} & I_1 = uv(u + v), & C_1 = 2uv^2. \end{array} \tag{5}$$

Since C_1 is different in structure from I_0, C_0, and I_1, by letting the five "analytical" (algebraic) operations work in the same manner upon I_1 and C_1 as upon I_0 and C_0, the same analytical expression would never be produced, as Gregory insisted in Proposition XI of his *Vera quadratura*. Therefore, he concluded, no analytical "terminatio" can exist.

As Huygens pointed out [4], one of the shortcomings of this alleged proof is that it has not been shown that (5) is the only way to represent the first terms rationally. There might be other representations which could perhaps open the way to an "analytical" expression for the limit. Gregory did not share this view but was unable to invalidate Huygens' argument.

3.

It is possible, however, to obtain more information from Gregory's double sequence by applying an idea of Gauss. In doing this, we shall see what potentially lay behind Gregory's starting point.

Gauss, in the course of his early mathematical studies, investigated the arithmetic-geometric mean in the 1790s [9]. It is defined as

$$a_{n+1} = \sqrt{a_n b_n} \ , \quad b_{n+1} = \frac{a_n + b_n}{2} \ . \tag{6}$$

For the limit $M(a_n, b_n)$ of this double sequence, he derived an infinite series. As the most useful representation, the one in which the law of the individual terms is most obvious, he found for the reciprocal of $M(1 + x, 1 - x)$ the series

$$\frac{1}{M(1 + x, 1 - x)} = 1 + \left(\frac{1}{2}\right)^2 x^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 x^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 x^6 + \cdots,$$

a special case of the hypergeometric series.

Working along the same lines we have for Gregory's sequence (3) that $M(kI_n, kC_n) = k \cdot M(I_n, C_n)$, and hence

$$\begin{aligned} M(I_0, C_0) &= I_0 \cdot M\left(1, \frac{C_0}{I_0}\right) = M(I_1, C_1) \\ &= I_1 \cdot M\left(1, \frac{C_1}{I_1}\right) \\ &= \sqrt{C_0 I_0} \cdot M\left(1, \frac{2\sqrt{C_0/I_0}}{\sqrt{C_0/I_0} + 1}\right). \end{aligned} \tag{7}$$

With

$$\sqrt{\frac{C_0}{I_0}} = 1 + y \tag{8}$$

one obtains the functional equation

$$M(1, 1 + y(y + 2)) = (1 + y) \cdot M(1, 1 + \frac{y}{y + 2}). \tag{9}$$

Assuming that $M(1, 1 + x)$ can be expanded into an infinite series,

$$M(1, 1 + x) = 1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots, \tag{10}$$

one can compare the series which arise from both sides of (9) when $x = y(y + 2)$ and $x = y/(y + 2)$ are substituted, respectively. This makes it possible to determine the coefficients α_n in (10):

$$\alpha_n = (-1)^{n+1} \cdot \frac{2}{(2n - 1)(2n + 1)} = (-1)^{n+1} \cdot \left(\frac{1}{2n - 1} - \frac{1}{2n + 1}\right). \tag{11}$$

Thus, the first result is that

$$M(1, 1 + x) = \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right) \arctan \sqrt{x} = \frac{1}{I_0} \cdot M(I_0, C_0). \tag{12}$$

Applied to a quarter of the unit circle where $I_0 = 1/2$, $C_0 = 1$, $y = \sqrt{2} - 1$, $x = (\sqrt{2} - 1)(\sqrt{2} + 1) = 1$, this implies

$$M(I_0,\ C_0) = M(\tfrac{1}{2},\ 1) = T = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots, \tag{13}$$

Leibniz' famous series for $\pi/4$!

In general, by reference to (1), (7), (8), (10), and (12), $\sqrt{x} = c/a = b/c = \tan\alpha$, and therefore

$$M(I_0,\ C_0) = T = I_0 \cdot M(1,\ 1 + x) = r^2 \arctan\sqrt{x} = r^2\alpha. \tag{14}$$

Gregory's double sequence hence is shown to converge to a limit T which, by expanding the limit function into an infinite series, reveals itself as the arctan function, with 2α the angle of the sector under consideration.

But this is not all! Instead of $M(I_n,\ C_n)$ we may consider, for example, $M(I_{n+1},\ C_n)$, for the limit or "terminatio" of this double sequence must be the same. Here we have

$$M(I_1,\ C_0) = I_1 \cdot M\left(1,\ \frac{C_0}{I_1}\right) = \sqrt{C_0 I_0} \cdot M\left(1,\ \sqrt{\frac{C_0}{I_0}}\right) = M(I_2,\ C_1) \tag{15}$$

$$= I_2 \cdot M\left(1,\ \frac{C_1}{I_2}\right) = \sqrt{C_0 I_0}\ \sqrt{\frac{2\sqrt{C_0/I_0}}{\sqrt{C_0/I_0} + 1}} \cdot M\left(1, \sqrt{\frac{2\sqrt{C_0/I_0}}{\sqrt{C_0/I_0} + 1}}\right). \tag{16}$$

With $\sqrt{\frac{C_0}{I_0}} = t$ the functional equation becomes

$$M(1,\ t) = \sqrt{\frac{2t}{t + 1}} \cdot M\left(1, \sqrt{\frac{2t}{t + 1}}\right). \tag{17}$$

Here, the idea suggests itself of repeating the functional relation several times:

$$M(1,\ t) = \sqrt{\frac{2t}{t + 1}} \cdot \sqrt{\frac{2\sqrt{2t/(t+1)}}{\sqrt{2t/(t+1)} + 1}} \cdot M\left(1, \sqrt{\frac{2\sqrt{2t/(t+1)}}{\sqrt{2t/(t+1)} + 1}}\right), \tag{18}$$

etc.

Specializing again to a quarter of the unit circle where $I_0 = 1/2$, $C_0 = 1$, $t = \sqrt{2}$, we obtain

$$M(I_1,\ C_0) = M\left(\sqrt{\frac{1}{2}},\ 1\right) = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{2\sqrt{2}}{\sqrt{2}+1}} \cdot \ \cdots$$

$$= \left(\frac{1}{2}\Big/\sqrt{\frac{1}{2}}\right) \cdot \left(1\Big/\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}\right) \cdot \left(1\Big/\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}}\right) \cdot \ \cdots$$

But

$$\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \ \cdots = \frac{2}{\pi}, \tag{19}$$

Viète's product! Thus we have

$$M(I_1,\ C_0) = M\left(\sqrt{\frac{1}{2}},\ 1\right) = \frac{\pi}{4} = \frac{1/2}{2/\pi}; \tag{20}$$

Viète's well-known infinite product for $2/\pi$ appears as the denominator in this representation.

It is therefore possible to derive both the arctan series (implying the Leibnizian series), and Viète's infinite product for $2/\pi$ from the double sequence. Two of the most important representations of the transcendental number π are implicitly contained in Gregory's construction!

This is, however, to consider the mathematics of 1667/1668 with the eyes of a mathematician of 1800, or of even later times if we take into account that Gauss with his ideas was penetrating like a pioneer into uncultivated wilderness. When the controversy between Gregory and Huygens broke out, infinite series had not yet been accepted, in fact had not even been used. For it was in the same year, 1668, that Mercator published his quadrature of the hyperbola by means of such a series, and this was the first example of such an approach that was made public. Newton, who at the same time was beginning to recognize the importance of infinite series as a new tool of mathematics, had not yet published any of his researches, and the Leibnizian series had not even been discovered yet.

In his fine characterization of Huygens' mathematical work, H. J. M. Bos [10] has emphasized that the Dutch mathematician was a geometer, not an analyst. Huygens mastered the classical methods in a superb way but never really grasped the new analytical methods of the calculus; he was a traditionalist in the best sense of the word, emphasizing the beauty and exactness of the classical methods.

Gregory, on the other hand, was trained in Italy in the school of Cavalieri, and was one of the wild young men who wanted to tear down the barriers of traditional mathematics at almost any price, who wanted to view hitherto uncultivated areas. Inspired by hopes for as yet unheard-of results, he freely introduced new methods while at times he neglected necessary care for details and exactness. Although, as we have known for nearly a century, Gregory's bold anticipation that π is not an "analytic" number was correct, it is fully understandable and justified that Huygens rejected his argumentation.

There is in fact a distinction to be made between Gregory's term "analytic" and our term "algebraic." The Scotsman's claim that a circle cannot be squared "analytically" is not exactly equivalent to the modern statement that π is a transcendental (i.e., nonalgebraic) number. Gregory's immediate assertion is less far reaching since his construction, in addition to the four elementary operations, involves only square roots. Gregory claimed the "terminatio" of the double sequence to be of a more complicated nature than any of the members of the sequence. Taken literally, this is equivalent to saying: the limit (i.e., the area of the circle) is not constructable by ruler and compass. This of course implies that the circular area is incommensurable with the square of the radius; in other words, that π is an irrational number (if the radius is taken to be a rational one).

In view of the fact, however, that Gregory seems to think not only of square roots but roots of an arbitrary order k, his assertion would seem to go a step further. Starting from the then still unshaken assumption that the solution of all algebraic equations can be expressed "analytically," he would doubtless have had in mind as limit a type of number lying outside this class. This then would have to be a nonalgebraic number in this special sense.

The application of Gaussian ideas opens the way to look at the problem from a later point of view. Gregory's double sequence, yielding an infinite series which turns out to be that of the arctan function, and an infinite product which is a special case of the product

$$\frac{\sin(2\alpha)}{2\alpha} = \cos\alpha \cdot \cos\frac{\alpha}{2} \cdot \cos\frac{\alpha}{2^2} \cdot \cos\frac{\alpha}{2^3} \cdot \cdots,$$

produces two of the central tools of 19th-century analysis. Both of these tools "transcend" the realm of algebraic equations; they both became indispensable for the investigation of transcendental problems and for such questions as Gregory had raised in 1667. Clearly he could not see the consequences that lay concealed in his construction. But he had an unerring sense of where they would lead, just as the young Gauss when he (at the age of 14!) began to investigate the arithmetic-geometric mean, could not have dreamed of the treasure he had begun to explore.

NOTES

1. Letter No. 1605 in [Huygens 1888-1950], Vol. 6 (1895), p. 154.

2. *Journal des Sçavans*, July 2, 1668, p. 361. Reprinted in [Huygens 1888-1950], Vol. 6, pp. 228-230, as letter No. 1647 from Huygens to Gallois; cf. also pp. 279-280, note 7.

3. *Philosophical Transactions*, No. 37, July 13/23, 1668. Reprinted in [Huygens 1888-1950], Vol. 6, pp. 240-243, as letter No. 1653 from Gregory to Oldenburg.

4. *Journal des Sçavans*, Nov. 12, 1668. Reprinted in [Huygens 1888-1950], Vol. 6, pp. 272-276, as letter No. 1669 from Huygens to Gallois.

5. Wallis to Brouncker, Nov. 14, 1668, published in [Huygens 1888-1950], Vol. 6, pp. 282-289, as No. 1672.

6. *Philosophical Transactions*, No. 44, Feb. 15/25, 1668/69. Reprinted in [Huygens 1888-1950], Vol. 6, pp. 306-311.

7. [Turnbull 1939], pp. 478-486: "James Gregory and Christiaan Huygens."

8. [Turnbull 1939], pp. 468-478: "On James Gregory's *Vera quadratura*."

9. Gauss' notes on the arithmetic-geometric mean, as published in [Gauss 1917], pp. 172-286, are very fragmentary. I am following Geppert's presentation in [Gauss 1927], pp. 27 ff.

10. [Bos et al. 1980], pp. 126-146: "Huygens and mathematics." See esp. p. 143.

REFERENCES

Bos, H. J. M., et al. 1980. *Studies on Christiaan Huygens*. Lisse: Swets & Zeitlinger.

Gauss, C. F. 1917. *Werke*. Zehnten Bandes erste Abteilung. Göttingen: Kgl. Gesellschaft der Wissenschaften; reprinted, Hildesheim/New York: Olms, 1973.

——— 1927. *Bestimmung der Anziehung eines elliptischen Ringes. Nachlass zur Theorie des arithmetisch-geometrischen Mittels und der Modulfunktion*. Übersetzt und herausgegeben von H. Geppert. Leipzig: Teubner (*Ostwald's Klassiker der exakten Wissenschaften*, Nr. 225).

Gregorius à St. Vincentio. 1647. *Opus geometricum quadraturae circuli et sectionum coni*. Antwerp.

Gregory, J. 1667. *Vera circuli et hyperbolae quadratura*. Padua; reprinted, Venice, 1668, as an appendix to *Geometriae pars universalis*.

——— 1668. *Exercitationes geometricae*. London. Excerpts are reprinted in [Huygens 1888-1950], Vol. 6, pp. 313-321, as No. 1684.

Heinrich, G. 1901. James Gregory's "Vera circuli et hyperbolae quadratura." *Bibliotheca Mathematica* (3) 2, 77-85.

Hofmann, J. E. 1950. Über Gregorys systematische Näherungen für den Sektor eines Mittelpunktkegelschnitts. *Centaurus* 1, 24-37.

Huygens, C. 1654. *De circuli magnitudine inventa.* Leiden; reprinted in [Huygens 1888-1950], Vol. 12.

——— 1888-1950. Oeuvres complètes. 22 vols. La Haye: Nijhoff.

Turnbull, H. W. (ed.) 1939. *James Gregory Tercentenary Memorial Volume.* London: Bell.